ıass never results in a naked
dden within a horizon. This
;hip *conjecture*. This conjec-
d singularities in the mathe-
t forbids only the formation
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Teukolsky, 1991). However,
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ɔle for $|a| \geq GM$, and a col-
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8.7 BLACK-HOLE THERMODYNAMICS; THE HAWKING PROCESS

The horizon, or one-way membrane, of a black hole acts as a perfect absorber--it permits anything to enter, but does not permit anything to leave. We therefore expect that it could serve as an ideal heat sink for the operation of a thermodynamic engine. We expect that when we dump heat on the horizon, in the form of thermal radiation at some given temperature, it will be completely absorbed. With such an ideal heat sink, we can operate a thermodynamic engine with 100% efficiency, that is, we can accomplish the complete conversion of heat into work. The following *Gedanken*-experiment describes a simple machine that attempts to accomplish this goal.

At large distance from a black hole, fill a box with thermal radiation at some temperature T (see Fig. 8.23). Close the box and slowly lower it toward the horizon, by means of a rope attached to a winch. As the box descends, the gravitational potential energy of the radiation is converted into useful mechanical work (the gravitational potential energies of the box and the rope are irrelevant, since we will ultimately have to lift the box and rope back to their initial positions). At the horizon of a Schwarzschild black hole, the gravitational potential energy of a mass m of radiation is $-mc^2$ (see the discussion following Eq. [7.131]); that is, all of the energy of the radiation will have been converted into useful work. We now open the bottom of the box, and dump the radiation into the black hole. Then we raise the empty box to its initial position, and thereby complete one cycle of the operation of our thermodynamic engine. The net result of this

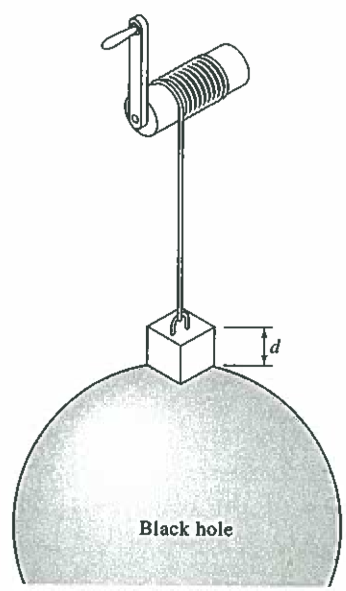

Fig. 8.23 A box attached to a winch is filled with thermal radiation and then lowered until its bottom reaches the horizon of the black hole.

cycle would seem to be the complete conversion of heat into mechanical work. Such a complete conversion would lead us to the conclusion that the thermodynamic temperature of the black hole is exactly zero.

However, this argument neglects to take into account the size of the box. Since thermal radiation of temperature T typically has a wavelength of about $\hbar c/kT$ (by Wien's law), a box that contains such thermal radiation must have a size d at least as large as the typical wavelength,

$$d \simeq \hbar c/kT \tag{69}$$

When the bottom of the box reaches the horizon, the center of gravity of the box will be at a distance of $d/2$ above the horizon, and the potential energy of the radiation will be $-mc^2 + mgd/2$, where g is the acceleration of gravity at the surface of the black hole. From Newtonian theory, we can estimate this acceleration of gravity as $g = GM/r_S{}^2 = c^4/4GM$ (by dumb luck, the exact relativistic value of g agrees with this Newtonian value; see Problem 7.18). The mechanical work we gain by dumping the radiation is therefore not mc^2, but

$$W = mc^2 - m\frac{c^4}{4GM}\frac{d}{2} \simeq mc^2\left(1 - \frac{\hbar c^3}{8GMkT}\right) \tag{70}$$

The efficiency of our thermodynamic engine is then

$$e = \frac{W}{mc^2} \simeq 1 - \frac{\hbar c^3}{8GMkT} \tag{71}$$

According to the definition of the thermodynamic temperature, this efficiency can be expressed as $1 - T_{BH}/T$, where T_{BH} is the temperature of the black hole. The formula [71] therefore leads to the conclusion that the temperature of the black hole is

$$T_{BH} \simeq \frac{\hbar c^3}{8GMk} \tag{72}$$

Note that this temperature is inversely proportional to the surface gravity of the black hole.

The result [72] for the temperature of a black hole was first obtained by Bekenstein (1973, 1974), who proposed to associate not only a temperature with the black hole, but also an entropy, so that the laws of thermodynamics can be applied to processes involving black holes. When our black hole absorbs an amount of heat Q, its entropy increases by Q/T_{BH}. The heat Q absorbed equals the increase of mass of the black hole, that is, $Q = \delta Mc^2$. Hence the entropy increase can be expressed as

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$$\delta S = \frac{\delta Mc^2}{T_{BH}} \simeq \frac{8kGM\delta M}{\hbar c} \qquad [73]$$

Alternatively, this can be expressed in terms of the change of the surface area of the black hole, $\delta A = 8\pi r_S \delta r_S = 32\pi G^2 M \delta M / c^4$:

$$\delta S \simeq \frac{kc^3}{4\pi\hbar G}\delta A \qquad [74]$$

This simple proportionality between δS and δA indicates that the entropy of a black hole must be proportional to its surface area,

$$S \simeq \frac{kc^3}{4\pi\hbar G}A \qquad [75]$$

The formation of a black hole therefore entails a large increase of entropy. Such an increase can be made plausible by the information-theoretic interpretation of entropy. When the black hole forms, or whenever we increase the size of the black hole by dumping any kind of matter into it, we lose information about the trapped matter. This loss of information corresponds to an increase of entropy (Bekenstein, 1973, 1974, 1980; Hawking, 1976).

We can now formulate the first law of thermodynamics for a black hole in the usual way: the increase δMc^2 in the internal energy equals the sum of the heat absorbed by the black hole and the mechanical work performed on the black hole by external forces. Since the black hole behaves like a rigid body, the only way to increase its internal mechanical energy is by spinning it up by an external torque. In this case $\delta W = \Omega \delta J$, where Ω is the angular velocity of the black hole and J its spin angular momentum.* Hence the first law of thermodynamics becomes

$$\delta Mc^2 = T_{BH}\,\delta S + \Omega \delta J \qquad [76]$$

The second law of thermodynamics for black holes is a direct consequence of Hawking's theorem for the increase of the surface area in any process involving one or more black holes. Since the entropy is proportional to the surface area, Hawking's theorem ensures the increase of entropy.

In the above discussion, we treated the black-hole temperature T_{BH} as a thermodynamic temperature. But it must also be a radiation temperature, that is, the black hole must emit thermal radiation of this characteristic temperature. If it did not, we could immerse the black hole in a bath of radiation of lower temperature, $T < T_{BH}$, and the

* In contrast to the notation of Section 8.4, we now use the symbol J for the spin.

black hole would then absorb this radiation without emitting any. Thus, while the radiation bath loses an amount of entropy $\delta Q/T$, the black hole would gain only a smaller amount of entropy $\delta Q/T_{BH}$, with a violation of the second law of thermodynamics.

The explicit proof that black holes emit thermal radiation was given by Hawking, who discovered that in the curved spacetime of a black hole, radiation is generated by a quantum process. Hawking found an elegant way to calculate the spectrum of the radiation from the behavior of quantum fields in curved spacetime (Hawking, 1974, 1975). Earlier attempts at such calculations foundered on the ambiguities of what boundary conditions to apply at the horizon and at the central singularity. Hawking cleverly bypassed these ambiguities by tracing the evolution of the quantum fields in time, from an initial, well-defined vacuum state before the beginning of gravitational collapse and before the formation of the black hole. He demonstrated that when the black hole forms and settles into its final, stationary state, the quantum fields settle into a state that involves a steady outward emission of radiation from the horizon toward infinity. The energy spectrum of this radiation is thermal, with a temperature

$$T_{BH} = \frac{\hbar c^3}{8\pi GMk} \qquad [77]$$

This exact result for the temperature of the black hole is consistent with the approximate result [72] obtained from thermodynamic arguments.

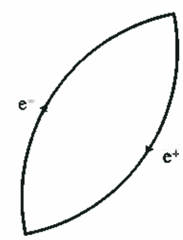

Fig. 8.24 Feynman diagram illustrating the spontaneous creation of a particle-antiparticle pair and its subsequent annihilation.

The Hawking emission process seems to contradict the fundamental property of the horizon--nothing should emerge from the horizon. Actually the thermal radiation does not come from inside the black hole, but it is created by quantum fluctuations at or near its surface. We know from quantum field theory that the vacuum is a restless and violent place, where particles are continuously created and destroyed. Fig. 8.24 shows such a creation and destruction event: a particle-antiparticle pair, such as an electron-antielectron pair, is spontaneously created at a spacetime point, and this pair is destroyed at a later point. If this event happens in the normal vacuum, far from a black hole, it merely produces an unobservable, small-scale fluctuation in the elec-

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tric current density. But if this event happens just outside the horizon of a black hole, the antiparticle (or the particle) might enter the horizon and fall into the singularity. If the antiparticle is in a state of negative energy $-E$, the particle will be left behind in a state of positive energy $+E$, and it will then be free to move outward, away from the black hole, and reach the detection instruments of an observer. The net effect is that a more or less steady stream of particles of positive energy flows outward from the region of the horizon, while the black hole absorbs a stream of antiparticles of negative energy, and therefore decreases its mass. An explicit calculation by Damour and Ruffini (1976) shows that the thermal spectrum of the emerging, liberated, particles arises from a "barrier-penetration" factor. The region in the immediate vicinity of the horizon strongly attenuates the ingoing antiparticle wave (in the calculation, the ingoing antiparticle wave of negative energy is treated as an outgoing particle wave of positive energy proceeding backward in time; see Fig. 8.25). In the limiting case of large energy ($E >> \hbar c^3/GM$), it turns out that the attenuation suffered by the antiparticle wave in crossing the horizon is $e^{-4\pi GME/\hbar c^3}$; thus, the probability for the antiparticle to penetrate into the black hole, and leave the particle liberated, is $e^{-8\pi GME/\hbar c^3}$. From quantum field theory, it is known that all quantum states contribute equally to the particle-antiparticle vacuum fluctuations, so the flux of antiparticles incident on the horizon is proportional to the number of

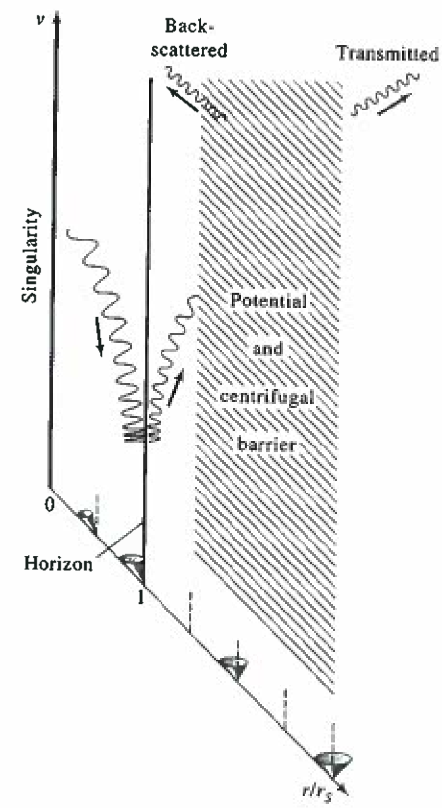

Fig. 8.25 In this diagram, the spacetime inside and outside a black hole is described by Eddington-Finkelstein coordinates. A particle-antiparticle pair is created spontaneously at the horizon. The antiparticle falls into the black hole, and the particle travels away. An antiparticle of negative energy can be treated as a particle of positive energy proceeding backward in time, from the singularity to the horizon. The net wavefunction, indicated schematically, is a wave of positive energy that proceeds from the singularity to the horizon, and from there to infinity. Near the horizon, this wave goes through an infinite number of oscillations and its amplitude is reduced by a penetration factor. (After Damour and Ruffini, 1976.)

quantum states, that is, $4\pi p^2 dp/h^3$. If we multiply this incident flux by the probability for penetration of the horizon, we obtain the outgoing flux of liberated particles:

$$[\text{outgoing flux of particles}] \propto \frac{4\pi p^2 dp}{h^3} e^{-8\pi GME/\hbar c^3} \qquad [78]$$

As expected, this has the form of a thermal spectrum, in the limiting case of large energy ($E >> kT$). The penetration factor $e^{-8\pi GME/\hbar c^3}$ plays the role of the Boltzmann factor $e^{-E/kT}$, with a temperature $T = T_{BH} = \hbar c^3/8\pi GMk$.

If the energy is not large compared with kT_{BH}, then the penetration factor is somewhat more complicated, and depends on whether the particles are fermions or bosons; but in any case, the resulting spectrum is thermal.

Actually, not all the particles liberated at the horizon manage to escape to infinity. Some of these particles are backscattered by the

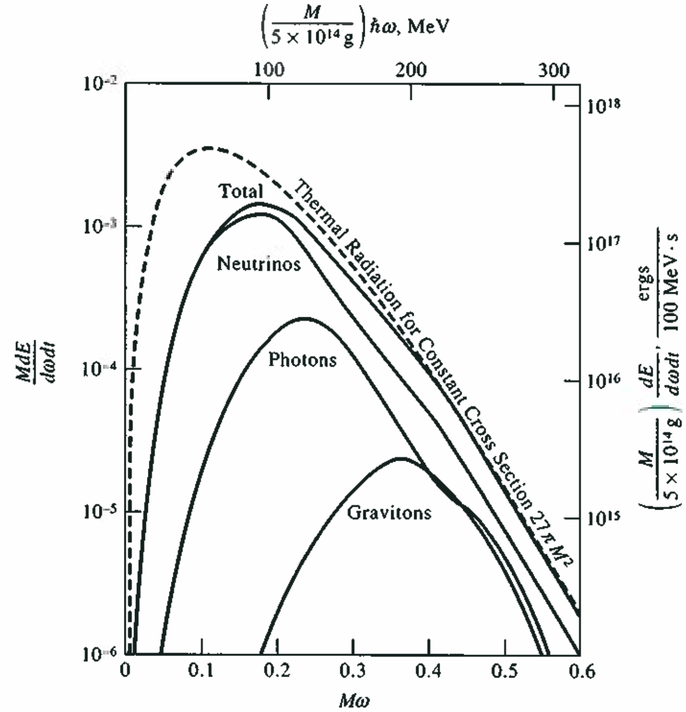

Fig. 8.26 Spectral distribution for different kinds of thermal radiation emitted by a black hole. The scales on the left and the bottom edges of the plot are expressed in units with $G = \hbar = c = 1$. (After Page, 1976a.)

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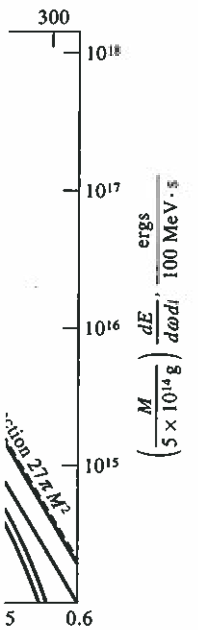

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gravitational potential surrounding the black hole, and they disappear into the black hole. This merely means that the black hole and its surrounding potential form a thermal radiator with a less than perfect emissivity, a thermal "gray" body rather than a thermal blackbody. The backscattering modifies the spectrum of the radiation that finally escapes to infinity, reducing it considerably below the spectrum of a perfect blackbody (see Fig. 8.26).

Numerically, the black-hole temperature can be expressed as

$$T_{BH} = (1.2 \times 10^{26}\ \mathrm{K})\ \frac{1}{M/(1\ \mathrm{g})} \qquad [79]$$

Thus, the temperature of a black hole of a mass equal to that of the Sun ($M_\odot = 2 \times 10^{33}$ g) is only 10^{-7} K. But a very small black hole, or minihole, of a mass of 10^{14} g, would have a temperature of 10^{12} K, and the typical thermal energy of the radiated particles would be about $kT_{BH} \simeq 100$ MeV. In general, the typical thermal energy determines what kinds of particles can be radiated. Particles of rest mass m cannot be radiated in significant numbers unless the typical thermal energy is of the order of mc^2. Thus, a black hole of mass equal to that of the Sun can radiate only particles of mass zero, that is, photons, neutrinos, and gravitons; but a minihole of mass 10^{14} g can also radiate electrons, mu mesons, and pions. Black holes of masses significantly smaller than a solar mass cannot be formed by the gravitational

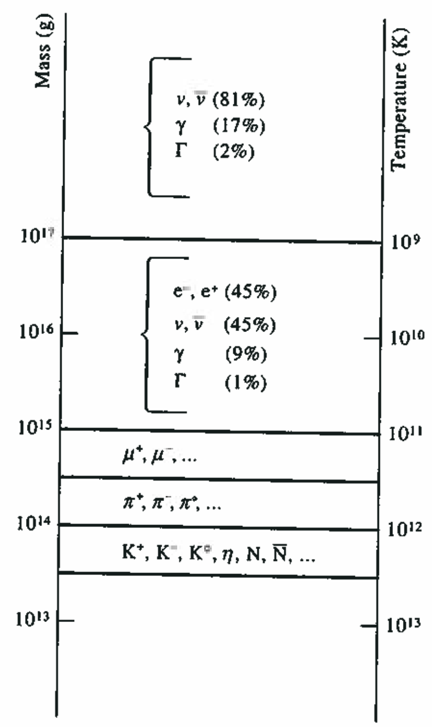

Fig. 8.27 Different kinds of particles radiated by a black hole as a function of its mass.

collapse of a star; such miniholes can form only in the early stages of the universe, from fluctuations in the very dense primordial matter. Fig. 8.27 lists the different kinds of particles radiated by a black hole as a function of its mass.

By Stefan's law the total power radiated by a black hole is of the order of

$$P \simeq [\text{area}] \times \sigma T_{BH}{}^4 \simeq 4\pi r_S{}^2 \times \sigma T_{BH}{}^4 \simeq \frac{10^{47}\ \text{erg/s}}{M^2/(1\ \text{g})^2} \qquad [80]$$

where σ is the Stefan-Boltzmann constant. The exact numerical factor in this equation depends on the various species of particles being radiated and on their backscattering (Page, 1976a,b).

From Eq. [80] we can calculate the rate of decrease of the mass of the black hole:

$$\frac{dM}{dt} = -\frac{P}{c^2} \simeq -\frac{10^{26}\ \text{g/s}}{(M/1\ \text{g})^2} \qquad [81]$$

As the mass decreases, the rate of radiation increases. When the mass becomes small, the rate of radiation becomes explosive. However, it is not known what happens when the mass reaches a value smaller than $\simeq \sqrt{\hbar c/G} \simeq 10^{-5}$ g (the Planck mass), where the time scale for the change of mass will be smaller than the Schwarzschild time, that is, $|dM/dt| \geq M/(r_S/c)$. Under these conditions, the spacetime geometry cannot be treated as a stationary or quasi-stationary background, and the dynamical changes in the geometry will begin to play a crucial role. Although it is widely believed that the minihole will release all its mass in a final explosion and disappear, the details are not understood.

From Eq. [81] we can estimate the lifetime of a black hole. If the initial mass is M, the time until it disappears or almost disappears is approximately

$$t \simeq \frac{M}{|dM/dt|} \simeq 10^{-26}\ \text{s} \times \frac{M^3}{(1\ \text{g})^3} \qquad [82]$$

According to this estimate, a black hole of initial mass 10^{14} g would have a lifetime of about 10^{10} years, equal to the age of the universe. Thus, such a black hole, formed during the early stage of the universe, would be reaching its final explosive phase today.

The observational search for radiation from black holes has concentrated on gamma rays. A black hole of 10^{14} g spends most of its lifetime at a temperature of about 10^{12} K, and at this temperature it produces gamma rays of about 100 MeV. If there is a more or less uniform distribution of such black holes all over the universe, we ı only in the early stages of ·y dense primordial matter. les radiated by a black hole

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$$\Gamma_{BH}{}^4 \simeq \frac{10^{47}\ \text{erg/s}}{M^2/(1\ \text{g})^2} \qquad [80]$$

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from black holes has con- f 10^{14} g spends most of its , and at this temperature it . If there is a more or less all over the universe, we

should see a diffuse background of 100-MeV gamma rays all over the sky. The observational limit on gamma rays sets an upper limit on the density of black holes of about 10^4 per $(\text{pc})^3$. A tighter limit has been set by examining the sky for localized bursts of high-energy gamma rays, 1000 MeV or more, produced during the final explosive phase of a black hole. The observations set an upper limit of 0.04 explosion per $(\text{pc})^3$. These upper limits on the abundance of miniholes imply that their contribution to the overall mass density in the universe is small-- no more than 1 part in 10^8 of the total mass is in the form of miniholes.

Although the negative observational evidence for radiation from miniholes is disappointing, we can draw some interesting conclusions from the absence of such miniholes. The preponderance of normal matter over miniholes tells us that conditions in the early universe were not favorable for the formation of miniholes. Since the miniholes formed from fluctuations, this means that either the fluctuations were not very violent or else the primordial matter offered strong resistance to compression (had a "stiff" equation of state).

8.8 GRAVITATIONAL COLLAPSE AND THE FORMATION OF BLACK HOLES

In normal stars, such as the Sun, the inward gravitational pull is held in equilibrium by the thermal pressure of the gas. This thermal pressure will be sufficient to resist the gravitational pull only if the star is hot enough. The star can therefore remain in equilibrium as long as the thermonuclear reactions in its core supply enough heat, that is, as long as the energy released in these reactions compensates for the energy lost by radiation at the surface. In a star that has exhausted its supply of nuclear fuel, the thermal pressure will ultimately disappear, and the star will collapse under its own weight. The collapse may be sudden (implosion) or gradual (contraction), but in any case it can be halted only if an alternative mechanism for generating sufficient pressure becomes available at high density.

In white-dwarf stars and in neutron stars such an alternative mechanism is available: these stars are so dense that the quantum mechanical zero-point pressure becomes dominant. Essentially, a degenerate Fermi gas of electrons supplies the equilibrium pressure in a white dwarf and a Fermi gas of neutrons that in a neutron star.

At the white-dwarf densities ($\rho \geq 10^5$ g/cm^3) the electrons are detached from their nuclei and move quite freely throughout the volume of the star. The star consists of interpenetrating gases of electrons and nuclei. The zero-point pressure of the electron gas gives the main contribution to the pressure, and the nuclei give the main contribution to the mass density. The equation of state (pressure as function of density) based on this model permits equilibrium configurations, pro-