The Rigidly Rotating Disk as the "Missing Link" in the History of General Relativity*

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1. Introduction

Working with the Einstein Archive at the Institute for Advanced Study has given me a chance to become familiar with some of the material in this most extensive repository of documents on the life and activities of Albert Einstein, collected indefatigably over the last quarter of a century by the Trustees of his Estate, Dr. Otto Nathan and Miss Helen Dukas; and organized by Miss Dukas, the Archivist of the collection, whose memory is undoubtedly its most important single resource. It has also made me realize how much the material held in the Archive can contribute towards the study of many problems in the history of modern physics, to say nothing of many cultural, social, and political topics.

As a small example, I shall discuss the question of the relativistic rigidly rotating disk, a topic that has been the subject of extensive—and intensive—discussion from the early days of the special theory of relativity to the present.[1] An examination of Einstein's treatment of this problem is of interest not only because it shows his way of treating the issues involved, but because it seems to provide a "missing link" in the chain of reasoning that led him to the crucial idea that a nonflat metric was needed for a relativistic treatment of the gravitational field.

2. Einstein's Treatment of the Rotating Disk

Einstein's first mention of rigidly rotating bodies that I have located in the Archive is in a letter of September 29, 1909, to Sommerfeld:

> The treatment of the uniformly rotating rigid body seems to me to be of great importance on account of an extension of the relativity principle to uniformly rotating systems along analogous lines of thought to those that I tried to carry out for uniformly accelerated translation in the last section of my paper published in the *Zeitschrift für Radioaktivität* (EA 21–377).[2]

Einstein is referring here to his first published attempt in 1907 to develop a relativistic treatment of gravitation based on the equivalence principle applied to a spatially uniform static gravitational field (Einstein 1907).

The occasion for Einstein's comment was probably the discussion of Born's paper (1909) at the Salzburg meeting of the German Society of Scientists and Physicians, which took place September 21–25, 1909. Born had presented his definition of rigid motions (and thus of rigid bodies, insofar as they are capable of existing) in special relativity. Sommerfeld (1909) had commented on Born's talk, and Born in a later paper (1910) noted that he and Einstein had discussed the rigid-body problem at Salzburg, and were puzzled "that a [rigid] body at rest can never be brought into uniform rotation"; this problem was discussed almost simultaneously by Paul Ehrenfest in a paper (1909) submitted September 29—the same day as Einstein's letter to Sommerfeld—and soon became known as "Ehrenfest's paradox." In spite of the importance he attached to the problem, and the intense discussion occasioned by Ehrenfest's paper,[3] Einstein published nothing directly on the question during the next years. His only contribution to the discussion was an answer to one of the points raised by Varičak (1911) in a comment on Ehrenfest's paradox.[4] Einstein's note (1911) made no reference to the rotating-disk problem but confined itself to rebutting Varičak's aspersions on the "reality" of the Lorentz contraction.

Einstein's first published reference to the rigidly rotating disk is a hesitant one. It occurs in the first of two papers on static gravitational fields written in 1912 during his stay in Prague (Einstein 1912a, 1912b). It dates from February 1912 and begins by reviewing his previous work on the uniformly accelerating coordinate system, pointing out that

> Such a system K, according to the equivalence principle, is strictly equivalent to a system at rest in which a matter-free static gravitational field of a certain kind exists. Let spatial measurements in K be made with measuring rods which—when compared with each other at rest at some point of K—have the same length; assume that the theorems of [Euclidean] geometry are valid for lengths measured in this way, and thus also for the relationship between the coordinates x, y, z and other lengths. This stipulation is not automatically permissible, but contains physical assumptions which ultimately could prove to be invalid. For example, they most probably do not hold in a uniformly rotating system, in which, on account of the Lorentz contraction, the ratio of the circumference of a circle to its diameter would have to differ from π using our definition of length. The measuring rod, as well as the coordinate axes, are to be treated as rigid bodies. This is permissible in spite of the fact that, according to [special] relativity theory, rigid bodies cannot really exist. For one can imagine the rigid measuring body replaced by a large number of small non-rigid bodies so aligned alongside each other that they do not exert any pressure forces on each other since each is separately held in place.

The tentative nature of his conclusions reflects Einstein's puzzlement during this period over the problem of the relationship between coordinates and measurements with rods and clocks, a point to which we shall later return.[5]

There are references to rotating frames of reference in several of Einstein's papers on the developing general theory of relativity,[6] as well as in the correspondence;[7] but the context is Einstein's interpretation of the equivalence principle: the explanation of the inertial forces occurring in such frames as equivalent to gravitational forces, and there is no direct reference to the rotating-disk problem. The next time that the rotating-disk argument occurs in Einstein's writings—this time without the tentative note—is in the 1916 review paper in which he presented the final version of the general theory, together with various arguments for it (Einstein 1916). Although the paper is well known and easily accessible, I quote the paragraph in full for the sake of completeness:

> In a space which is free of gravitational fields we introduce a Galilean system of reference $K(x, y, z, t)$, and also a system of co-ordinates $K'(x', y', z', t')$ in uniform rotation relatively to K. Let the origins of both systems, as well as their axes of Z, permanently coincide. We shall show that for a space-time measurement in the system K' the above [special relativistic] definition of the physical meaning of lengths and times cannot be maintained. For reasons of symmetry it is clear that a circle around the origin in the X, Y plane of K may at the same time be regarded as a circle in the X', Y' plane of K'. We suppose that the circumference and diameter of this circle have been measured with a unit measure infinitely small compared with the radius, and that we have the quotient of the two results. If this experiment were performed with a measuring-rod at rest relatively to the Galilean system K, the quotient would be π. With a measuring rod at rest relatively to K', the quotient would be greater than π. This is readily understood if we envisage the whole process of measuring from the "stationary" system K, and take into consideration that the measuring rod applied to the periphery undergoes a Lorentzian contraction, while the one applied along the radius does not. Hence Euclidean geometry does not apply to K'. The notion of co-ordinates defined above, which presupposes the validity of Euclidean geometry, therefore breaks down in relation to the system K'. So, too, we are unable to introduce a time corresponding to physical requirements in K', indicated by clocks at rest relatively to K'. To convince ourselves of this impossibility, let us imagine two clocks of identical constitution placed, one at the origin of co-ordinates, and the other at the circumference of the circle, and both envisaged from the "stationary" system K. By a familiar result of the special theory of relativity, the clock at the circumference—judged from K—goes more slowly than the other, because the former is in motion and the latter at rest. An observer at the common origin of co-ordinates, capable of observing the clock at the circumference by means of light, would therefore see it lagging behind the clock beside him. As he will not make up his mind to let the velocity of light along the path in question depend explicitly on the time, he will interpret his observations as showing that the clock at the circumference "really" goes more slowly than the clock at the origin. So he will be obliged to define time in such a way that the rate of a clock depends upon where the clock may be.
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> We therefore reach this result:—In the general theory of relativity, space and time cannot be defined in such a way that the differences of the spatial co-ordinates be directly measured by the unit measuring-rod, or differences in the time co-ordinate by a standard clock. (Einstein 1916, pp. 115–117)

Note that the "rigidly rotating disk" is not actually referred to in these considerations. However, that Einstein had it in mind—or at least was not averse to its consideration in this context—is made clear by the expanded discussion of the topic that he gave in his *Relativity* book of 1916 (Einstein 1917), chapter 23 of which is devoted to the topic "Behavior of Clocks and Measuring-Rods on a Rotating Body of Reference." In this amplified discussion, he adds: "In order to fix our ideas, we shall imagine K' to be in the form of a plane circular disk, which rotates uniformly in its own plane about its centre," and thereafter phrases his discussion in terms of the disk. He also adds an element not made explicit in his previous discussion, asserting that an observer at rest on the disk is entitled to regard "the force acting on himself, and in fact on all other bodies which are at rest relative to the disk ... as the effect of a gravitational field." Thus, he links up his treatment of the rotating disk with his earlier treatment of rotating reference frames; but we shall continue to confine ourselves to the disk aspect of his discussion.

He draws the conclusion "that the propositions of Euclidean geometry cannot hold exactly on the rotating disk, nor in general in a gravitational field, at least if we attribute the length 1 to the [measuring] rod in all positions and in every orientation." He immediately follows this discussion with a discussion in chapter 24 of the "Euclidean and Non-Euclidean Continuum," and in chapter 25 with a discussion of the use of "Gaussian Co-ordinates" to treat non-Euclidean continua mathematically. Since the book is still in print and easily accessible, I shall omit this long discussion.

In *The Meaning of Relativity*, based upon his 1921 Princeton lectures, Einstein again gives a similar discussion of the rotating disk. This time I shall quote his conclusions, since they briefly summarize the material in chapters 23, 24, and 25 just mentioned:

> Space and time, therefore, cannot be defined with respect to K' as they were in the special theory of relativity with respect to inertial systems. But, according to the principle of equivalence, K' may also be considered as a system at rest, with respect to which there is a gravitational field (field of centrifugal force, and force of Coriolis). We therefore arrive at the result: the gravitational field influences and even determines the metrical laws of the space-time continuum. If the laws of configuration of ideal rigid bodies are to be expressed geometrically, then in the presence of a gravitational field the geometry is not Euclidean.
>
> The case that we have been considering is analogous to that which is presented in the two-dimensional treatment of surfaces. It is impossible in the latter case also to introduce co-ordinates on a surface (e.g., the surface of an ellipsoid) which have a simple metrical significance, while on a plane the Cartesian co-ordinates, x_1, x_2, signify directly lengths measured by a unit measuring rod. Gauss overcame this difficulty, in his theory of surfaces, by introducing curvilinear co-ordinates which, apart from satisfying conditions of continuity, were wholly arbitrary, and only afterwards these co-ordinates were related to the metrical properties of the surface. In an analogous way we shall introduce in the general theory of relativity arbitrary co-ordinates, x_1, x_2, x_3, x_4, which shall number uniquely the space-time points, so

that neighboring events are associated with neighboring values of the co-ordinates; otherwise, the choice of co-ordinates is arbitrary. We shall be true to the principle of relativity in its broadest sense if we give such a form to the laws that they are valid in every such four-dimensional system of co-ordinates, that is, if the equations expressing the laws are co-variant with respect to arbitrary transformations.

The most important point of contact between Gauss's theory of surfaces and the general theory of relativity lies in the metrical properties upon which the concepts of both theories, in the main, are based. (Einstein 1922a, pp. 60–61)

The topics are presented in the same order as in the *Relativity* book (Einstein 1917): first the discussion of the disk, leading to the conclusion that a non-Euclidean geometry holds on the disk, and therefore space (and time) coordinates cannot be given a direct physical meaning as in the special theory. The two-dimensional Gaussian theory of curved surfaces is recalled, based upon the possibility of introducing entirely arbitrary coordinate systems, and then using the metric tensor to describe the metrical properties of the surface. The analogy is then drawn with the use of arbitrary space-time coordinates and the metric tensor to characterize the gravitational field mathematically. Later on, I shall comment upon the possible historical significance of this order of presentation. But first I shall finish the account of Einstein's discussions of the disk.

In *The Evolution of Physics*, his popular book written with Leopold Infeld, Einstein again reverts to the example of the rotating disk to show the necessity to introduce non-Euclidean geometry if one wants to generalize the principle of relativity so that it applies to noninertial frames of reference (Einstein and Infeld 1938, pp. 226–234).

I have not found any other discussion of the rotating disk in Einstein's published writings (in a far from exhaustive search, I must add); but there are a number of letters in which Einstein refers to the subject, giving a much more detailed discussion than in any of the printed sources, and explicitly replying to some of the objections that were offered to his treatment. He also makes several comments on the importance of the problem for his development of the general theory of relativity, which will be of great value for my later historical discussion.

The first such letter I have come across is one to Joseph Petzoldt, the well-known positivist philosopher, and the author of several early essays claiming special relativity theory as a triumph of the positivistic approach (see, for example Petzoldt 1912). On July 26, 1919, Petzoldt wrote Einstein a letter (EA 19-055) in which he raised the objection to Einstein's treatment of the rotating disk (an objection that will not be unknown to the connoisseur of the literature on this subject) that the Lorentz contraction of the rotating rods on the circumference implies that the circumference of the rotating disk should be *shorter* than 2π times the radius. Einstein replied at length in a letter of August 19, 1919, which has been published in German; I translate it here:

As concerns the rotating disk, I cannot agree with you at all. It is well to remark that a rigid circular disk at rest must break up if it is set into rotation, on account of the Lorentz contraction of the tangential fibers and the noncontraction of the radial ones. Similarly, a rigid disk in rotation (produced by casting) must explode as a consequence of the inverse changes in length, if one attempts to bring it to the rest state. If you fully take into account this state of affairs, your paradox vanishes.

Now you believe that a rigidly rotating circular line must have a circumference that is less than $2r\pi$ because of the Lorentz contraction. The basic error here is that you instinctively set the radius r of the rotating circular line equal to the radius r_0 that the circular line has in the case when it is at rest. This however, is not correct; because of the Lorentz contraction rather $2\pi r = 2\pi r_0\sqrt{1-(v^2/c^2)}$.

The treatment of the metric of the circular disk runs as follows in detail. Let U_0 be the circumference, r_0 the radius of the rotating disk, considered from the standpoint of K_0 [that is, the rest frame]; then, on account of ordinary Euclidean geometry,

$$U_0 = 2\pi r_0 \tag{1}$$

U_0 and r_0 naturally are to be thought of as measured with nonrotating measuring rods, i.e., at rest relative to K_0.

Now let me imagine corotating measuring rods of rest length 1 laid out on the rotating disk, both along a radius as well as the circumference. How long are these, considered from K_0? Let us imagine, in order to make this clearer to ourselves, a "snapshot" taken from K_0 (definite time t_0). On this snapshot the radial measuring rods have the length 1, the tangential ones, however, the length $\sqrt{1-(v^2/c^2)}$. The "circumference" of the circular disk (considered from K) is nothing but the number of tangential measuring rods that are present in the snapshot along the circumference, whose length considered from K_0 is U_0. Therefore,

$$U = U_0/\sqrt{1-(v^2/c^2)}. \tag{2}$$

On the other hand, obviously

$$r = r_0 \tag{3}$$

(since the snapshot of the radial unit measuring rod is just as long as that of a measuring rod at rest relative to K_0).

Therefore, from (2), (3), $U/r = U_0/r_0(1/\sqrt{1-(v^2/c^2)})$, or on account of (1) $= 2\pi/\sqrt{1-(v^2/c^2)}$.

You make the analogous error for clocks as for measuring rods. The *rotating observer notes very well that, of his two equivalent clocks, that placed on the circumference runs slower than that placed at the center.* We again prove this by considering the entire process from K_0. Let U_z be the clock at the center, U_p the one at the periphery. Considered from K_0, U_p goes slower than U_z; a corotating observer placed next to U_z therefore also sees U_p as going slower than U_z. For it is clear that—judged from K_0—the time between the occurrence of a position of the hands of the clock and its perception by our observer is constant (independent of the time). I hope this explanation will suffice. (EA 19-069; Thiele 1971, pp. 71–73)

Apparently, Petzoldt did not find this explanation fully satisfactory, and in a missing letter must have objected to the introduction of rigid bodies into the argument (again, an objection not unknown to the connoisseur). Einstein replied in a second letter, of August 23, 1919, which has also been published in the original German:

> I also think that only a personal discussion can produce real clarity. I request that you therefore visit me soon (after making an appointment by telephone). In the meantime, the following on the matter: I know quite well, naturally, that rigid bodies cannot exist according to relativity theory. But one can proceed with advantage as if such did exist; i.e., it is a question of an idealization that can be applied in certain considerations without any contradiction. The considerations of my letter are to be understood in this sense.
>
> You have incorrectly set the radius of the rotating "rigid" circular line equal to r_0. Because the circumference, *thought of as materialized by itself*, contracts because of the Lorentz transformation. It would be otherwise, if only the *radii* were thought of as materialized, *but not the tangential connections of their endpoints.*
>
> What you say about peripheral measuring rods and clocks is quite untenable. It is a question of the unjustified taking over of results of special relativity to accelerated reference systems (relative to the inertial systems). Freundlich and Schlick are absolutely correct here. By your sort of reasoning one could just as well conclude that every light ray must propagate rectilinearly with respect to an arbitrary rotating system, etc. Your misunderstanding is quite fundamental. (EA 19-072; Thiele 1971, p. 73)

As late as 1951 Einstein again gave a detailed discussion of the disk, in his draft reply (EA 25-482) to a letter from an Australian medical student named Leonard Champion, who had been teaching himself general relativity but could not find anyone on the staff of Melbourne University able to answer all of his questions.[8] In particular, he had run across the account of the rotating disk by Whittaker (1949), who mentioned that Lorentz and Eddington regarded the geometry of the disk as Euclidean, while others, including Einstein, took it as non-Euclidean. Attempting to resolve the conflict between the sources, Champion made a calculation that amounted essentially to taking the metric of the disk to be given by the line element orthogonal to the world lines of the disk; i.e.,

$$d\sigma^2 = g_{ij} - g_{0i}g_{0j}/g_{00},$$

where the world lines of the points of the disk are given by $x^i = \text{const}$ ($i = 1, 2, 3$). Einstein's reply agreed fully with Champion's calculation. He stated that one had to assume the existence of rigid infinitesimal rods, which implies that if two such rods once agreed in length, when compared, they would always do so, no matter what sort of gravitational field each might afterwards have been subjected to; and a similar assumption must be made for clocks. That is, physical objects that measure the metrical interval are assumed to exist. (Einstein points out that his assumption could be wrong, even though the gravitational field equations were correct.) It follows that the length of an elementary measuring rod is the orthogonal interval between the world lines of the endpoints of the rod.

He then states that he does not know what Eddington meant by claiming the geometry on the rotating disk is flat. While four-dimensional Minkowski space is naturally flat, no matter what coordinates are used, this is not the case for the geometry of the disk as measured with measuring rods rotating with the disk.

He points out that to set up a rigidly rotating disk one would first have to melt a disk at rest, then set the molten disk into rotation and solidify it while it rotates. He admits that there are not really any completely rigid bodies, since if there were one could signal with superluminal velocities; but he maintains that the use made of rigid bodies in his argument seems justified. He states that this example of the disk was of "decisive importance" to him in setting up the general theory of relativity because it showed that a gravitational field (here equivalent to the centrifugal field) causes non-Euclidean arrangements of measuring rods, and thus compelled a generalization of Euclidean space. He emphasizes that the behavior of the rotating measuring rods can be obtained from special-relativistic considerations, since everything is considered from the nonrotating frame of reference.

In this letter Einstein thus brings together in summary form all of his considerations on the rigidly rotating disk, together with his answers to many objections to his treatment.

Although this paper is primarily historical in its aim, it is perhaps worth noting one epistemological feature of Einstein's argument, because it is of some importance for the historical discussion. Einstein sees the argument for the necessity of non-Euclidean metrical relations on the rigidly rotating disk to be based upon three premises:

1. Special relativity holds in a global inertial frame, in which no gravitational field is present.
2. Any coordinate system may be used, and indeed not only mathematically, but may be interpreted as a physical frame of reference, provided that the appropriate gravitational (cum-inertial) field is introduced.
3. A small measuring rod does not change its length in any gravitational field.

The first assumption represents Einstein's conviction that special relativity theory retained its validity within any gravitational theory as the important limiting case in which no noninertial gravitational field occurs. The second assumption is finally embodied in general relativity in the postulate that the metric tensor is the appropriate mathematical representation of the gravitational field potentials. The third assumption, in the context of the metric interpretation of the second, gives physical significance to the metrical interval. That the third assumption really is an independent one is a point that Einstein emphasized a number of times.

In addition, he makes a most significant remark for the history of the development of general relativity, about the importance of his considerations in convincing him of the need to go over to non-Euclidean geometries in his treatment of the gravitational field. While I have not found any discussion of such a role for the rotating disk in Einstein's published writings on the origins of the general theory of relativity (Einstein 1921, 1933, 1949), it is not the first time that the claim is made in his correspondence. In a letter (EA 26-351) presumed to date from the winter of 1939–40 to Hyman Levy, the English Marxist mathematician and author of a number of books on modern science and philosophy for popular audiences, Einstein comments on Levy's latest book (Levy 1939). After stating how pleased he was with much of the book, he recommends that Levy correct one glaring error in later editions. Levy had stated on page 595 that observers on the disk would verify Euclidean geometry. Einstein points out that just the opposite is the case, and adds that it was just the recognition that non-Euclidean geometry holds on the rotating disk which convinced him, at the time he was working on his gravitation theory, that Euclidean geometry could not hold for rigid bodies in the presence of a gravitational field.

There are other references to the rotating disk problem in the correspondence; but we now have essentially all of Einstein's basic ideas connected with the problem. I will try to use these ideas to help solve a problem in the history of general relativity.

3. The Rotating Disk as a "Missing Link"

In his discussion of the development of general relativity in the "Autobiographical Notes," Einstein points out that the significance of the equivalence principle in requiring a generalization of the special theory was clear to him in 1908. He then adds

> Why were another seven years required for the construction of the general theory of relativity? The main reason lies in the fact that it is not so easy to free oneself from the idea that co-ordinates must have an immediate physical meaning. (Einstein 1949, p. 67)

In the context of the development of the general theory of relativity as a theory of gravitation (leaving aside the question of possible generalized unified field theories), I think it is clear that what is meant is that only the coordinates-cum-metric tensor in some coordinate system have a physical meaning.

In trying to trace Einstein's journey from the special to the general theory, the following difficulty presents itself.[9] In the papers up to and including those published in 1912, there is no mention of the need for a nonflat space-time, much less of the metric tensor as mathematical representation of the gravitational field. Yet the first paper of 1913 presents us with a full-fledged argument for the representation of the gravitational field by $g_{\mu\nu}$, together with the development of four-dimensional tensor analysis on a Riemannian manifold, the Riemann tensor, etc. (Einstein and Grossmann 1913). Of course, the problem of the correct field equations for the metric tensor was not resolved until late in 1915; but once the crucial step of the correct mathematical description of the gravitational field had been taken, it was only a matter of time until the right field equations were found. I shall argue that the consideration of the rotating disk is a "missing link" in the crucial developments which must have taken place in late 1912.

By the end of March 1912, Einstein had completed his work on the static gravitational field, which he treated by introducing the concept of a variable speed of light, which played the role of gravitational potential (Einstein 1912a, 1912b). At this point he even felt compelled to give up the symmetry between space and time that had characterized the special theory of relativity, especially in Minkowski's formulation of space-time.[10] He had also arrived at the conclusion that his previous formulations of the equivalence principle only held locally. Rather than quote his papers on this question, we shall summarize his account in a letter to Ehrenfest, since this letter also takes us an important step forward beyond the papers. In this letter (EA 9-333; undated, but marked by Ehrenfest as received July 7, 1912), he states that his papers on the static gravitational field (Einstein 1912a, 1912b) indicate that the equivalence hypothesis can only hold for infinitesimally small fields. He notes that his discussion of static gravitational fields corresponds to the electrostatic case in electromagnetic theory; while what he calls "the general static case" would include the analogue of magnetostatic fields. He mentions the "rotating ring" as an example of a system that will generate such a nonstatic but time-independent field.

Thus, by July 1912 Einstein, pursuing his step-by-step approach to the problem, was ready to attack what he called the "general static case"—what we would today call the case of stationary gravitational fields.[11] This presumably led him to look again at the rotating-disk problem—the simplest case of a stationary gravitational field—which he had already tentatively discussed early in 1912, as mentioned earlier. There exist some notes, unfortunately undated, in Einstein's notebooks from roughly this period that may preserve evidence of this study. At this point the argument quoted herein could have occurred to him: from special-relativistic considerations (which he certainly did not doubt hold in the absence of a gravitational field) plus the hypothesis that any coordinate system may be used, provided it is treated as a frame of reference with a corresponding gravitational field (which was a leitmotiv of his search for a general theory of relativity), and the assumption that a unit measuring rod always measures the same length in any gravitational field, he would have concluded that "in a gravitational field Euclidean geometry could not hold with respect to the arrangement of rigid bodies," as he put it in the letter to Levy. The results obtained could also have helped to shake him free from any lingering idea that "co-ordinates must have an immediate metrical meaning." As we have seen, this was one of the points implicit in his earliest printed discussion of the rotating disk. Together with the idea that the equivalence principle only holds infinitesimally, this may have reminded him of the use of Gaussian coordinates to describe the line element of curved surfaces, in which the idea that Euclidean geometry holds infinitesimally plays such a major role. We have evidence that Einstein, in his studies at the ETH, had become familiar with the Gaussian formula for the line element through the lectures on infinitesimal geometry given by Professor Geiser,[12] lectures which stood out in Einstein's memory fifty years later, when he described them as "true masterpieces of pedagogical art that later helped me very much in wrestling with general relativity" (Einstein 1955). At any rate, we have Einstein's words to assure us that

> I first had the decisive idea of the analogy of mathematical problems connected with the theory and Gauss's theory of surfaces in 1912 after my return to Zurich [which took place in August 1912] without knowing at that time Riemann's and Ricci's or Levi-Civita's work. (Einstein 1922b)[13]

Minkowski's four-dimensional formulation played an important role in Einstein's considerations at this point, as he tells us (Einstein 1955), and he soon saw that what was needed was a four-dimensional generalization of Gauss's two-dimensional surface theory, and that the flat metric tensor of Minkowski's formulation of special relativity had to be generalized to a nonflat metric. At this point, with the mathematical problem already well formulated, he approached Marcel Grossmann for help,[14] with the well-known results, which must have been largely attained by October 29, 1912, when he wrote Sommerfeld

> I am now occupying myself exclusively with the problem of gravitation and believe that, with the aid of a local mathematician who is a friend of mine [Grossmann], I'll now be able to master all difficulties. But one thing is certain, that in all my life I have never struggled as hard and that I have been infused with great respect for mathematics, the subtler parts of which, in my simple-mindedness, I had considered pure luxury up to now! Compared to this problem the original relativity theory [i.e., special relativity] is child's play (EA 13-082; Hermann 1968, p. 26)[15]

Three years were actually to elapse before Einstein was truly able to "master all the difficulties" of the general theory—but that is another story! Meanwhile, if my reconstruction is approximately accurate, we can see that Einstein's recapitulations of the rotating-disk story in the 1916 paper and the popular book (Einstein 1916, 1917), as well as in the Princeton lectures (Einstein 1922a) and the Einstein-Infeld book (Einstein and Infeld 1938), would not only represent a certain logical order of presentation of the material leading up to the recognition of the need for a nonflat space-time structure to describe the gravitational field, but would also represent a fairly accurate historical reconstruction of Einstein's own journey. Since Einstein notes in the preface of the 1916 book that he has attempted to present his ideas "on the whole, in the sequence and connection in which they actually originated" (Einstein 1917), this is perhaps not so surprising.[16] What is more surprising, if our reconstruction is more or less correct, is the lack of mention of the rotating-disk problem in any of his papers on gravitational theory from 1907 through 1915. We can hazard the guess that the reason for this is the amazingly brief period—some time between mid-July and mid-October 1912—when the problem played its role; and the fact that the next critical step of generalizing

the result to the four-dimensional metric tensor was taken almost immediately afterwards. It was this latter great leap forward which provided the starting point for the Einstein-Grossmann investigations, and Einstein started his section of their paper (1913) with an account of this step. Thus the disk problem, having played its role of a "missing link," modestly stayed in the background; but that role was never forgotten by Einstein, as his later letters show.

Notes

1 Bibliographies of the literature on the rotating disk may be found in Arzeliès 1966 and Grøn 1975. A recent paper by Grünbaum and Janis (1977) gives additional references.

2 "Die Behandlung des gleichförmig rotierenden starren Körpers scheint mir von grosser Wichtigkeit wegen einer Ausdehnung des Relativitätsprinzips auf gleichförmig rotierende Systeme nach analogen Gedankengängen, wie ich sie im letzten § meiner in der *Zeitschr. f. Radioaktivit.* publizierten Abhandlung für gleichförmig beschleunigte Translation durchzuführen versucht habe." This letter (EA 21-377) is not included in the published volume of the Einstein-Sommerfeld correspondence (Hermann 1968).

3 See Klein 1970, pp. 152–154, for an account of this discussion, with references.

4 In a letter of April 12, 1911, to Ehrenfest (EA 9-316), Einstein suggested that Ehrenfest reply, noting that a short reply was needed to avoid confusion; however, he took on the job himself.

5 In a reply to an attack by Abraham a few months later, Einstein notes "One sees already from the previously treated highly special case of the gravitation of masses at rest that the space-time coordinates lose their simple physical interpretation; and it still cannot be foreseen what form the general space-time transformation equations may take. I should like to ask all colleagues to have a try at this important problem!" (Einstein 1912c, pp. 1063–1064). Curiously enough, a paper using a definition of spatial distances on the disk equivalent to Einstein's and actually deriving the metric of the rotating disk had been published two years earlier by Theodor Kaluza (1910). The paper was to have been delivered by Kaluza at the 1910 Naturforscherversammlung in Königsberg, where he was then working; but he took sick and only the published version appeared under the title "Zur Relativitätstheorie," which gave no idea of its contents. I have found no evidence that Einstein—or anyone else in the long history of the rotating-disk problem for that matter—was aware of the existence of Kaluza's work.

6 See, for example, pp. 1031–1032 of Einstein's summary survey of the state of general relativity theory (Einstein 1914). The fact that rotating reference frames did not satisfy the field equations of the Einstein-Grossmann theory, while they satisfy the generally covariant field equations of general relativity, played an important role in motivating Einstein's abandonment of the former in favor of the latter, when he discovered this in 1915, as has been pointed out in Earman and Glymour 1978b.

7 See, for example, Einstein's letter to Mach (EA 17-454), thought to date from late 1913, published in Herneck 1966.

8 No record exists in the Archive to indicate that a reply was actually sent.

[9] I have consulted on this topic Illy 1977; Lanczos 1972; Guth 1970; and Earman and Glymour 1978a, 1978b. I am extremely grateful to Dr. Illy, of the Institute of Isotopes of the Hungarian Academy of Sciences, for making his work available to me.

[10] For example, he states in a letter to Smoluchowski of March 24, 1912; "The simple schema of the equivalence of the four dimensions does not hold here in the way it does with Minkowski" (EA 20-597; Teske 1969).

[11] In another letter to Ehrenfest, undated but surely from a little earlier in 1912, Einstein speaks of his work on the static case as finished, and states that he is considering the "dynamic case" now, "again proceeding from the special to the more general case" (EA 9-321).

[12] Marcel Grossmann's notes of Geiser's lectures have been preserved and are now in the ETH Library HS 421:15. They may have been used by Einstein to study for his examinations (see Einstein 1955). They contain a discussion of curvilinear coordinates and the Gaussian line element for the plane (private communication from Professor Res Jost).

[13] "Den entscheidenden Gedanken von der Analogie des mit der Theorie verbundenen mathematischen Problems mit der Gauss'schen Flächentheorie hatte ich allerdings erst 1912 nach meiner Rückkehr nach Zurich, ohne zunächst Riemanns und Riccis, sowie Levi-Civitas Forschungen zu kennen."

[14] Both Einstein 1955 and Einstein 1922b state explicitly that the mathematical problems to be solved were formulated by Einstein before he approached Grossmann for help in their solution. On the other hand, neither of them makes any reference to Georg Pick or any mathematical help received from him in Prague in the formulation of the problem. Indeed, I have quoted the passage from the Preface to the Czech edition of Einstein 1917 to the effect that the "decisive idea" occurred to Einstein after his return to Zurich. This stands in contrast to Philip Frank's account in his biography of Einstein (Frank 1947). Since Frank is usually very careful in his account, and moreover was Einstein's successor in Prague and presumably had a chance to speak to Pick, the discrepancy is puzzling.

[15] For all but the last sentence, I have used the translation in McCormmach 1976, p. xxviii.

[16] Max Wertheimer, in his discussion of the origins of special relativity based on discussions with Einstein, states "In the course of one of his books he did report some steps in the process" (Wertheimer 1945, p. 168), and later makes it clear that the book was Einstein 1917. On page 174 he says: "For what now followed in Einstein's thinking we can fortunately report paragraphs from his own writing [then follows a reference to pages 14 ff. of the German edition of Einstein 1917]. He wrote them in the form of a discussion with the reader. What Einstein says here is similar to the way his thinking proceeded. . . ." Wertheimer's account of the development of the special theory has been attacked recently as unreliable, but not on this point; see Miller 1975.

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Einstein's Search for General Covariance, 1912–1915*

JOHN STACHEL

1. Introduction

Einstein listed the stages of his search for a generally covariant theory of gravitation in a biographical note written in 1916:

1907 Basic idea for the general theory of relativity.
1912 Recognition of the non-Euclidean nature of the metric and of the physical determination of the latter by gravitation.
1915 Field equations of gravitation. Explanation of the perihelion motion of Mercury. (EA 11-196)[1]

There were thus three key moments in Einstein's development of the general theory of relativity:

1. Adoption of the principle of equivalence as the crucial element in a relativistic theory of gravitation (1907).
2. Recognition that the gravitational field must be characterized mathematically via a four-dimensional (pseudo-) Riemannian metric tensor (1912).
3. Discovery of the final form of the field equations relating the metric tensor to the sources of the gravitational field (1915).

Elsewhere, I have discussed this story in broad outlines (Stachel 1979a, 1979b), and have tried to contribute to a more detailed understanding of the decisive second step (Stachel 1980). Most discussions of the development of general relativity have focused on the third step, in particular on the puzzling question: Why—once the decisive step of representing the gravitational field by the metric tensor had been taken—did Einstein take so long to arrive at the final form of the field equations? (See Earman and Glymour 1978a, 1978b; Hoffmann 1972; Lanczos 1972; Mehra 1973; Vizgin and Smorodinskiî 1979.)

I shall suggest an answer that is (no doubt) still incomplete, but that differs from the existing accounts in several respects. In particular, I shall try to explain why it took Einstein over two years to return to general covariance after rejecting it in 1913: