Special Relativity Solutions 1:
S P A C E T I M E D I A G R A M S (I)

In addition to the solutions described here, Maple worksheets are also provided.

1. The front of a building facing a street is half as high as it is wide. A high-speed spacecraft is traveling parallel to the building at $0.87\,c$. What does the building look like to the passengers on the spacecraft?

   Length contraction only occurs in the direction of motion. The passengers of the spacecraft will measure the same height of the building but not the same width. Due to their high speed, the width will be contracted.

   Let's say that in the rest frame of the building the width is L. This means that the height is $\frac{L}{2}$. In the rest frame of the spacecraft the height will also be $\frac{L}{2}$.

   Now we need to find the width in the rest frame of the spacecraft. We start by calculating gamma for the spacecraft.

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = 2.0$$

$$width = \frac{L}{\gamma} = \frac{L}{2}$$

We find that in the frame of the spacecraft, the width and the height of the building are both $\frac{L}{2}$. The passengers of the spacecraft see the building as square rather than rectangular.

2. The Milky Way galaxy is approximately 10^5 light years in diameter. A proton is traveling at 0.99999c relative to the rest frame of the galaxy. How long will it take the proton to traverse the galaxy as measured in

a. the rest frame of the galaxy?
b. the rest frame of the proton?

a. In the rest frame of the galaxy, the time would be approximately 10^5 years, because the proton is traveling at very nearly the speed of light.

b. We found that moving clocks run slow. Therefore, in the rest frame of the proton, we have:

$$\Delta t_p = \Delta t\sqrt{1-\frac{v^2}{c^2}} = 450\ years$$

How can the proton cover a distance of 10^5 ly in only 450 years?

In the rest frame of the proton, the diameter of the galaxy is *not* 10^5 ly but, rather:

$$\Delta l_p = \Delta l\sqrt{1-\frac{v^2}{c^2}} = 450\ ly$$

Thus, the proton sees a galaxy 450 ly in diameter moving at $0.99999c$ relative to the rest frame of the proton.

3. A box at rest in the laboratory has dimensions given by $L_o \times W_o \times H_o$. The box is completely filled with a fluid of density $\rho_o = 1.5 \times 10^3\ kg/m^3$. If the box is given a velocity of 0.6c along its L_o dimension, what will be the measured value of ρ in the laboratory frame?

Let us call the density ρ equal to $M/_V$ in whatever frame we happen to be measuring it. In the rest frame of the box,

$$\rho_o = \frac{M_o}{V_o} = \frac{M_o}{L_o \times W_o \times H_o}$$

In the rest frame of the laboratory,

$$\rho = \frac{m}{V} = \frac{m_o \Big/ \sqrt{1 - \frac{v^2}{c^2}}}{L_o\sqrt{1 - \frac{v^2}{c^2}} \times W_o \times H_o}$$

Then

$$\frac{\rho}{\rho_o} = \frac{1}{1 - \frac{v^2}{c^2}}$$

$$\rho = \frac{\rho_o}{1 - \frac{v^2}{c^2}} = \frac{1.5}{1 - 0.6^2}$$

$$\rho = 2.34 \times 10^3\ kg/m^3$$

4. In the rest frame of the Earth the distance between Edmonton and Montreal is about 4000 km. By how much is the distance shortened when observed by a jet, the space shuttle and a cosmic ray. (Hint: You will need to make estimations).

A jet can travel this distance in roughly 5 hours. Therefore, its speed in units of c is approximately

$$v_{jet} = \frac{distance}{time} = \frac{4 \times 10^6}{(5 \times 24 \times 60 \times 60)} \times \frac{c}{3 \times 10^8} = 3 \times 10^{-8}\ c$$

The space shuttle orbits the Earth in 90 minutes and the radius of the Earth is 6500 km. Therefore its speed in units of c is approximately

$$v_{shuttle} = \frac{distance}{time} = \frac{4 \times 10^6}{(90 \times 60)} \times \frac{c}{3 \times 10^8} = 2.5 \times 10^{-5}\ c$$

A cosmic ray has a speed very close to c.

$$v_{cosmic} = 0.9c$$

To calculate the difference in the distance measured in the Earth frame to the other, we use:

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

And

$L = \frac{L_o}{\gamma}$ where L_o is the distance in the Earth frame.

To find the difference we subtract L from L_o such that

$L_o - L = L_o - \frac{L_o}{\gamma} = \left(1 - \frac{1}{\gamma}\right)L_o$

Although this expression can be evaluated directly, there are situations in which v may differ from c by no more than say, one part in 10^{10}.
In such a case the limitations of the rounding errors on typical calculators will prevent a solution. Therefore, we will take a slightly different tack, which can be used when v is much less than c.

In the Math Primer we learned about the binomial approximation. Namely $(1 + x)^n \approx 1 + nx$ when x is small. Let's use the binomial approximation here to approximate γ and simplify our expression for the difference in measured distance.

$$L - L_o \approx \frac{v^2}{2c^2} L_o$$

Evaluating this for each of the three cases we find:

For the jet:

$$L - L_o \approx 2nm$$

For the shuttle:

$$L - L_o \approx 1.3\ mm$$

In the case of the cosmic ray, we cannot use the approximation, because $\frac{v}{c}$ is not small. Therefore we use the original expression

$L_o - L = L_o - \frac{L_o}{\gamma} = \left(1 - \frac{1}{\gamma}\right)L_o$

$$L - L_o = 2300\ km$$