Math Primer Solutions 2:
INFINITE SERIES

[easy] a) Approximate $y = \sin(2)$ by using 4 terms.

$$\begin{aligned}\sin(2) &= 2 - \frac{1}{6}(2)^3 + \frac{1}{120}(2)^5 - \frac{1}{5040}(2)^7 \\ &= 2 - \frac{8}{6} + \frac{32}{120} - \frac{128}{5040} \\ &\approx 0.9079\end{aligned}$$

b) Approximate $y = \sin(90°)$ by using 4 terms.

$$\begin{aligned}\sin(90°) &= \sin\left(\frac{\pi}{2}\right) \\ &= \frac{\pi}{2} - \frac{1}{6}\left(\frac{\pi}{2}\right)^3 + \frac{1}{120}\left(\frac{\pi}{2}\right)^5 - \frac{1}{5040}\left(\frac{\pi}{2}\right)^7 \\ &= \frac{\pi}{2} - \frac{\pi^3}{48} + \frac{\pi^5}{3840} - \frac{\pi^7}{645120} \\ &\approx 0.9998\end{aligned}$$

[medium] Write out an approximation series for $y = \sin(2x)$ and approximate using 2 terms.

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

$$\begin{aligned}\sin(2x) &= (2x) - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \ldots \\ &= 2x - \frac{8x^3}{3!} + \frac{32x^5}{5!} - \frac{128x^7}{7!} \\ &\approx 2x - \frac{4}{3}x^3\end{aligned}$$

.

Write and simply $2\sin(x)\cos(x)$ to 2 terms by using only the 2-term approximation of $\sin(x)$ and $\cos(x)$. What did you notice?

$$
\begin{aligned}
2\sin(x)\cos(x) &= 2\left(x-\frac{1}{3!}x^3\right)\left(1-\frac{1}{2!}x^2\right)\\
&= 2\left(x-\frac{1}{2!}x^3-\frac{1}{3!}x^3+\frac{1}{3!2!}x^5\right)\\
&= 2x-x^3-\frac{2}{3}x^3+\frac{1}{3!}x^5\\
&\approx 2x-\left(\frac{4}{3}\right)x^3
\end{aligned}
$$

[hard] a)

$$
\begin{aligned}
P(x) &= \sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}(x)^k\\
&= \frac{f(0)}{0!}(x)^0+\frac{f'(0)}{1!}(x)^1+\frac{f''(0)}{2!}(x)^2+\frac{f'''(0)}{3!}(x)^3+\ldots\\
&= f(0)+f'(0)\cdot x+\frac{f''(0)\cdot x^2}{2!}+\frac{f'''(0)\cdot x^3}{3!}+\frac{f^{(4)}(0)\cdot x^4}{4!}+\ldots\\
&= \cos(0)-\sin(0)\cdot x-\frac{\cos(0)\cdot x^2}{2!}+\frac{\sin(0)\cdot x^3}{3!}+\frac{\cos(0)\cdot x^4}{4!}+\ldots\\
&= 1-0-\frac{x^2}{2!}+0+\frac{x^4}{4!}+\ldots\\
&= 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}+\ldots
\end{aligned}
$$

b) $$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots$$

$$
\begin{aligned}
\frac{d}{dx}\left(e^x\right) &= \frac{d}{dx}\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots\right)\\
&= 0+1+\frac{2x}{2!}+\frac{3x^2}{3!}+\frac{4x^3}{4!}+\ldots\\
&= 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots
\end{aligned}
$$