

Rainbow cycles for families of matchings

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Abstract

Given a graph G and a coloring of its edges, a subgraph of G is called *rainbow* if its edges have distinct colors. The *rainbow girth* of an edge coloring of G is the minimum length of a rainbow cycle in G . A generalization of the famous Caccetta-Häggkvist conjecture (CHC), proposed by the first author, is that if G has n vertices, G is n -edge-colored and the size of every color class is k , then the rainbow girth is at most $\lceil \frac{n}{k} \rceil$. In the only known example showing sharpness of this conjecture, that stems from an example for the sharpness of CHC, the color classes are stars. This suggests that in the antipodal case to stars, namely matchings, the result can be improved. Indeed, we show that the rainbow girth of n matchings of size at least 2 is $O(\log n)$, as compared with the general bound of $\lceil \frac{n}{2} \rceil$.

1 Introduction

The *girth* $g(G)$ of a graph G is the minimal length of a cycle in it. Given a graph G and a (not necessarily proper) coloring of its edges, a subgraph of G is called *rainbow* if its edges have distinct colors. The *rainbow girth* $rg(G)$ of G (actually, of its edge-coloring) is the minimum length of a rainbow cycle. All the above definitions apply to both the directed and undirected cases, where in the directed case the cycles are assumed to be directed.

The famous Caccetta-Häggkvist conjecture (below - CHC) is that any digraph G on n vertices satisfies $g(G) \leq \lceil \frac{n}{\delta^+(G)} \rceil$, where $\delta^+(G)$ is the minimal out-degree of a vertex. A possible generalization was proposed by the first author in [2]. It is that for any n -vertex undirected graph G and edge coloring of G with n colors such that each color class has size at least r , $rg(G) \leq \lceil n/r \rceil$. The sharpness of this conjecture is shown by the same example showing the sharpness of CHC. In it the i -th color class is the “star” centered at i consisting of the edges $\{i, i+1\}, \{i, i+2\}, \dots, \{i, i+r\}$ for $i = 1, 2, \dots, n$ (indices taken modulo n). Devos et. al. [4] proved this conjecture for $r = 2$, and in [1] the harmonic mean version was proved when all sets are of size 1 or 2.

The above “stars” example is the only extremal one known for the rainbow girth conjecture, so it is natural to guess that in the antipodal case, when the sets of edges are matchings, the conjecture can be strengthened. (“Antipodal” is with respect to the covering number, which is 1 in a star, and the number of edges in a matching.) Indeed, a simple observation is that for the first open case of CHC, that of $\delta^+ = \frac{n}{3}$ (in which a directed triangle is conjectured to exist) the rainbow undirected version is trivial when the sets of edges are matchings. In this case, it is enough that the arithmetic mean of the sizes of the sets is larger than $\frac{n}{4}$, because then by Mantel’s theorem there exists a triangle contained in the union of the sets, and if the sets are matchings then a triangle is necessarily rainbow.

Our main result is a corroboration of the intuition that sets of matchings have small rainbow girth. We prove that the rainbow girth of n matchings of size at least 2 is $O(\log n)$, as compared with the general bound of $\lceil \frac{n}{2} \rceil$.

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2 Rainbow cycles for matchings

Theorem 2.1. *There exists a constant C such that for any n -vertex graph G and edge coloring of G with n colors, if each color class is a matching of size 2, then the rainbow girth of G is at most $C \log n$.*

Remark 2.2. *The assumption that G is a graph and not a multigraph breeds no loss of generality, since a double edge is a rainbow digon, meaning that the rainbow girth is 2.*

A key ingredient in the proof is a result by Bollobás and Szemerédi [3] on the girth of sparse graphs.

Theorem 2.3. *For $n \geq 4$ and $k \geq 2$, every n -vertex graph with $n+k$ edges has girth at most*

$$\frac{2(n+k)}{3k}(\log k + \log \log k + 4).$$

The logarithms are on base 2. Theorem 2.1 will follow from this result, and the following:

Theorem 2.4. *There exist universal $c, \delta > 0$, such that for any large enough n , given an n -vertex graph G and an edge coloring of G such that each color class is a matching of size 2, there exists a subset S of $V(G)$ of size at most cn containing the edges of a rainbow set of edges of size at least $(c+\delta)n$.*

Note that the last condition entails $c + \delta \leq 1$. Once this is proved, Theorem 2.1 will follow by applying Theorem 2.3 with $k = \delta n$.

We shall use the following two well-known concentration inequalities.

Theorem 2.5 (Chernoff). *Let X be a binomial random variable $\text{Bin}(n, p)$. For any $0 < \epsilon < 1$, we have*

$$\mathbb{P}(X \geq (1 + \epsilon)\mathbb{E}X) \leq \exp(-\epsilon^2 \mathbb{E}X/3).$$

Theorem 2.6 (Chebyshev's inequality). *Let X be a random variable. For any $\epsilon > 0$, we have*

$$\mathbb{P}(|X - \mathbb{E}X| \geq \epsilon \mathbb{E}X) \leq \text{Var } X / (\epsilon \mathbb{E}X)^2,$$

where $\text{Var } X$ is the variance of X .

Proof of Theorem 2.4. Denote the i -th color class (which, by our assumption, consists of two disjoint edges) by M_i . Our assumption that G is a graph and not a multigraph implies that the matchings M_i are disjoint.

Let S be a random vertex subset of $V(G)$, in which each vertex of G is included independently with probability p , for some constant p close to 1, to be determined later. Then

$$\mathbb{E}|S| = np. \tag{1}$$

For $1 \leq i \leq n$ let X_i be the indicator random variable that an edge of color i is contained in S , i.e., $X_i := \mathbb{1}_{\{\text{an edge of color } i \text{ is contained in } S\}}$. By inclusion-exclusion we have

$$\mathbb{E}X_i = 2p^2 - p^4.$$

Let

$$X := \sum_{i=1}^n X_i. \tag{2}$$

We have

$$\mathbb{E}X = n(2p^2 - p^4). \tag{3}$$

By Theorem 2.5, for fixed $0 < p < 1$ and $\epsilon > 0$, we have

$$\mathbb{P}(|S| \geq (1 + \epsilon)np) \leq \exp(-\Omega(n)). \tag{4}$$

So, with probability tending to 1 as n tends to infinity,

$$|S| \leq (1 + \epsilon)np. \tag{5}$$

Writing $p - (2p^2 - p^4) = p(p-1)(p^2+p-1)$, we see that for $\frac{-1+\sqrt{5}}{2} < p < 1$, we have $p < 2p^2 - p^4$, yielding the separation between $\mathbb{E}X$ and $\mathbb{E}|S|$, needed for the application of Theorem 2.3.

Claim 2.7. *There exist constants $p \in (\frac{-1+\sqrt{5}}{2}, 1)$ and $\epsilon > 0$ such that*

$$(1 - \epsilon)(2p^2 - p^4) - (1 + \epsilon)p \geq [(2p^2 - p^4) - p]/3 > 0, \quad (6)$$

and with probability tending to 1 as n tends to infinity,

$$X \geq (1 - \epsilon)n(2p^2 - p^4). \quad (7)$$

To prove (7) (with p, ϵ chosen below), we shall apply Chebyshev's inequality. For this purpose we have to estimate $\text{Var } X$. With a look at (2), we have

$$\text{Var } X = \mathbb{E}X^2 - (\mathbb{E}X)^2 = \sum_{i,j} (\mathbb{E}X_i X_j - \mathbb{E}X_i \mathbb{E}X_j). \quad (8)$$

Note that if the edges in the color classes i, j are vertex-disjoint, then X_i and X_j are independent and $\mathbb{E}X_i X_j - \mathbb{E}X_i \mathbb{E}X_j = 0$.

Since the matchings M_j are disjoint, for every $i \in [n]$ at most $6 = \binom{4}{2}$ matchings M_j can have an edge contained in $\bigcup M_i$. This means that there exist at most $2 \cdot 6n$ pairs (M_i, M_j) such that an edge from M_j is contained in $\bigcup M_i$, or vice versa. Thus the contribution of such pairs to $\text{Var } X$ is at most $O(n)$, which is $o((\mathbb{E}X)^2)$ as $\mathbb{E}X = \Omega(n)$ (see (3)).

Apart from vertex-disjointness, there are two more possible forms of $M_i \cup M_j$:

- I. three connected components: one 2-path and two disjoint edges, or
- II. two vertex-disjoint 2-paths.

Examine Case I. Let $M_i = \{a, b\}$, where $a = xy$, $b = uv$, and let $M_j = \{c, d\}$, where $c = xz$ and $d = st$. Then the contribution of M_i, M_j to $\mathbb{E}X_i X_j$, namely the probability that $X_i = X_j = 1$, is the sum over the following settings:

- $a, c \subseteq S$ and $b, d \not\subseteq S$, for which the probability is $p^3(1 - p^2)^2$.
- $a, d \subseteq S$ and $b, c \not\subseteq S$, for which the probability is $p^4(1 - p)(1 - p^2)$.
- $b, c \subseteq S$ and $a, d \not\subseteq S$, for which the probability is $p^4(1 - p)(1 - p^2)$.
- $b, d \subseteq S$ and $a, c \not\subseteq S$, for which the probability is $p^4(p(1 - p)^2 + (1 - p))$ (explanation: p^4 is the probability that $u, v, s, t \in S$, $p(1 - p)^2$ is the probability that $x \in S$ and $y, z \notin S$ and $1 - p$ is the probability that $x \notin S$).
- $a, b, c \subseteq S$ and $d \not\subseteq S$, for which the probability is $p^5(1 - p^2)$.
- $a, b, d \subseteq S$ and $c \not\subseteq S$, for which the probability is $p^6(1 - p)$.
- $a, b, c, d \subseteq S$, for which the probability is p^7 .

Therefore in case I,

$$\begin{aligned} \mathbb{E}X_i X_j &= p^3(1 - p^2)^2 + p^4(1 - p)(1 - p^2) + p^4(1 - p)(1 - p^2) \\ &\quad + p^4(p(1 - p)^2 + (1 - p)) + p^5(1 - p^2) + p^6(1 - p) + p^7 \\ &= p^3 + 3p^4 - 3p^5 - 3p^6 + 3p^7. \end{aligned}$$

In case II, analyzing in a similar way, we have

$$\begin{aligned} \mathbb{E}X_i X_j &= 2 \left(p^3(p(1 - p)^2 + (1 - p)) + p^4(1 - p)^2 + p^5(1 - p) \right) + p^6 \\ &= 2p^3 + 2p^4 - 6p^5 + 3p^6. \end{aligned}$$

Note that $\mathbb{E}X_i \mathbb{E}X_j = (2p^2 - p^4)^2 = 4p^4 - 4p^6 + p^8$. Let $f(p) := \mathbb{E}X_i X_j$ and $g(p) := \mathbb{E}X_i \mathbb{E}X_j$. Then, in Case I, $f(1) = g(1) = 1$, while $f'(1) = 3, g'(1) = 0$, so for p close enough to 1, $f(p) < g(p)$, namely $\mathbb{E}X_i X_j < \mathbb{E}X_i \mathbb{E}X_j$. A similar calculation shows the same inequality in Case II. Summing, we have

$$0 \leq \text{Var } X = o((\mathbb{E}X)^2).$$

Then by Theorem 2.6, taking p close enough to 1 and $\epsilon > 0$ small enough so as to satisfy (6), we get

$$\mathbb{P}\left(X \leq (1 - \epsilon)n(2p^2 - p^4)\right) \leq \text{Var } X / (\epsilon \mathbb{E}X)^2 = o(1). \quad (9)$$

This proves Claim 2.7.

Combining (9) with (4), we have that with probability tending to 1 as n tends to ∞ ,

$$|S| \leq (1 + \epsilon)np < (1 - \epsilon)n(2p^2 - p^4) \leq X.$$

Theorem 2.4 now follows from (6), upon taking $c := (1 + \epsilon)p$ and $\delta := [(2p^2 - p^4) - p]/3$. \square

In the original CHC, each set of edges is a star associated with a vertex, hence it was natural that there are n sets. In the rainbow undirected case there is no natural choice of the number of sets. Indeed, the main theorem is valid also with fewer than n sets.

Theorem 2.8. *For any constant $\alpha > \frac{3\sqrt{6}}{8}$, there exists a constant C such that for any n -vertex graph G and edge coloring of G with αn colors, if each color class is a matching of size 2, then the rainbow girth of G is at most $C \log n$.*

Claim 2.9. *There are $O(n)$ many pairs of i, j of types I and II.*

This follows by examining the arguments proving Theorem 2.4. For p large enough if the pair M_i, M_j is of type I or II then $\mathbb{E}X_i X_j - \mathbb{E}X_i \mathbb{E}X_j < 0$, so the claim follows from the facts that there are $O(n)$ pairs of the other types (in which, as recalled, $\bigcup M_i \cap \bigcup M_j = \emptyset$, or an edge of one matching is contained in the union of the other), and $\text{Var } X \geq 0$.

Corollary 2.10. $\text{Var } X = o((\mathbb{E}X)^2)$ for any constant $0 \leq p \leq 1$.

Proof of Theorem 2.8. By the corollary, for the argument in the proof of Theorem 2.4 to work with αn colors, we need to have $p < \alpha(2p^2 - p^4)$ for some p (which would imply separation between $\mathbb{E}|S| = pn$ and $\mathbb{E}X = \alpha n(2p^2 - p^4)$).

Thus we need to find a minimal $0 < \alpha_0 < 1$ such that $p = \alpha_0(2p^2 - p^4)$ for some $p \in (0, 1)$. This will happen when the two curves $y(p) = p$ and $y(p) = \alpha_0(2p^2 - p^4)$ are tangent, namely $(\alpha_0(2p^2 - p^4))' = p' = 1$. The above two constraints and $\alpha_0 p \neq 0$ imply that $\alpha_0 = \frac{3\sqrt{6}}{8}$ and the only feasible p is $\frac{\sqrt{6}}{3}$. \square

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