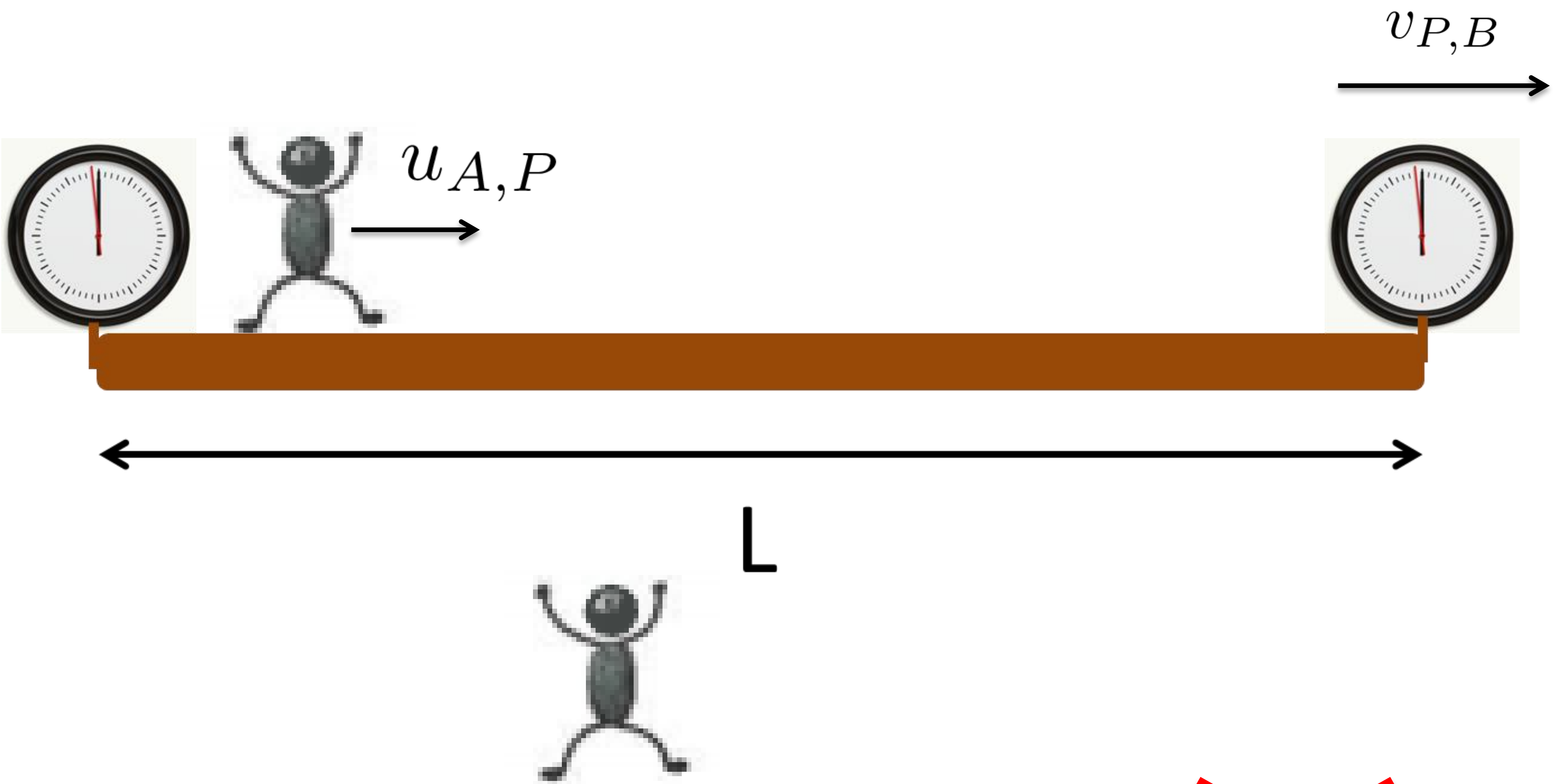
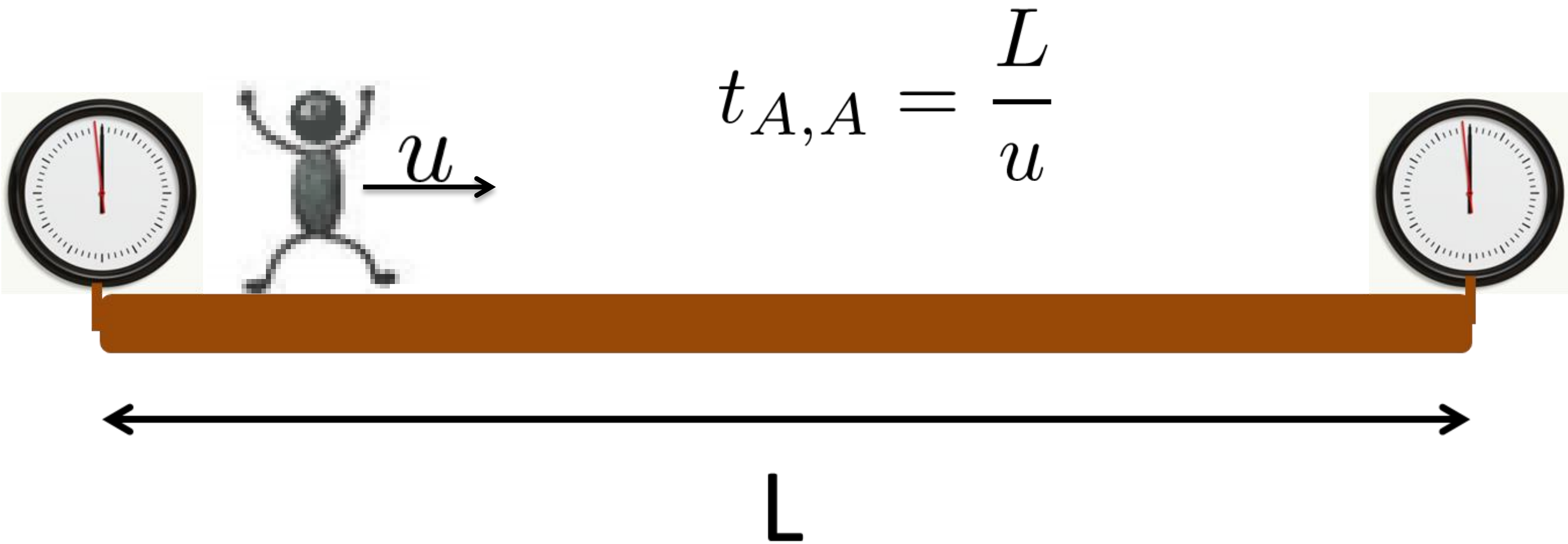
Velocity Addition



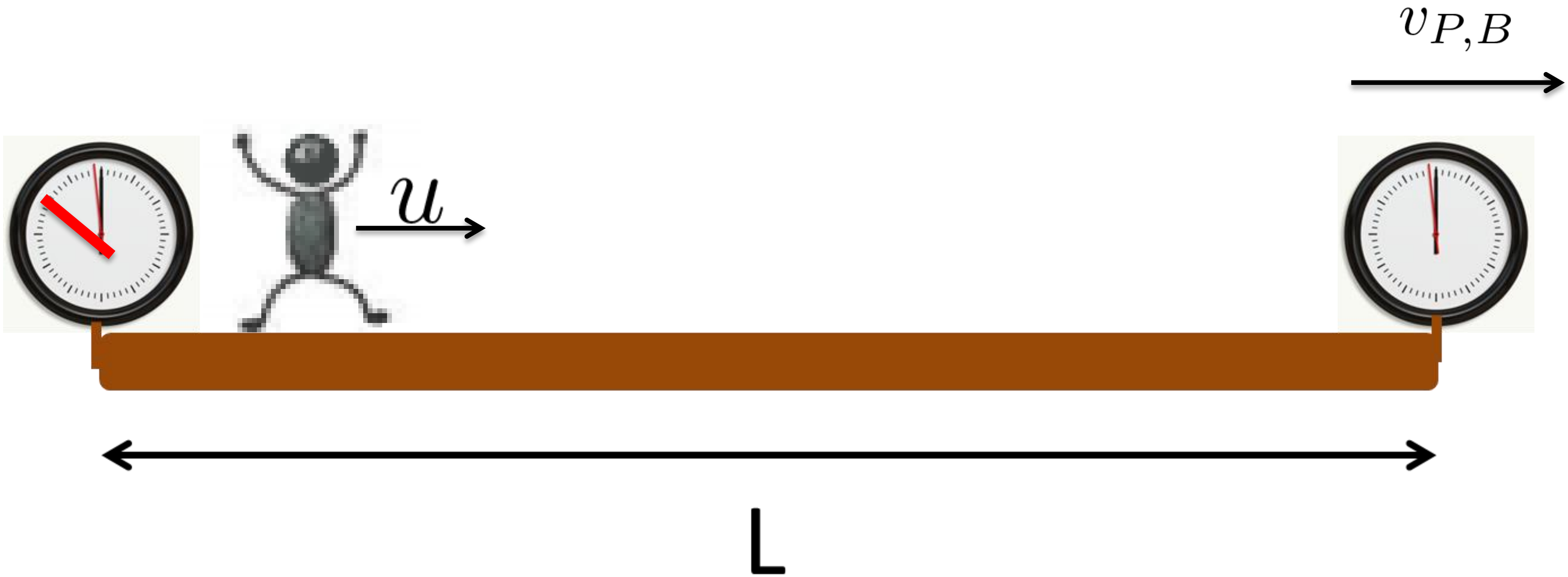
$$v_{A,B} = u_{A,P} + \cancel{v_{P,B}}$$

Alice’s Frame



$$t_{A,A} = \frac{L}{u}$$

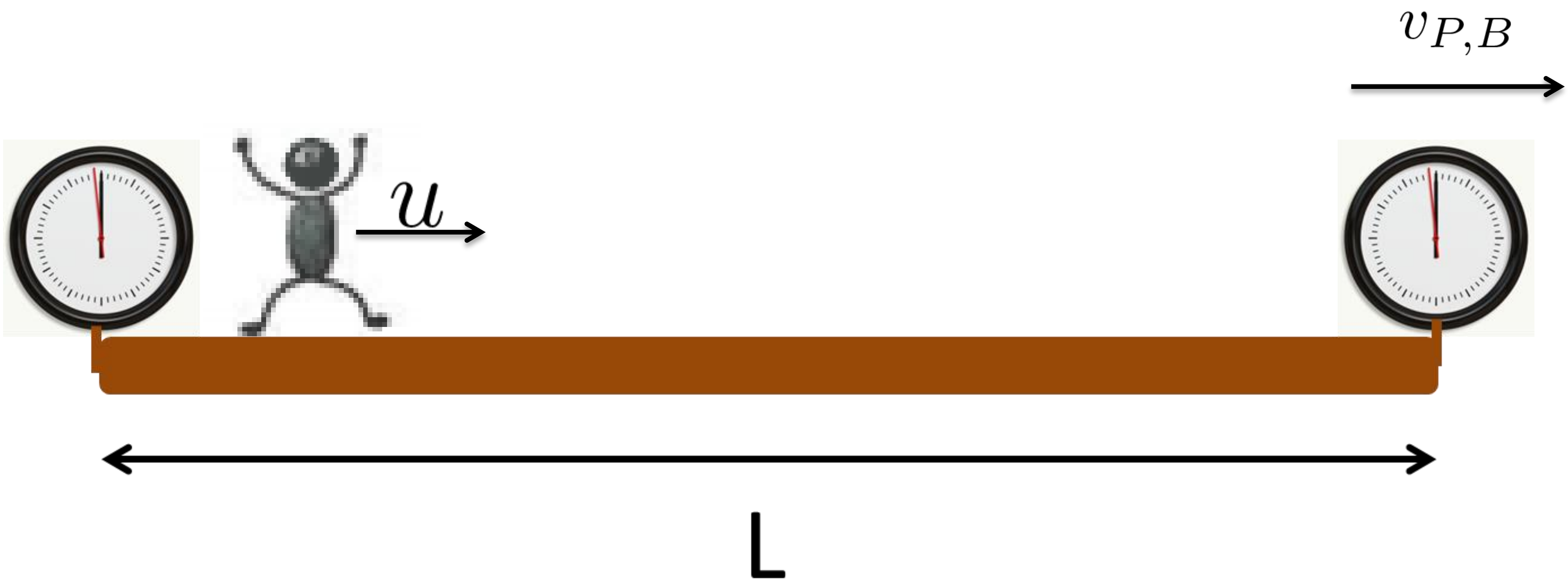
Bob's Frame



The clocks are not synchronized according to Bob!

Bob sees the clock at the back actually reading $\frac{Lv}{c^2}$ seconds the instant the front clock starts

Bob's Frame



Bob sees Alice's clock run slow by a factor: $\sqrt{1-\frac{v^2}{c^2}}$

Time of Alice's Walk in Bob's Frame

- Two effects to consider:
  - Lack of synchronicity
  - Time Dilation

- Bob sees Alice arrive at front when the front clock reads

$$L/u$$

- But back clock reads

$$L/u + Lv/c^2$$

Time of Alice's Walk in Bob's Frame

- We also need to account for time dilation

$$\sqrt{1 - \frac{v^2}{c^2}}$$

- Bob measures:

$$t_{walk} = \frac{L/u + Lv/c^2}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Distance Walked by Alice in Bob's Frame

- According to Bob, Alice walks the **length of the train** plus the **distance the train has moved** in the time it takes her to complete the walk.

- To Bob, the train is length contracted so the length of the train is:

$$L\sqrt{1-\frac{v^2}{c^2}}$$

- To Bob, the distance travelled by Alice is:

$$d=\frac{\frac{Lv}{u}+\frac{Lv^2}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}+L\sqrt{1-\frac{v^2}{c^2}} \longrightarrow d=\frac{L(1+v/u)}{\sqrt{1-\frac{v^2}{c^2}}}$$

Alice’s Speed Relative to Bob

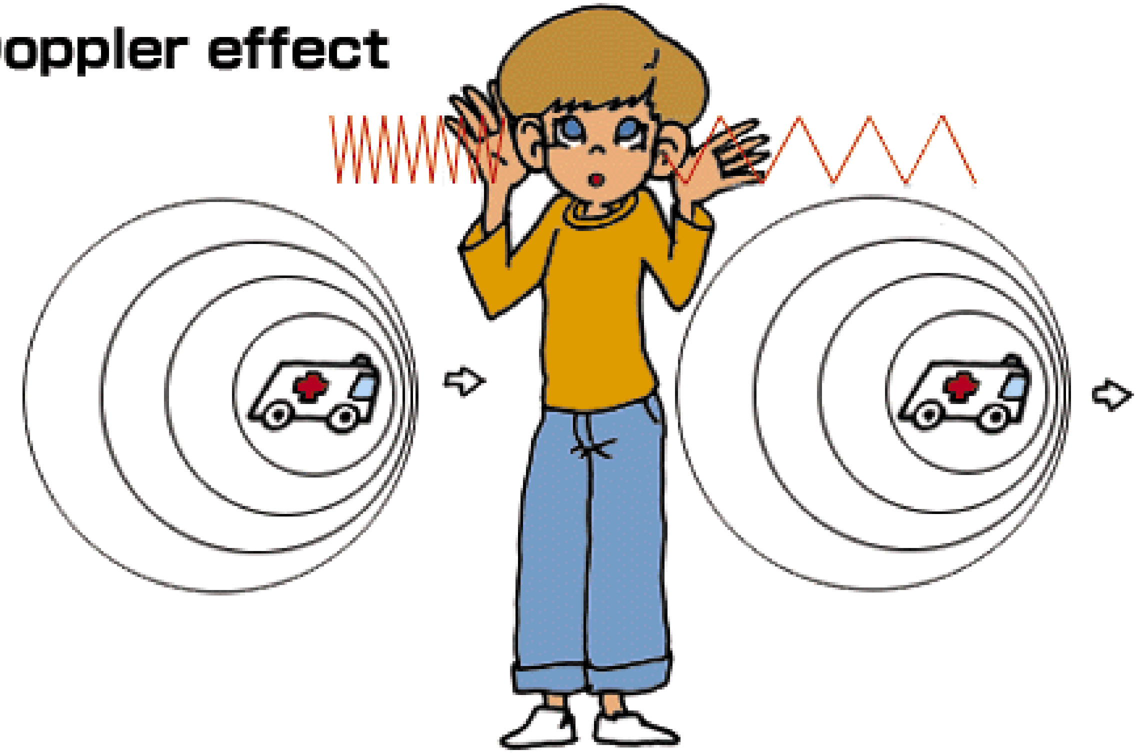
- Will be the distance she travels as measured by Bob, divided by the time as measured by Bob

$$v_{AB} = \frac{d}{t} = \frac{u + v}{1 + \frac{uv}{c^2}}$$

Comparisons

- When speeds are much less than c the results are the same.
- After the second rocket fires her speed is 0.8c. After the third rocket fires her speed is 0.93c. Her speed approaches c, but never reaches.

Doppler effect



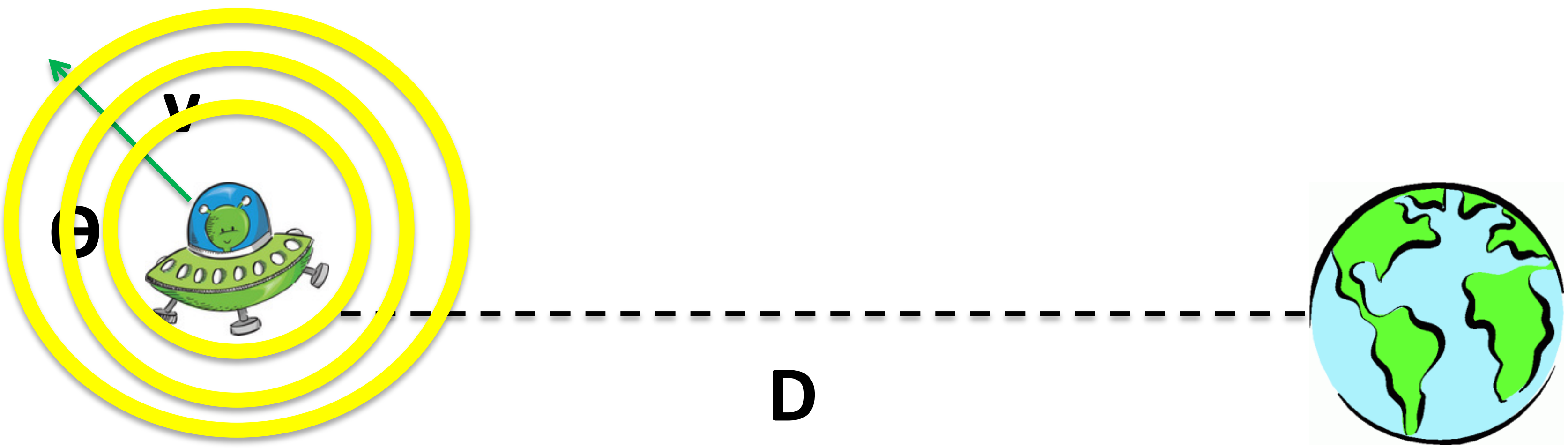
Doppler Shift

- Sound, light, water waves emitted at some frequency by a moving object are perceived at a different frequency by a stationary observer

$$f' = f \cdot \frac{1}{1 \pm \frac{v}{c}} \qquad \lambda' = \lambda \cdot (1 \pm \frac{v}{c})$$

-ve if approaching
+ve if receeding

Relativistic Doppler Shift



Alien emits pulses of light at intervals of time $\Delta\tau_e$

If the light pulse emitted at time 0 arrives at Earth at time t=D/c, how much later does the next pulse arrive?

Interval Between Receiving Pulses

- Moving clocks run slow

- Next pulse is emitted at: $\Delta t_e = \gamma \Delta \tau_e$

- However, the alien has moved away by:

$$\Delta x = v \Delta t_e \cos\theta$$

- The flash takes an additional time: $\dfrac{\Delta x}{c}$

Interval Between Receiving Pulses

- The time interval between receiving pulses is:

$$\Delta t_r = \Delta t_e + \frac{v}{c}\Delta t_e \cos\theta = (1 + \beta\cos\theta)\gamma\Delta\tau_e$$

$$\beta = \frac{v}{c}$$

Crests of a Light Wave

- Instead of flashes consider successive crests of an EM wave.

$$\Delta t_r = (1 + \beta \cos\theta)\gamma\Delta\tau_e$$

- If the observed period is longer than the rest frame period, the observed frequency is lower and the light is shifted to the red.

Redshift

$$(1+z) \equiv \frac{\Delta t_r}{\Delta \tau_e} = \gamma(1+\beta\cos\theta)$$

$$\theta = 0$$

$$(1+z) = \gamma(1+\beta) = \sqrt{\frac{1+\beta}{1-\beta}}$$

$$\theta = \pi$$

$$(1+z) = \gamma(1-\beta) = \sqrt{\frac{1-\beta}{1+\beta}}$$

Blueshifted

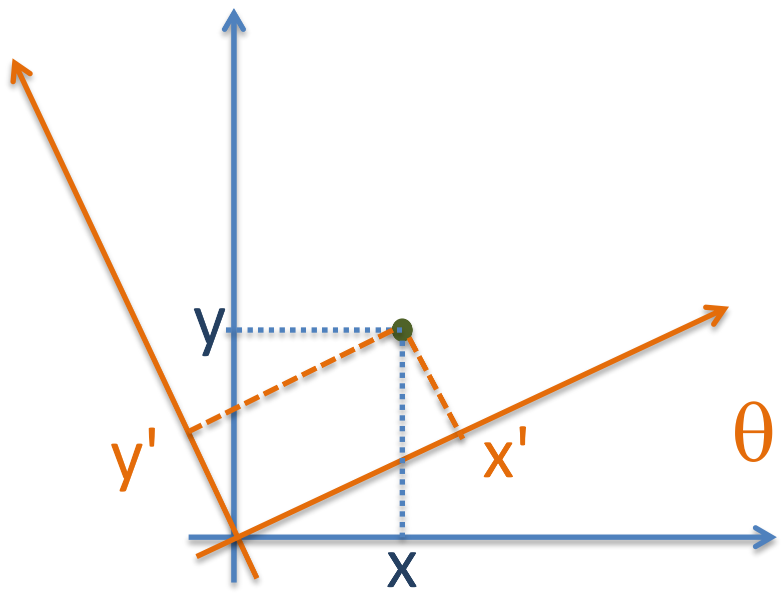
Redshift

$$(1+z) \equiv \frac{\Delta t_r}{\Delta \tau_e} = \gamma(1+\beta\cos\theta)$$

$$\theta = \frac{\pi}{2}$$

We still get a ɣ-factor! This is different from ordinary redshift and is known as **second order redshift**.
It has been observed with pulsars.

Changing Coordinate Systems



$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

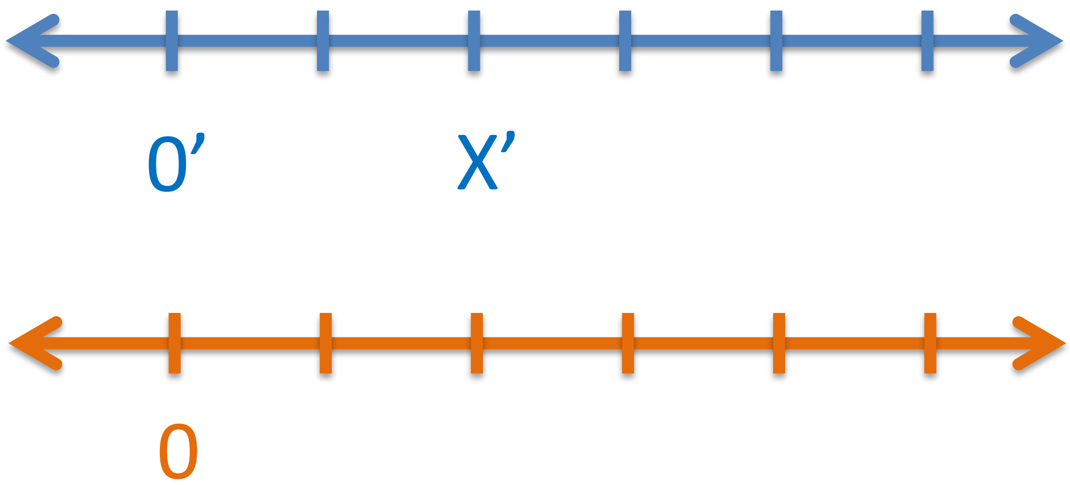
So the coordinates (x',y') of the point (x,y)

$$x' = x\cos\theta - y\sin\theta,$$
$$y' = x\sin\theta + y\cos\theta.$$

Can we develop a coordinate transformation that would take us from one frame to another?

Can we find a coordinate transformation to take us from the frame at rest to the frame in motion?

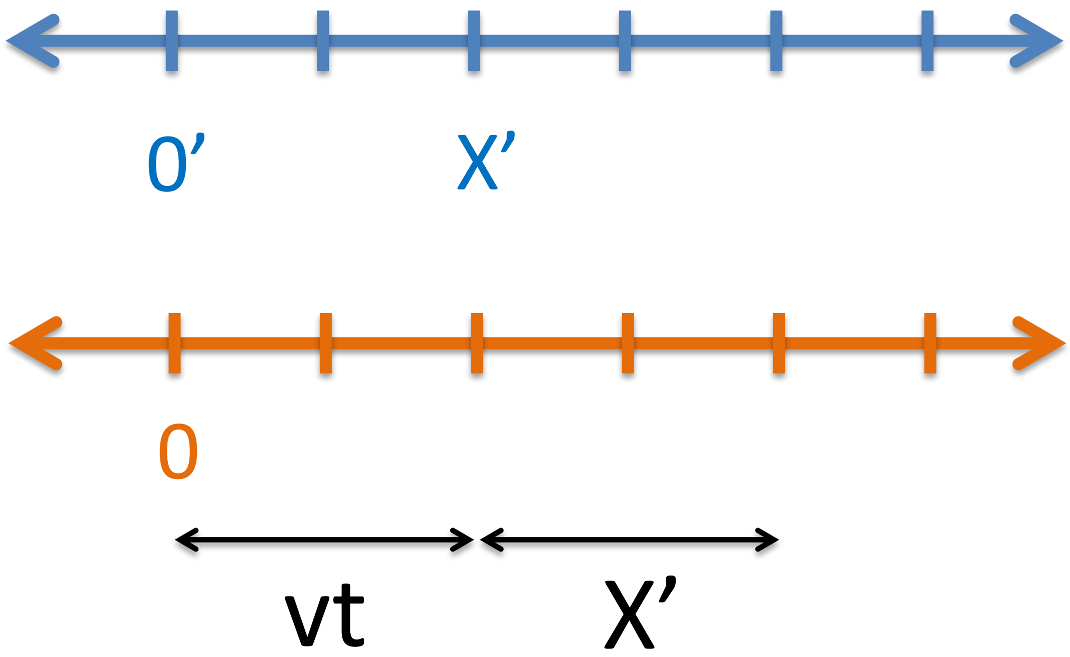
Galileo & Newton



Galileo & Newton

$t = t'$

$x = x' + vt$



Galileo & Newton

$$t' = t$$

$$x' = x - vt$$

~~Length~~

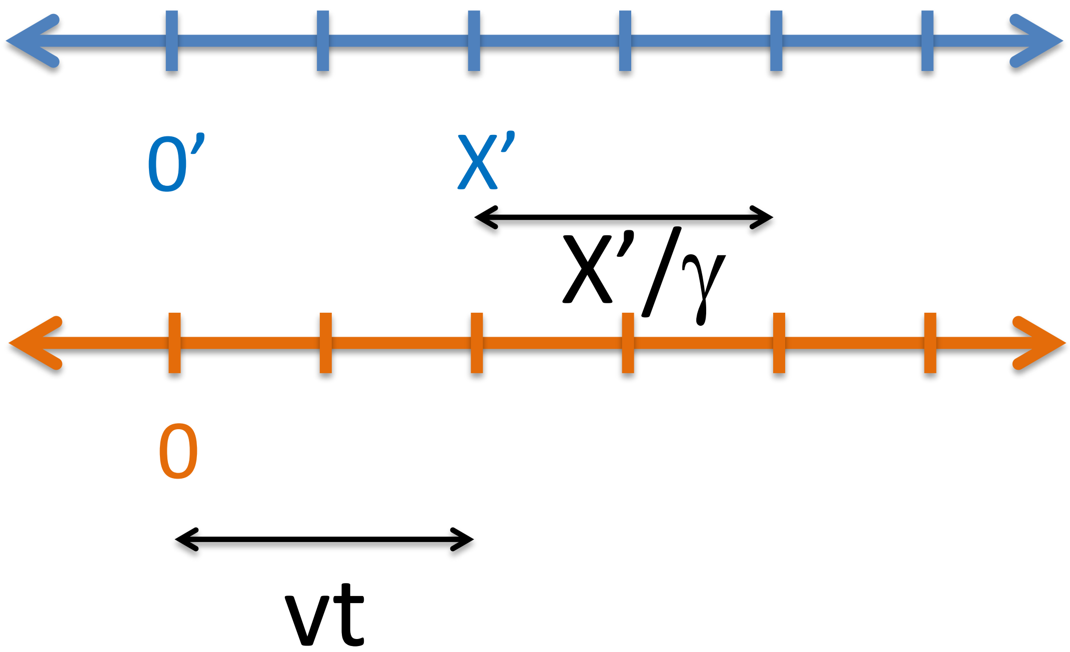
~~Time~~

~~Simultaneity~~

~~Synchronicity~~

Einstein - x

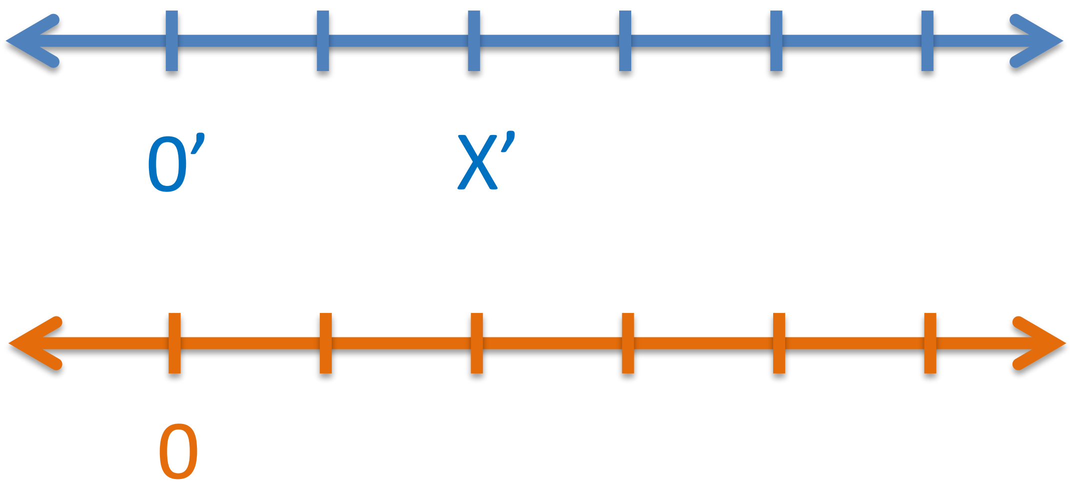
$$x = x'/\gamma + vt$$



Galileo & Newton	**Einstein**
$t' = t$	
$x' = x - vt$	$x' = \gamma(x - vt)$

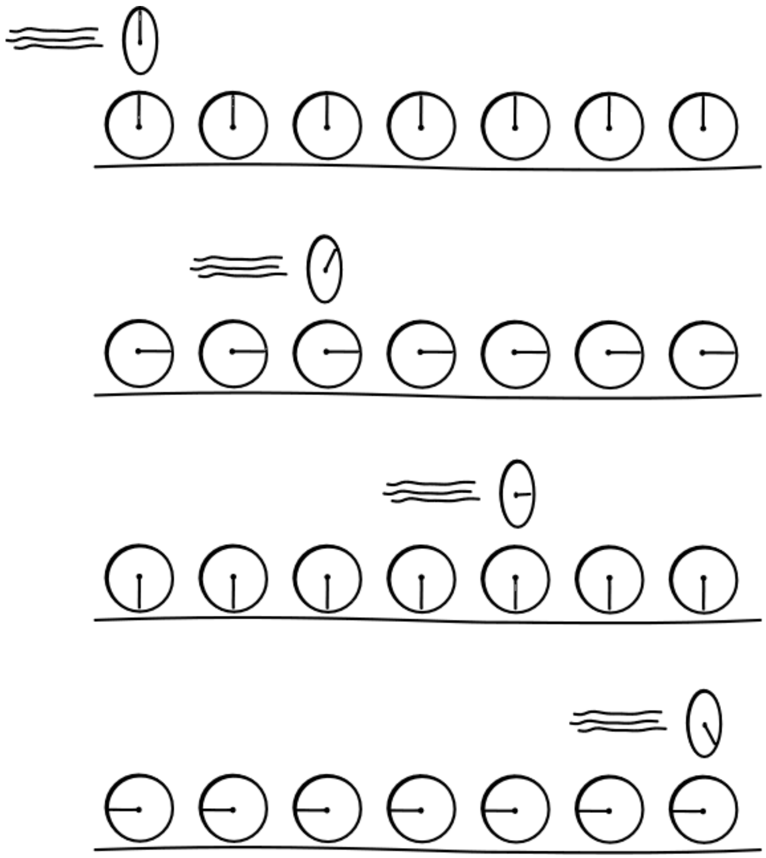
Einstein - t

The clocks are not synchronized so we need to specify where the clocks are!



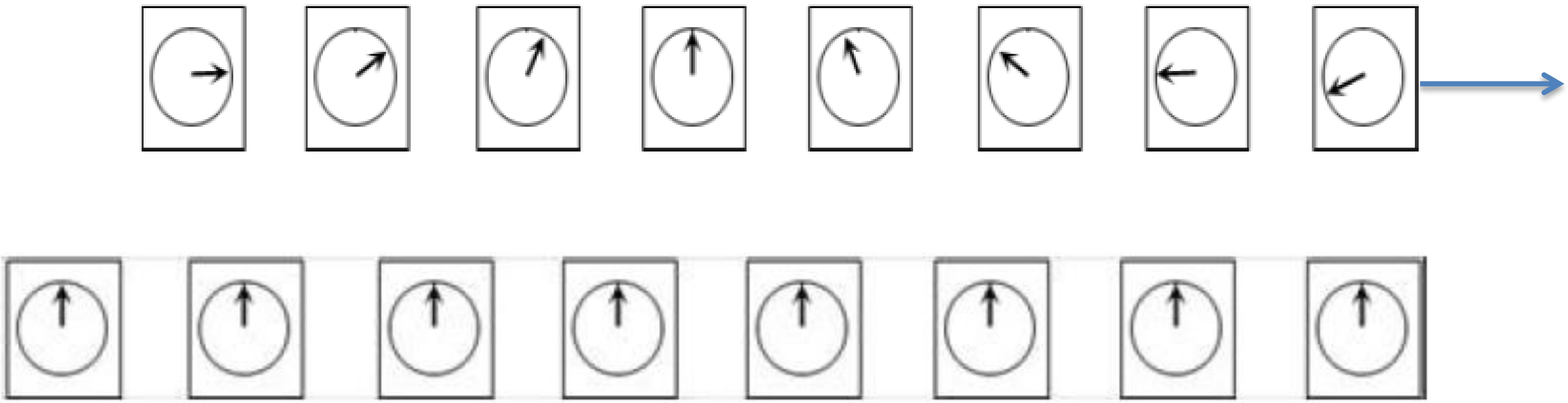
Time Dilation

$$t_{origin} = \gamma t'_{origin}$$



Lack of Synchronicity

$$t'_{x'} = t'_{origin} - \frac{vx'}{c^2}$$



Einstein - t

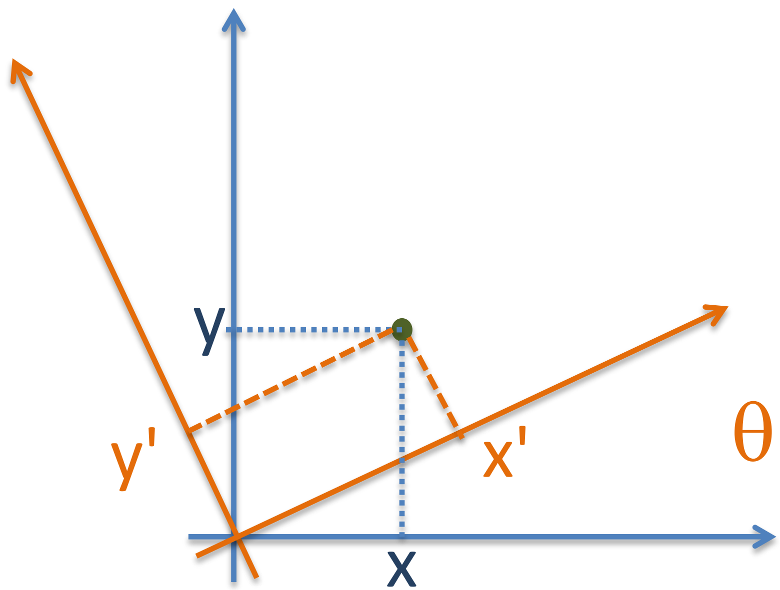
$$t'_{x'} = t'_{origin} - \frac{vx'}{c^2} \longrightarrow t_{origin} = \gamma t'_{origin}$$

$$t_{origin} = \gamma(t' + \frac{vx'}{c^2})$$

$$t' = \gamma(t - \frac{vx}{c^2})$$

Galileo & Newton	**Einstein**
t' = t	$t' = \gamma(t - \frac{vx}{c^2})$
x' = x - vt	$x' = \gamma(x - vt)$

We found the relation….
But what does it look like?



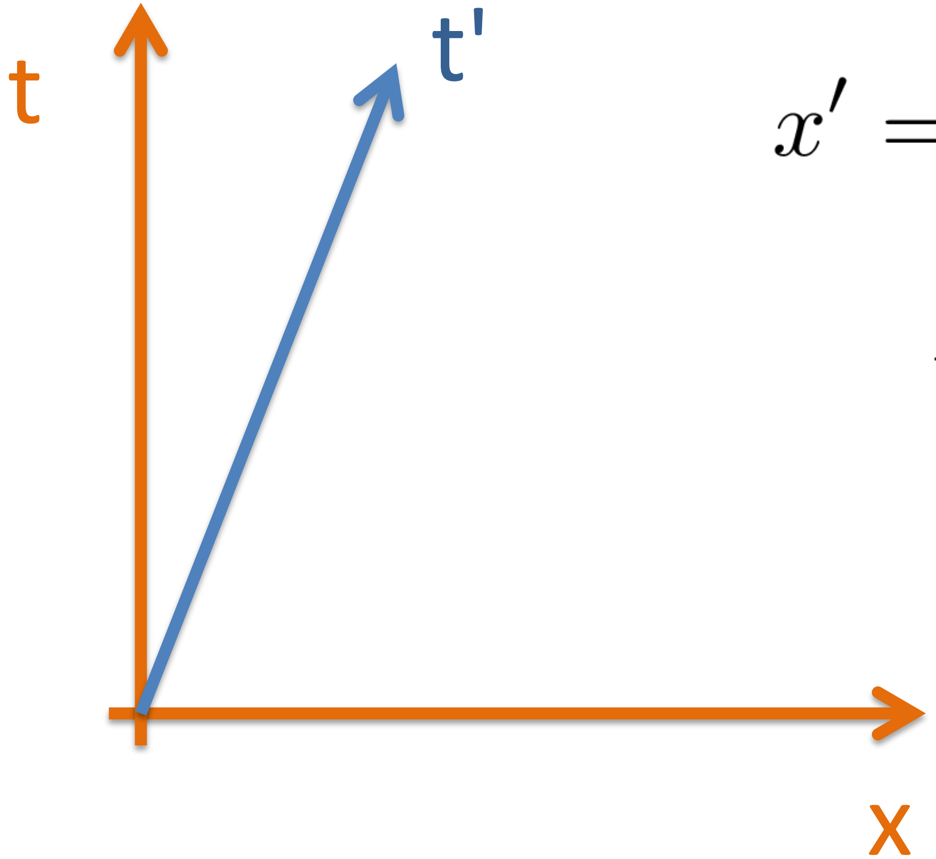
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

So the coordinates (x',y') of the point (x,y)

$$x' = x\cos\theta - y\sin\theta,$$
$$y' = x\sin\theta + y\cos\theta.$$

t’ axis

c = 1

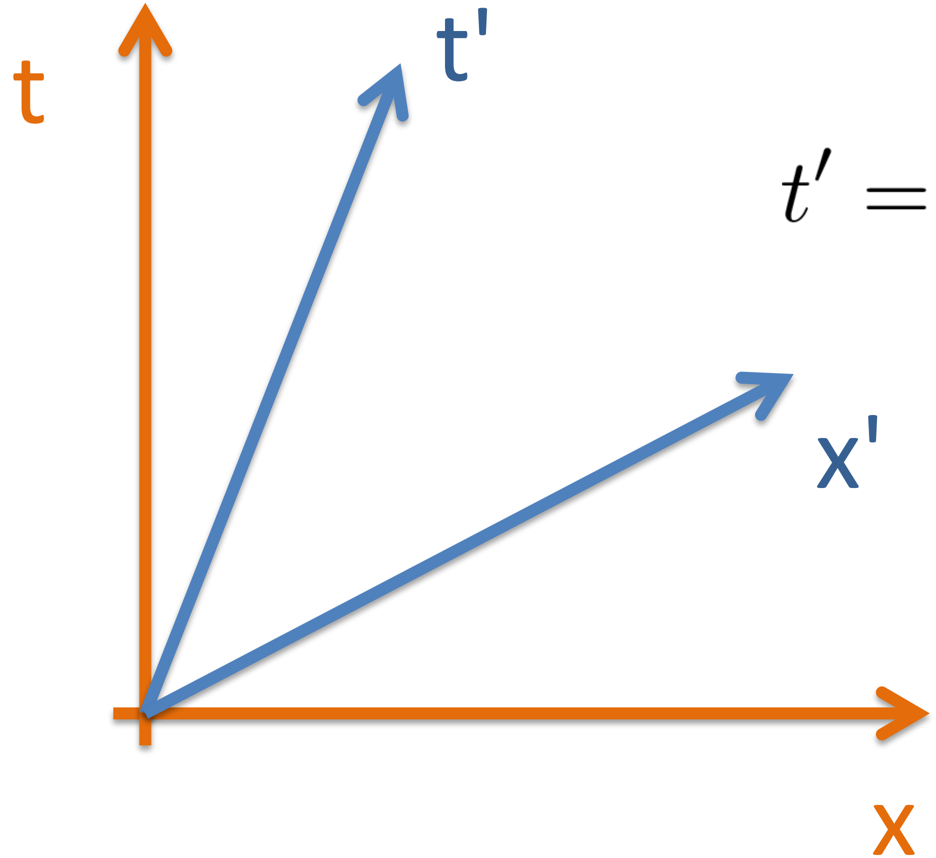


$$x' = \gamma(x - vt)$$

$$t = \frac{x}{v}$$

x’ axis

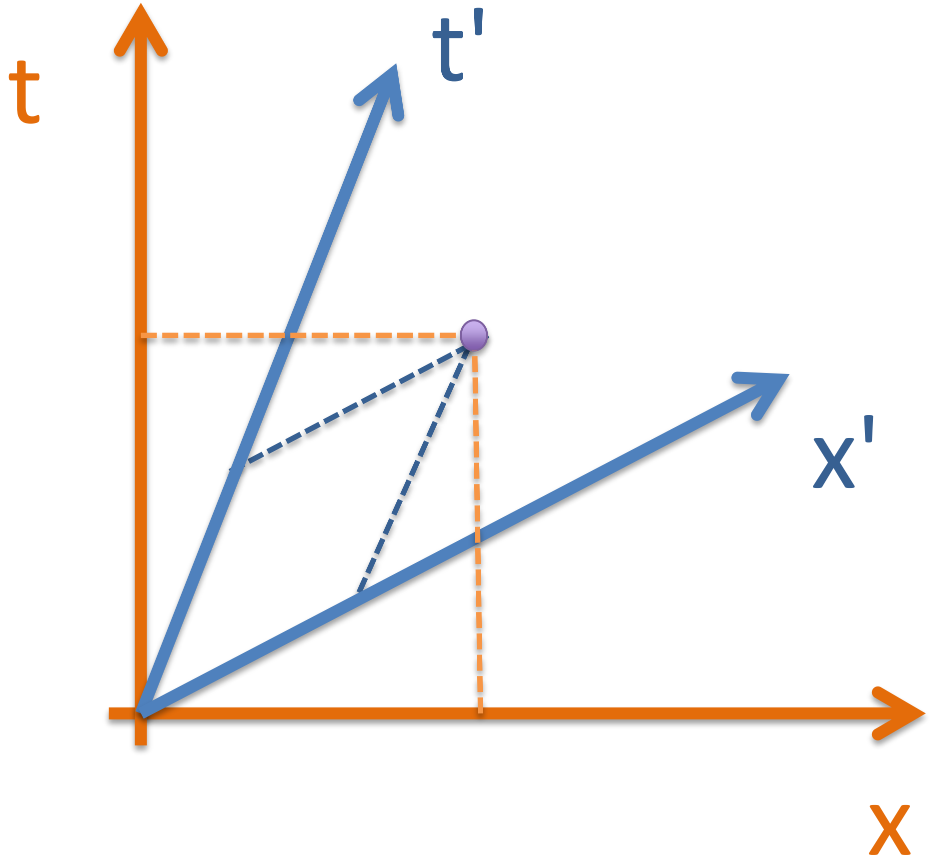
c = 1

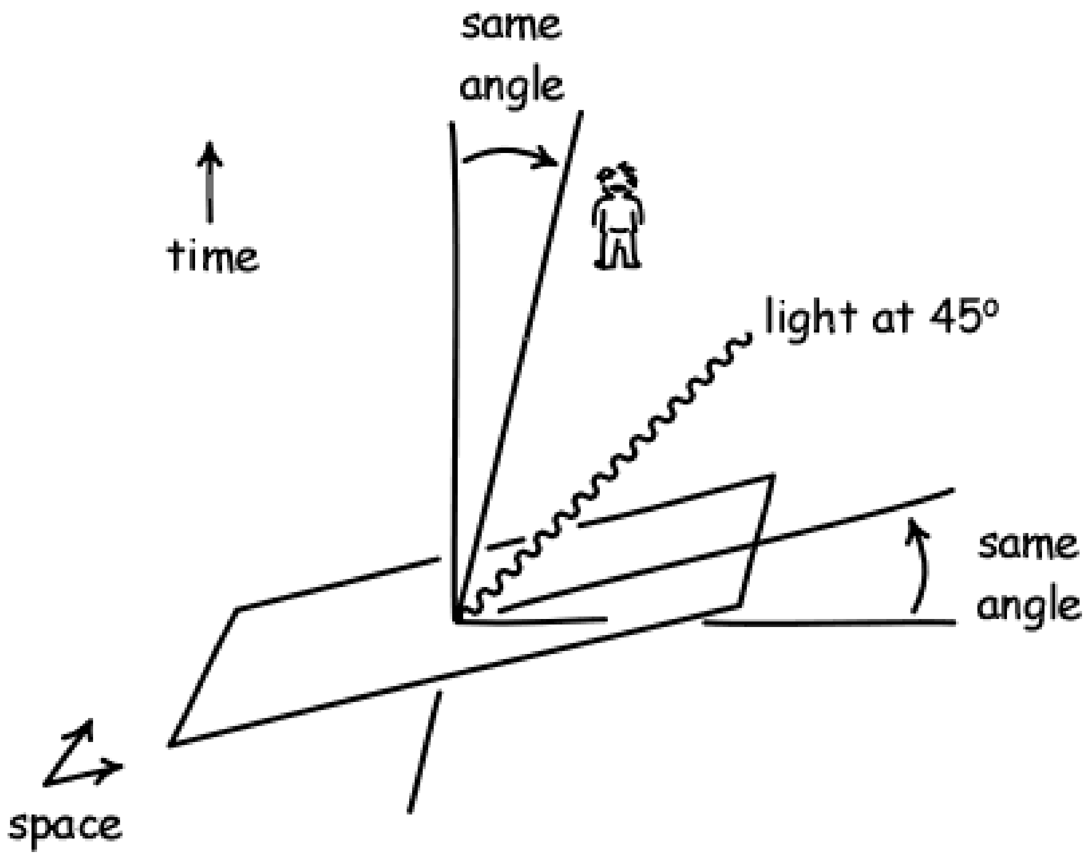


$$t' = \gamma(t - \frac{vx}{c^2})$$

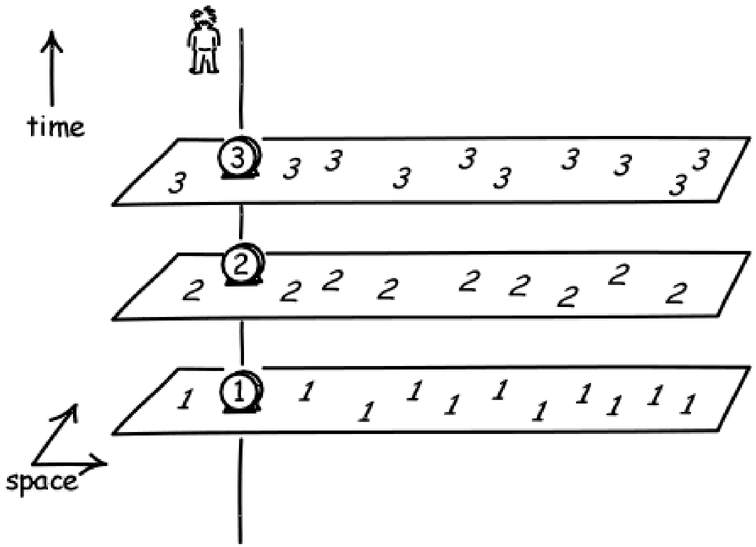
$$t = vx$$

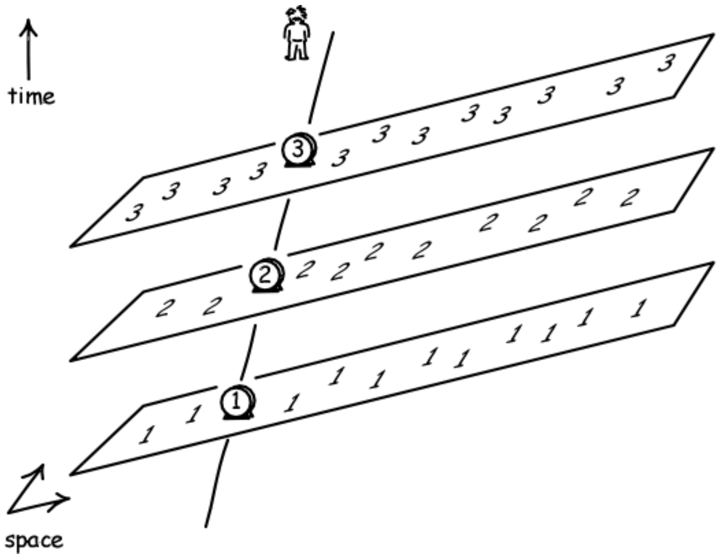
Shear




same
angle
time
light at 45°
same
angle
space

Hypersurfaces of Simultaneity





Galileo & Newton

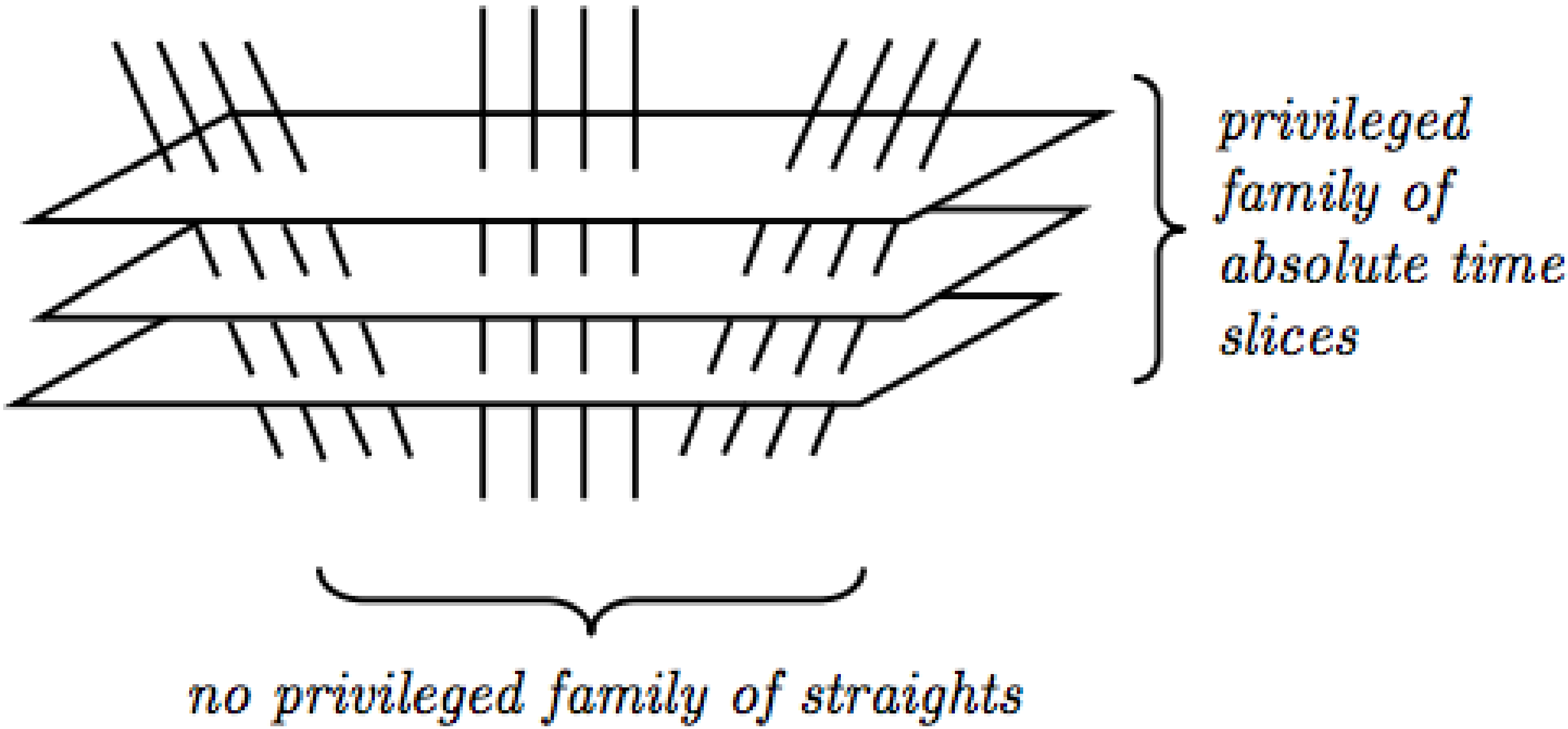


Figure “stolen” from J. Bain

Minkowski

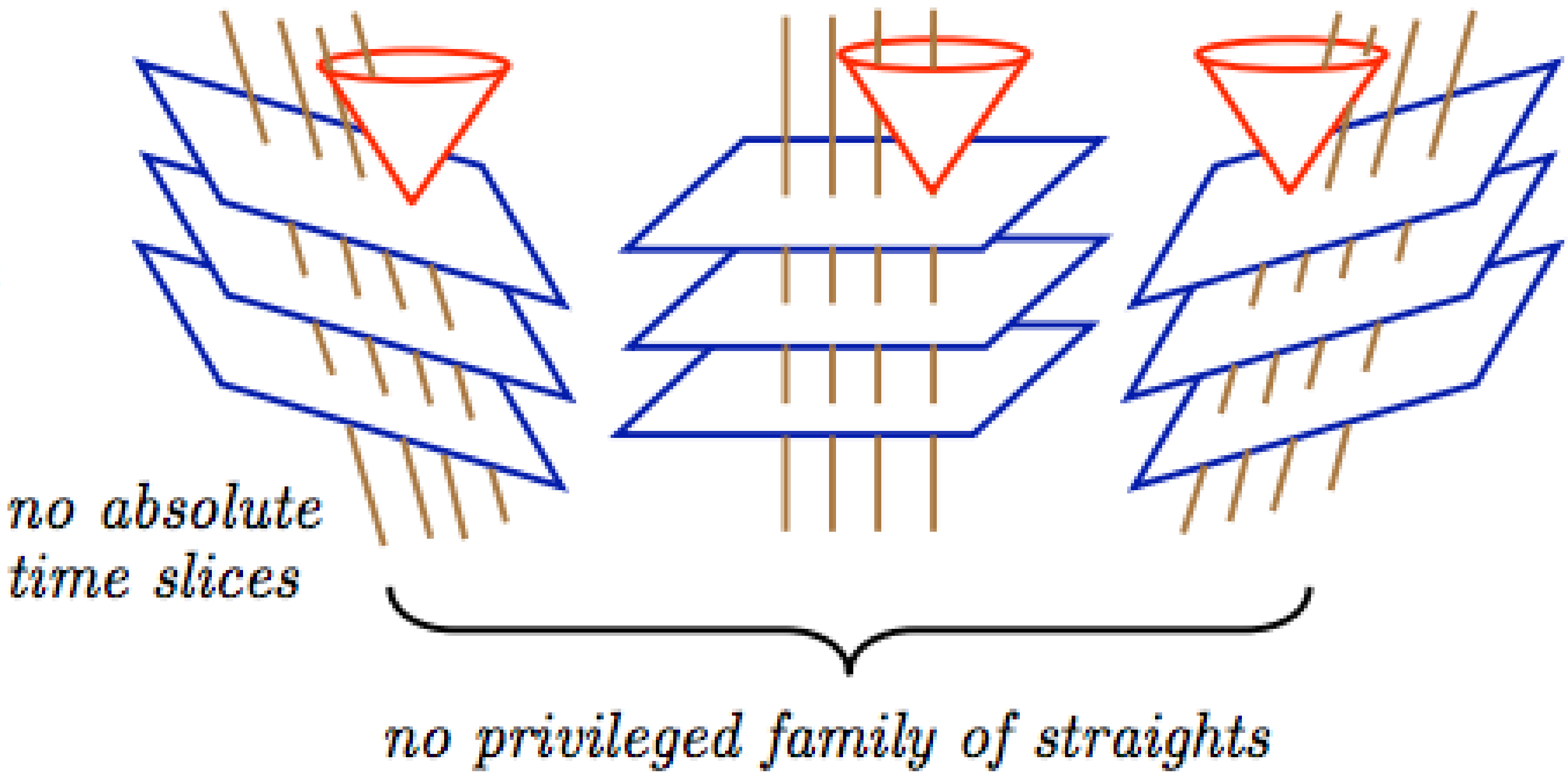
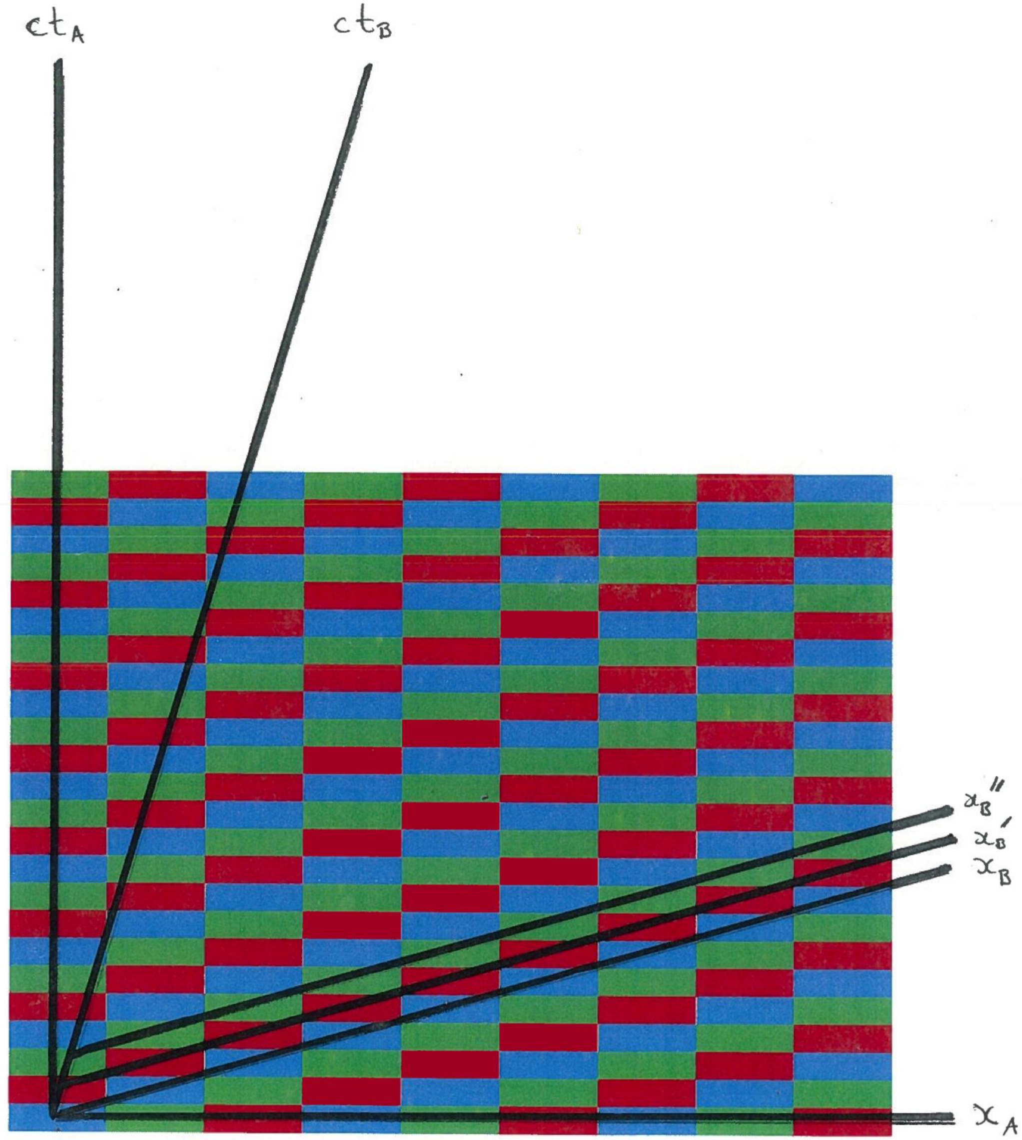


Figure “stolen” from J. Bain



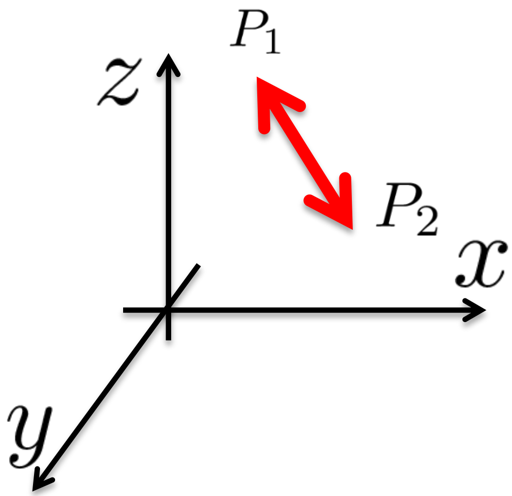
ct_A
ct_B
x_B''
x_B'
x_B
x_A

Can we agree on anything?

- Length X
- Time X
- Simultaneity X
- Synchronicity X

Is there anything that is frame-independent?

Euclidean Space - Invariant



$$(\Delta r)^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

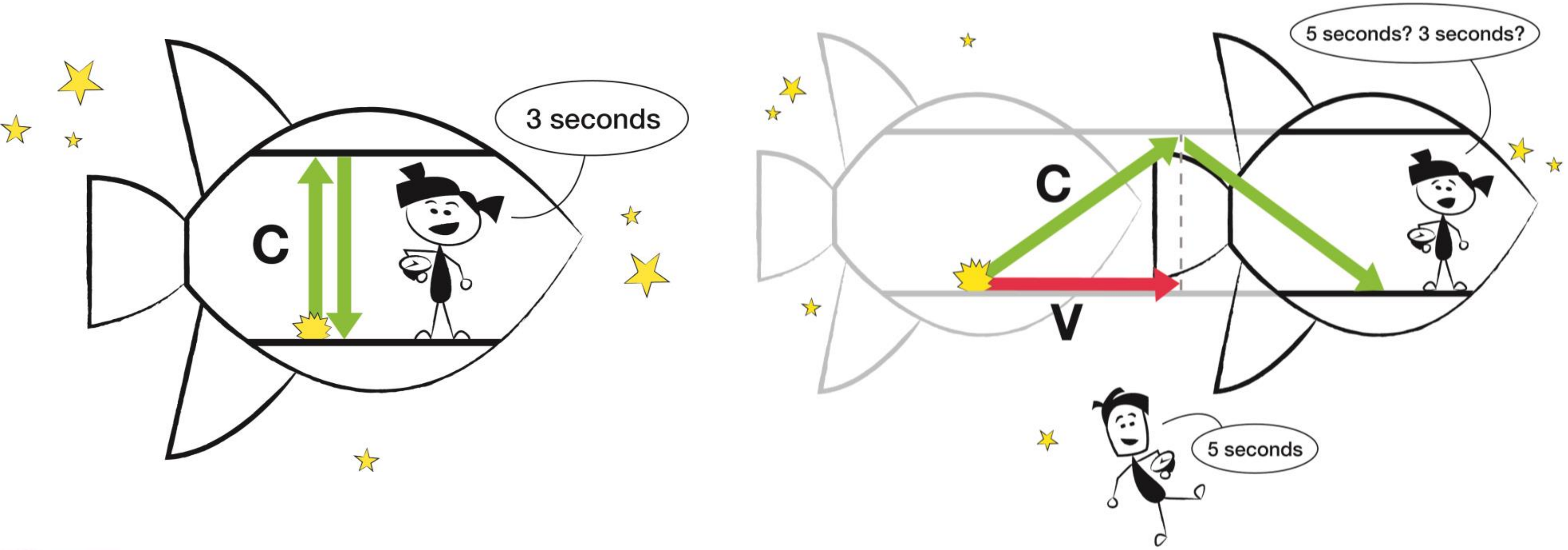
$$(\Delta r)^2 = (\Delta x')^2 + (\Delta y')^2 + (\Delta z')^2$$

In Euclidean space, everyone agrees on the distance between two points!

Invariant

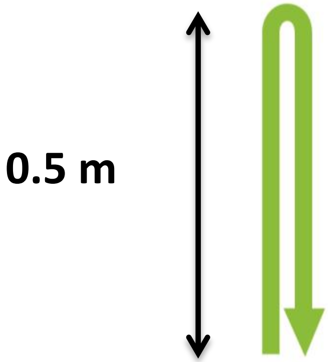
$$(\Delta s)^2 \equiv (c\Delta t)^2 - (\Delta r)^2$$

Alice & Bob

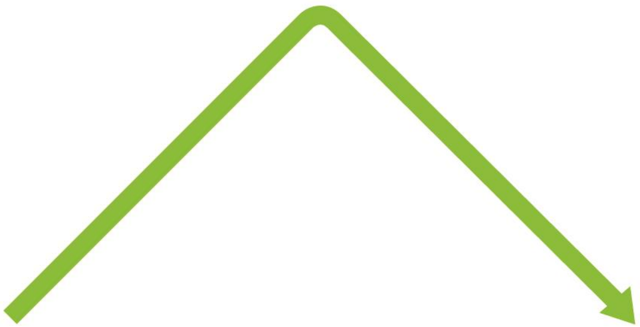


C=1

Alice



Bob



$$(\Delta s)^2 \equiv (c\Delta t)^2 - (\Delta r)^2$$

$$(c\Delta t) = 1$$

$$(\Delta x) = 0$$

$$(\Delta s)^2 = 1$$

$$(\Delta s')^2 \equiv (c\Delta t')^2 - (\Delta r')^2$$

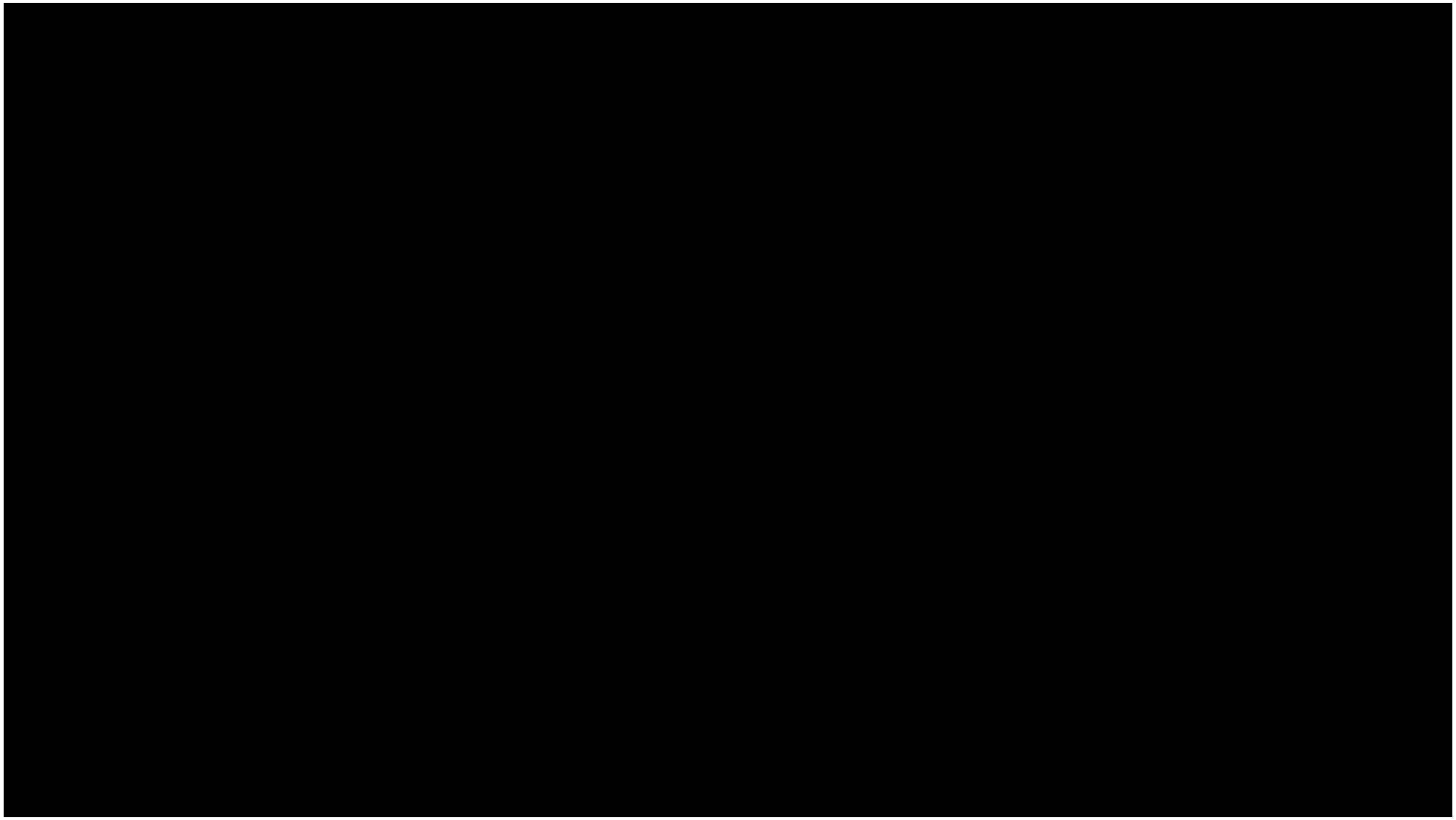
$$(c\Delta t') = \gamma 1$$

$$(\Delta x') = vt' = \frac{\gamma v(1)}{c}$$

$$(\Delta s)^2 = \gamma^2 (1 - \frac{v^2}{c^2})(1) = 1$$

Invariant

- Euclidean versus Minkowski
- Proper time
- Proper length
- “light-like”



Proper Time

- *The time experienced by an observer in whose frame the events take place at the same place.*

For Alice, "send" and "receive" take place at the same place.

Proper Time

- *The time experienced by an observer in whose frame the events take place at the same place.*

- *If the invariant is **positive**:*
  - $(\Delta S)^2 > 0$
  - *There is **<u>always frame</u>** where the events take place at the same place.*
- *If the invariant is **negative**:*
  - $(\Delta S)^2 < 0$
  - *It is still invariant, but there is **<u>no frame</u>** where the events take place at the same place.*

Proper Length

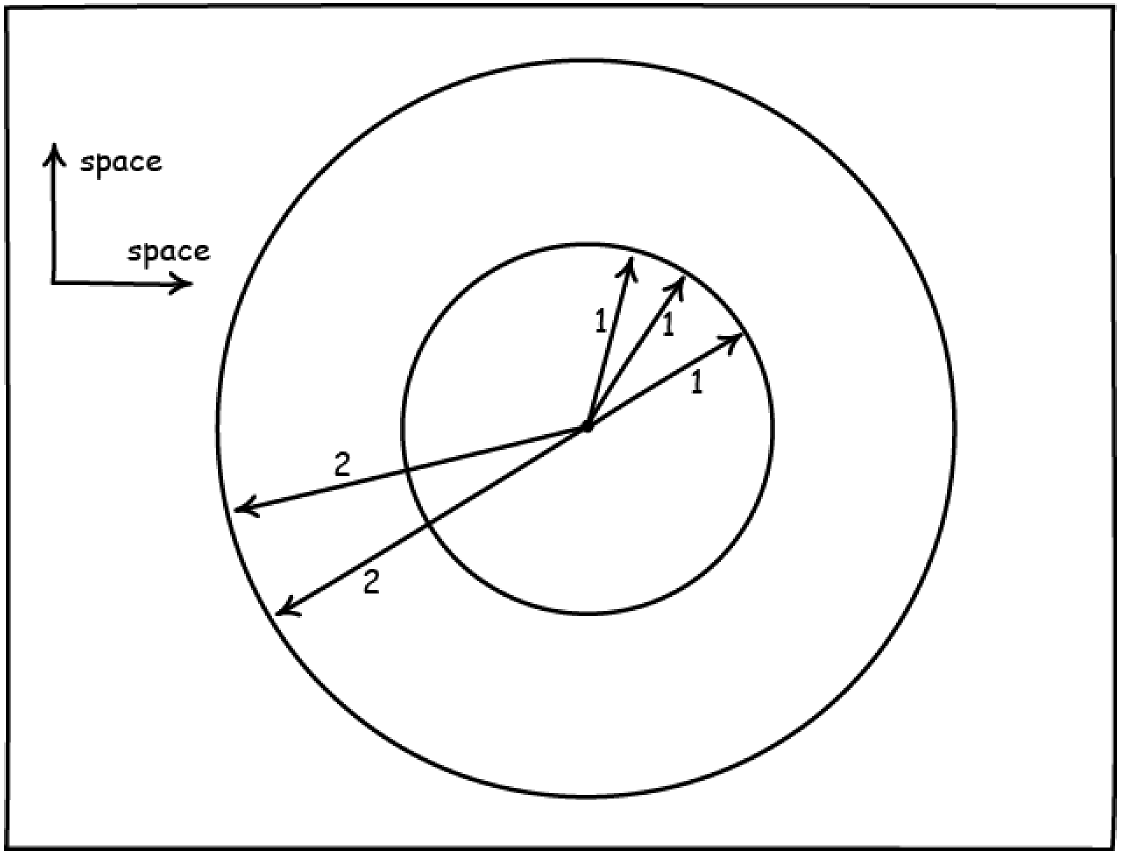
- *The distance between two events in a frame where they occur at the same time!*

Proper Length

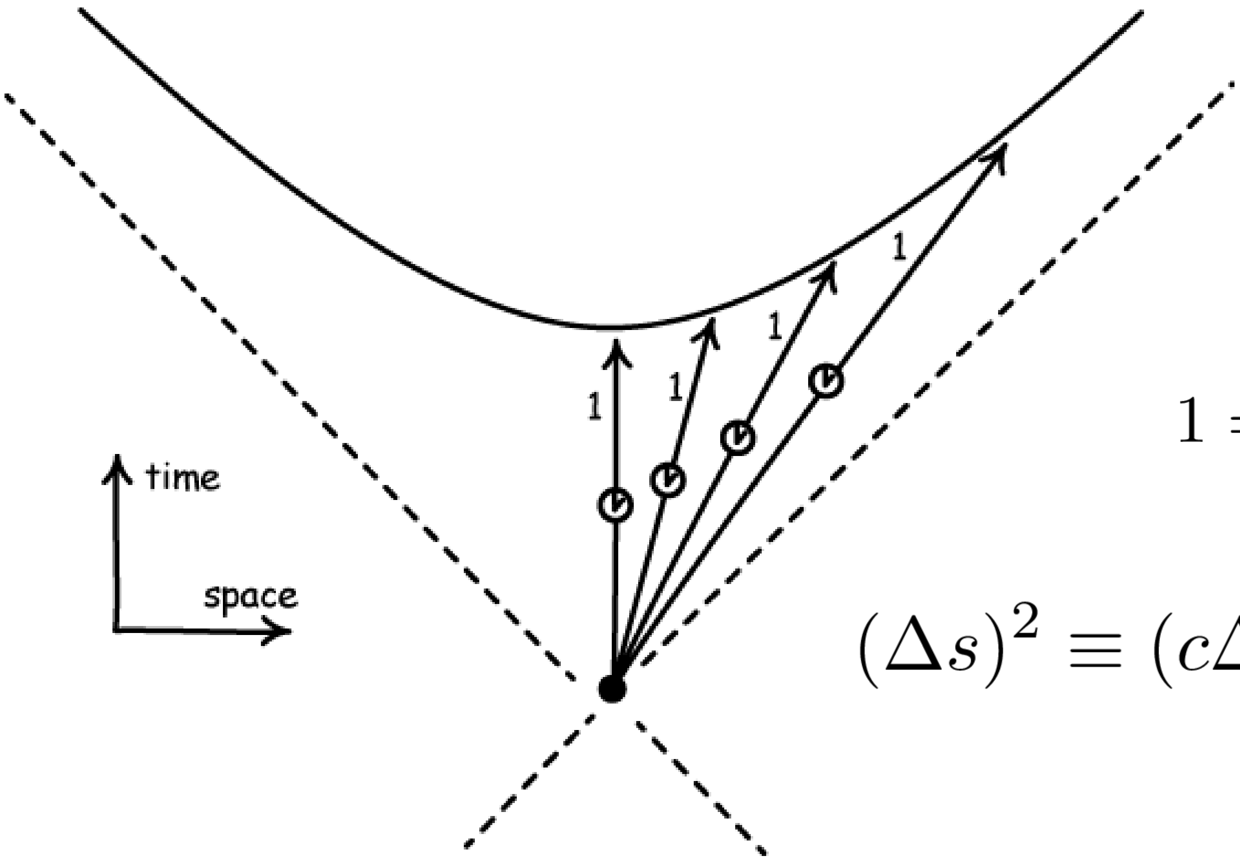
- *The distance between two events in a frame where they occur at the same time!*

- *This interval must be negative!*

Euclidean Geometry



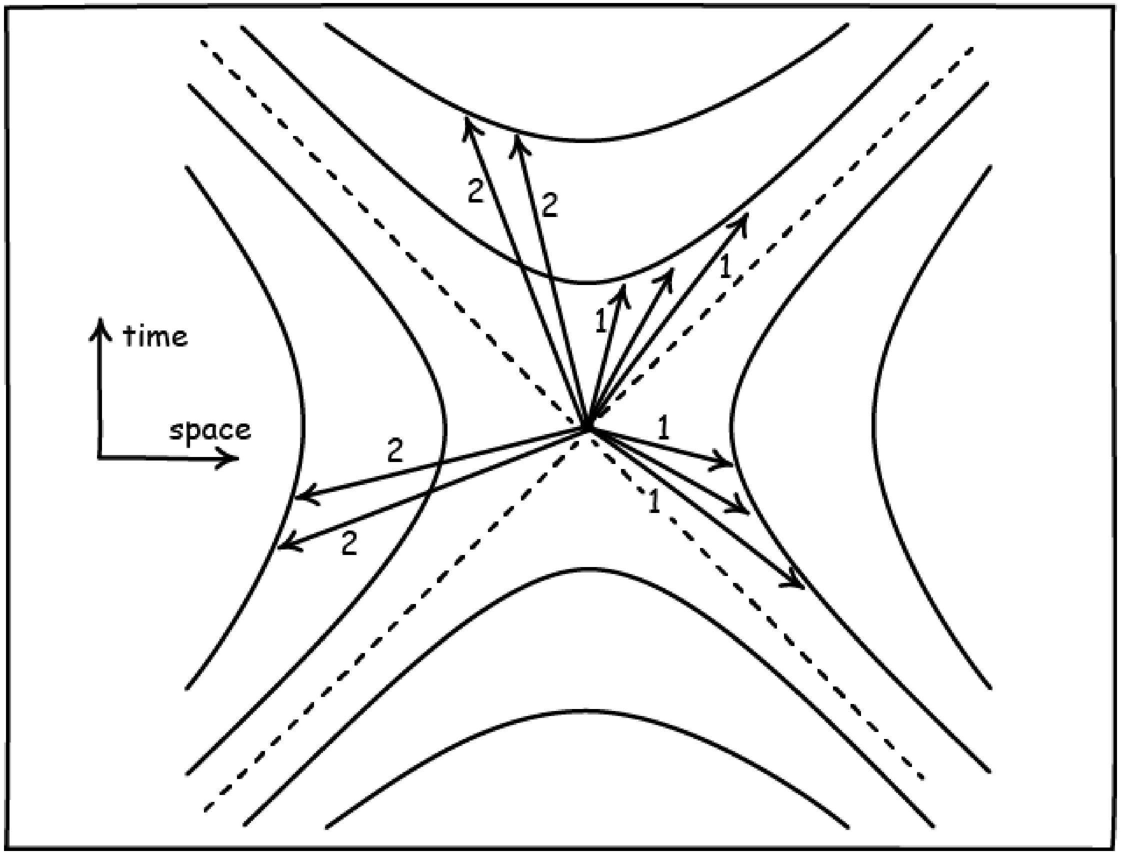
Minkowski Geometry



$$1 = \frac{y^2}{a} - \frac{x^2}{b}$$

$$(\Delta s)^2 \equiv (c\Delta t)^2 - (\Delta r)^2$$

Minkowski Geometry



What happens when the invariant =0 ?

- $(\Delta S)^2 = 0$

LIGHTLIKE

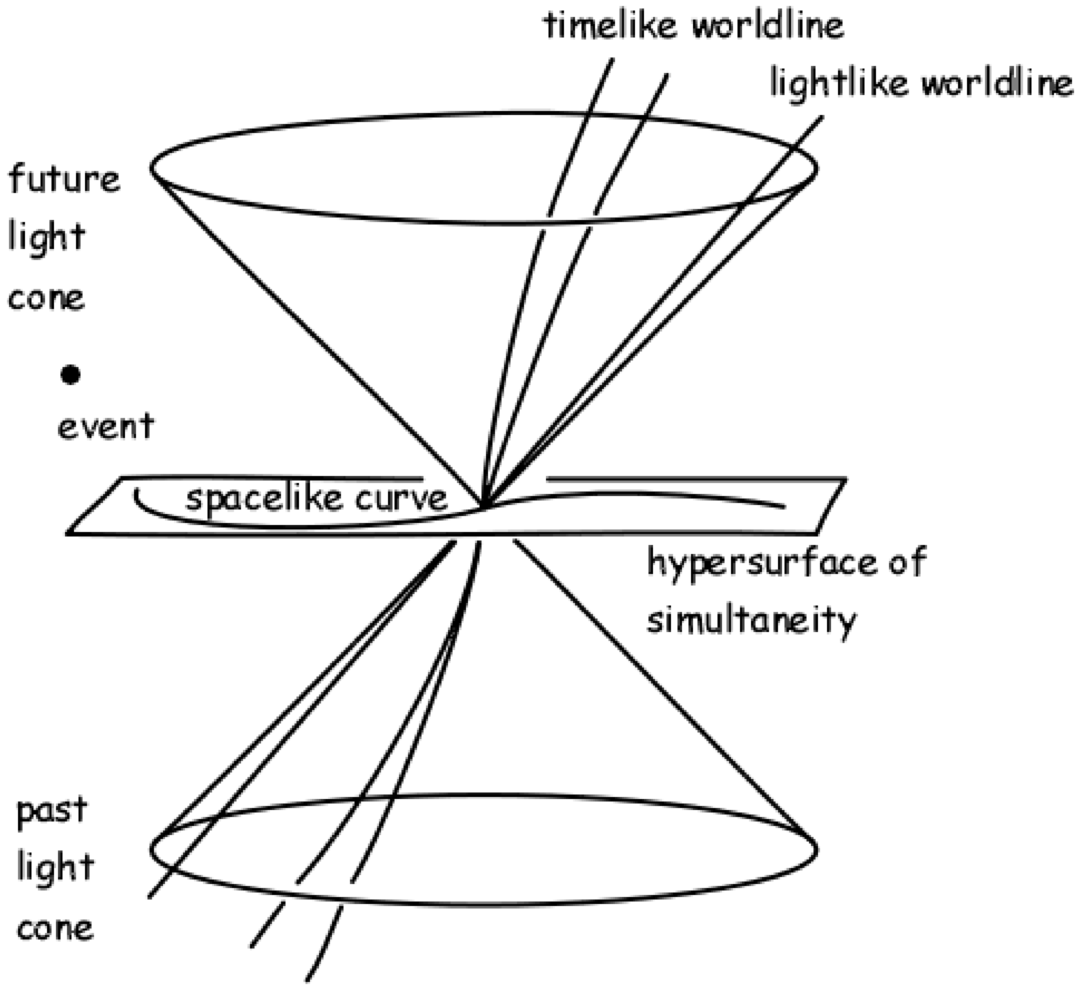
$$(\Delta S)^2 = 0$$

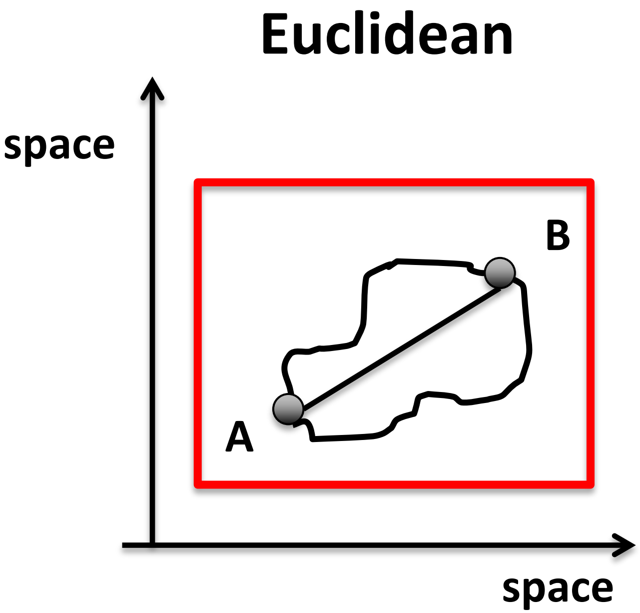
TIMELIKE

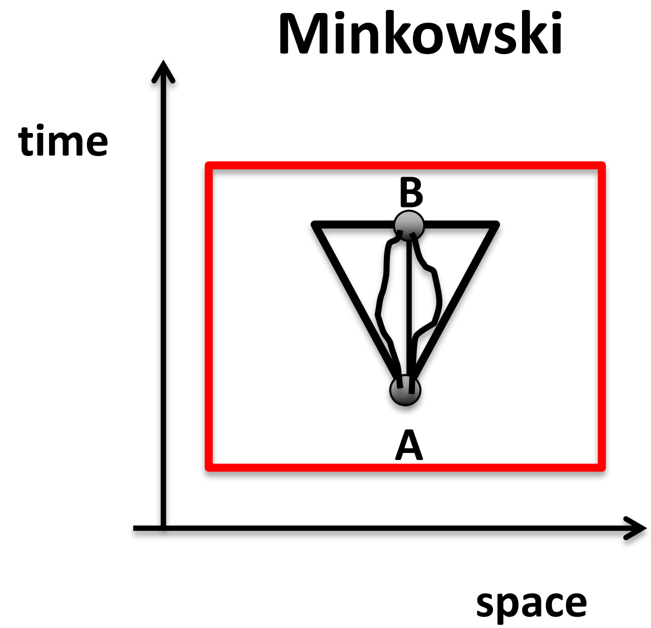
$$(\Delta S)^2 > 0$$

SPACELIKE

$$(\Delta S)^2 < 0$$



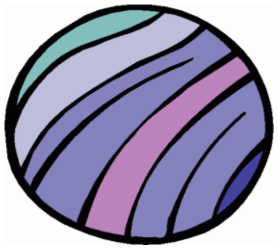




straightest line is the shortest length

straightest line is the *longest* proper time

Alice & Bob – Twin ~~Paradox~~



Fraternal Twins

- Bob stays on Earth and Alice travels to a distant exoplanet 5.2 light years away at speed $\sqrt{0.75}c$
- As soon as she arrives she heads back home
- We ignore acceleration and deceleration and assume she attains the speed instantaneously.

Twin Paradox

- Since Alice is moving and moving clocks run slow she should age less than Bob (by how much?)
- But in Alice's frame, Bob is moving away from her, so shouldn't he age less?
- They both see each other age less! How???

Bob's Frame

- How long does it take Alice to go out and back?

$$TotTime_{Bob} = \frac{5.2 \times 2}{\sqrt{0.75}} = 12$$

What does Bob say Alice measures?

- What time interval does Alice measure?

- Alice is moving so the time interval will be smaller, because her clocks run slow.

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = 2$$

$$TotTime_{Alice} = \frac{TotTime_{Bob}}{\gamma} = 6$$

Distance to Planet Measured by Bob

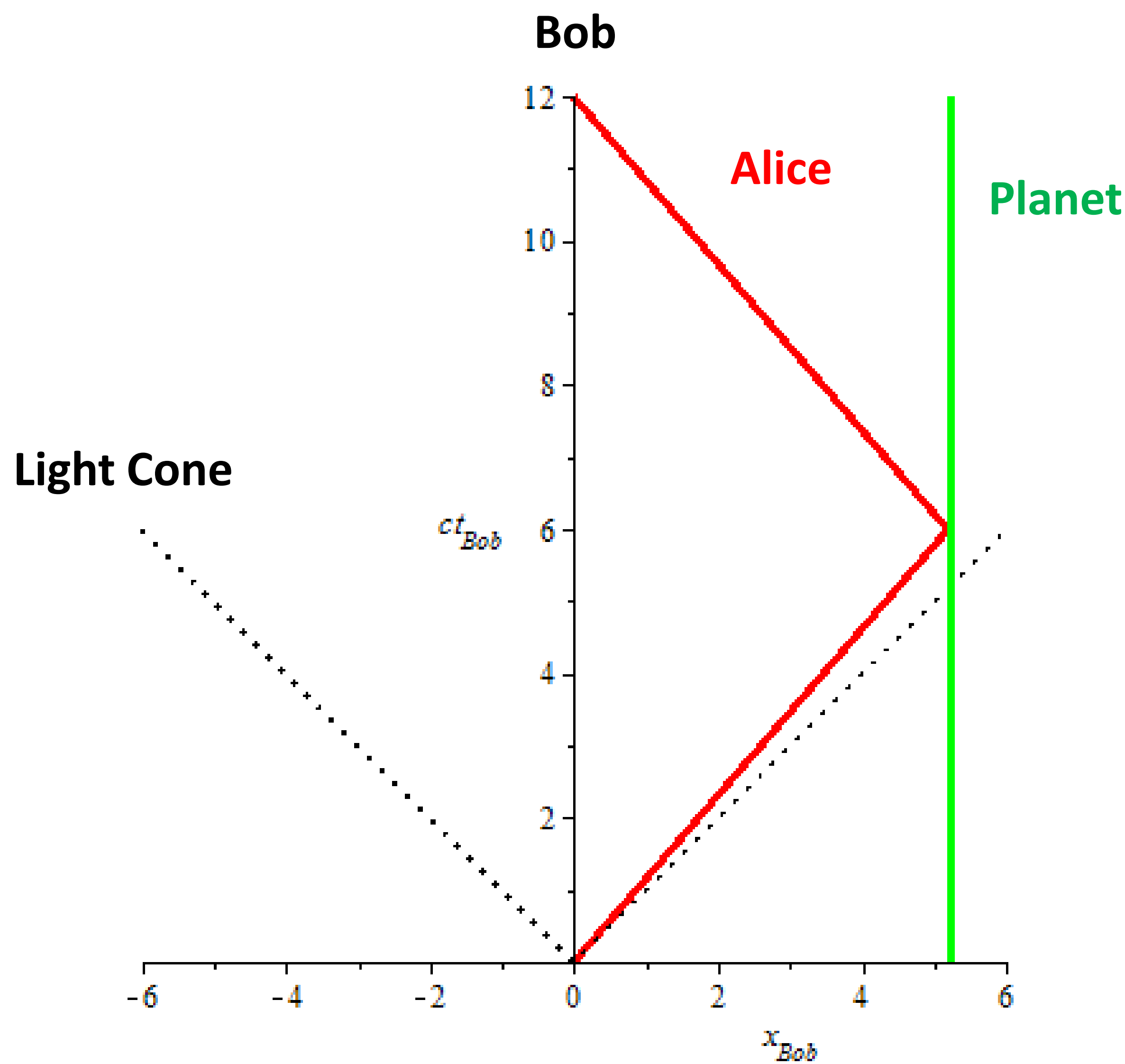
$$x_{Bob,turn} = v \cdot TurnTime_{Bob} = \sqrt{0.75}c \cdot 6 = 5.2$$

Distance to Planet Measured by Alice

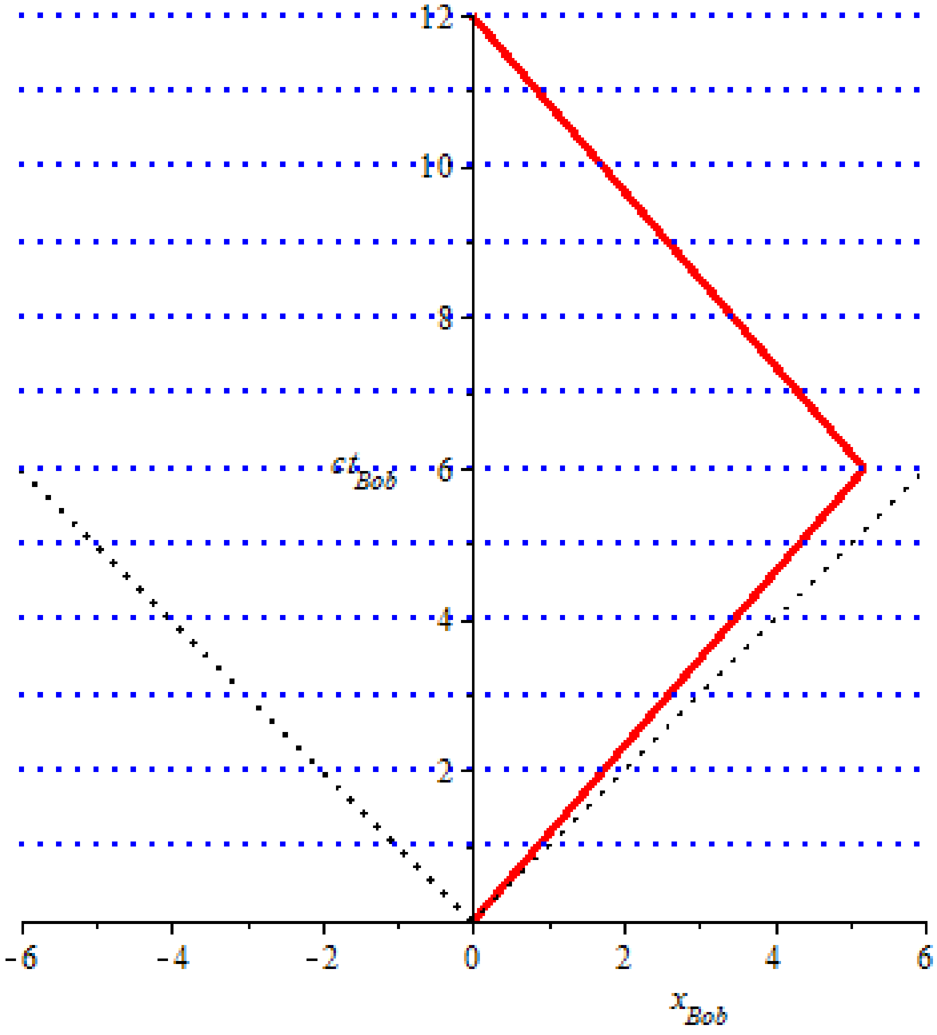
- Alice measures a different distance due to length contraction

$$x_{Alice,turn} = \frac{x_{Bob,turn}}{\gamma} = 2.6$$

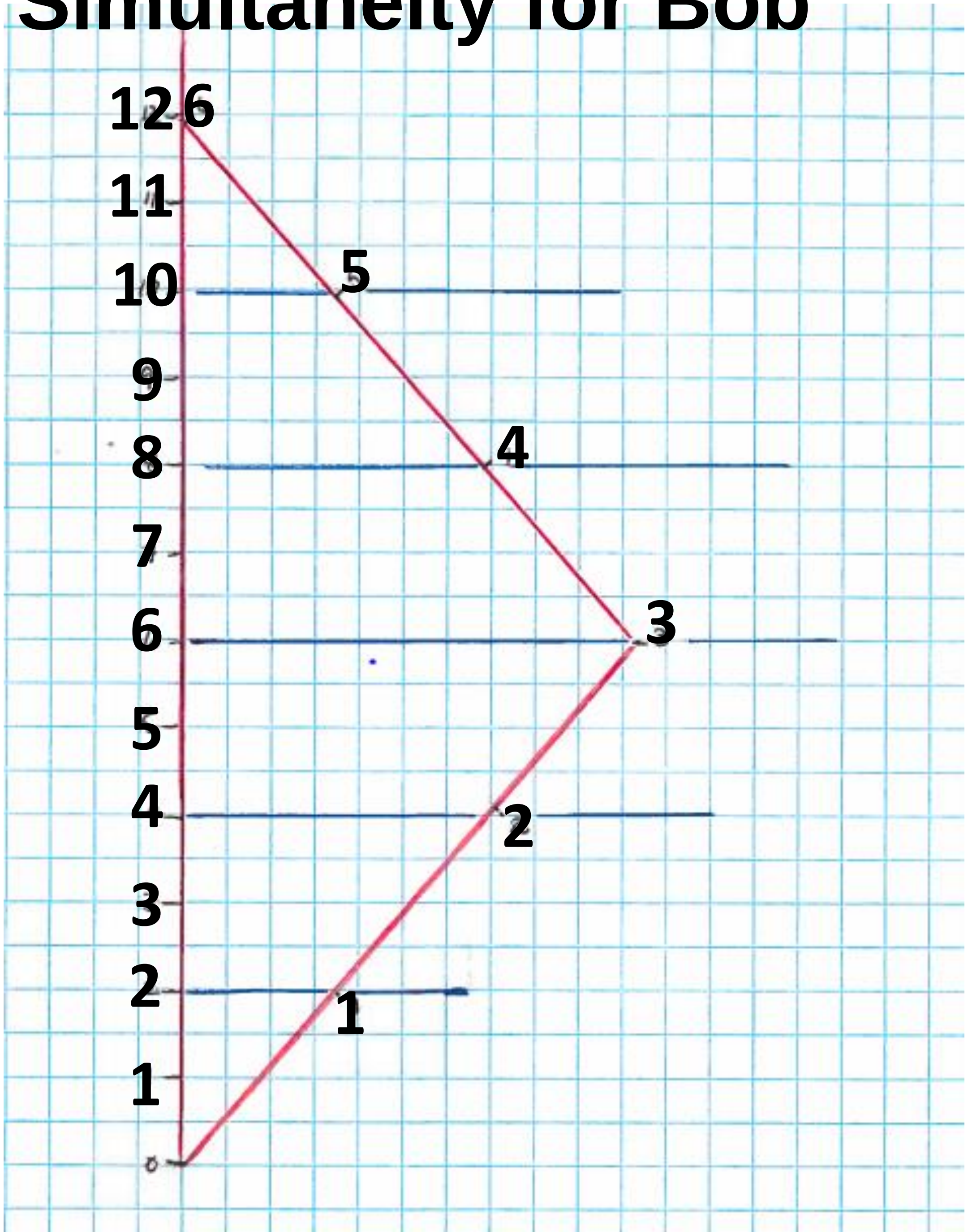
Bob’s Rest Frame Spacetime Diagram



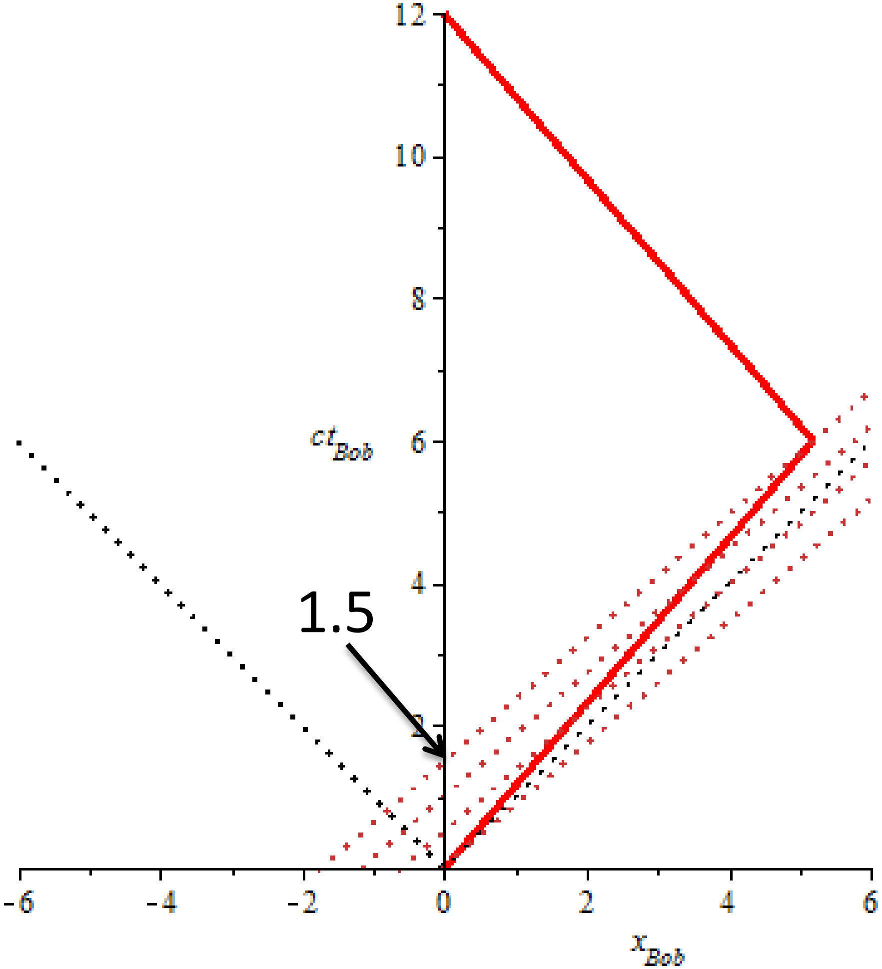
Lines of Simultaneity for Bob

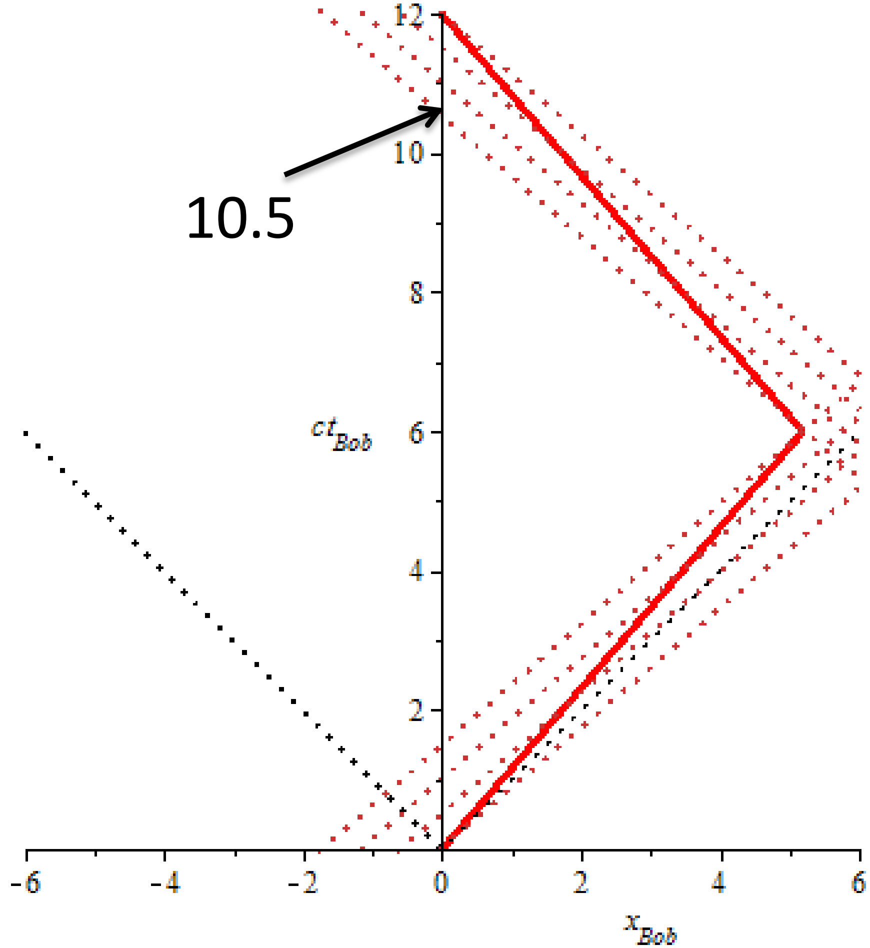


Lines of Simultaneity for Bob

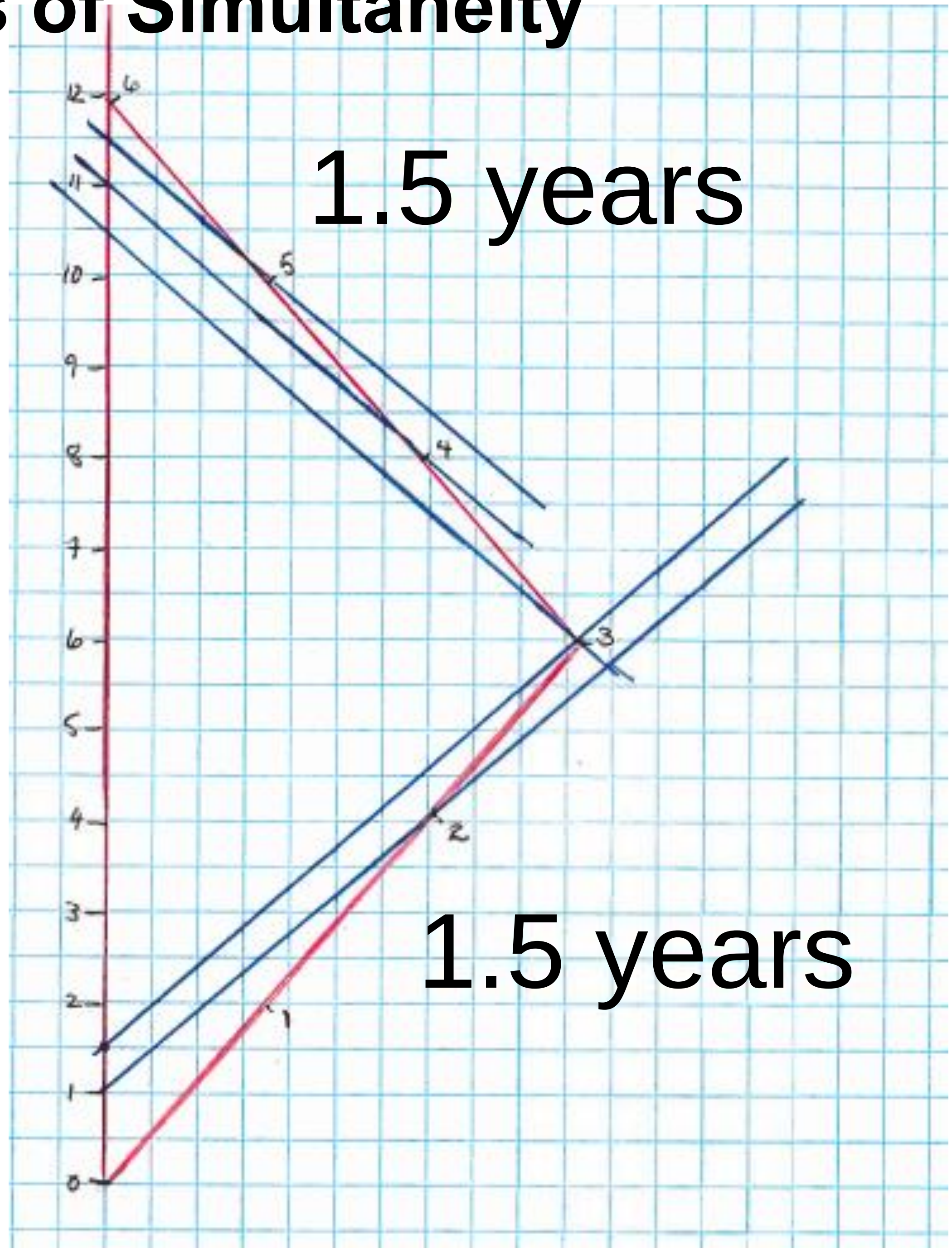


Lines of Simultaneity for Alice

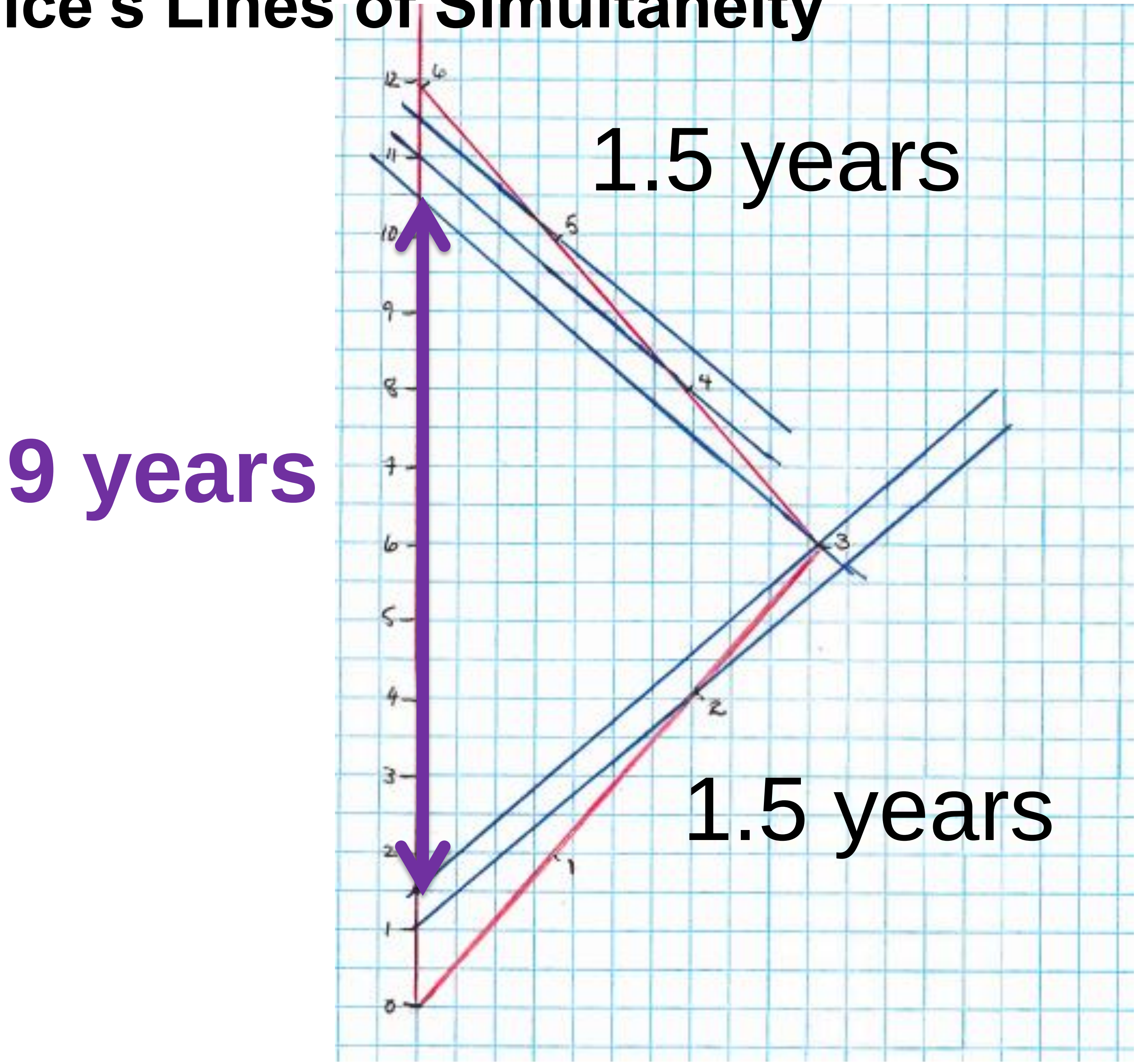




Alice's Lines of Simultaneity

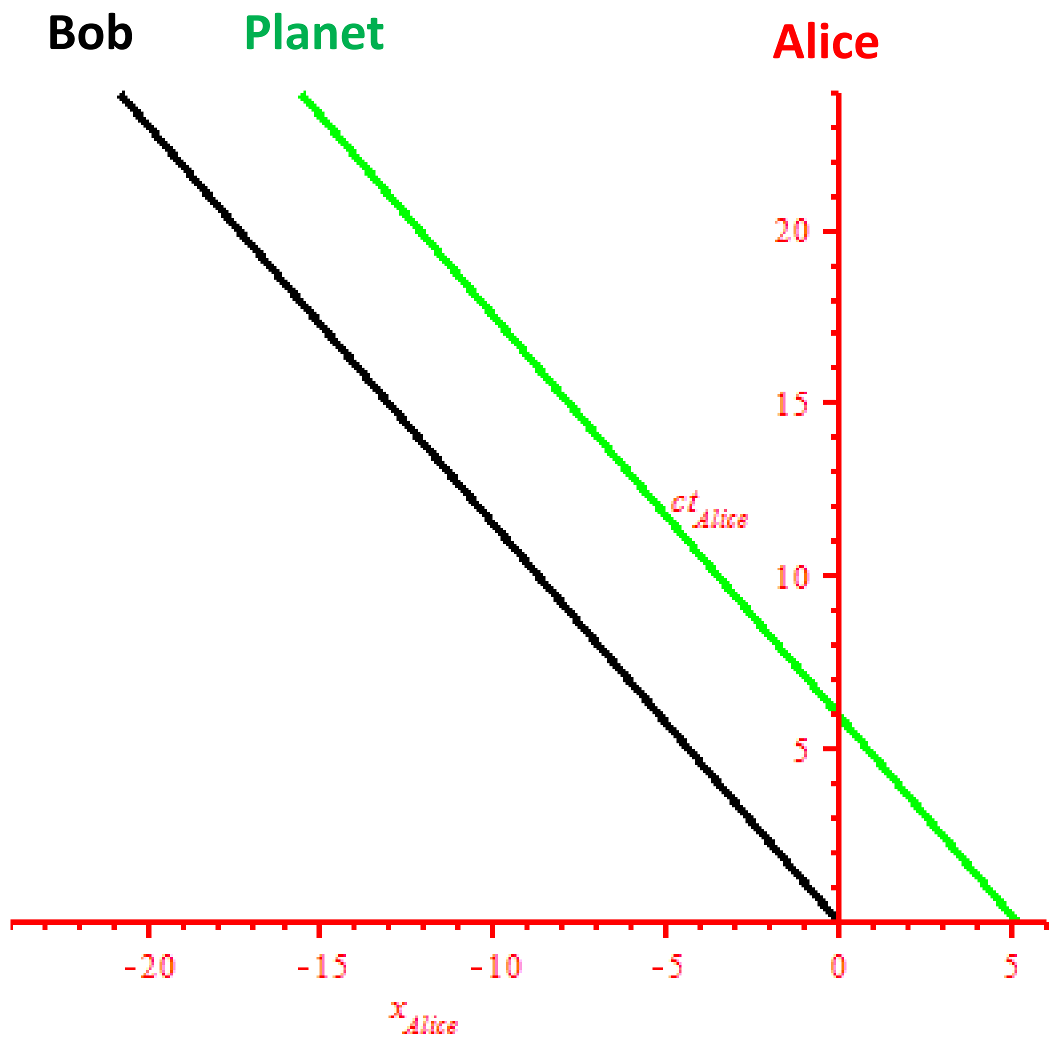


Alice’s Lines of Simultaneity

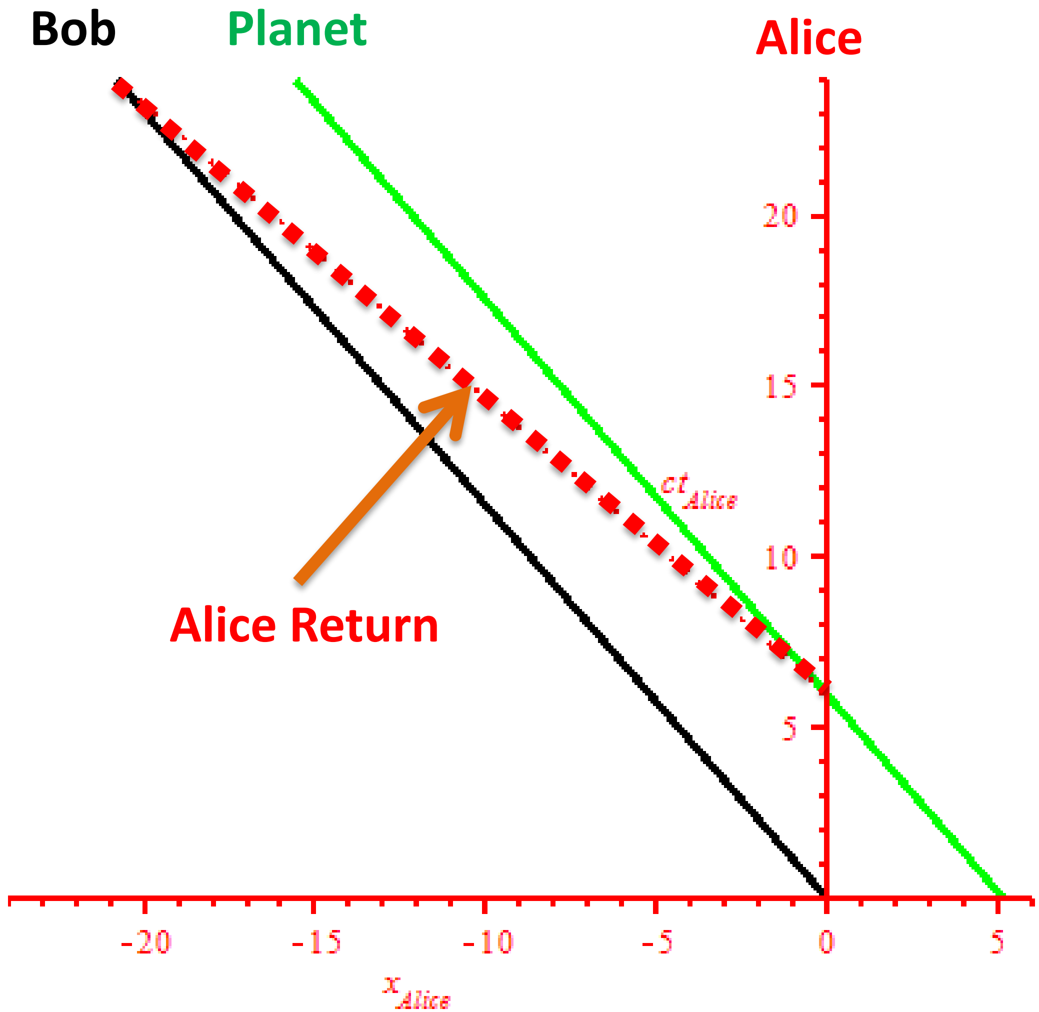


Draw the trip in terms of Alice's (traveling twin) rest frame for the first part of the journey.

Alice's Outward Journey Frame



Alice's Outward Journey Frame



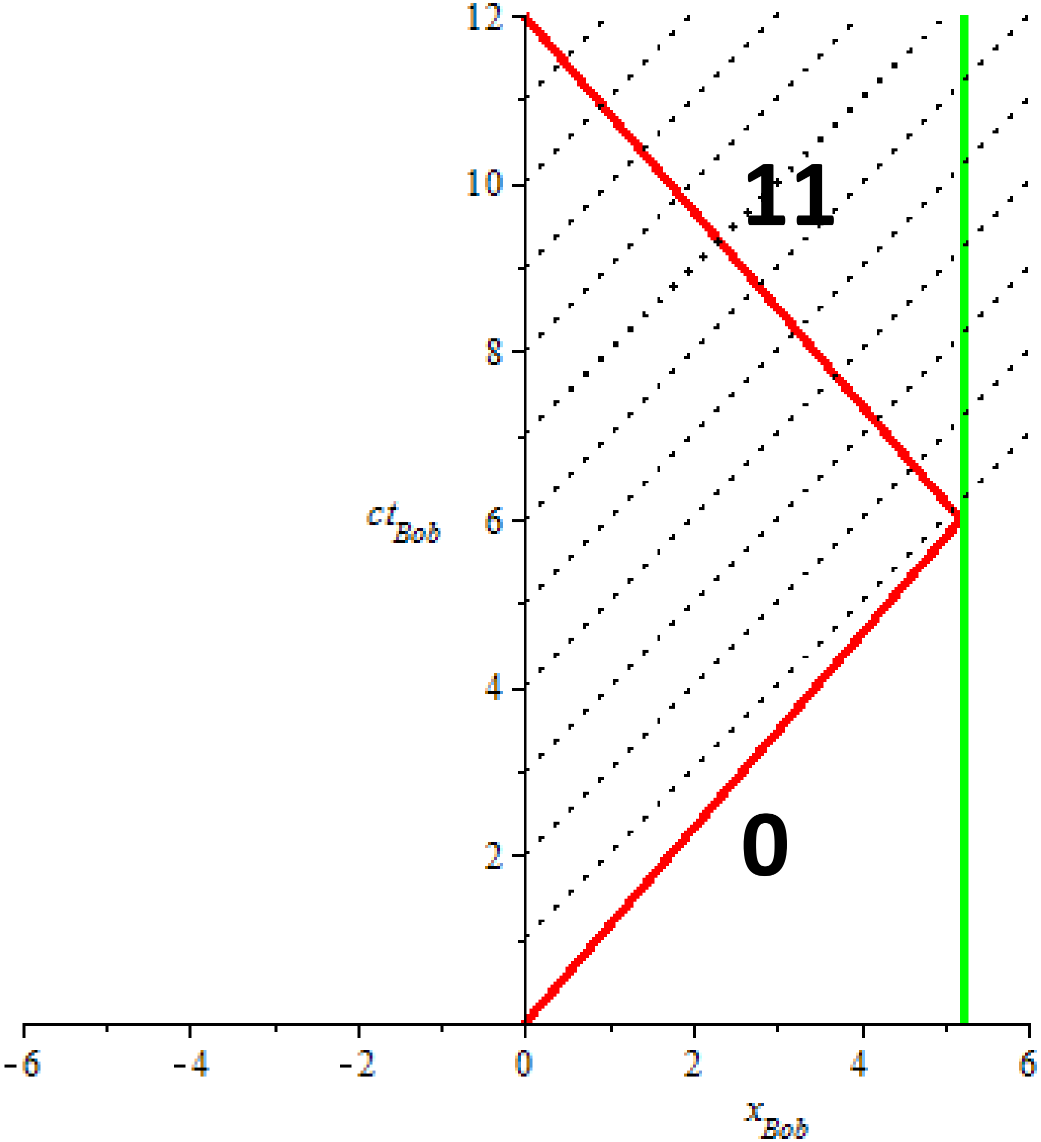
Birthday Messages

Each twin decides to send a message (radio) at their birthday.

Draw the messages sent by Bob (stay-at-home twin)

How many messages does Alice receive on the outward journey? How many on the inward?

Birthday Messages Sent by Bob

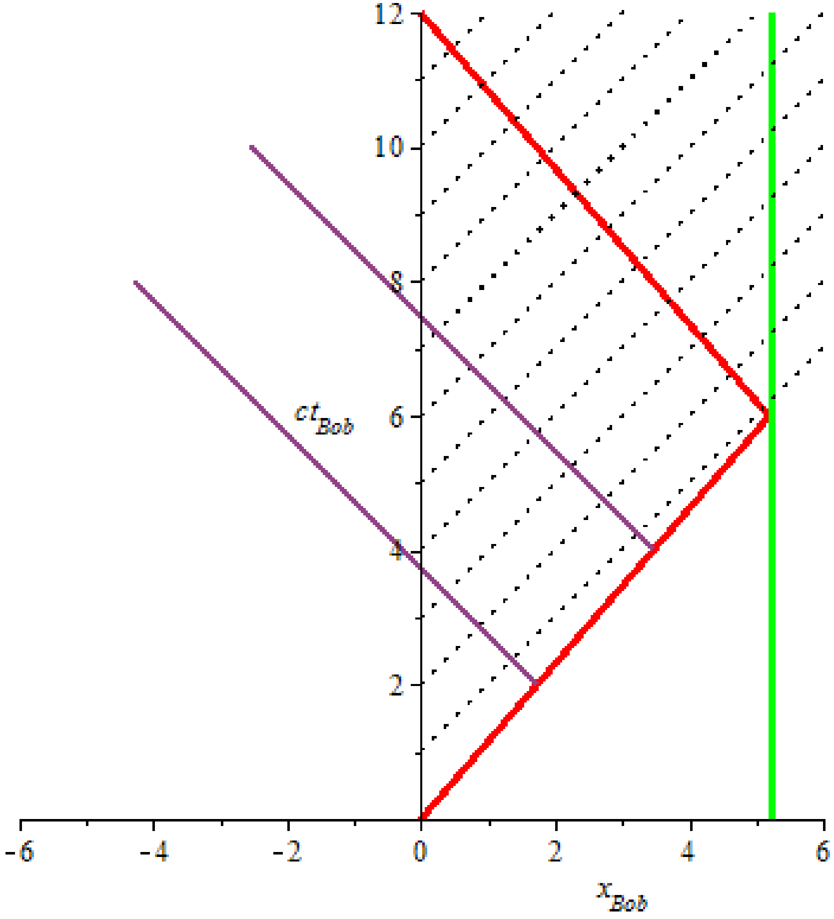


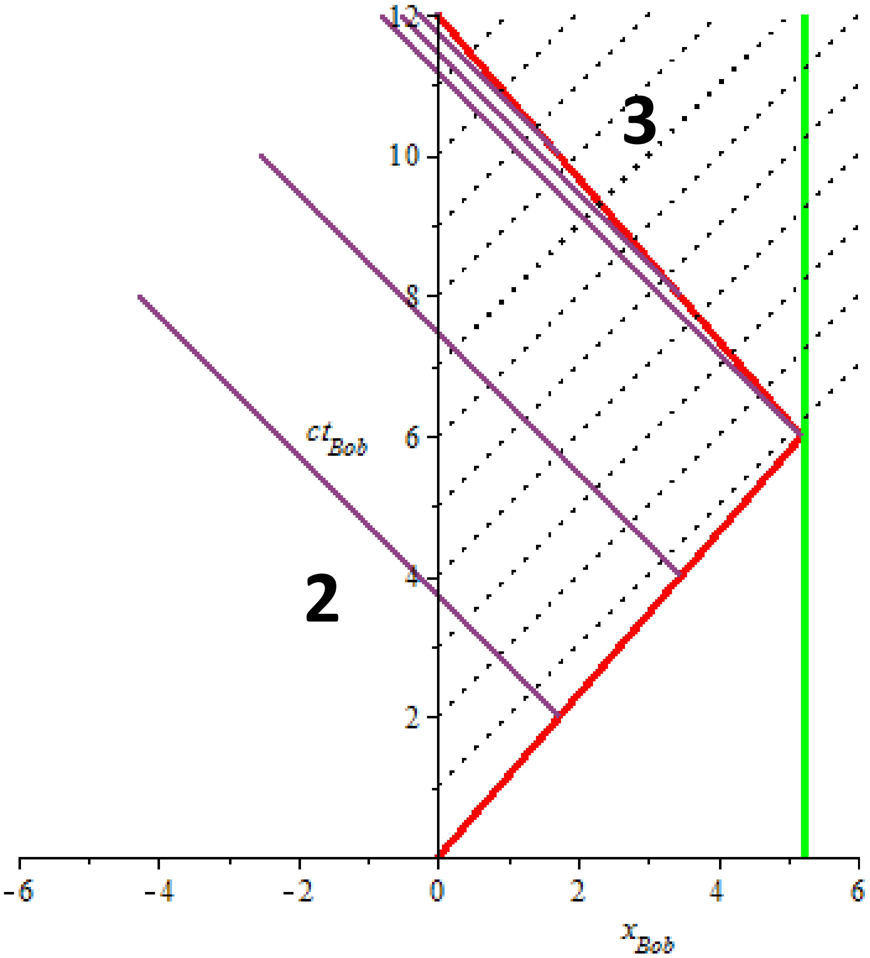
Birthday Messages

Draw the messages sent by Alice (traveling twin).

How many messages does Bob receive during Alice's outward journey? How many on the inward?

Birthday Messages Sent by Alice





Ratio of Signals Sent and Received

- Bob sends Alice 11 signals on her return journey

- Bob receives 3 signals from Alice on her return journey

- $\frac{signals\ sent}{signals\ received} = \frac{11}{3} = 3.7$

Relativistic Doppler Effect

Calculate the ratio of frequencies based on the relative velocities using the relativistic Doppler shift.

Remember: Alice travels at 0.866c

$$\frac{f_s}{f_r} = \sqrt{\frac{1+v/c}{1-v/c}}$$

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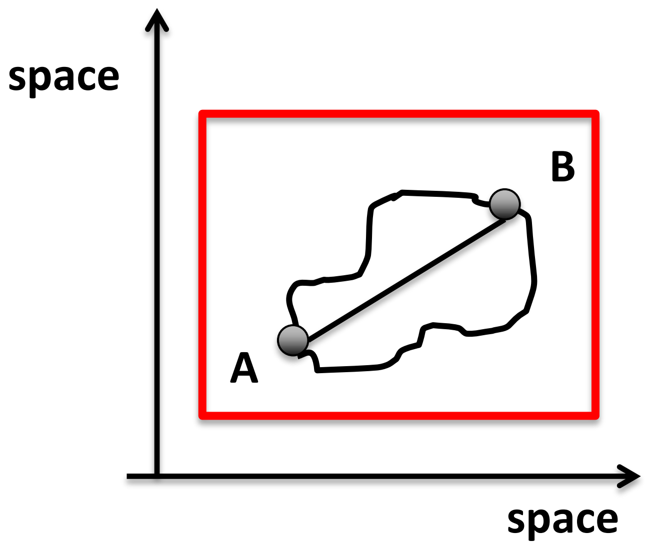
Nothing Paradoxical

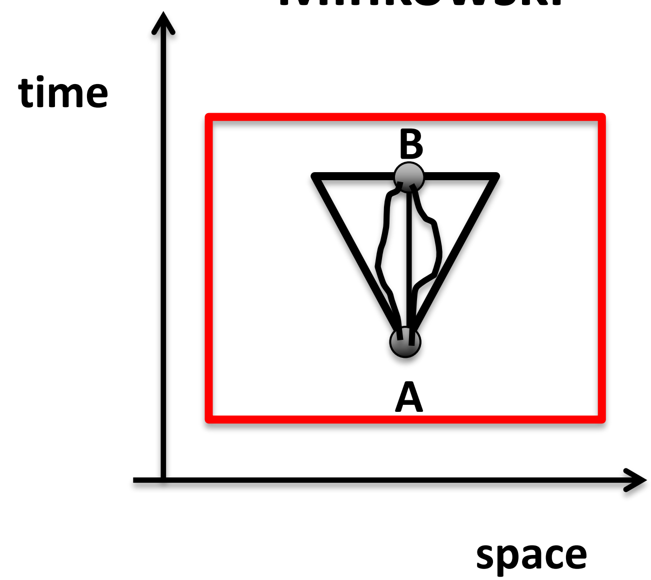
- The traveling twin had a crooked worldline and the stay-at-home twin did not.
- There is not a perfect symmetry between the two, so why expect symmetrical effects?
- The effect of the change in motion is to completely alter the traveler's judgement of simultaneity
- The traveler judges the stay at home twin's clock to have jumped suddenly.

Euclidean



straightest line is the shortest length

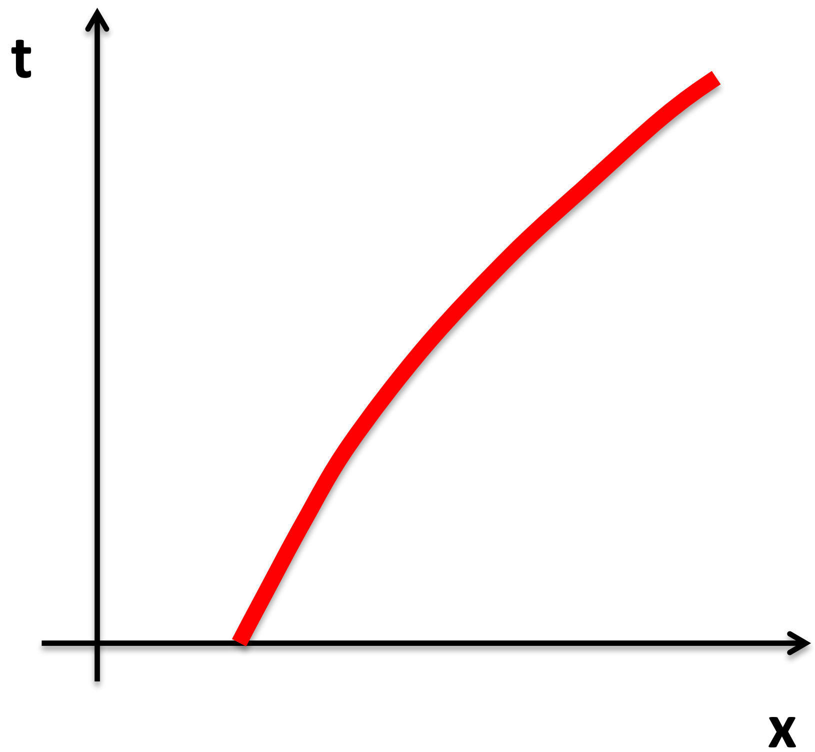
Minkowski



straightest line is the *longest* proper time

Can Special Relativity "handle" acceleration?

Newtonian Uniform Acceleration

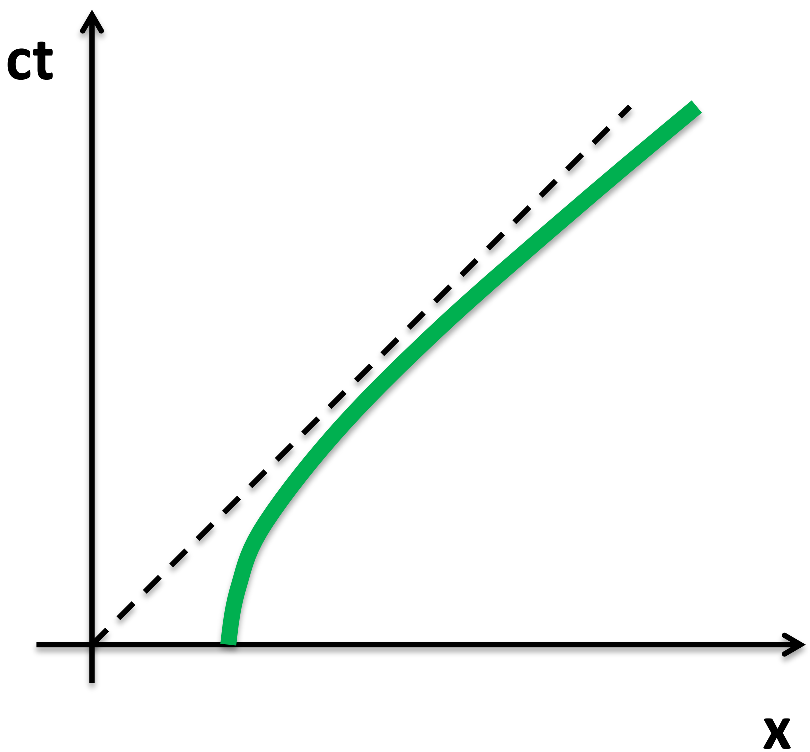


$$a = \frac{\Delta v}{\Delta t}$$

$$v = v_o + at$$

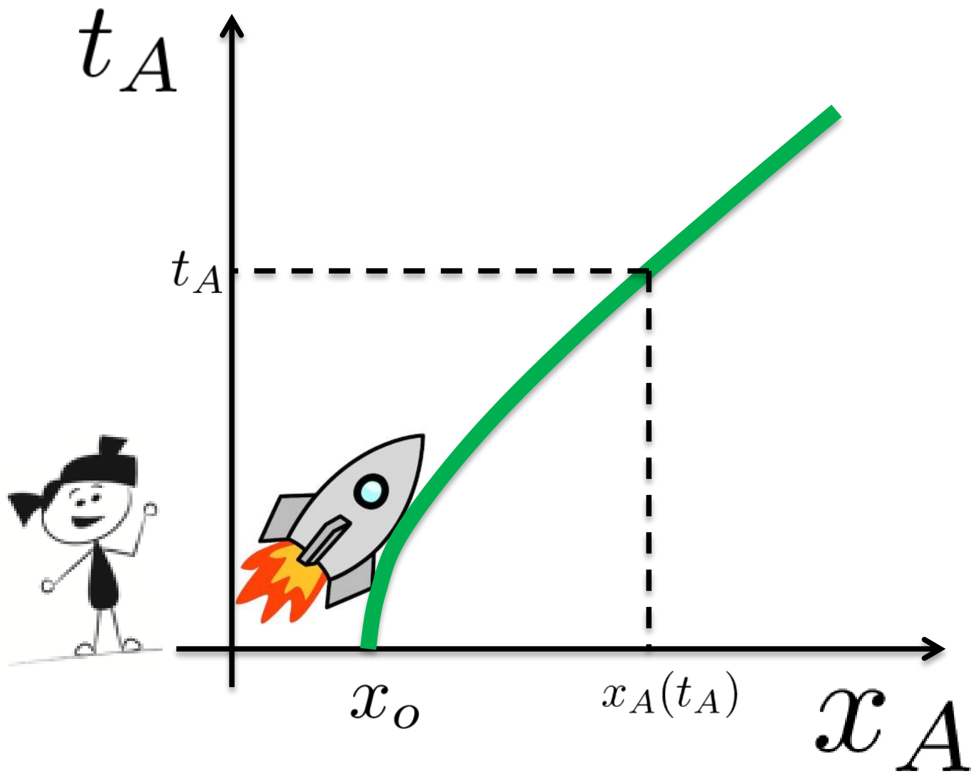
$$x = x_o + v_o t + \frac{at^2}{2}$$

Our guess for acceleration in SR ….

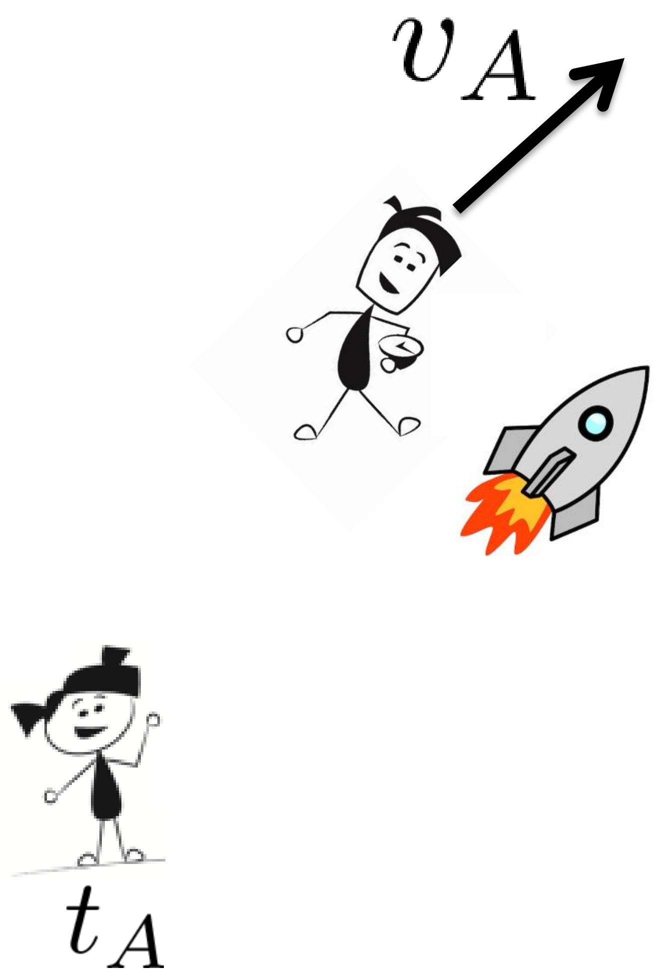


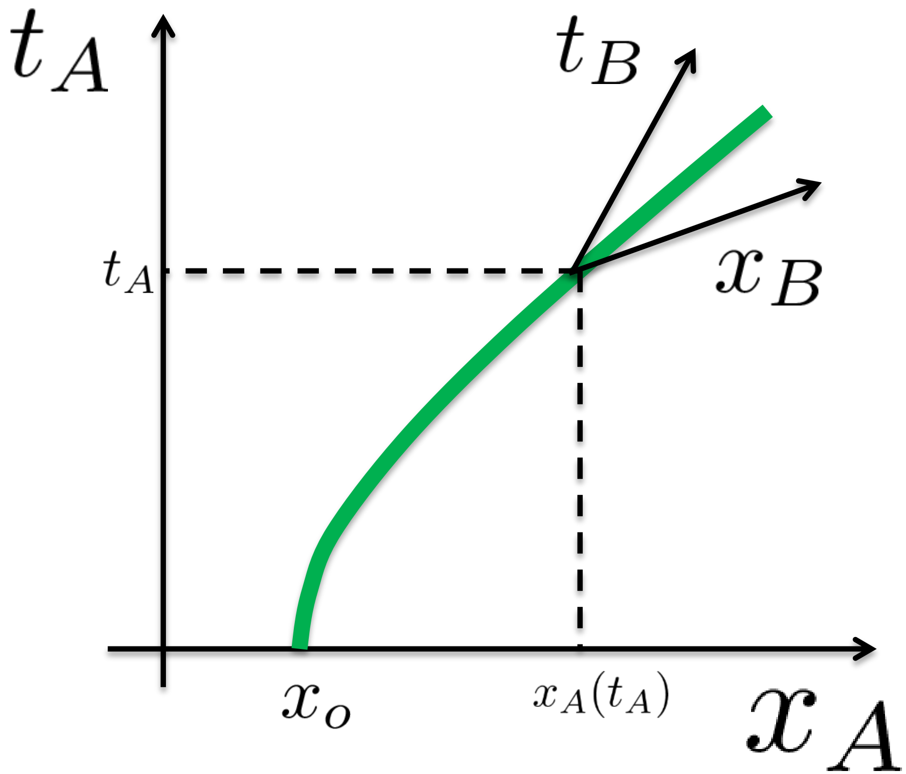
Similar trajectory, but slope should never reach the speed of light

Hyperbola!



$$\frac{dt_A}{dx_A} = \frac{1}{v_A}$$


v_A
t_A

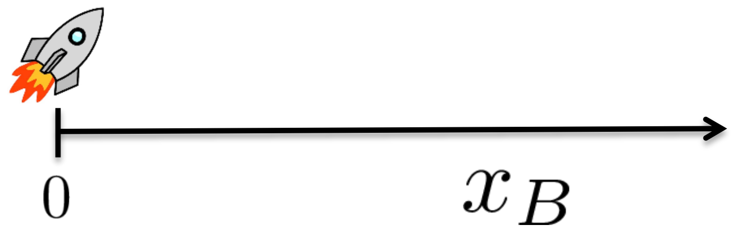


t_A
t_B
t_A
x_B
x_o
$x_A(t_A)$
x_A

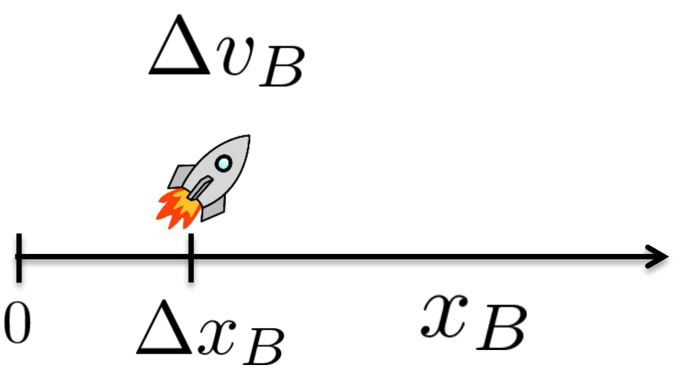
Bob's Frame

Regardless of how fast the rocket is moving in Alice's frame, it is hardly moving in Bob's!

$$t_B = 0$$

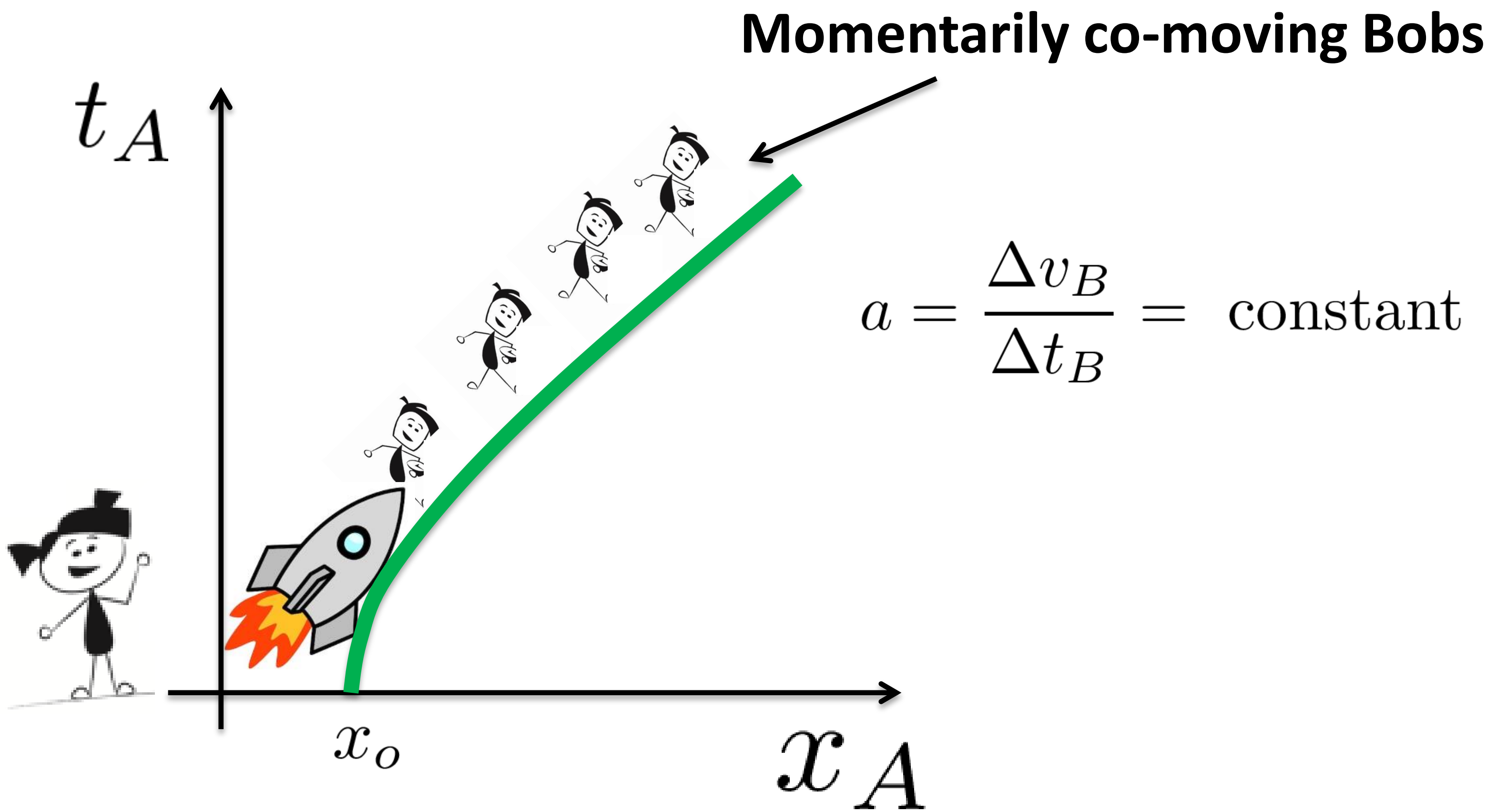


$$t_B = \Delta t_B$$



$$a = \frac{\Delta v_B}{\Delta t_B}$$

Setup many Bobs all along the path…



$$a = \frac{\Delta v_B}{\Delta t_B} = \text{constant}$$

What acceleration does Alice see?

$$v_A + \Delta v_A \equiv$$ **Velocity of rocket relative Alice (after time Δt_B has elapsed for Bob)**

$$\frac{u+v}{1+\frac{uv}{c^2}}$$

$$= \frac{v_A + \Delta v_B}{1 + \frac{v_A \Delta v_B}{c^2}}$$

$$\approx (v_A + \Delta v_B)(1 - \frac{v_A \Delta v_B}{c^2})$$

$$\approx v_A + (1 - \frac{v_A^2}{c^2})\Delta v_B$$

$$\Delta v_A = (1 - \frac{v_A^2}{c^2})\Delta v_B$$

What acceleration does Alice see?

$$\Delta v_A = (1 - \frac{v_A^2}{c^2})\Delta v_B$$

$$a_A = \frac{\Delta v_A}{\Delta t_A}$$

What acceleration does Alice see?

$$\Delta v_A = (1 - \frac{v_A^2}{c^2})\Delta v_B$$

Time elapsed for Bob for rocket to reach speed Δv_B

$$a_A = (1 - \frac{v_A^2}{c^2})\frac{\Delta t_B}{\Delta t_A} a$$

Time for Alice

$$\Delta t_B = \sqrt{1 - \frac{v_A^2}{c^2}}\Delta t_A$$

What acceleration does Alice see?

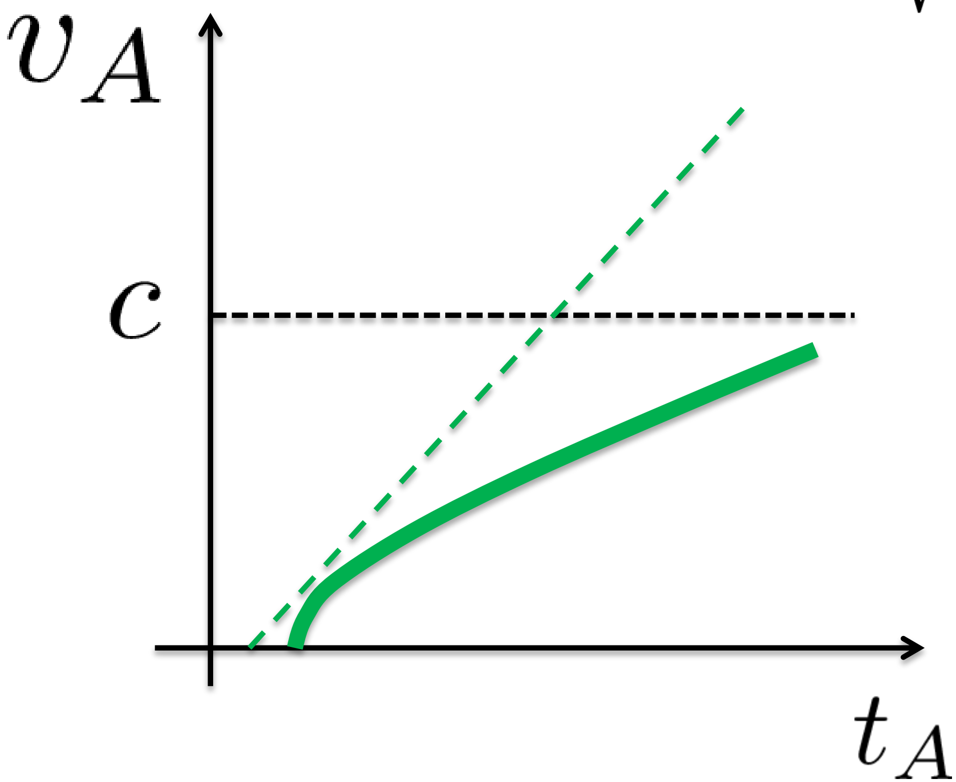
$$a_A \equiv \frac{dv_A}{dt_A} = (1 - \frac{v_A^2}{c^2})^{3/2} a$$

Not Constant

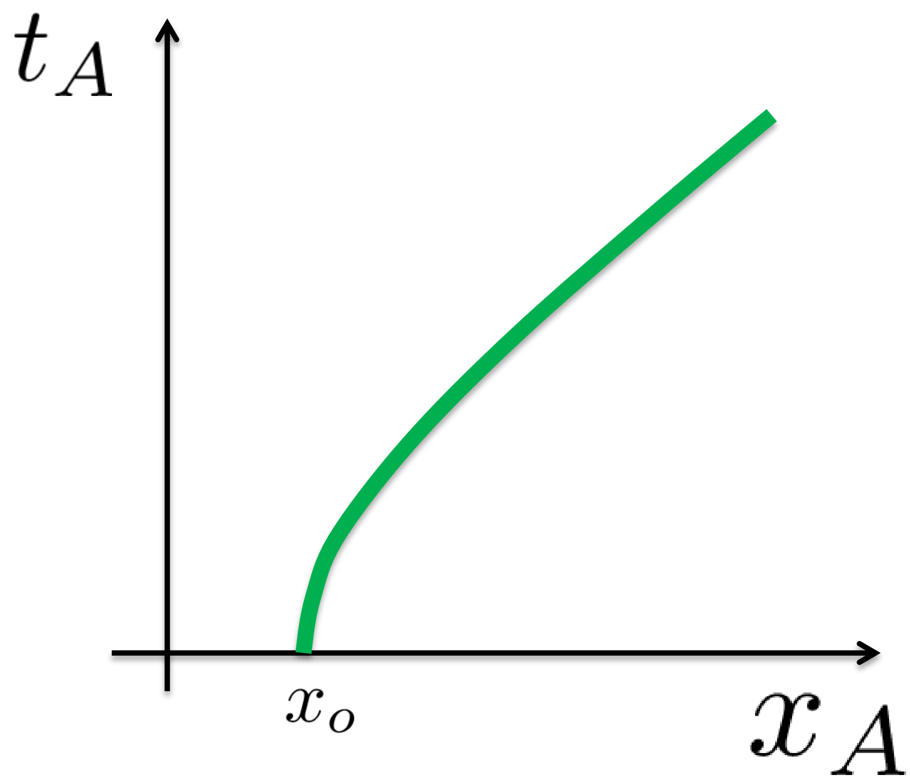
Constant

a_A *decreases*, as v_A *increases*

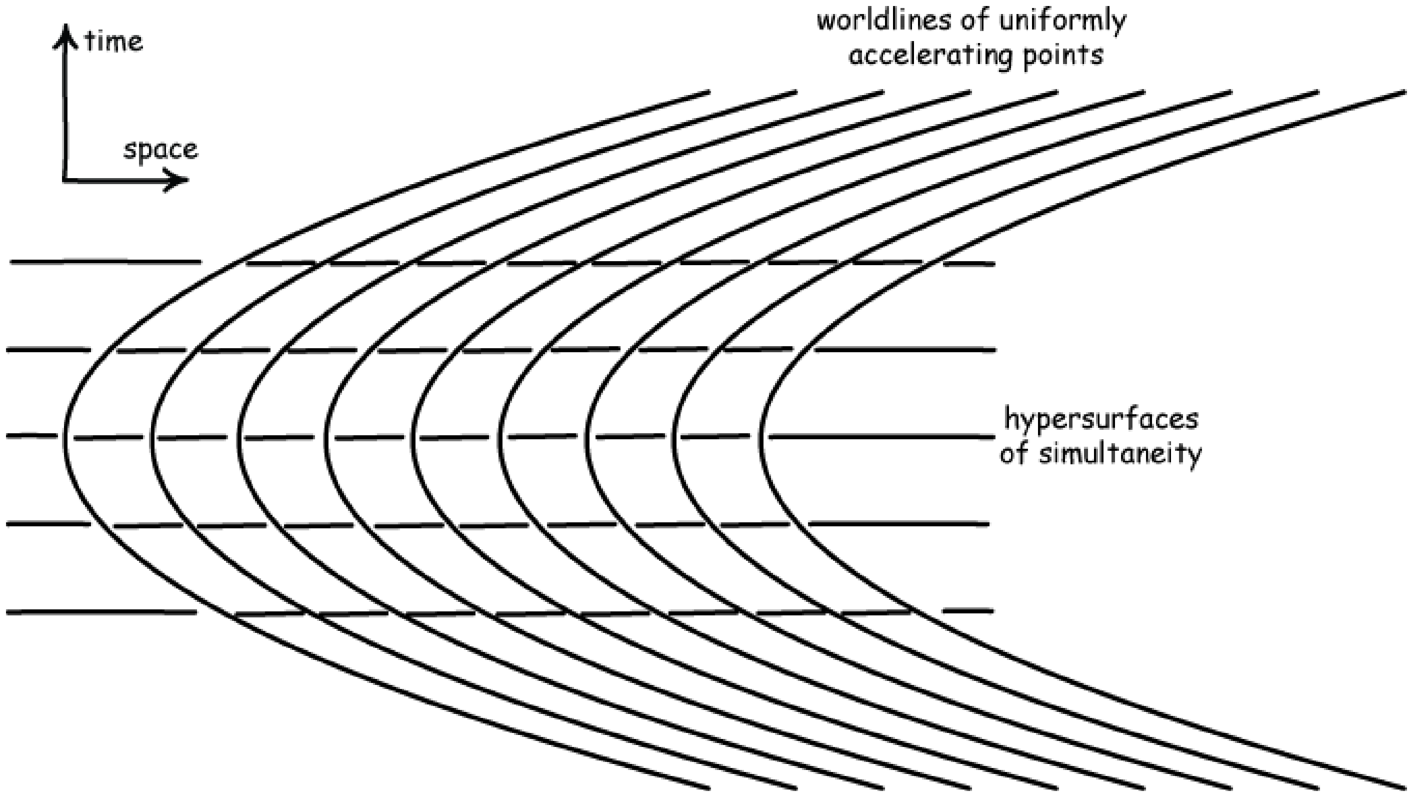
$$v_A(t_A) = \frac{at_A}{\sqrt{1 + \frac{(at_A)^2}{c^2}}}$$



Hyperbola

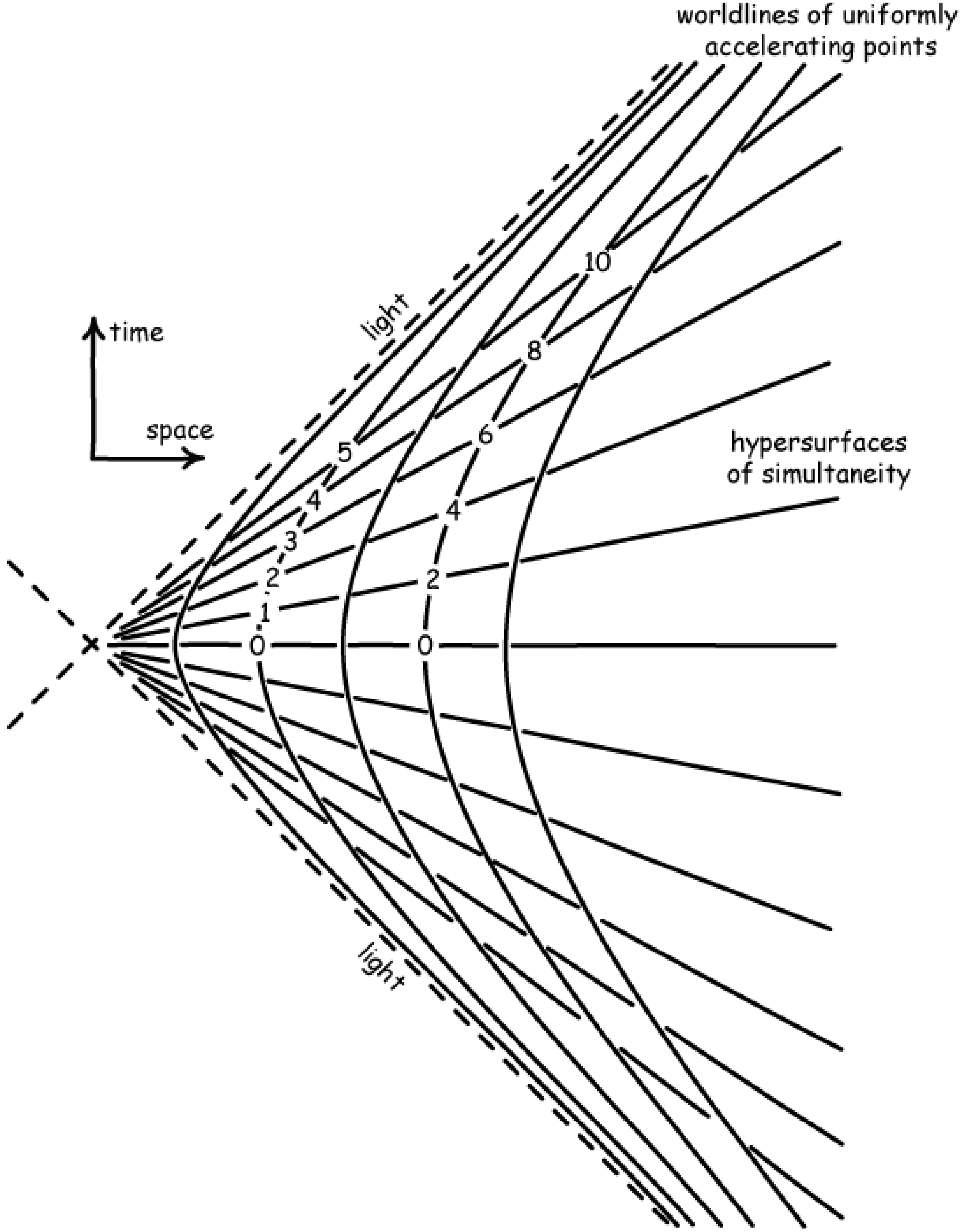


Newton



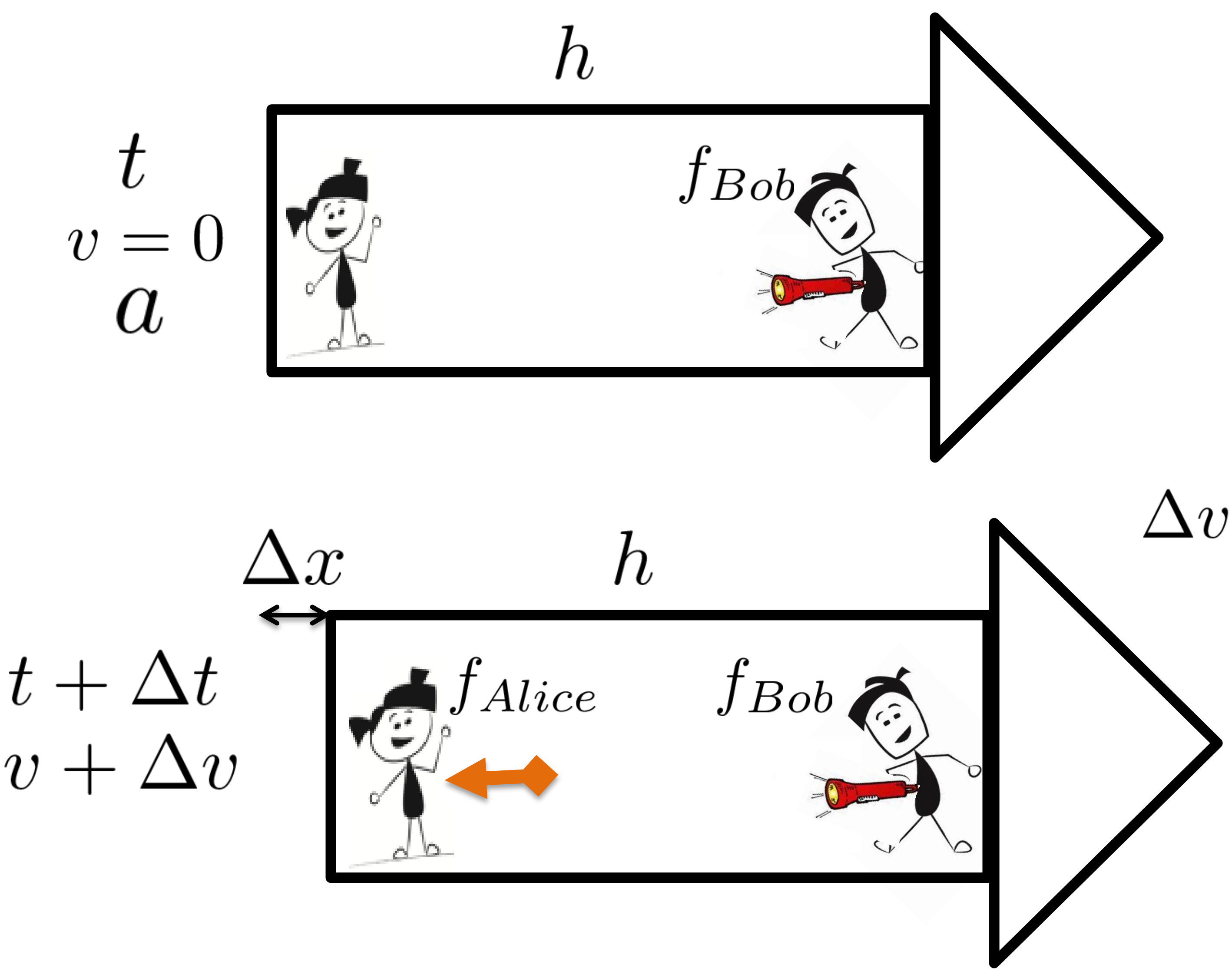
time
space
worldlines of uniformly
accelerating points
hypersurfaces
of simultaneity

Einstein



worldlines of uniformly accelerating points
time
space
light
hypersurfaces of simultaneity
light
10
8
6
5
4
4
3
2
2
1
0
0

Light Pulse in an Accelerated Rocket



$$\Delta t \approx \frac{h}{c}$$

$$\Delta v \approx a\Delta t = \frac{ah}{c}$$

Doppler Shift

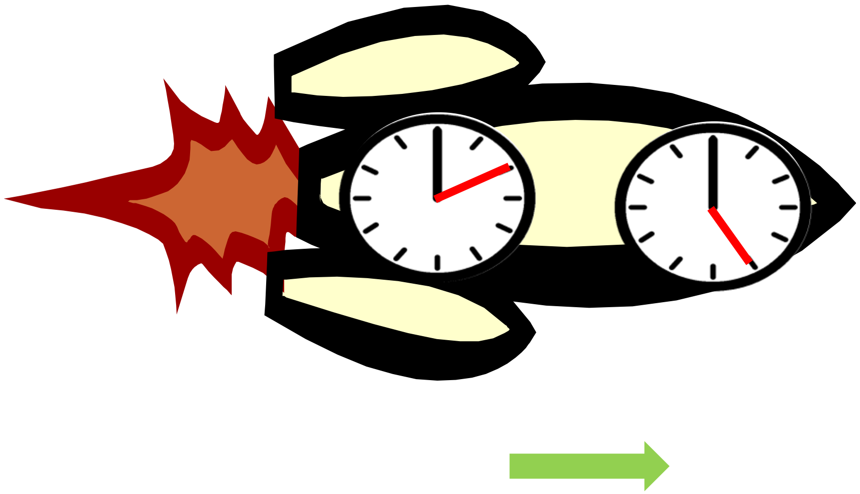
- When the light is turned on, both Alice & Bob are moving at the same velocity

- By the time the light reaches Alice, she is effectively moving toward the source with a relative velocity Δv (due to the acceleration)

$$\frac{f_A}{f_B} = \sqrt{\frac{1+\frac{\Delta v}{c}}{1-\frac{\Delta v}{c}}} \approx \sqrt{(1+\frac{\Delta v}{c})(1+\frac{\Delta v}{c})} = 1 + \frac{\Delta v}{c} = 1 + \frac{ah}{c^2}$$

Acceleration Affects Time

Observers at the front have clocks that tick faster!



$$\Delta t_{back} = (1 - \frac{ah}{c^2})\Delta t_{front}$$

Kinematics & Dynamics

- **Kinematics:** study of motion in space and time
  - *WHEN is a moving body WHERE*

- **Dynamics:** study of what causes motion
  - Force
  - Energy
  - Momentum

- Momentum: Measure of the quantity of motion
- Force: Rate of transfer of momentum
- If we apply a constant force we gain momentum and energy
- $\Delta p = F\Delta t$
- $\Delta E = F\Delta d$

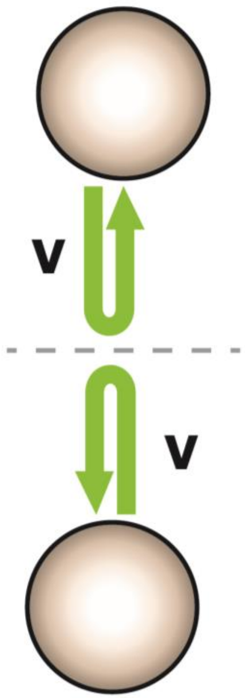
Conservation Laws

- For a closed system:
  - Energy conservation $E_{final} = E_{initial}$
  - Momentum Conservation $p_{final} = p_{initial}$

Consider the collision of two Super Balls

Each has speed ***v*** going in and out of collision.

How do their masses compare?

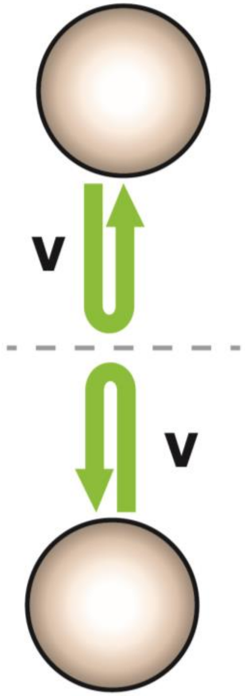


Momentum Conservation

$$m_1 v - m_2 v = m_2 v - m_1 v$$

$$m_1 - m_2 = m_2 - m_1$$

$$m_1 = m_2$$



Consider the collision of two Super Balls

One has speed ***v*** going in and out, the other has a greater speed ***V*** going in and out of the collision.

How do their masses compare?

Momentum Conservation

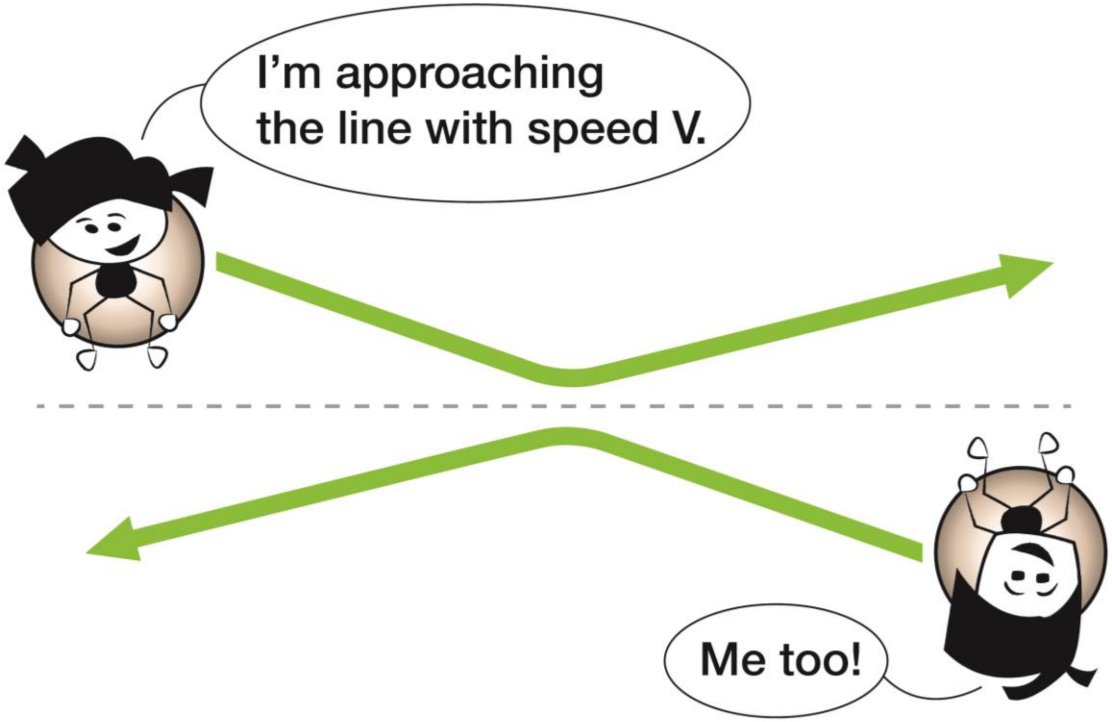
$$m_1 v - m_2 V = m_2 V - m_1 v$$

$$2m_2 V = 2m_1 v$$

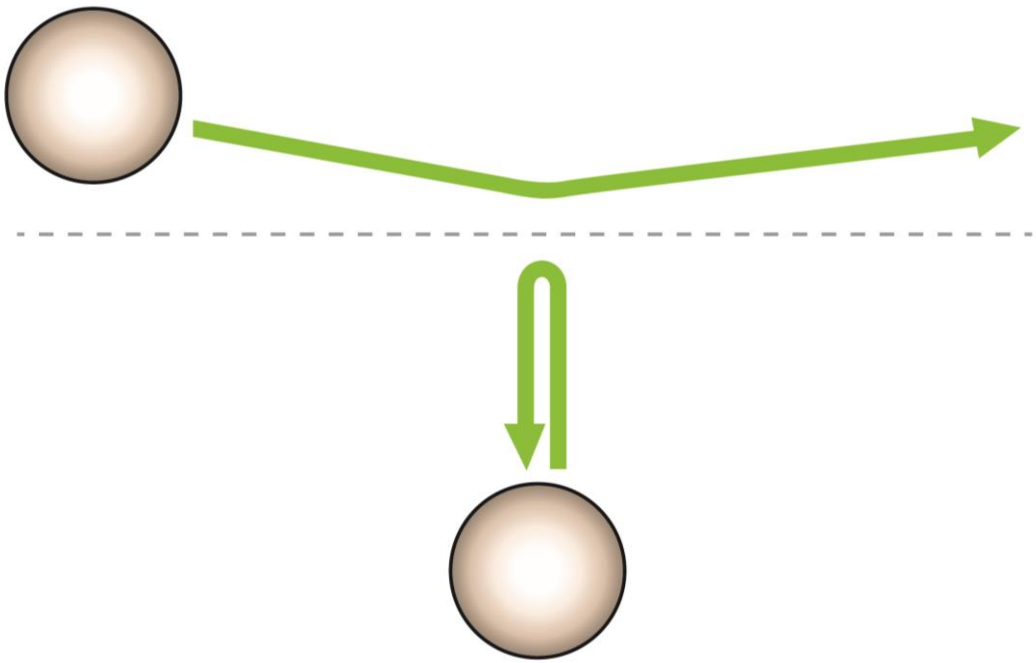
$$\frac{m_2}{m_1} = \frac{v}{V}$$

The one with the higher velocity has the lower mass.

Now consider a glancing collision…



Running parallel to Bob…



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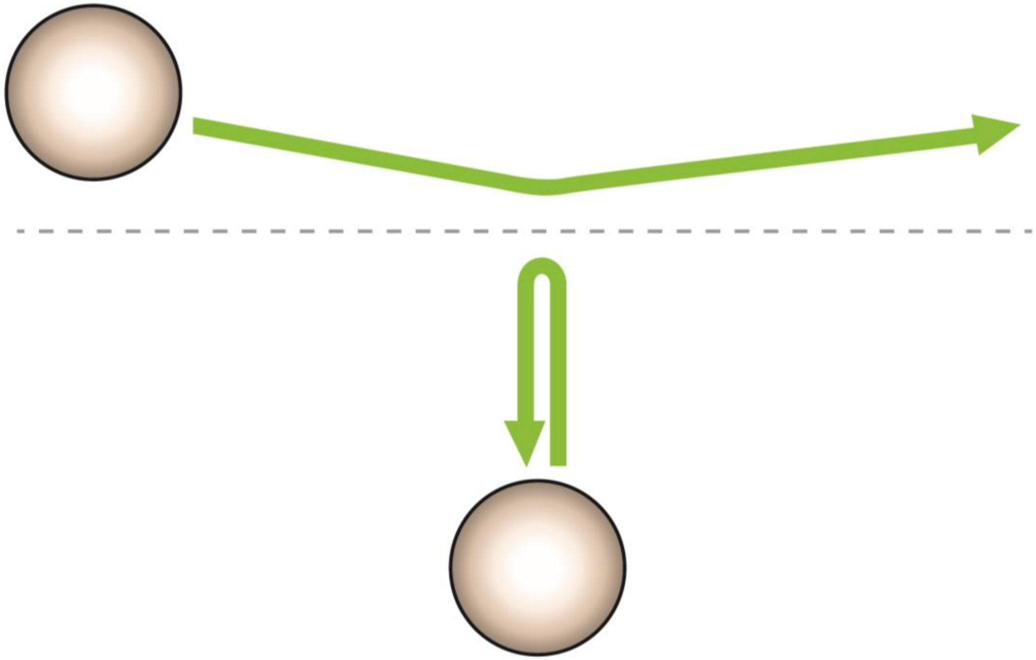
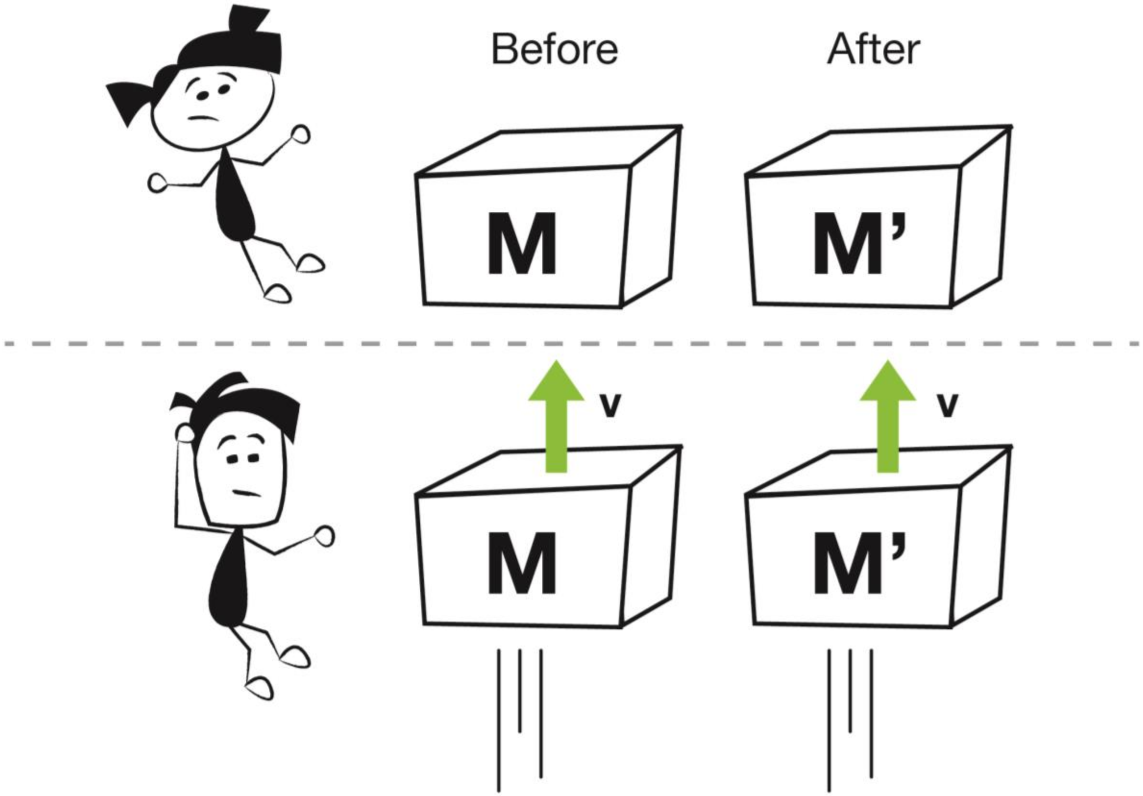
Effective Inertia

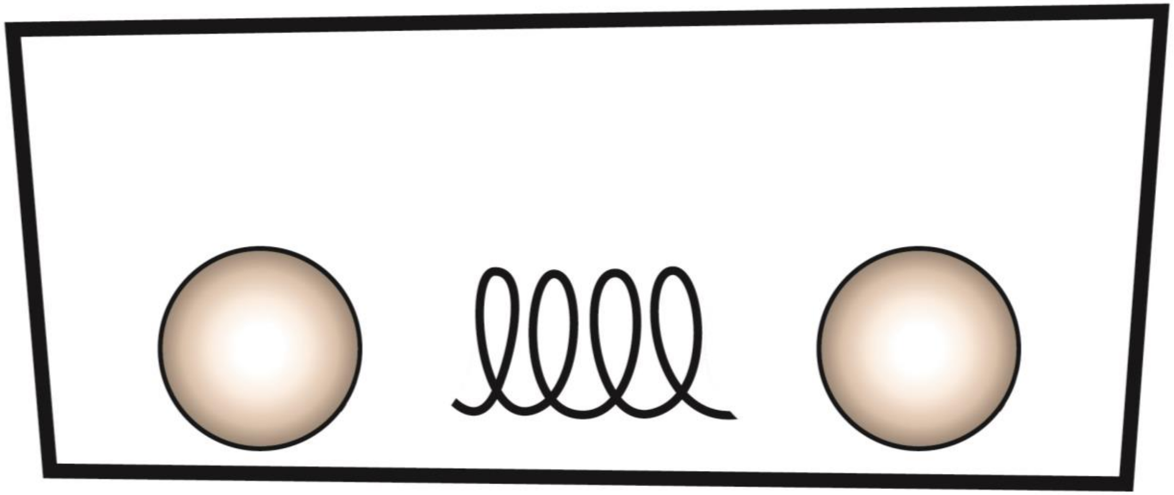
$$M = \gamma m$$



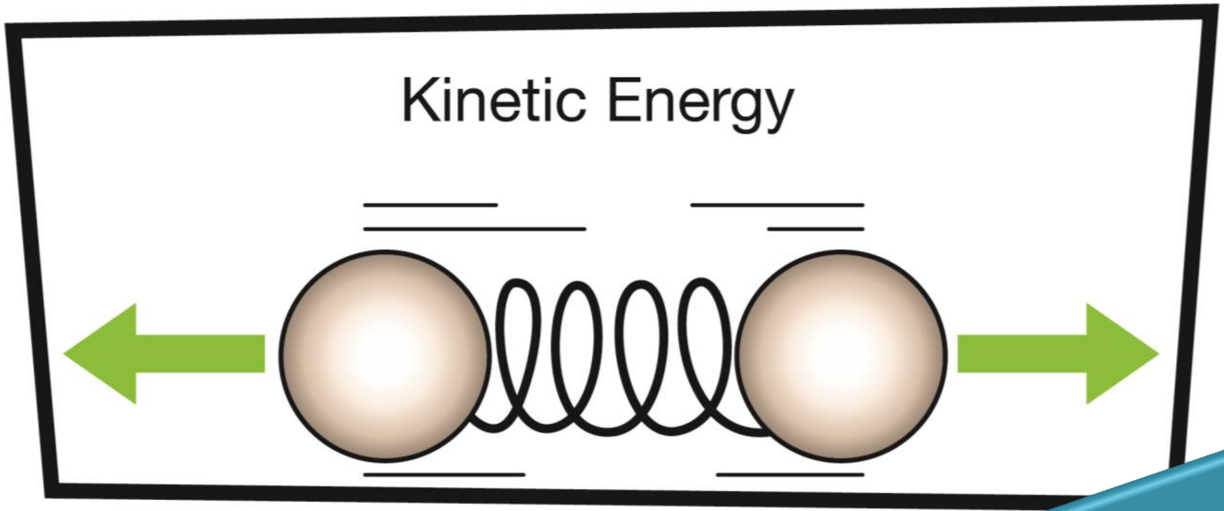
Can the mass of a box suddenly change?



Open the box to find two heavy spheres and a very special spring…



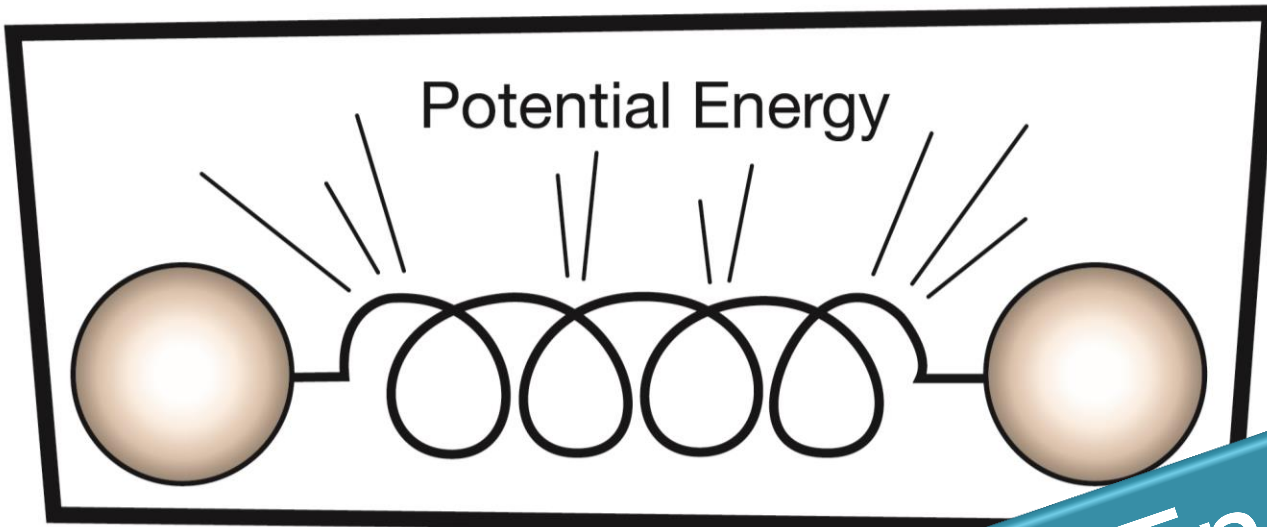
Vibrates at high speed…



What happens to th

Kinetic Energy has Inertia

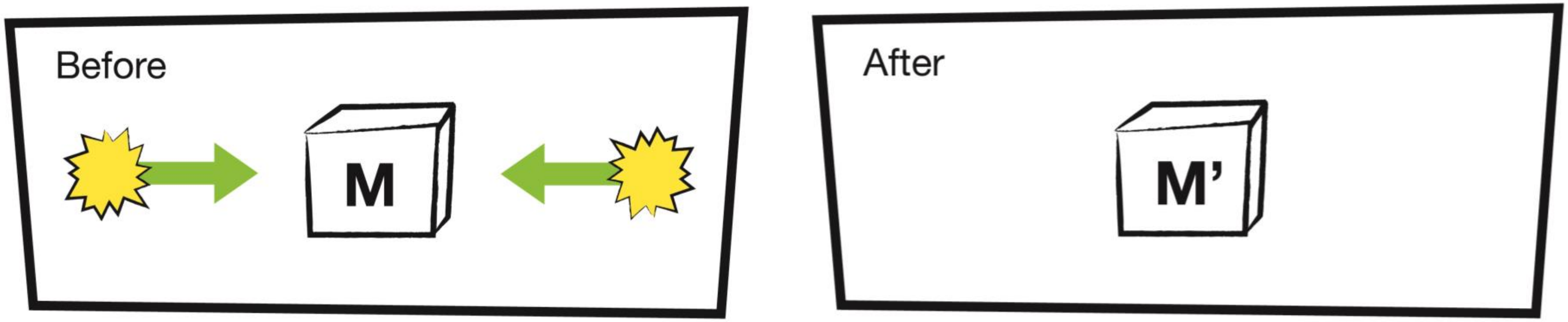
When the spring is fully stretched…



What happens to th

Potential Energy
has Inertia

Shine light on a brick.



What happens to the mass of th ?

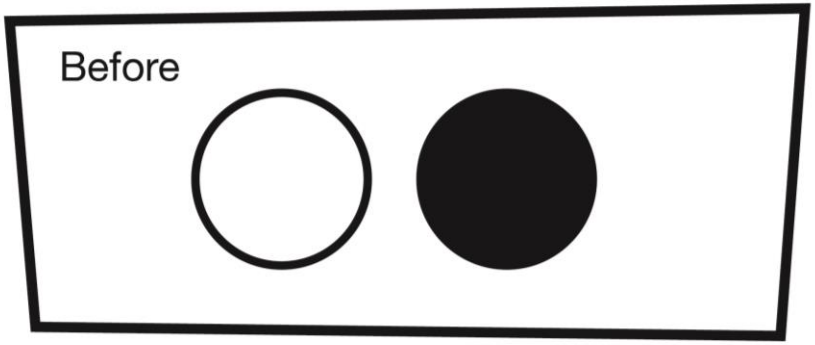
Light Energy has Inertia

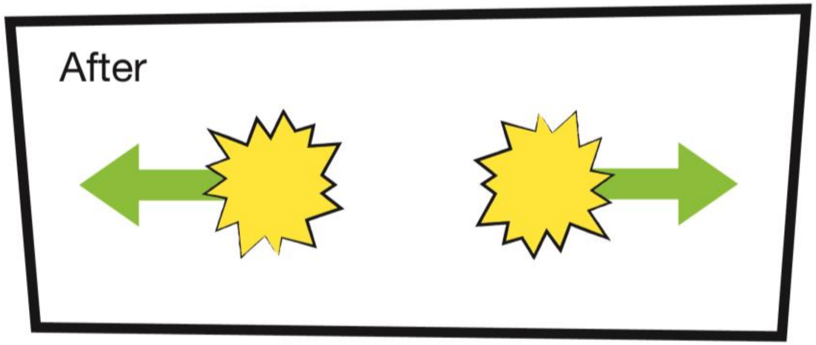
ENERGY

has

INERTIA

Matter and antimatter annihilate…





What happens to the mass of the box?

Mass is a form of energy…

$$E = mc^2$$

This equation is a special case of a more general expression…

Energy-Momentum Equivalence

$$E^2 = m^2c^4 + p^2c^2$$

where $\quad p = \gamma m v$

For objects that are “at rest”, p=0 and

$$E = mc^2$$

For massless objects, m=0 and

$$E = pc$$

NB: Momentum of light is a classical result from Maxwell's equations…

Relativistic Energy & Momentum

- We have seen that

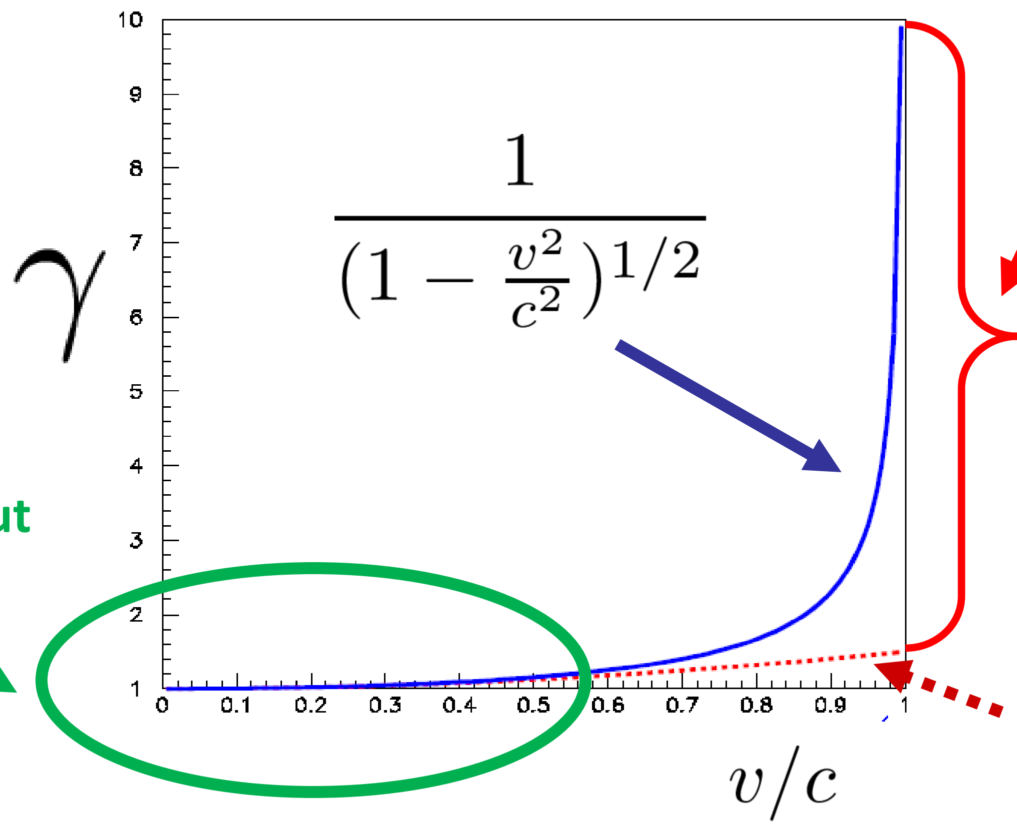
$$m(v) = m_o\gamma = \frac{m_o}{(1 - \frac{v^2}{c^2})^{1/2}}$$

- Try expanding γ via the Taylor Expansion

Taylor Expansion of Gamma:

Other Stuff

$$\gamma = \frac{1}{(1-\frac{v^2}{c^2})^{1/2}} = 1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \ \ldots$$



The "other stuff" is pretty huge near v=c

What about here?

$$1 + \frac{1}{2}\frac{v^2}{c^2}$$

When v/c is small:

$v/c < 0.1$

$$\gamma = 1 + \frac{1}{2}\frac{v^2}{c^2}$$

$$\gamma - 1 = \frac{1}{2}\frac{v^2}{c^2}$$

This covers all velocities ever encountered in 1905

$$m(v) = m_o\gamma = \frac{m_o}{(1-\frac{v^2}{c^2})^{1/2}}$$

$$m = m_o(1+\frac{1}{2}\frac{v^2}{c^2})$$

$$\Delta m = m - m_o = \frac{1}{2}\frac{v^2}{c^2}m_o$$

$$\Delta mc^2 = \frac{1}{2}v^2 m_o$$

Kinetic Energy

Einstein's hypothesis:

$$mc^2 = m_0c^2 + \text{kinetic energy}$$

holds for all velocities
(not only for v<0.1c)

$$\text{Kinetic energy} = (m - m_0)c^2 = \tfrac{1}{2}m_0v^2 + \text{“other stuff”}$$

Since this is only significant for huge speeds, Newton, etc had no way of knowing about it

Mass-Energy Equivalence

$$mc^2 = m_o c^2 + \text{ Kinetic Energy}$$

This must be the energy at rest!

This must be the total energy!

Momentum $p = mv = m_o \gamma v$

$$\begin{aligned} E^2 - p^2c^2 &= m_o^2\gamma^2c^4 - m_o^2\gamma^2v^2c^2 \\ &= m_o^2\gamma^2c^4(1 - \frac{v^2}{c^2}) \end{aligned}$$

$$E^2 - p^2c^2 = m_o^2c^2$$

ENERGY-MOMENTUM INVARIANT