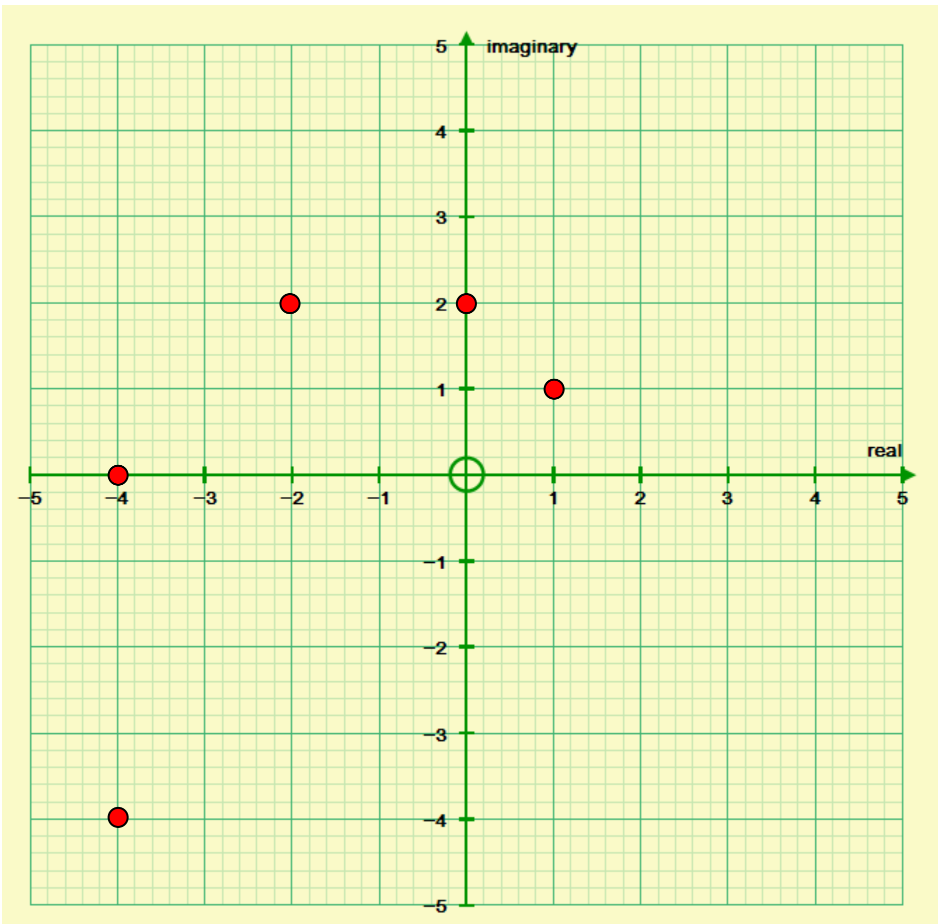
Math Primer Solutions 3 :
IMAGINARY & COMPLEX NUMBERS

[easy] First let's determine the complex numbers z, z^2, z^3, z^4, z^5:

$$z = 1+i$$
$$z^2 = (1+i)(1+i) = 2i$$
$$z^3 = (1+i)(1+i)(1+i) = -2+2i$$
$$z^4 = (1+i)(1+i)(1+i)(1+i) = -4$$
$$z^5 = (1+i)(1+i)(1+i)(1+i)(1+i) = -4-4i$$



[medium] Write $\sin(\theta)$ and $\cos(\theta)$ in terms of e.

Hint: use $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ and $e^{-i\theta} = \cos(-\theta) + i\sin(-\theta)$

We note that

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$
$$e^{-i\theta} = \cos(\theta) - i\sin(\theta)$$

Add and subtract these and we obtain

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}, \ \sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

[hard] Using $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, show that $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$ and that $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$.

$$\begin{aligned} e^{i2\theta} &= \left(e^{i\theta}\right)^2 \\ &= \left(\cos(\theta) + i\sin(\theta)\right)^2 \\ &= \cos^2(\theta) + 2i\cos(\theta)\sin(\theta) + i^2\sin^2(\theta) \\ &= \cos^2(\theta) - \sin^2(\theta) + i2\cos(\theta)\sin(\theta) \end{aligned}$$

For $\cos(2\theta)$, we take the real component of the above result, i.e., $\cos^2(\theta) - \sin^2(\theta)$.

For $\sin(2\theta)$, we take the imaginary component of the above result, i.e., $2\cos(\theta)\sin(\theta)$.

How about cos(A+B) and sin(A+B)?

$$\begin{aligned} e^{i(A+B)} &= \left(e^{iA}\right)\left(e^{iB}\right) \\ &= \left(\cos(A) + i\sin(A)\right)\left(\cos(B) + i\sin(B)\right) \\ &= \cos(A)\cos(B) + i\cos(A)\sin(B) + i\sin(A)\cos(B) + i^2\sin(A)\sin(B) \\ &= \cos(A)\cos(B) - \sin(A)\sin(B) + i\left(\cos(A)\sin(B) + \sin(A)\cos(B)\right) \end{aligned}$$

For $\cos(A+B)$ we have $\cos(A)\cos(B) - \sin(A)\sin(B)$

For $\sin(A+B)$ we have $\cos(A)\sin(B) - \sin(A)\cos(B)$

[hard] For those of you with some Calculus experience:

Show that $\frac{d}{d\theta}\left(\sin(\theta)\right)=\cos(\theta)$ and $\frac{d}{d\theta}\left(\cos(\theta)\right)=-\sin(\theta)$ by doing the derivative of $e^{i\theta}$ and remembering that $\cos(\theta)$ is the real component and $\sin(\theta)$ is the imaginary component.

$$\begin{aligned}\frac{d}{d\theta}\left(\sin(\theta)\right)&=\frac{d}{d\theta}\left(e^{i\theta}\right)\\&=ie^{i\theta}\\&=i\left(\cos(\theta)+i\sin(\theta)\right)\\&=-\sin(\theta)+i\cos(\theta)\end{aligned}$$

$$\begin{aligned}\frac{d}{d\theta}\left(\cos(\theta)\right)&=\frac{d}{d\theta}\left(e^{i\theta}\right)\\&=ie^{i\theta}\\&=i\left(\cos(\theta)+i\sin(\theta)\right)\\&=-\sin(\theta)+i\cos(\theta)\end{aligned}$$

[hard] Determine $\int e^{-x}\cos(x)$ by using $e^{i\theta}=\cos(\theta)+i\sin(\theta)$, and recalling that $\cos(\theta)=\mathrm{Re}\left(e^{i\theta}\right)$ (the real component of $e^{i\theta}$).

$$\begin{aligned}\int e^{-x}\cos(x)\,dx&=\int e^{-x}e^{ix}dx\\&=\int e^{-x+ix}dx\\&=\int e^{(-1+i)x}dx\\&=\frac{1}{-1+i}e^{(-1+i)x}\\&=\frac{-1-i}{(-1+i)(-1-i)}e^{(-1+i)x}\\&=\frac{-1-i}{2}e^{(-1+i)x}\\&=\frac{-1-i}{2}e^{-x}e^{ix}\\&=\frac{-1-i}{2}e^{-x}\left(\cos(x)+i\sin(x)\right)\end{aligned}$$

Now we want the real component of the above

$$
\begin{aligned}
&= \frac{-1-i}{2} e^{-x} \left(\cos(x) + i\sin(x)\right) \\
&= -\frac{e^{-x}}{2}(1+i)\left(\cos(x) + i\sin(x)\right) \\
&= \frac{e^{-x}}{2}\left(\sin(x) - \cos(x)\right)
\end{aligned}
$$