Quantum Mechanics Exercises 3:

PARTICLE IN A BOX

In this exercise we will calculate the allowed standing waves (allowed quantum wave-functions) corresponding to a particle of mass **m** confined to a one-dimensional box of length **L**. We will also look at the corresponding quantized energy levels and the zero point energy.

[easy] 1. We saw in a previous problem (Quantum Exercise: Standing Waves Real) that a standing wave of wavelength $\boldsymbol{\lambda}$ and period **T** can be represented by the wavefunction: $\psi(x,t)=\cos\left(\frac{2\pi t}{T}\right)\sin\left(\frac{2\pi x}{\lambda}\right)$, which we also saw is equivalent to a superposition of a right- and a left-moving traveling wave. Let us try to use this standing wave to describe a particle in a box, which stretches from **x = 0** to **x = L**. To begin with, let us ignore the time dependence of the wavefunction, and just consider its dependence on **x**, i.e., consider the wavefunction: $\psi(x)=\sin\left(\frac{2\pi x}{\lambda}\right)$. Since the particle is definitely inside the box (definitely not outside), we know that its wavefunction must be zero outside the box, i.e., $\psi(x)=0$ for x < 0 and x > L. It turns out that quantum wavefunctions must be ***continuous***, i.e., they cannot have any "jumps". Since the wavefunction is zero outside the box, this means it must also be zero at the two ends of the box, x = 0 and x = L. Because sin(0) = 0 we see that our wavefunction $\psi(x)=\sin\left(\frac{2\pi x}{\lambda}\right)$ is already zero at x = 0 for any value of λ. Now we just need it to be zero at x = L.

What does this tell us about the allowed values of the wavelength, λ? (To answer this question, first answer the following question: for what values of θ is $\sin(\theta)=0$?)

[medium] 2. Sketch the wavefunction, $\psi(x)$, and the corresponding probability function, $P(x)=|\psi(x)|^2$, for the first few allowed wavelengths.

[hard] 3. Using the de Broglie relation, determine the corresponding allowed values for the magnitude of the particle momentum, **p**. Using the fact that the particle's energy (which is just kinetic energy) is given by $E = \frac{p^2}{2m}$, determine the corresponding allowed values for the particle energy. Observe that only certain discrete values of energy are allowed: the energy is quantized. What is the lowest allowed energy (called the zero point energy)? As emphasized in the lectures, the fact that this lowest allowed energy is non-zero is of huge significance to the nature of our universe. As the box is made smaller, observe that the lowest allowed energy increases. Discuss with your colleagues how this makes sense. Hint: to make the box smaller you must squeeze it.