ISSYP

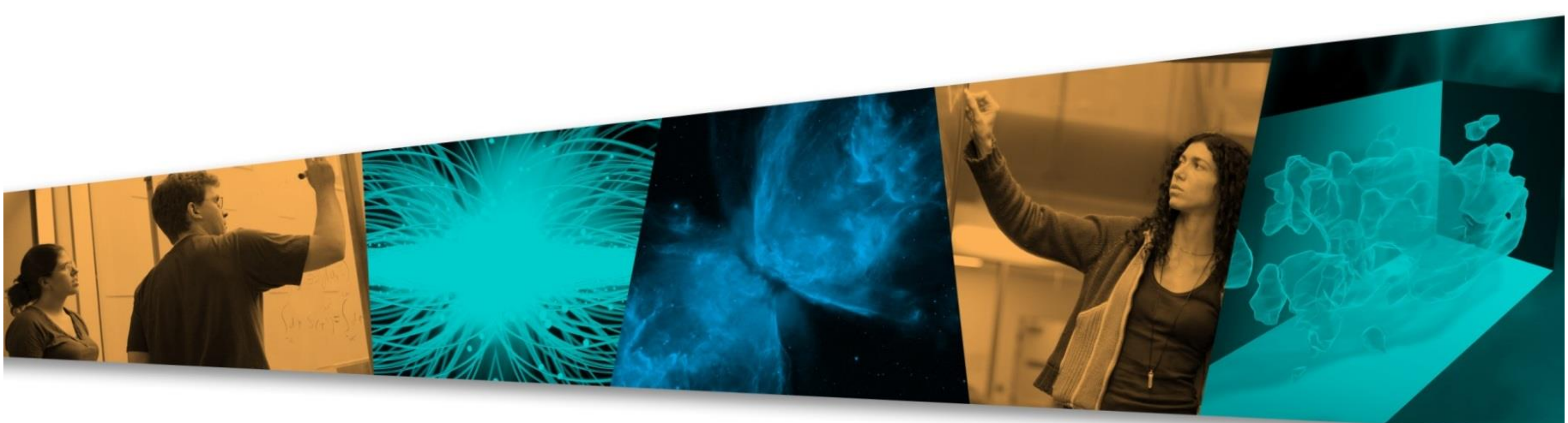
Math Primer

Mathematics is the language with which God has written the universe.

– Galileo Galilei

Do not worry about your difficulties in Mathematics. I can assure you mine are still greater.

– Albert Einstein



Introduction

You are probably familiar with the expression “mathematics is the language of physics” and some of you might be worried that you aren’t fluent enough in mathematics to understand the physics that will be presented at ISSYP. Relax. The physics lessons are designed so the required mathematics is minimal, although there will be plenty of opportunity to stretch yourself.

Consider the analogy of travelling to a foreign country. Learning the local language is not essential to enjoying the beautiful scenery and sampling new foods, but a little knowledge of the language will allow you to be more adventurous and to dig a little deeper into the culture. Our goal at ISSYP is to introduce you to the beautiful scenery of modern physics and to give you the chance to dig as deeply into the mysteries of nature as you can. Some of you are well on your way to becoming fluent so you are ready to dig deeper than others but all of you will be able to enjoy the beauty.

This primer is meant to give you the rudimentary language that will help you in the physics lessons. Please look through these concepts and familiarize yourself with them. Do not worry if you haven’t studied them yet in school, mathematics is best learned in context and we will not abandon you.

The key concepts that we will develop during the lessons are:

- ☐ **Binomial Series**
- ☐ **Series Expansion**
- ☐ **Complex Numbers**
- ☐ **Real and Complex Waves**
- ☐ **Matrices**

You might be surprised to see that there is no calculus in the list. More advanced mathematics allows you to dig deeper but are not needed to understand the key concepts. You will have the opportunity to really flex your mathematical muscles in the mentor projects and some of the extension activities.

Binomial Series

One of Isaac Newton's greatest achievements in mathematics was the development of a technique for analyzing the area under curves, now known as the Binomial Theorem—turns out that this technique is very useful for lots of things. Let's develop it together and see where it takes us.

Start with the expression: $(a+b)^n = ? \text{ for } n = 0,\ 1,\ 2,\ 3,\ \ldots$

Factor and simplify: $\left[a\left(1+\dfrac{b}{a}\right)\right]^n = a^n\left(1+\dfrac{b}{a}\right)^n = ?$

Isolate the interesting bit: $\left(1+\dfrac{b}{a}\right)^n = ?$

Generalize to $(1+x)$ to the power of n: $(1+x)^n = ?$

Try a bunch of values for n and look for a pattern:

$(1+x)^0 = 1$ Coefficients: 1

$(1+x)^1 = 1+x$ 1 1

Pascal's triangle

$(1+x)^2 = 1+2x+x^2$ 1 2 1

$(1+x)^3 = 1+3x+3x^2+x^3$ 1 3 3 1

$(1+x)^4 = 1+4x+6x^2+4x^3+x^4$ 1 4 6 4 1

Can you predict what $(1+x)^5$ equals?

HINT: Look for patterns in the coefficients for each term.

- first and last coefficients are always 1

- coefficient for x increases by 1 each time $(1, 2, 3, 4, \ldots)$

- next line in triangle is produced by adding together adjacent terms from previous line

$$(1+x)^5 = 1+5x+10x^2+10x^3+5x^4+x^5$$

Generalize: $$\boxed{(1+x)^n = 1 + \frac{n}{1}x + \frac{n}{1}\left(\frac{n-1}{2}\right)x^2 + \frac{n}{1}\left(\frac{n-1}{2}\right)\left(\frac{n-2}{3}\right)x^3 + \ldots}$$

Check that the factors involving n give the correct coefficients in Pascal's triangle.

When $n = 0, 1, 2, 3, \ldots$ the series will terminate (i.e., the series is finite in length).

What happens when $n = -1$?

$$(1+x)^{-1} = 1 + \frac{-1}{1}x + \frac{-1}{1}\left(\frac{-1-1}{2}\right)x^2 + \frac{-1}{1}\left(\frac{-1-1}{2}\right)\left(\frac{-1-2}{3}\right)x^3 + \ldots = 1 - x + x^2 - x^3 + \ldots \rightarrow \text{ an infinite series}$$

Check by substituting $x = -\frac{1}{2}$

L.H.S: $(1+x)^{-1} = \left(1 - \frac{1}{2}\right)^{-1} = 2$ R.H.S: $\left(1 - \frac{1}{2}\right)^{-1} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots \rightarrow 2$ ✓

What happens when $n = \frac{1}{2}$?

$$(1+x)^{1/2} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \ldots$$

Check to see what happens when this is multiplied by itself.

the higher order terms all cancel

We should get: $(1+x)^{1/2}(1+x)^{1/2} = 1 + x$

$$\begin{aligned}
\left(1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \ldots\right)\left(1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \ldots\right) &= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \ldots \\
&\quad + \frac{1}{2}x + \frac{1}{4}x^2 - \frac{1}{16}x^3 + \ldots \\
&\quad - \frac{1}{8}x^2 - \frac{1}{16}x^3 + \ldots \\
&\quad + \frac{1}{16}x^4 + \ldots \\
&\quad + \ldots \\
&= 1 + x
\end{aligned}$$

Try this for a decimal number:

$$\begin{aligned}\sqrt{1.1} &= (1+0.1)^{1/2} \\ &= 1+\frac{1}{2}(0.1)-\frac{1}{8}(0.1)^2+\frac{1}{16}(0.1)^3+\ldots \\ &= 1+0.05-0.00125+0.0000625+\ldots \\ &\approx 1.05\end{aligned}$$

For very small values of x, higher order terms can be ignored. Check if this is true for $n=2$.

$$(1+x)^2 = 1+2x+x^2$$

Ignoring this term creates less than 1% error

For $x=0.1$, $(1+0.1)^2 = 1+2(0.1)+(0.1)^2 = 1+0.2+0.01 = 1.21$

For $x=0.01$, $(1+0.01)^2 = 1+2(0.01)+(0.01)^2 = 1+0.02+0.0001 = 1.0201$

Ignoring this term creates about 0.01% error

So, for small x, $(|x| \ll 1)$: $(1+x)^2 \approx 1+2x$

In general, for small x, $(|x| \ll 1)$ we can use the binomial approximation:

$$\boxed{(1+x)^n \approx 1+nx}$$

Examples: $\sqrt{1+x} = (1+x)^{1/2} \approx 1+\frac{1}{2}x$

$$\sqrt{1.001} \approx 1+\frac{1}{2}(0.001) = 1.0005$$

Check: $(1.0005)^2 = 1.00100025$

the approximation introduces an error of 0.000025%

Now let's use this to gain some insight into energy…

According to Einstein's Special Relativity, the general relationship between the energy of a moving object (of "rest mass", m) and its momentum is:

$$\boxed{E^2 = m^2c^4 + p^2c^2}$$

where m is the rest mass (a.k.a. "invariant" mass; the "intrinsic" mass of the object at rest),
E is the energy (a.k.a. "relativistic" energy, i.e., "valid at all speeds up to c"),
p is the momentum (a.k.a. "relativistic" momentum, i.e., "valid at all speeds up to c"),
c is the speed of light.

Let's rearrange this expression to discover something interesting about mass:

$$E = \sqrt{m^2c^4 + p^2c^2} = \sqrt{m^2c^4\left(1 + \frac{p^2c^2}{m^2c^4}\right)} = mc^2\left(1 + \frac{p^2c^2}{m^2c^4}\right)^{\frac{1}{2}}$$

For objects moving much slower than the speed of light ($v \ll c$) we can use the classical (a.k.a. "non-relativistic") definition for momentum ($p \approx mv$) and thus:

$$x = \frac{p^2c^2}{m^2c^4} \approx \frac{m^2v^2c^2}{m^2c^4} \approx \left(\frac{v}{c}\right)^2 \ll 1$$

Example: Even a very fast-moving object like the space shuttle, which circles the planet once every 90 minutes, is slow when compared to light.

$$\frac{v}{c} \approx \frac{8000\ {}^{m}\!/\!{}_{s}}{3\text{x}10^8\ {}^{m}\!/\!{}_{s}} \approx 0.000027 \qquad\qquad x \approx \left(\frac{v}{c}\right)^2 \approx 0.0000000007 \ll 1$$

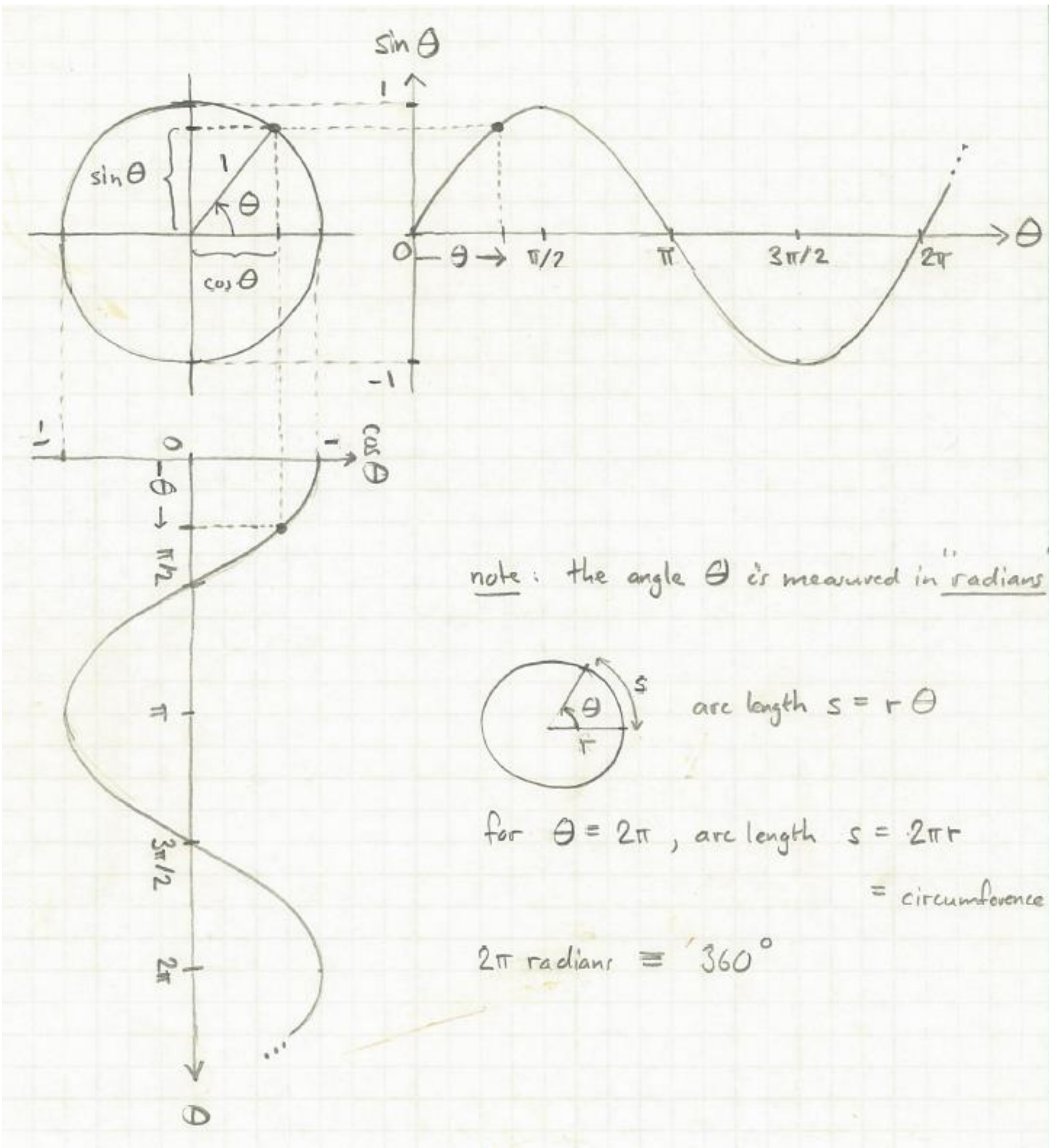
Thus, to a very good approximation we can use the binomial approximation in the relation between energy and momentum:

$$E = mc^2\left(1 + \frac{p^2c^2}{m^2c^4}\right)^{\frac{1}{2}} \approx mc^2\left(1 + \frac{1}{2}\frac{p^2}{m^2c^2}\right) \approx mc^2 + \frac{p^2}{2m} \approx \overbrace{mc^2}^{\text{“rest energy”}} + \overbrace{\frac{1}{2}mv^2}^{\text{“kinetic energy”}}$$

For slow-moving objects, Einstein's relation between energy and momentum reduces to the sum of "rest energy" and "kinetic energy". Prior to Einstein, physicists did not know about this extremely important (and much larger) rest energy—the vast amount of energy contained in the rest mass of an object.

Series Expansions

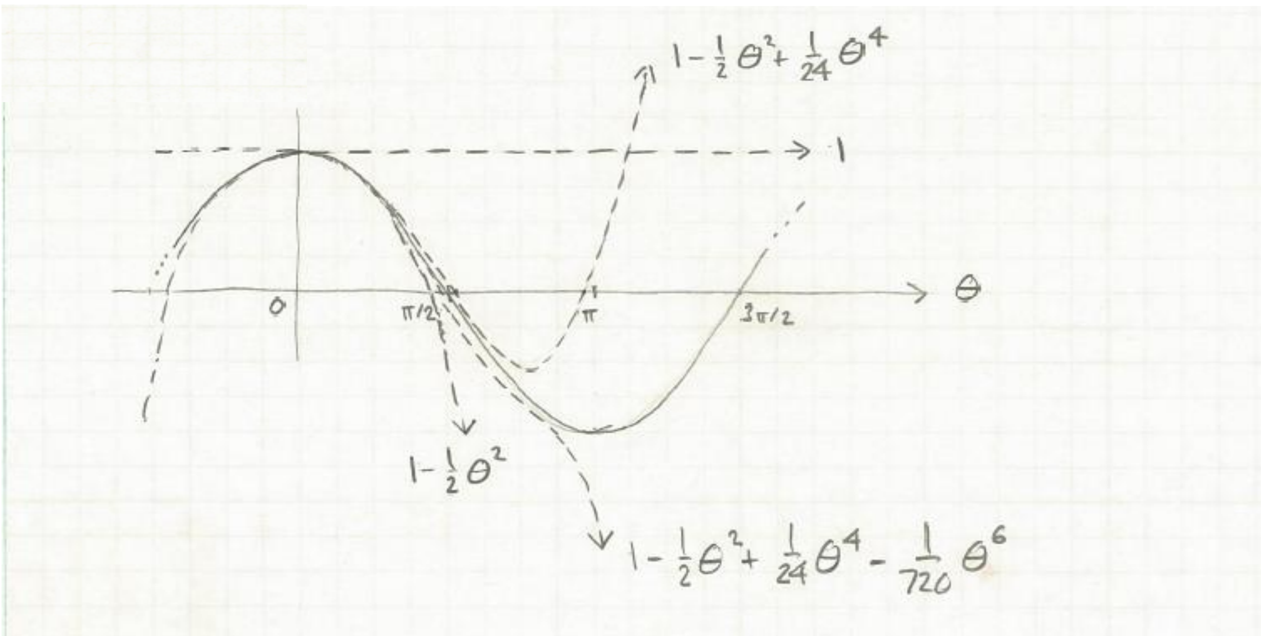
Let’s use a unit circle to develop an expression for sine and cosine. As we move around the circle sine and cosine are projections onto the two axes. When you complete one full rotation on the circle you will note that the sine and cosine functions also go through one complete cycle and begin to repeat.



Question: For a given θ, how does a calculator determine $\cos\theta$?

Answer: $$\cos\theta = 1 - \frac{1}{2}\theta^2 + \frac{1}{24}\theta^4 - \frac{1}{720}\theta^6 + \ldots \quad \text{an infinite number of terms}$$

This series expression for cosine comes from calculus, but let's see how it works graphically. Below is a series of sketches, each adding another term in the series. Notice how each term is a small correction to the previous sketch and as more terms are included the sketch begins to resemble the cosine function.



Example: $\cos\frac{\pi}{2} = 0 \qquad (exactly)$

$$\cos\frac{\pi}{2} \approx 1 - \frac{1}{2}\left(\frac{\pi}{2}\right)^2 = -0.23370055\ldots \qquad \leftarrow \textit{close to zero}$$

$$\cos\frac{\pi}{2} \approx 1 - \frac{1}{2}\left(\frac{\pi}{2}\right)^2 + \frac{1}{24}\left(\frac{\pi}{2}\right)^4 = +0019968957\ldots \qquad \leftarrow \textit{closer to zero}$$

$$\cos\frac{\pi}{2} \approx 1 - \frac{1}{2}\left(\frac{\pi}{2}\right)^2 + \frac{1}{24}\left(\frac{\pi}{2}\right)^4 - \frac{1}{720}\left(\frac{\pi}{2}\right)^6 = -0.000894522\ldots \qquad \leftarrow \textit{even closer to zero}$$

We get closer and closer to zero (the exact value) as we include more terms in the series expansion. Including ***all*** the terms will give us the ***exact*** value. Notice how the sketches overlap nicely for small values of θ ?

For small θ, we can use an approximation: $$\boxed{\cos\theta \approx 1 - \frac{1}{2}\theta^2}$$

<u>Similarly</u>: $\sin\theta = \theta - \frac{1}{6}\theta^3 + \frac{1}{120}\theta^5 - \frac{1}{5040}\theta^7 + \ldots$ an infinite number of terms

Try making a series of sketches like we did for cosine. You will notice that for small values of θ the sketches overlap nicely.

For small θ, we can use the approximation: $\boxed{\sin\theta \approx \theta - \frac{1}{6}\theta^3}$

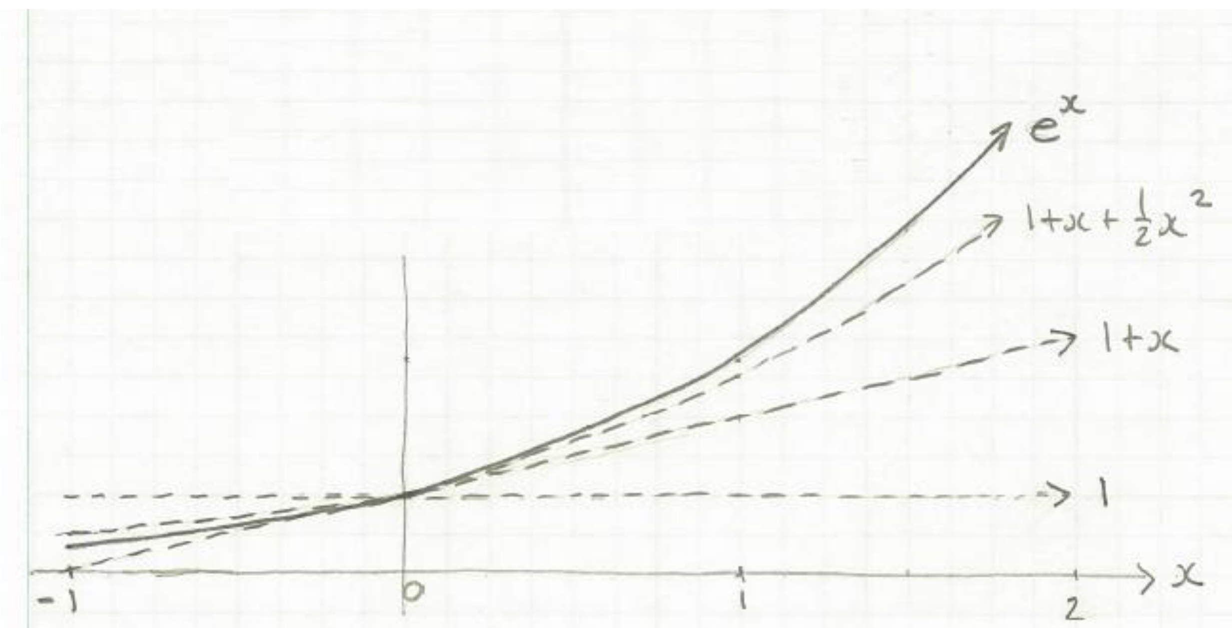
There is a similar series expansion for the exponential function (here the number $e = 2.713\ldots$):

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 + \ldots \quad \text{an infinite number of terms}$$

$$2 = 2\cdot 1$$
$$6 = 3\cdot 2\cdot 1$$
$$24 = 4\cdot 3\cdot 2\cdot 1$$
$$120 = 5\cdot 4\cdot 3\cdot 2\cdot 1$$
$$720 = 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1$$

This pattern is called a factorial and it is given the special symbol '!'.

e.g., $4! = 4\cdot 3\cdot 2\cdot 1 = 24$

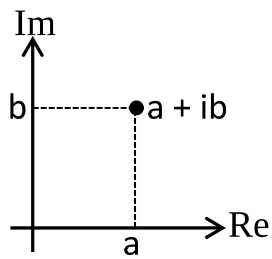


Note how the lines overlap for small values of x.

For small x, we can use the approximation: $\boxed{e^x \approx 1 + x}$

Complex Numbers

Complex numbers allow us to solve equations that cannot be solved using real numbers alone. You can think of complex numbers as an extension of real numbers, like adding a dimension. Instead of solving problems along a 1-D number line we will use a 2-D number plane, called the “complex plane”.



The imaginary unit is defined as:

$$i^2 = -1$$

Complex numbers can be manipulated just like ordinary numbers.

ADDITION:

$$(a+ib)+(c+id)=(a+c)+i(b+d)$$

MULTIPLICATION:

$$\begin{aligned}(a+ib)(c+id) &= ac+iad+ibc+i^2bd \\ &= (ac-bd)+i(ad+bc)\end{aligned}$$

Something very cool happens when we put complex numbers into the exponential series:

$$e^x = 1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+\ldots$$

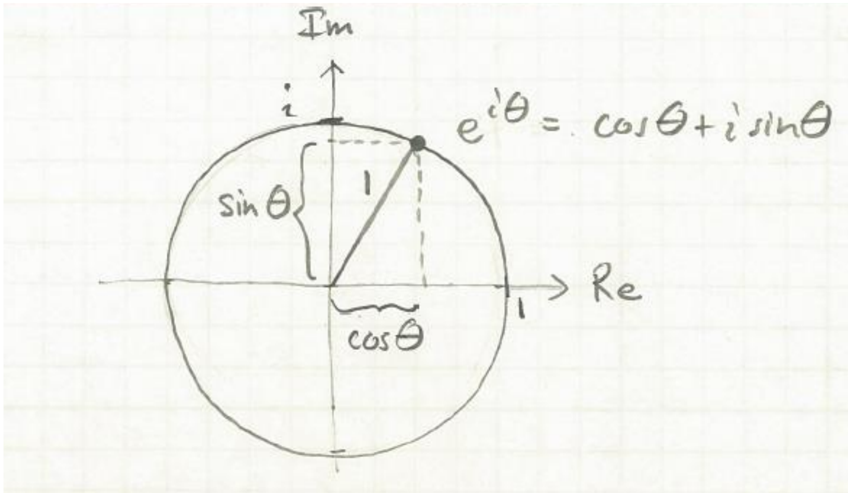
Let $x = i\theta$,

$$\begin{aligned}e^{i\theta} &= 1+i\theta+\frac{1}{2!}i^2\theta^2+\frac{1}{3!}i^3\theta^3+\frac{1}{4!}i^4\theta^4+\ldots \\ &= 1+i\theta-\frac{1}{2!}\theta^2-\frac{1}{3!}i\theta^3+\frac{1}{4!}\theta^4+\ldots \\ &= \left\{1-\frac{1}{2!}\theta^2+\frac{1}{4!}\theta^4+\ldots\right\}+i\left\{\theta-\frac{1}{3!}\theta^3+\frac{1}{5!}\theta^5+\ldots\right\} \\ &= \{\qquad \cos\theta \qquad\}+i\{\qquad \sin\theta \qquad\}\end{aligned}$$

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

extremely important !!

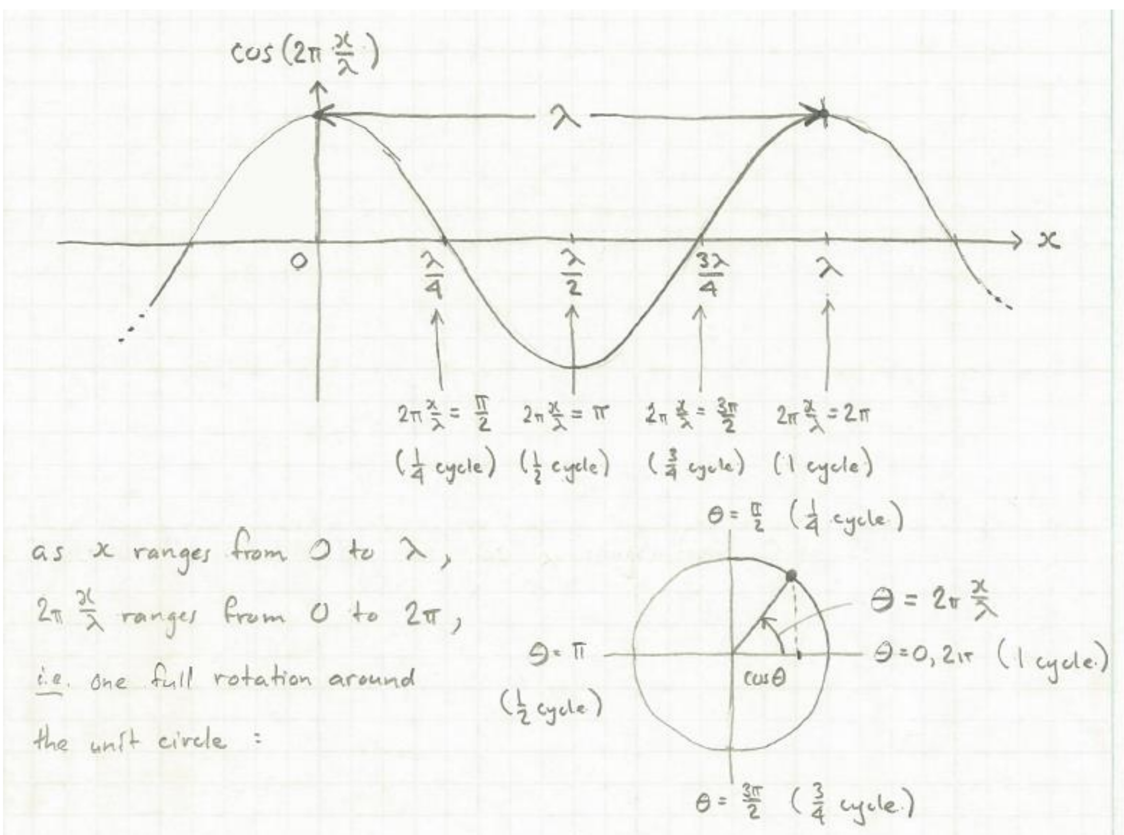
This incredibly useful result can be pictured by putting a unit circle onto the complex plane.



It turns out that this is exactly the type of mathematics needed to describe quantum waves.

Real and Complex Waves

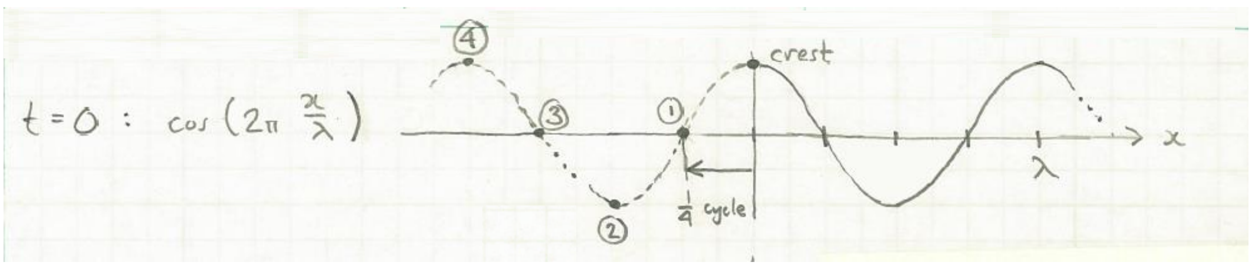
Consider a static wave of wavelength λ:



To make this a <u>traveling</u> wave—a wave that moves to the right as time passes—we add a time dependence to the argument of the cosine function:

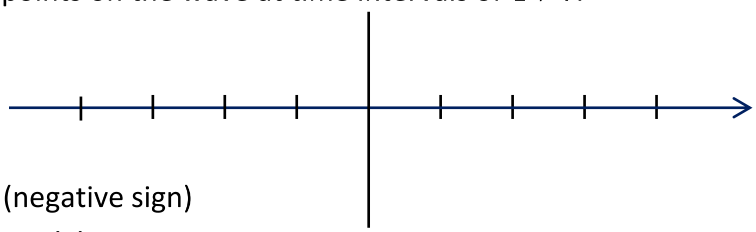
$$\cos\left(2\pi\frac{x}{\lambda}\right) \to \cos\left(2\pi\frac{x}{\lambda} - 2\pi\frac{t}{T}\right)$$

where T is the period of the wave (the time it takes a wave to move through a distance λ).



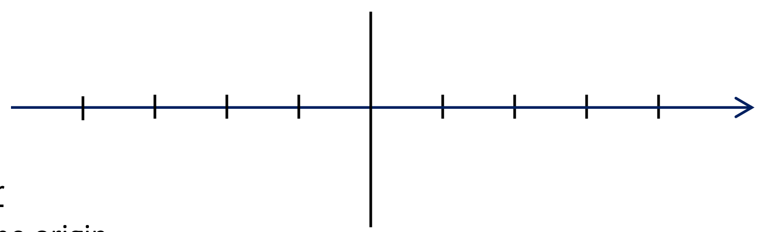
Try this for yourself! Start with a cosine function at $t = 0$ (above sketch) and then track the forward motion of the labeled points on the wave at time intervals of $T/4$.

$t = \frac{T}{4}$: $\cos\left(2\pi\frac{x}{\lambda} \underbrace{- \frac{\pi}{2}}\right)$



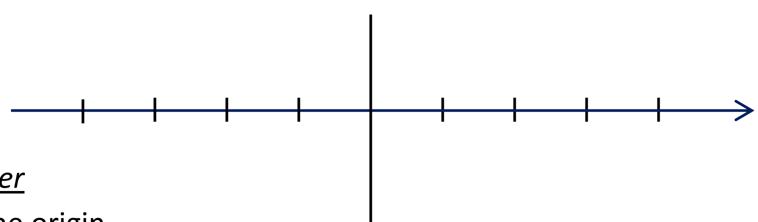
Start wave ¼ cycle *<u>earlier</u>* (negative sign)
Point "1" has moved to the origin

$t = \frac{T}{2}$: $\cos\left(2\pi\frac{x}{\lambda} \underbrace{- \pi}\right)$



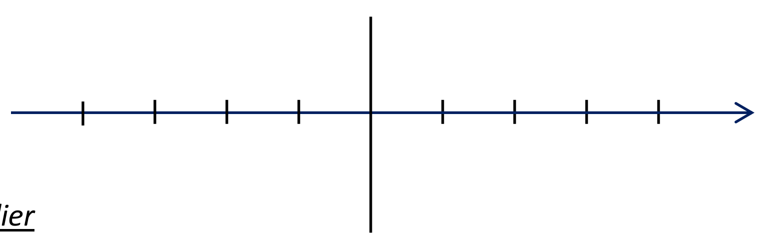
Start wave ½ cycle *<u>earlier</u>*
Point "2" has moved to the origin

$t = \frac{3T}{4}$: $\cos\left(2\pi\frac{x}{\lambda} \underbrace{- \frac{3\pi}{2}}\right)$



Start wave 3/4 cycle *<u>earlier</u>*
Point "3" has moved to the origin

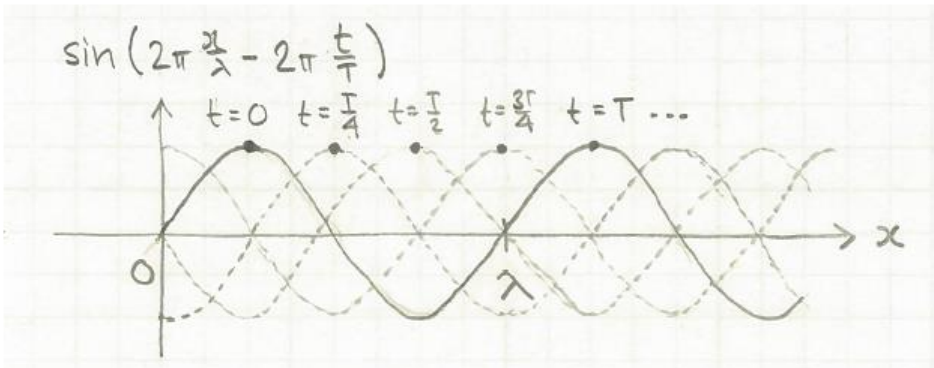
$t = T$: $\cos\left(2\pi\frac{x}{\lambda} \underbrace{- 2\pi}\right)$



Start wave one cycle *<u>earlier</u>*
Point "4" has moved to the origin

As t ranges from 0 to T (one full period), the point on the initial wave marked "crest" has moved to the right from $x = 0$ to $x = \lambda$ (one full wavelength). The wave is traveling!

We can do the same thing with a sine wave:



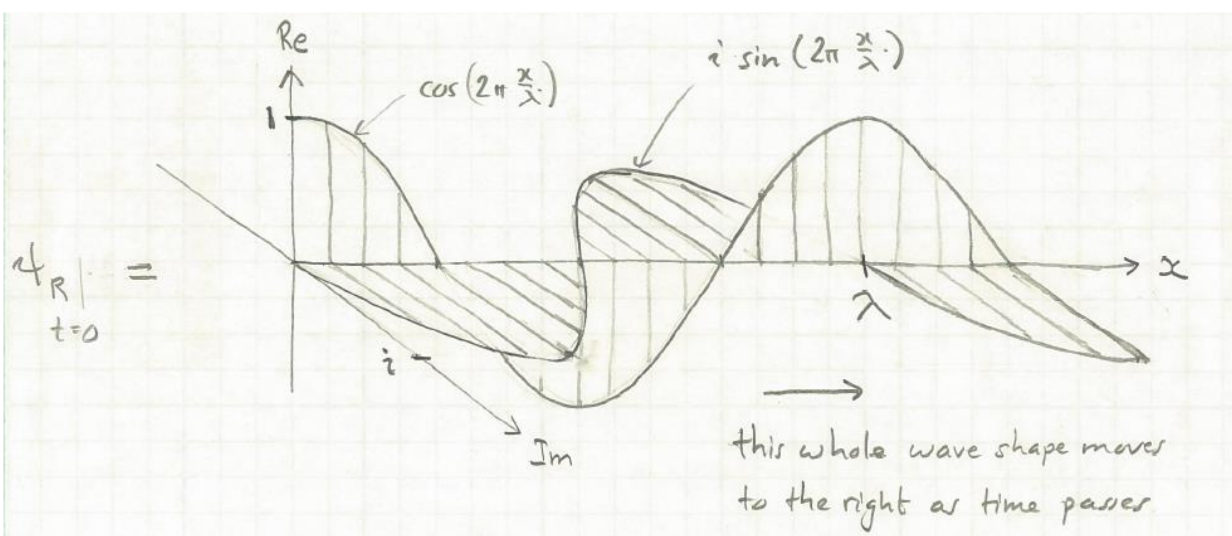
Such traveling waves have an important application in quantum (wave) mechanics, not as *real* cosine and sine waves, but as complex waves.

Recall: $e^{i\theta} = \cos\theta + i\sin\theta$

Set $\theta = 2\pi \dfrac{x}{\lambda} - 2\pi \dfrac{t}{T}$ and we have a complex traveling wave:

$$\psi_R \equiv e^{i2\pi\left(\frac{x}{\lambda}-\frac{t}{T}\right)} = \cos 2\pi\left(\frac{x}{\lambda}-\frac{t}{T}\right) + i\sin 2\pi\left(\frac{x}{\lambda}-\frac{t}{T}\right)$$

R is for Right-traveling wave



The sum of the cosine and sine waves is a *spiral* wave!

In quantum mechanics we learn that a particle with momentum p and energy E is associated with a complex traveling wave of exactly this form, where

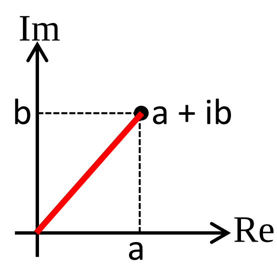
$$\lambda = \frac{h}{p} \quad \text{and} \quad T = \frac{1}{f} = \frac{h}{E} \qquad \left(f = \text{frequency} = \frac{1}{\text{period}} \right)$$

or: $$\frac{2\pi}{\lambda} = \frac{p}{\hbar} \quad \text{and} \quad \frac{2\pi}{T} = \frac{E}{\hbar} \quad \text{, where } \hbar = \frac{h}{2\pi}$$

hence: $\psi_R = e^{\frac{i}{\hbar}(px - Et)}$ describes a particle with momentum p and energy E moving to the *right*.

To get a *left* moving particle we just change the sign of the momentum (reverse the velocity):

$$\psi_L = e^{\frac{i}{\hbar}(-px - Et)}$$



Note: A complex number $a+ib$ has *magnitude* $|a+ib|$ equal to the length of the hypotenuse (see red line in figure), i.e., $|a+ib|^2 = a^2 + b^2$ by Pythagoras's theorem. In quantum mechanics we learn that $|\psi|^2$ is associated with the probability of finding a particle.

Since $|e^{i\theta}|^2 = \cos^2\theta + \sin^2\theta = 1$, we see that $|\psi_R|^2 = |\psi_L|^2 = 1$ for all x and t.

Meaning?

ψ_R (ψ_L) describes a particle moving to the right (left) with definite momentum p $(-p)$.

Definite $p \Rightarrow \Delta p = 0 \Rightarrow \Delta x = \infty$ $\left[\text{use } \Delta x \geq \frac{\hbar}{2\Delta p} \text{ with } \Delta p = 0 \right]$

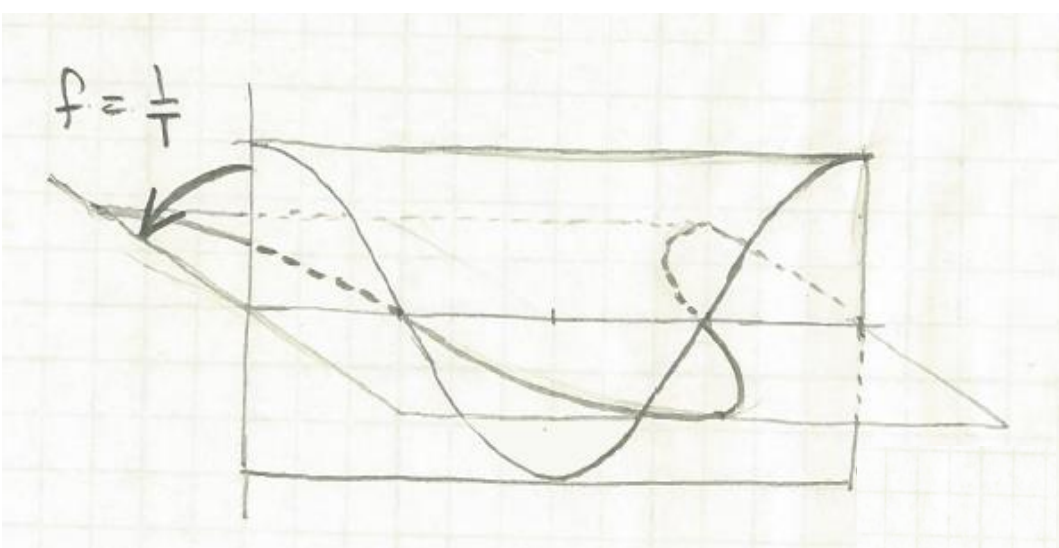
$\Delta x = \infty$ means the particle is *equally likely* to be found *anywhere* along the x-axis. If we are certain of the particle's momentum we are completely uncertain of its position!!

Complex Standing Waves

A particle confined within a box is in a superposition (sum) of right and left moving states:

$$
\begin{aligned}
\psi &= \psi_R + \psi_L \\
&= e^{\frac{i}{\hbar}(px - Et)} + e^{\frac{i}{\hbar}(-px - Et)} \\
&= e^{-\frac{i}{\hbar}Et}\left(e^{\frac{i}{\hbar}px} + e^{-\frac{i}{\hbar}px}\right) \\
&= e^{-\frac{i}{\hbar}Et}\left(\cos\frac{px}{\hbar} + i\sin\frac{px}{\hbar} + \cos\frac{px}{\hbar} - i\sin\frac{px}{\hbar}\right) \\
&= 2e^{-\frac{i}{\hbar}Et}\cos\frac{px}{\hbar} \\
&= 2e^{-i2\pi\frac{t}{T}}\cos\left(2\pi\frac{x}{\lambda}\right)
\end{aligned}
$$

This function describes a cosine wave shape $\left[\cos\left(2\pi\frac{x}{\lambda}\right)\right]$ that ***rotates*** in the complex plane with frequency $f = \frac{1}{T}$ $\left[e^{-i2\pi\frac{t}{T}}\right]$.



Matrices

A matrix is a simple way to organize information. Matrices are very useful for solving complex physics problems, especially ones involving vectors. In fact, many of you have done matrix mathematics without even realizing it as you learned how to manipulate vectors.

Matrices are defined by their dimensions, which is just how many rows and columns they have. For example,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = 2\times 2 \text{ "square" matrix}, \begin{bmatrix} i & j & k \end{bmatrix} = 1\times 3 \text{ "row" matrix}, \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = 2\times 1 \text{ "column" matrix.}$$

Matrices have their own set of rules, or algebra.

ADDITION: You can only add matrices together if they have the same dimensions. The sum of two matrices is the sum of the individual entries. For example,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$$

Notice that order does not matter when ADDING matrices together.

SCALAR MULTIPLICATION: You can multiply a matrix by a scalar quantity. The scalar acts on each individual entry. For example,

$$k\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$$

MATRIX MULTIPLICATION: You can multiply two vectors together but only if their dimensions match. The number of columns in the first vector must match the number of rows in the second (and vice-versa). The reason for this is evident when you see how we perform the operation.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{x} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

$$\begin{bmatrix} a\,b\,c \\ d\,e\,f \end{bmatrix} \text{x} \begin{bmatrix} m & n \\ o & p \\ q & r \end{bmatrix} = \begin{bmatrix} am+bo+cq & an+bp+cr \\ dm+eo+fq & dn+ep+fr \end{bmatrix}$$

Notice that the order DOES matter when multiplying. $AB \neq BA$.