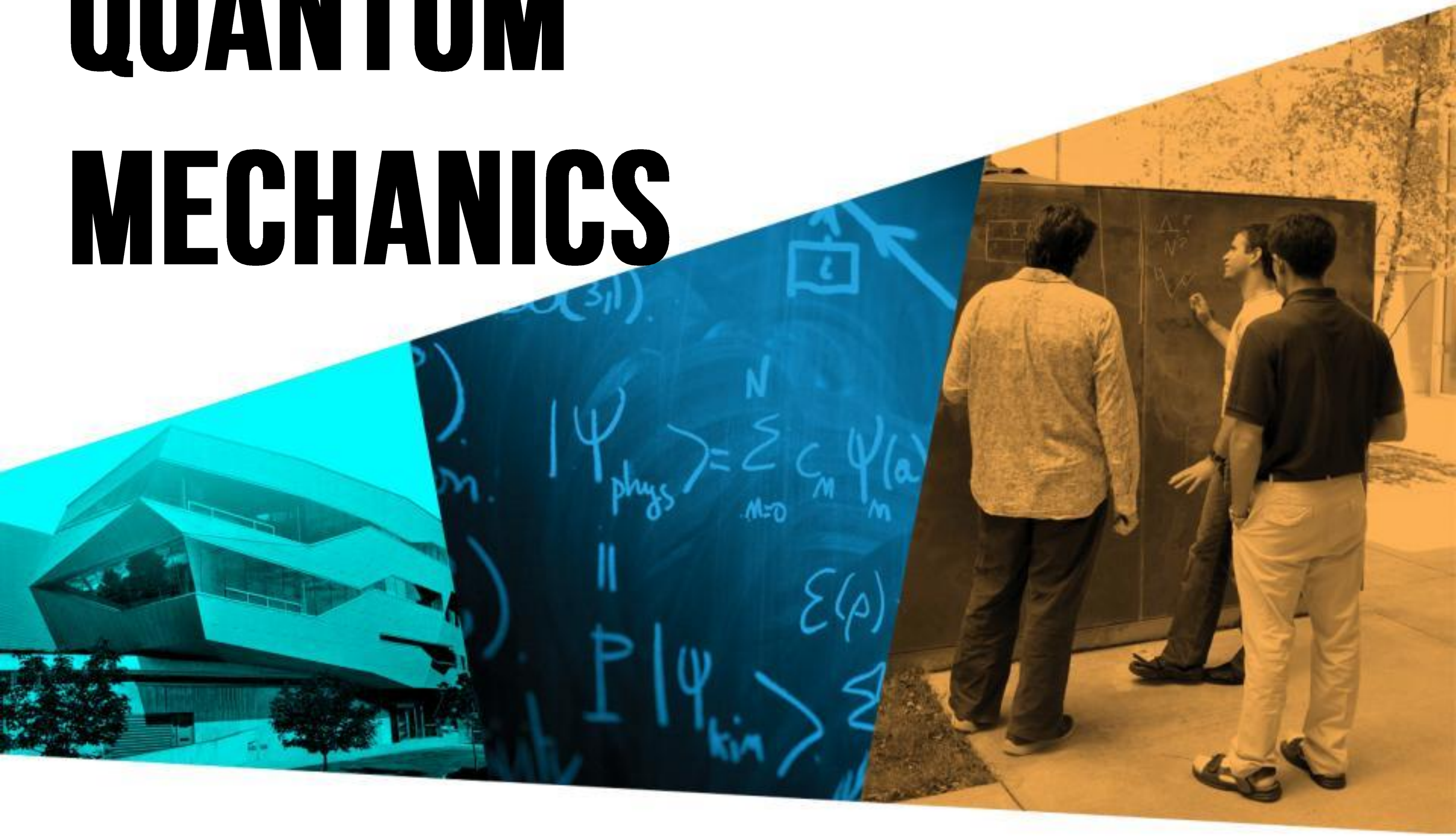
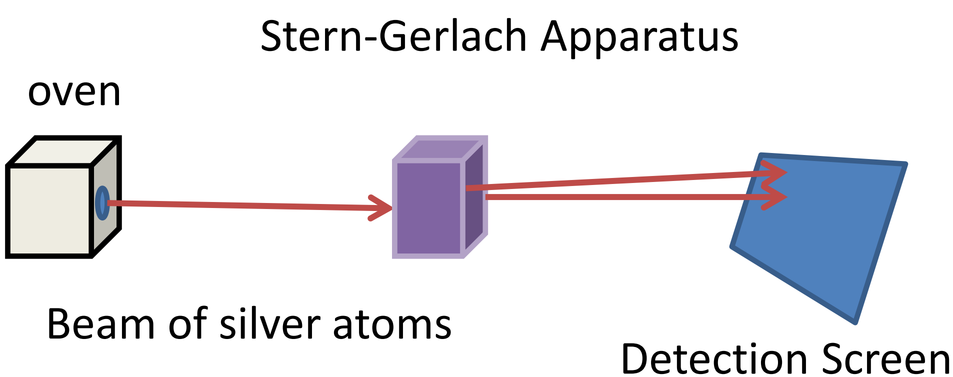
QUANTUM MECHANICS



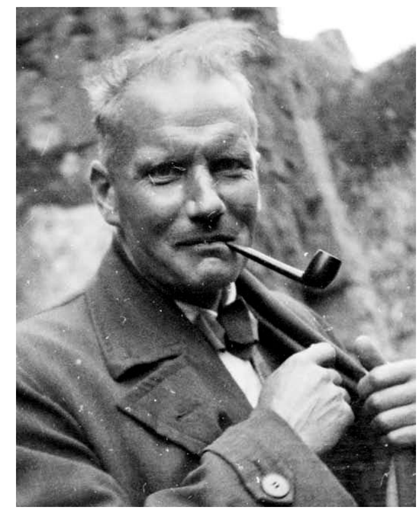
PERIMETER INSTITUTE FOR THEORETICAL PHYSICS

The Physics of Spin

1922 Stern-Gerlach Experiment



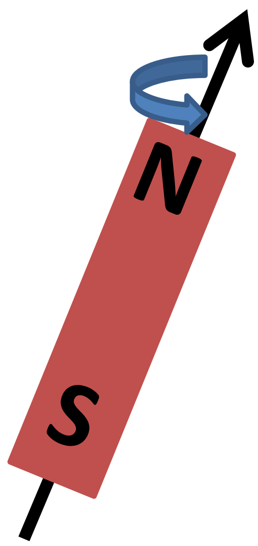
Otto Stern



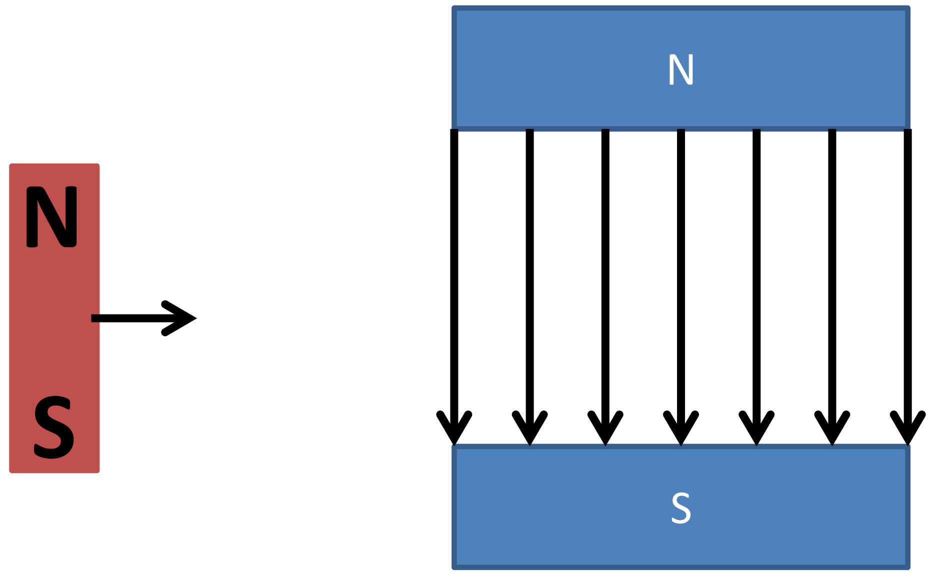
Walter Gerlach

Electron Spin

- Electron is like a spinning top.
- Spinning bar magnet

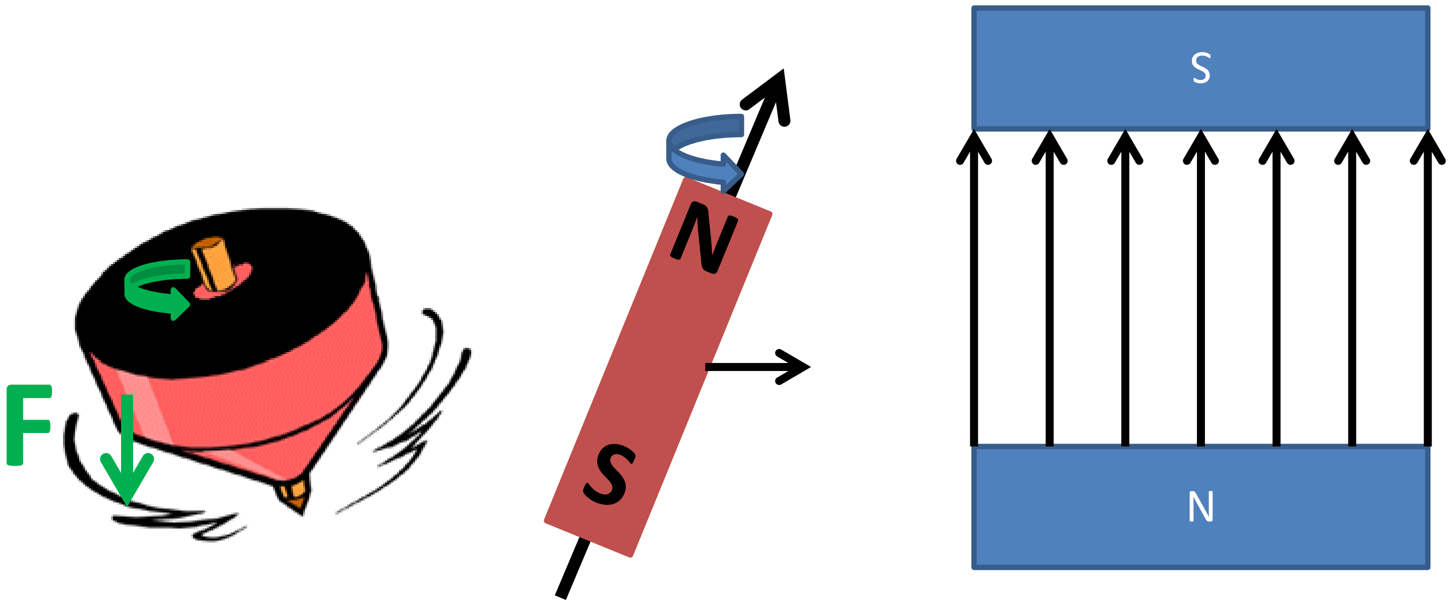


What happens to a bar magnet in a uniform magnetic field?



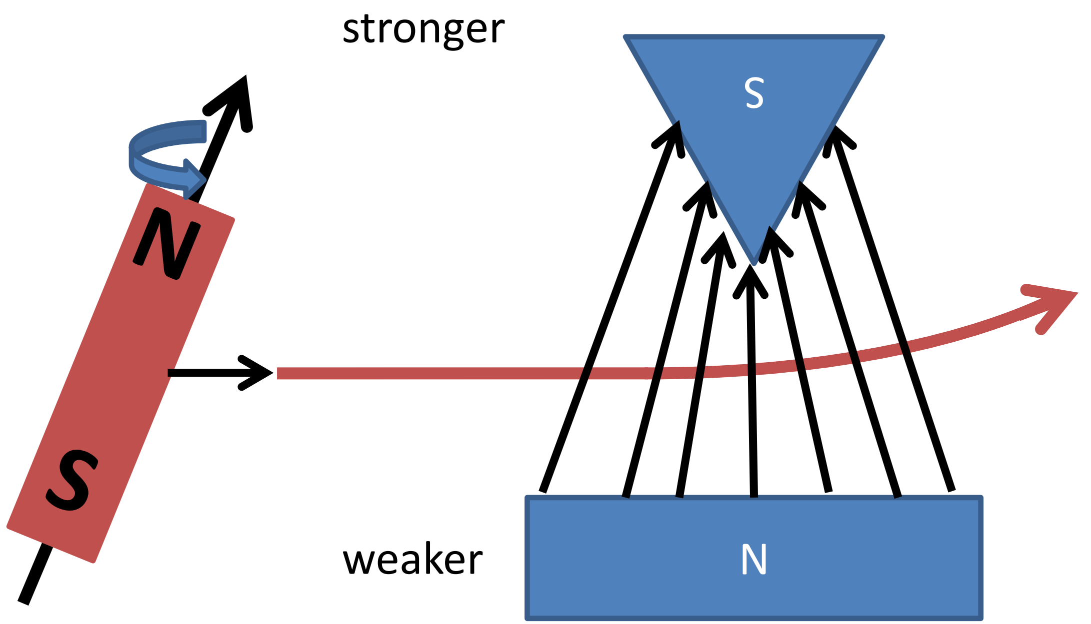
The little magnet will feel a torque (a twisting force).
It will align with the magnetic field.

What happens if the bar magnet is spinning?



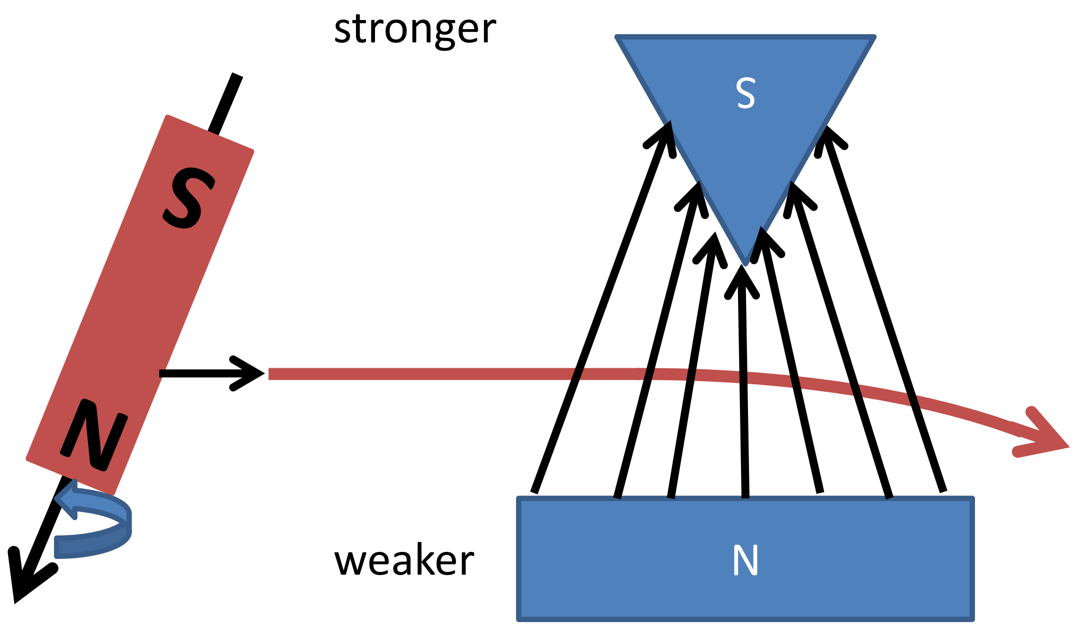
The magnet will precess like a gyroscope.

What if the field is not uniform?



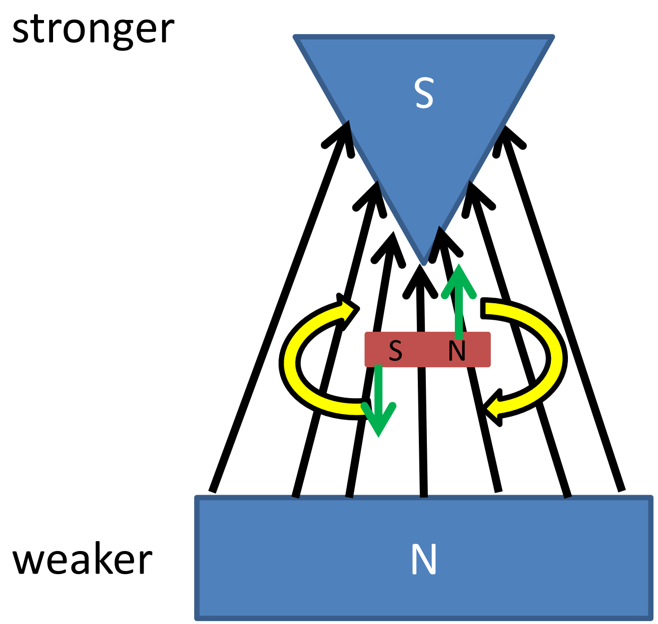
In this case, it will be deflected upward.

What if the field is not uniform?



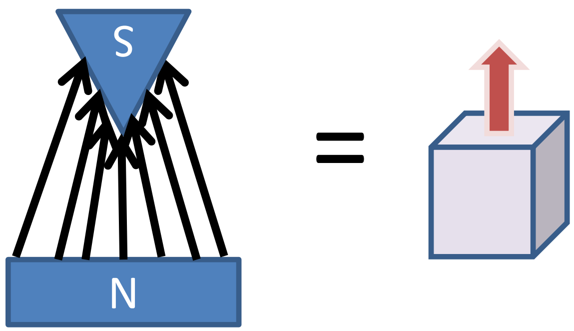
In this case, it will be deflected downward.

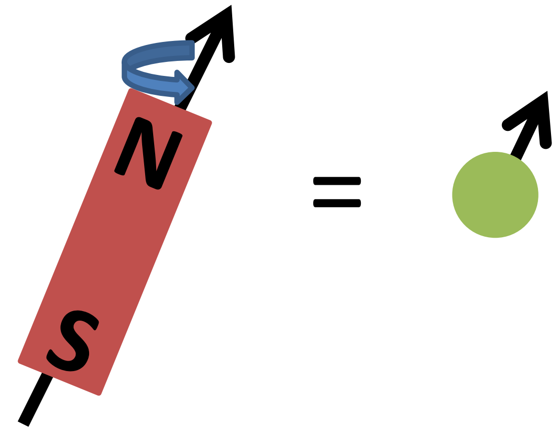
What if it is horizontal?

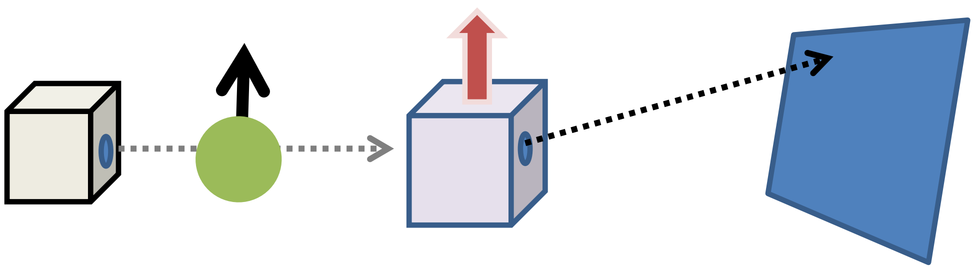


No Deflection!

Experiment #1

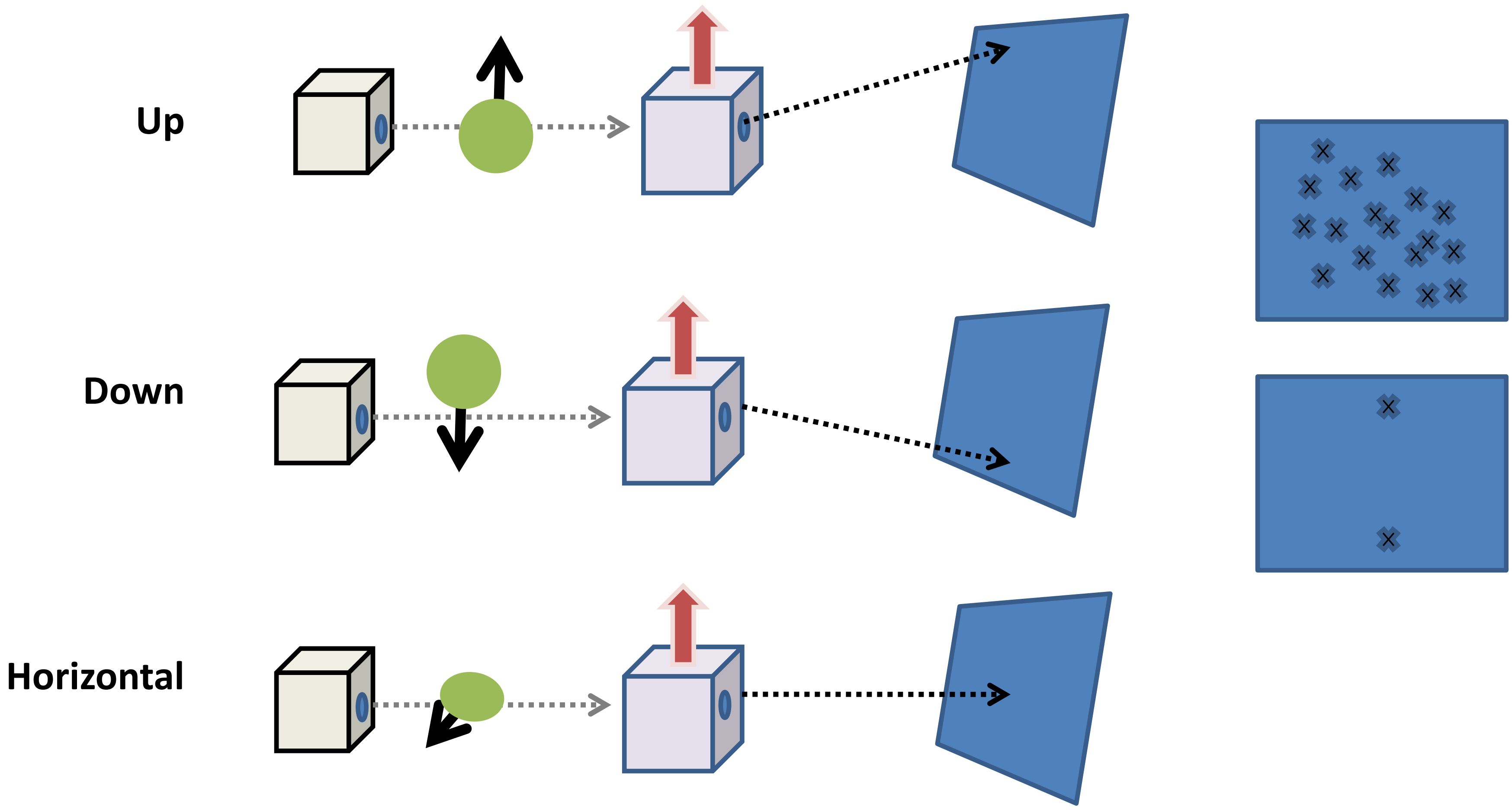




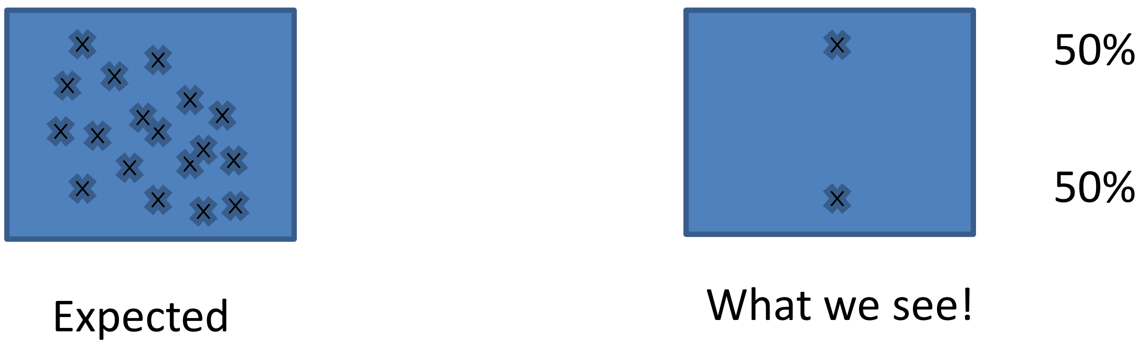


Experiment #1

We should see hits everywhere with equal likelihood!

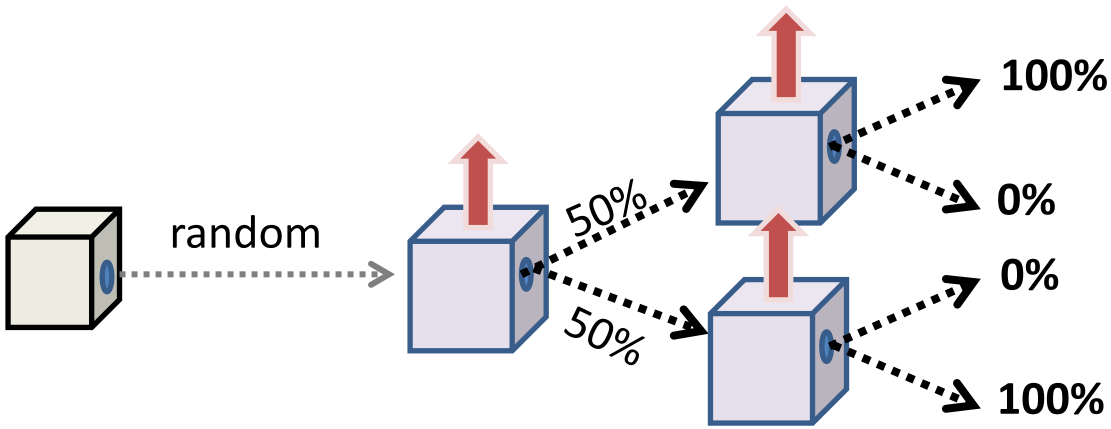


Experiment #1 Results



We can’t explain this using Newton’s Laws!

Experiment #2

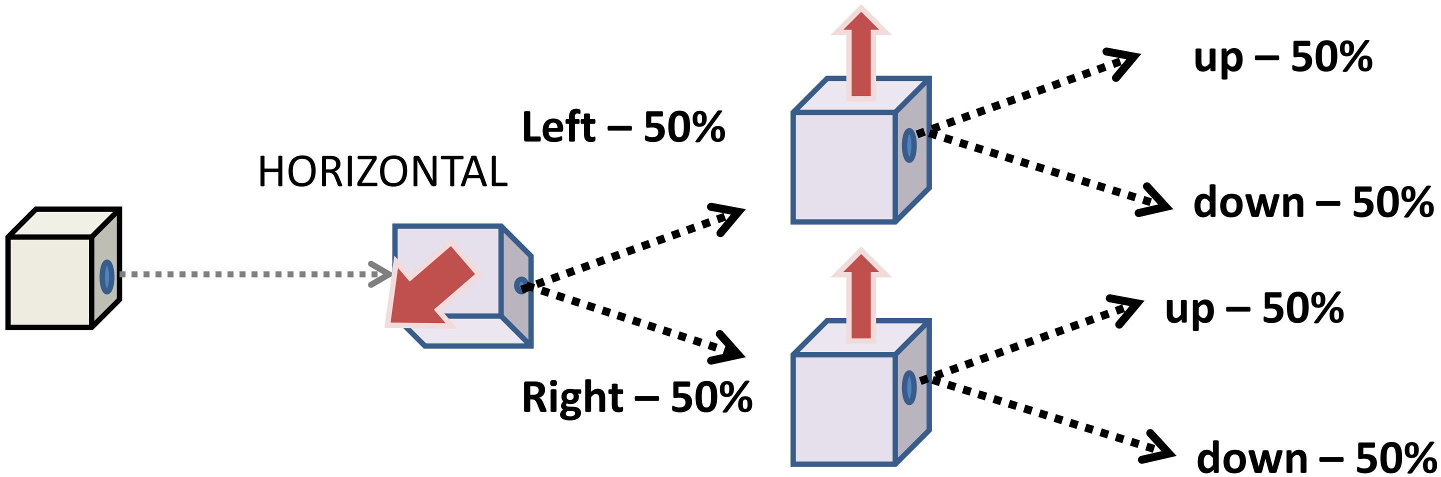


After passing through the Stern-Gerlach apparatus all of the orientations are transformed from random to up and down only.

The upper deflected beam is all spin up.

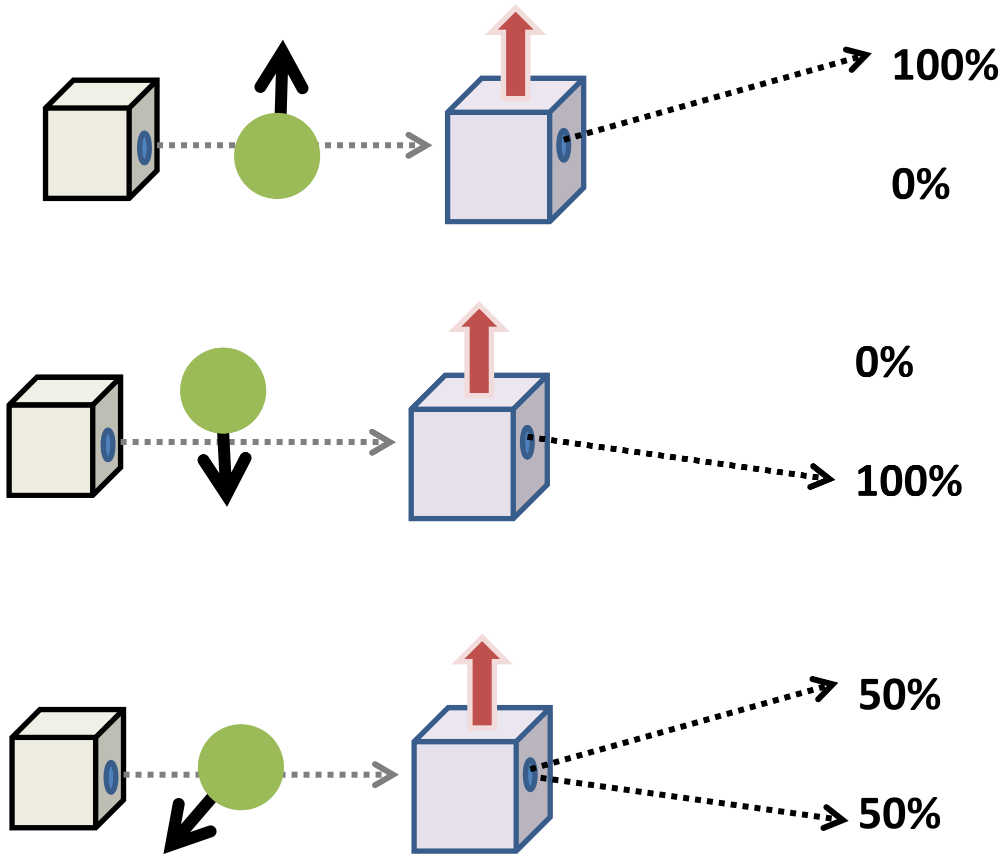
The lower deflected beam is all spin down.

Experiment #3



A horizontal atom has a 50/50 chance of coming out up or down!

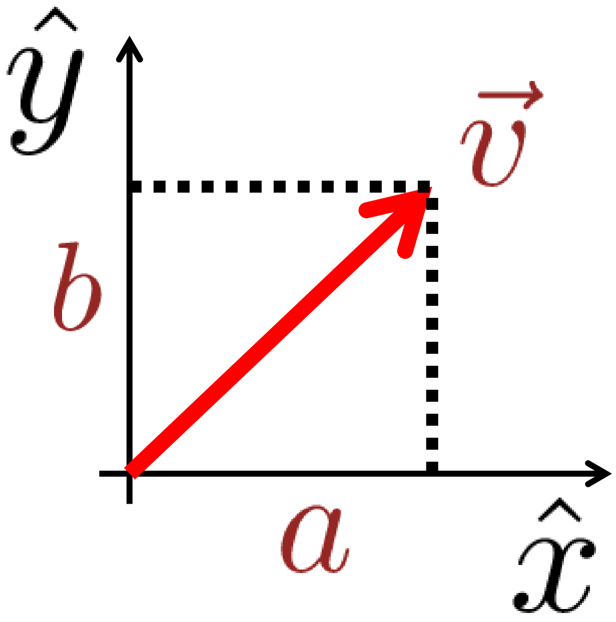
Summary



1. Quantization of spin

2. Probability

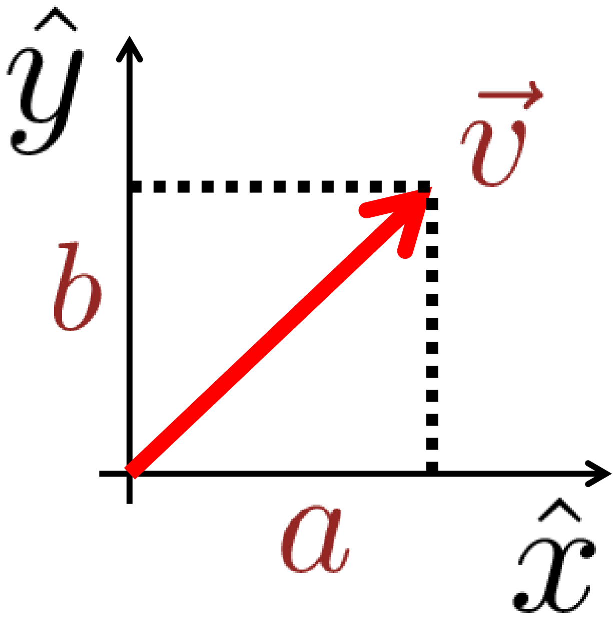
Mathematics of Spin



$$\vec{v} = a\hat{x} + b\hat{y}$$

$$\text{Length} = \sqrt{a^2 + b^2}$$

Mathematics of Spin



If we have a unit vector then:

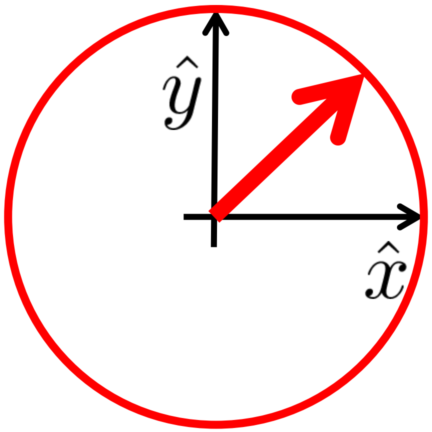
$$a^2 + b^2 = 1$$

$$a = \cos\theta$$

$$b = \sin\theta$$

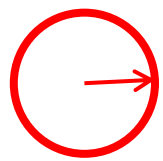
$$\vec{v} = \cos\theta\hat{x} + \sin\theta\hat{y}$$

Mathematics of Spin



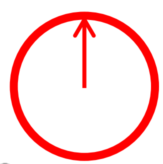
UNIT CIRCLE

$$\vec{v} = \cos\theta\,\hat{x} + \sin\theta\,\hat{y}$$



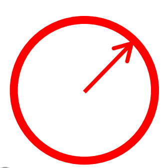
$\theta = 0$

$$\vec{v} = \hat{x}$$



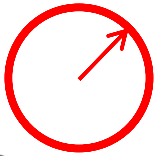
$\theta = 90^\circ$

$$\vec{v} = \hat{y}$$



$\theta = 45^\circ$

$$\vec{v} = \frac{1}{\sqrt{2}}\hat{x} + \frac{1}{\sqrt{2}}\hat{y}$$



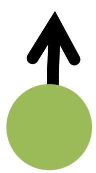
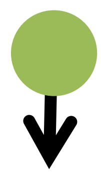
$$\theta = 45^\circ$$

$$\vec{v} = \frac{1}{\sqrt{2}}\hat{x} + \frac{1}{\sqrt{2}}\hat{y}$$

$$a^2 + b^2 = (\frac{1}{\sqrt{2}})^2 + (\frac{1}{\sqrt{2}})^2 = 1$$

Let’s apply this to spin!

- What are the analogues of the unit basis vectors $\hat{x}$ and $\hat{y}$?

- Up  or Down 

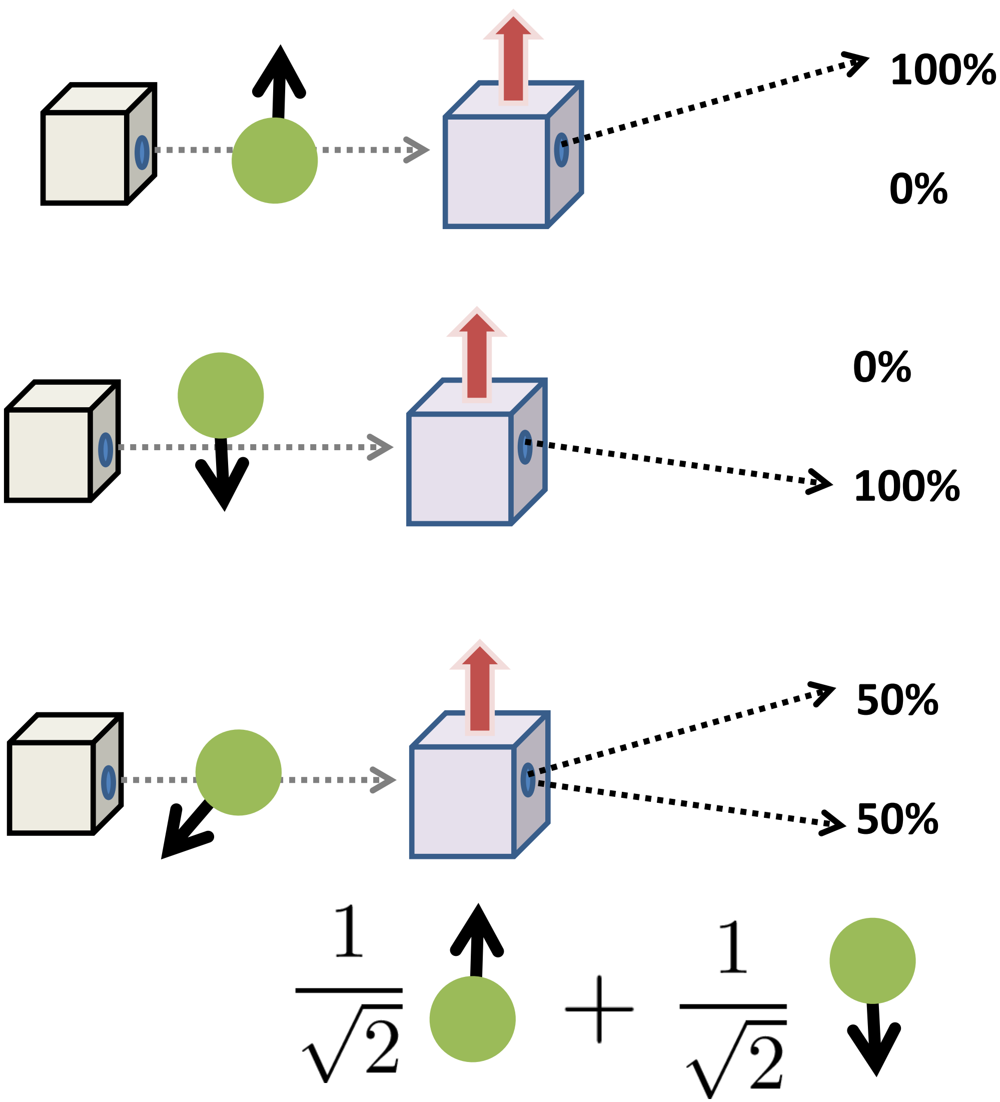
- The quantization of spin suggests the unit vectors are UP and DOWN

- Probability suggests a natural interpretation for

$$a^2 + b^2 = 1$$

- No matter what a and b are:

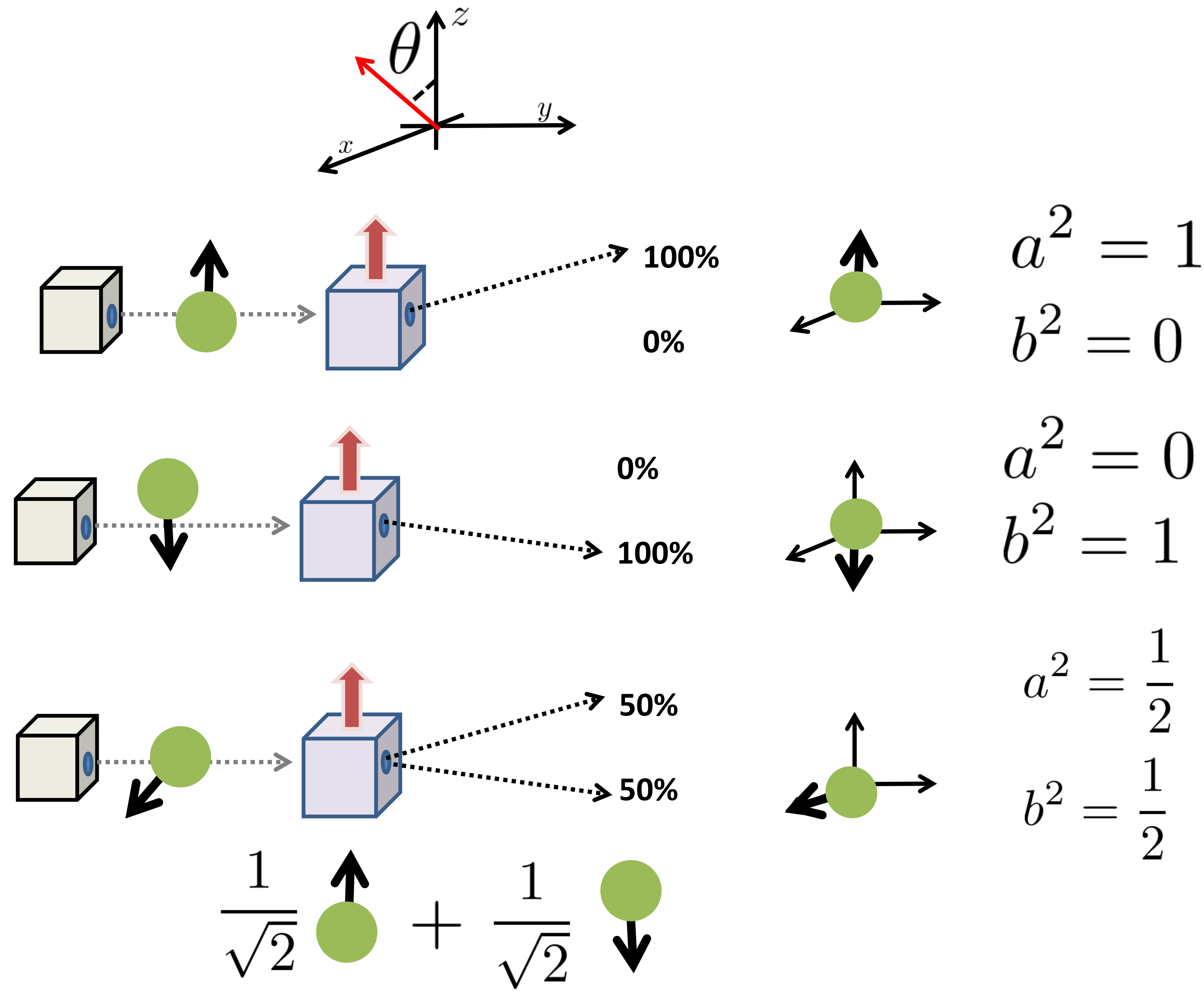
$$a^2 + b^2 = 100\%$$



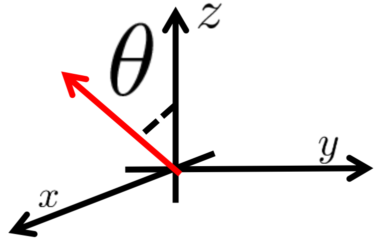
$$a^2 = 1$$
$$b^2 = 0$$

$$a^2 = 0$$
$$b^2 = 1$$

$$a^2 = \frac{1}{2}$$
$$b^2 = \frac{1}{2}$$

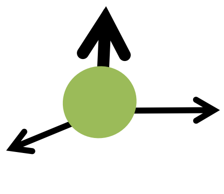


θ
z
y
x
100%
0%
$a^2 = 1$
$b^2 = 0$
0%
100%
$a^2 = 0$
$b^2 = 1$
50%
50%
$a^2 = \frac{1}{2}$
$b^2 = \frac{1}{2}$
$\frac{1}{\sqrt{2}}$
$+$
$\frac{1}{\sqrt{2}}$

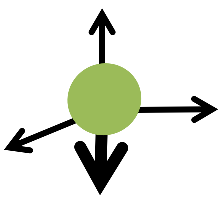


$$= ?\uparrow + ?\downarrow$$

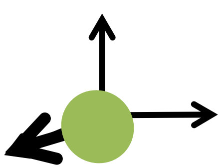
$$\cancel{= \cos\theta \uparrow + \sin\theta \downarrow}$$

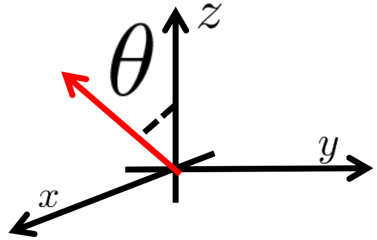


$$= 1 \uparrow + 0 \downarrow$$



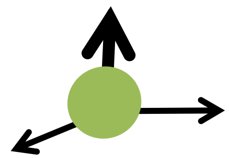
$$= -1 \uparrow + 0 \downarrow$$





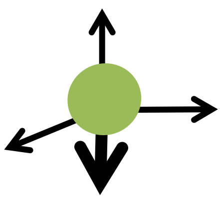
$$= ?\uparrow + ?\downarrow$$

$$= \cos k\theta \uparrow + \sin k\theta \downarrow$$

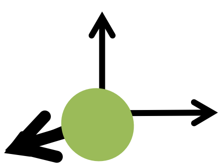


$$= 1 \uparrow + 0 \downarrow$$

$$\boldsymbol{k} = ?\frac{1}{2}$$



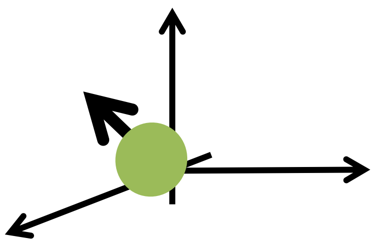
$$= 0 \uparrow + 1 \downarrow$$



$$= \frac{1}{\sqrt{2}} \uparrow + \frac{1}{\sqrt{2}} \downarrow$$

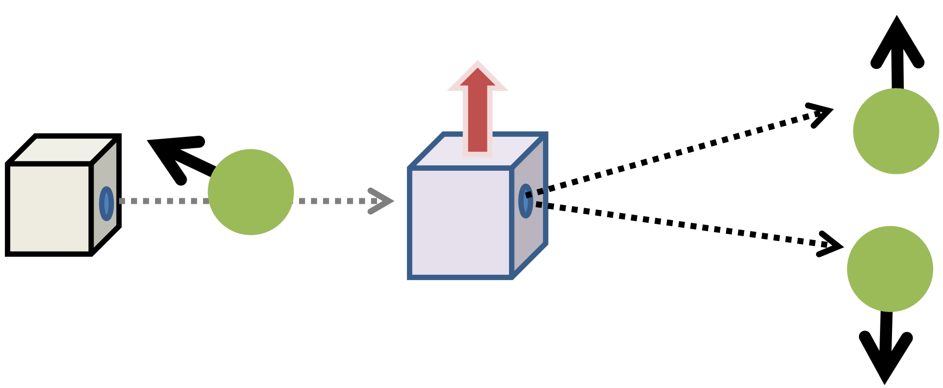
$$= \cos\frac{\theta}{2}\uparrow + \sin\frac{\theta}{2}\downarrow$$

$$\theta = 60^\circ$$



$$= \frac{\sqrt{3}}{2}\uparrow + \frac{1}{2}\downarrow$$

$$(\frac{\sqrt{3}}{2})^2 + (\frac{1}{2})^2 = 1$$

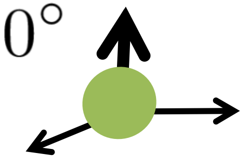


$$(\frac{\sqrt{3}}{2})^2 = 75\%$$

$$(\frac{1}{2})^2 = 25\%$$

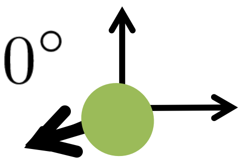
Does the math predict anything new?

0°



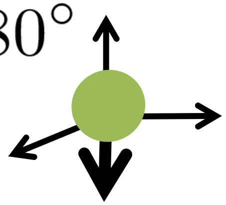
$$= 1 \uparrow + 0 \downarrow$$

90°



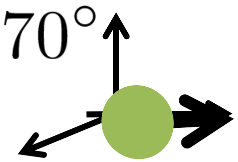
$$= \frac{1}{\sqrt{2}} \uparrow + \frac{1}{\sqrt{2}} \downarrow$$

180°



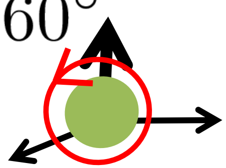
$$= 0 \uparrow + 1 \downarrow$$

270°



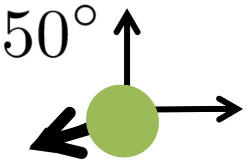
$$= -\frac{1}{\sqrt{2}} \uparrow + \frac{1}{\sqrt{2}} \downarrow$$

360°



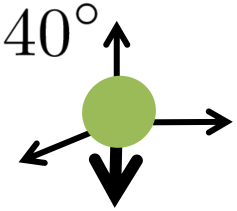
$$= -1 \uparrow + 0 \downarrow$$

450°



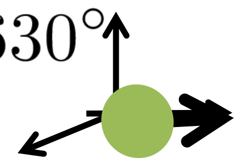
$$= -\frac{1}{\sqrt{2}} \uparrow - \frac{1}{\sqrt{2}} \downarrow$$

540°



$$= 0 \uparrow - 1 \downarrow$$

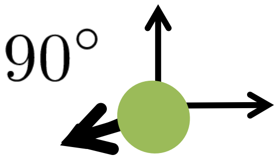
630°



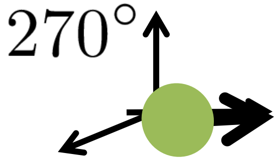
$$= \frac{1}{\sqrt{2}} \uparrow - \frac{1}{\sqrt{2}} \downarrow$$

720°

$$= 1 \uparrow + 0 \downarrow$$



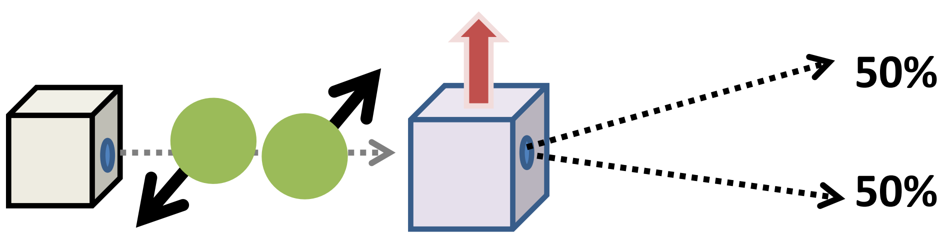
$$= \frac{1}{\sqrt{2}} \uparrow + \frac{1}{\sqrt{2}} \downarrow$$



$$= -\frac{1}{\sqrt{2}} \uparrow + \frac{1}{\sqrt{2}} \downarrow$$

- The minus sign distinguishes the –x spin direction from the +x spin direction

- But it does not affect the probabilities!

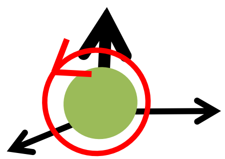


$$a^2 = \frac{1}{2}$$

$$b^2 = \frac{1}{2}$$

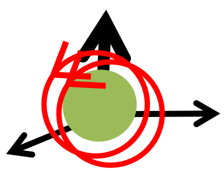
The Stern-Gerlach apparatus does not "see" the minus sign!

Does the math predict anything new?



$$= -1 \uparrow + 0 \downarrow$$

One rotation

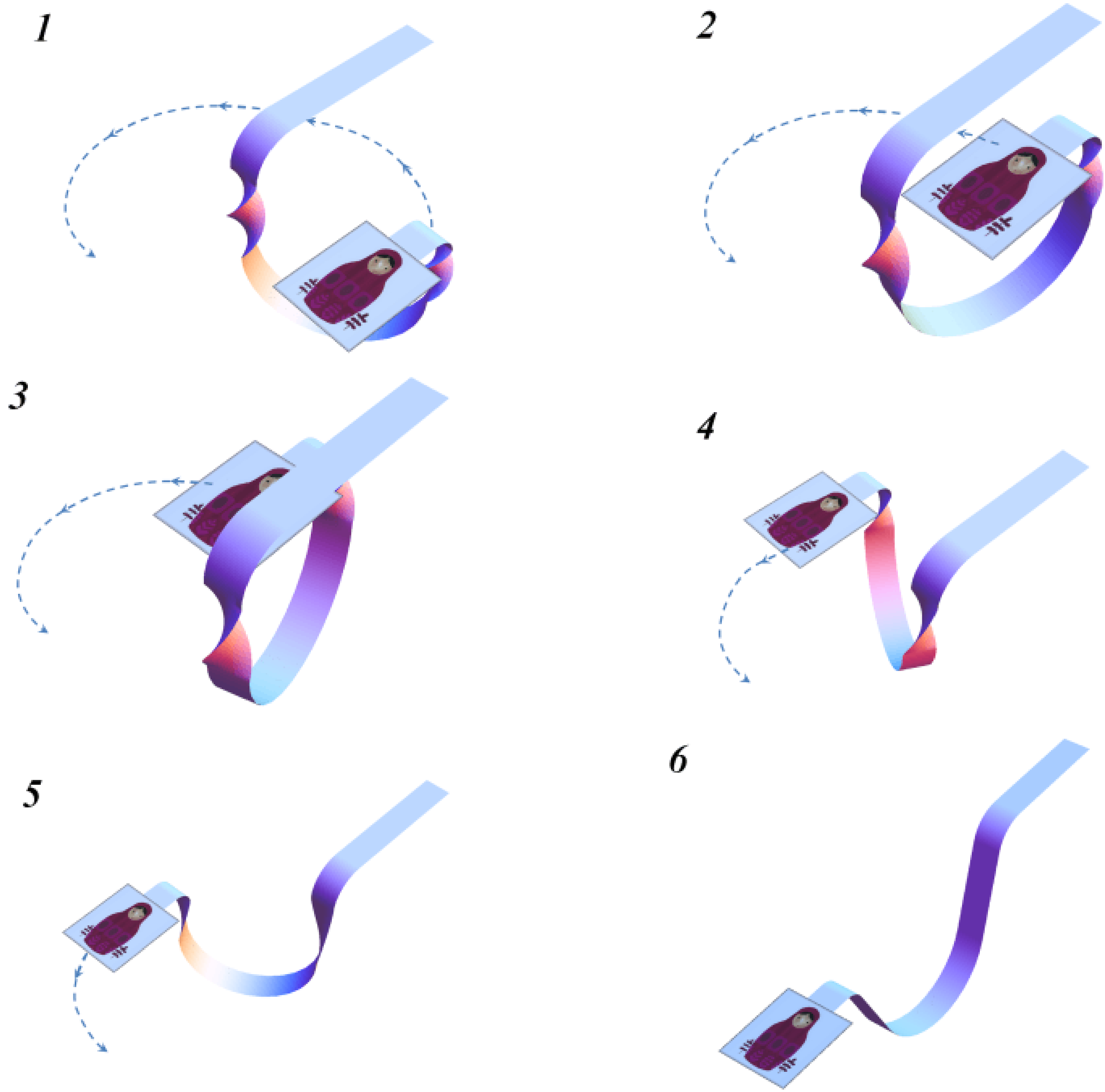


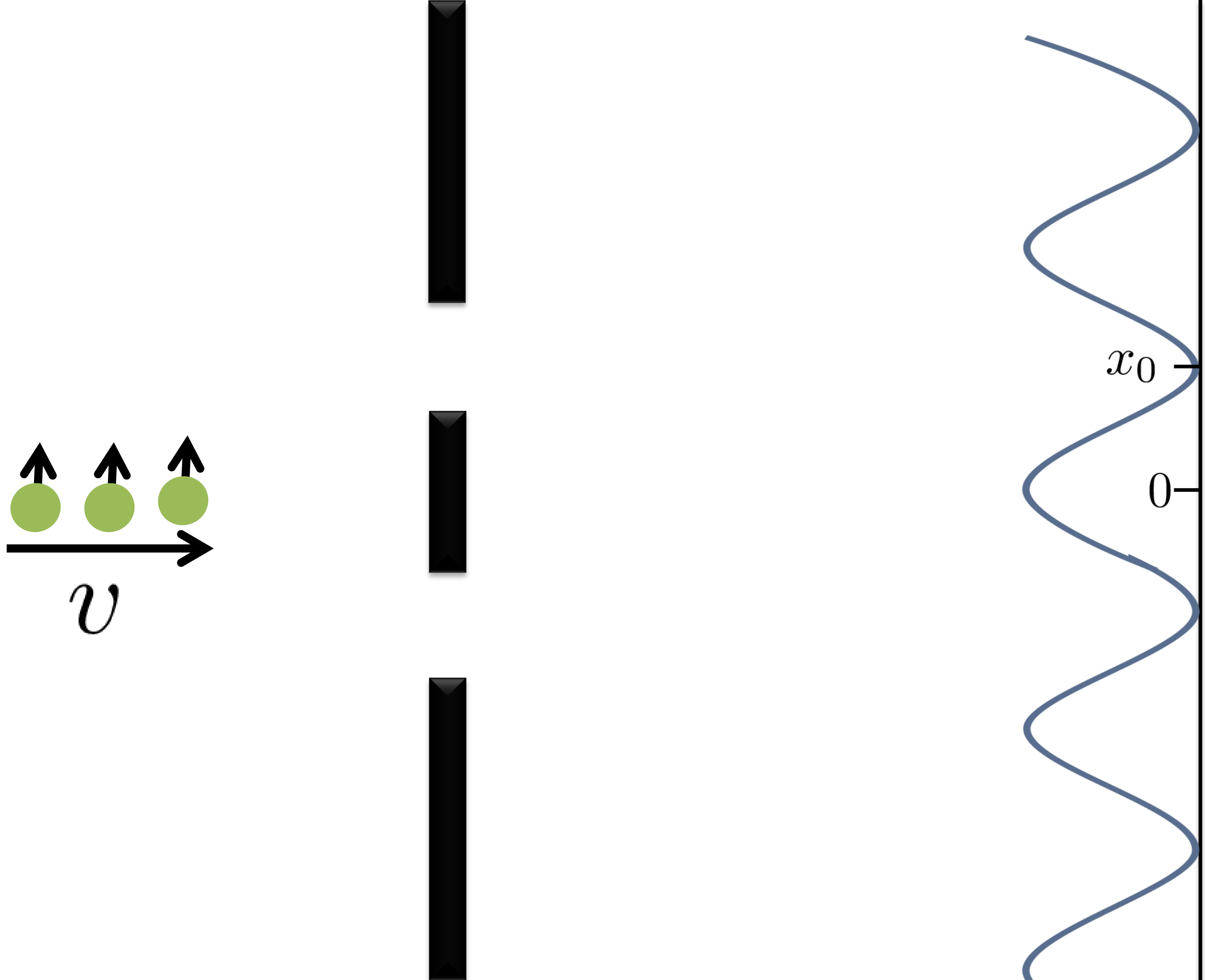
$$= 1 \uparrow + 0 \downarrow$$

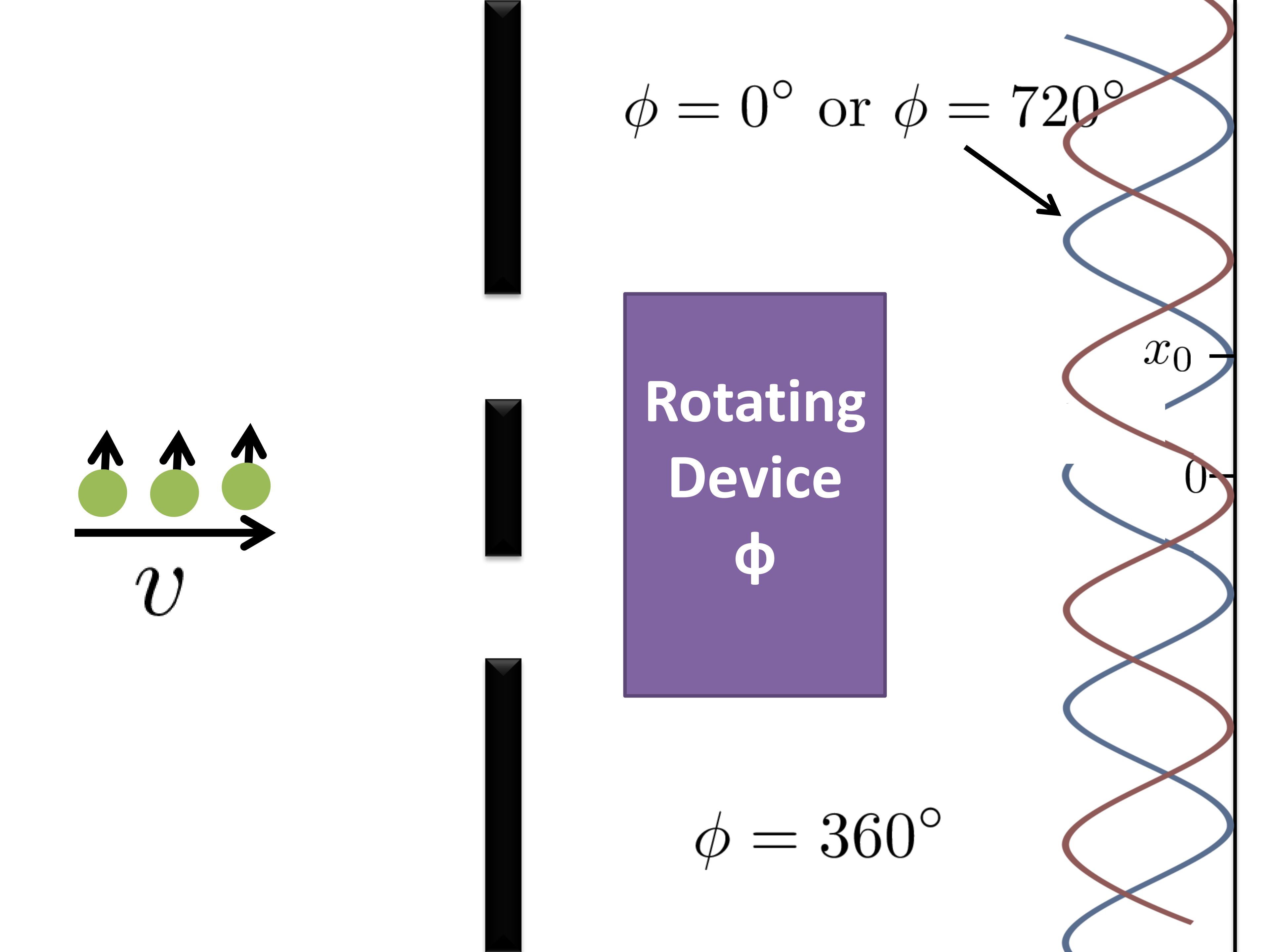
Two rotations

This is not like a spinning top! If it makes a 360 degree rotation it doesn't look the same.

You need to rotate it twice to get back where you started!


1
2
3
4
5
6


x_0
0
v



$\phi = 0^\circ$ or $\phi = 720^\circ$
x_0
0
Rotating Device
ϕ
v
$\phi = 360^\circ$

Experimental Results

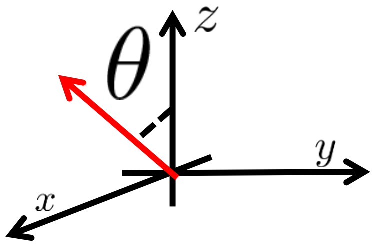
- When $\phi = 0^\circ$ we get the same pattern of hits as before the rotator (obvious!)

- When $\phi = 360^\circ$ the pattern of hits ***shifts***

- When $\phi = 720^\circ$ the pattern of hits returns to the original!

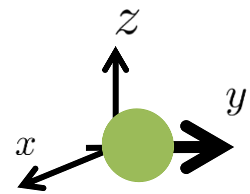
Another deeply mysterious property of quantum mechanics!

The Right Track … but!

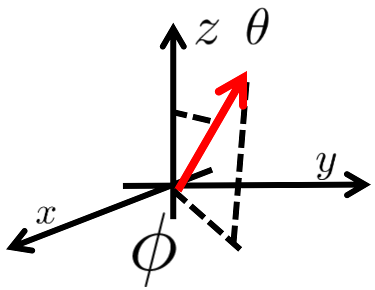


$$= \cos\frac{\theta}{2}\uparrow + \sin\frac{\theta}{2}\downarrow$$

- So far our equation is predicting results well
- But, it only covers spin directions lying in the x-z plane
- What about something like this?



Yes! But we need φ and i!

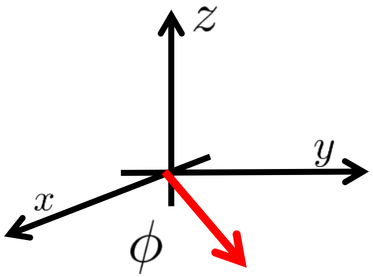


$$= e^{-i\frac{\phi}{2}} \cos\left(\frac{\theta}{2}\right)(\uparrow) + e^{i\frac{\phi}{2}} \sin\left(\frac{\theta}{2}\right)(\downarrow)$$

Φ=0 is the case we have been considering up until now.

We see here again the role of ***complex numbers*** in quantum mechanics!

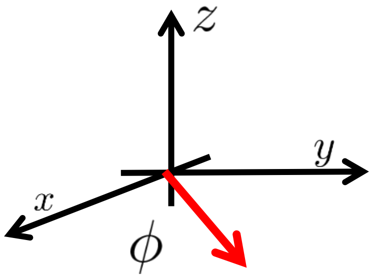
Write down the description of an atom where the spin lies at an angle ϕ in the x-y plane (i.e. Θ=90º).



$$= \underbrace{e^{-i\frac{\phi}{2}}\frac{1}{\sqrt{2}}}_{a}(\uparrow) + \underbrace{e^{i\frac{\phi}{2}}\frac{1}{\sqrt{2}}}_{b}(\downarrow)$$

$$|e^{\pm i\frac{\phi}{2}}| = |\cos\frac{\phi}{2} \pm i\sin\frac{\phi}{2}| = \sqrt{\cos^2\frac{\phi}{2} + \sin^2\frac{\phi}{2}} = 1$$

Write down the description of an atom where the spin lies at an angle ϕ in the x-y plane (i.e. ϴ=90º).



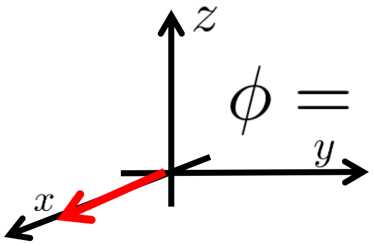
$$= \underbrace{e^{-i\frac{\phi}{2}}\frac{1}{\sqrt{2}}}_{a}(\uparrow) + \underbrace{e^{i\frac{\phi}{2}}\frac{1}{\sqrt{2}}}_{b}(\downarrow)$$

$$|a|^2 = |b|^2 = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2} = 50\%$$

Regardless of the angle ϕ, the atom will come out in either the up or down state (50/50)!

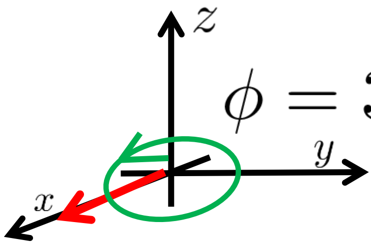
Starting with the spin "vector" in the +x direction (i.e. Θ=90° and Θ =0) rotate the atom through 360° and then through 720°.

$$e^{-i\frac{\phi}{2}}\frac{1}{\sqrt{2}}(\uparrow)+e^{i\frac{\phi}{2}}\frac{1}{\sqrt{2}}(\downarrow)$$



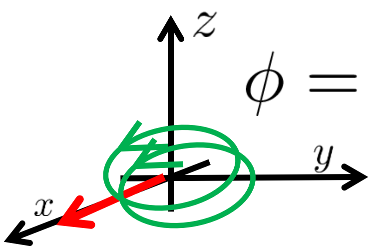
$$\phi = 0^\circ$$

$$=\frac{1}{\sqrt{2}}(\uparrow)+\frac{1}{\sqrt{2}}(\downarrow)$$



$$\phi = 360^\circ$$

$$=-\frac{1}{\sqrt{2}}(\uparrow)-\frac{1}{\sqrt{2}}(\downarrow)$$



$$\phi = 720^\circ$$

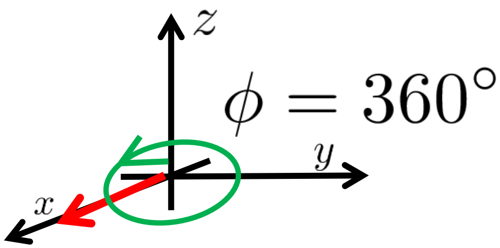
$$=\frac{1}{\sqrt{2}}(\uparrow)+\frac{1}{\sqrt{2}}(\downarrow)$$

Starting with the spin "vector" in the +x direction (i.e. ϴ=90ᵒ and ϴ =0) rotate the atom through 360ᵒ and then through 720ᵒ.

$$e^{\pm i \frac{360^\circ}{2}} = e^{\pm i 180^\circ}$$

$$= \cos(180^\circ) \pm i \sin(180^\circ)$$

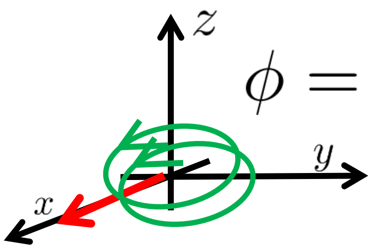
$$= -1$$



$$= -\frac{1}{\sqrt{2}}(\uparrow) - \frac{1}{\sqrt{2}}(\downarrow)$$

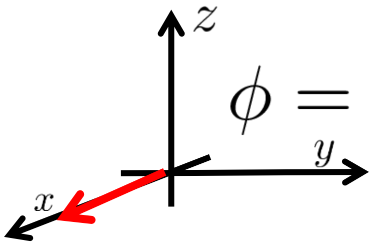
Starting with the spin "vector" in the +x direction (i.e. ϴ=90° and ϴ =0) rotate the atom through 360° and then through 720°.

$$e^{\pm i\frac{720^\circ}{2}} = e^{\pm i360^\circ} = \cos(360^\circ) \pm i\sin(360^\circ) = +1$$

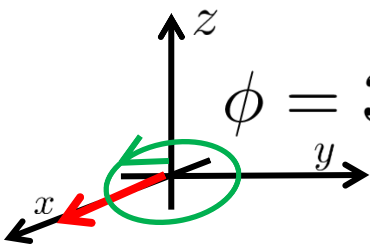


$$\phi = 720^\circ \qquad = \frac{1}{\sqrt{2}}(\uparrow) + \frac{1}{\sqrt{2}}(\downarrow)$$

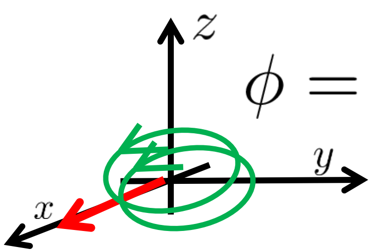
Starting with the spin "vector" in the +x direction (i.e. Ө=90° and Ө =0) rotate the atom through 360° and then through 720°.



$$\phi = 0^\circ \qquad = \frac{1}{\sqrt{2}}(\uparrow) + \frac{1}{\sqrt{2}}(\downarrow)$$

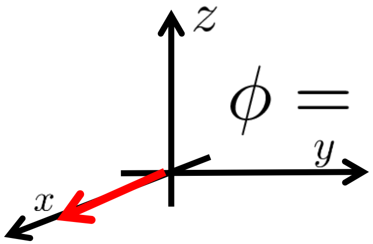


$$\phi = 360^\circ \qquad = -\frac{1}{\sqrt{2}}(\uparrow) - \frac{1}{\sqrt{2}}(\downarrow)$$

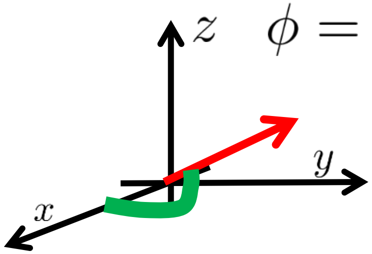


$$\phi = 720^\circ \qquad = \frac{1}{\sqrt{2}}(\uparrow) + \frac{1}{\sqrt{2}}(\downarrow)$$

Starting with the spin "vector" in the +x direction (i.e. ϴ=90° and ϴ =0) rotate the atom in two different ways to the –x direction. Is there a difference?

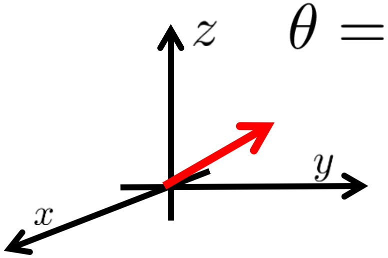


$$\phi = 0^\circ \qquad = \frac{1}{\sqrt{2}}(\uparrow) + \frac{1}{\sqrt{2}}(\downarrow)$$



$$\phi = 180^\circ$$

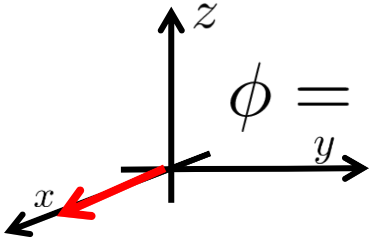
$$e^{\pm i\frac{180^\circ}{2}} = e^{\pm i 90^\circ}$$
$$= \cos(90^\circ) \pm i\sin(90^\circ)$$
$$= \pm i$$



$\theta = 90^\circ$ to $\theta = 270^\circ$

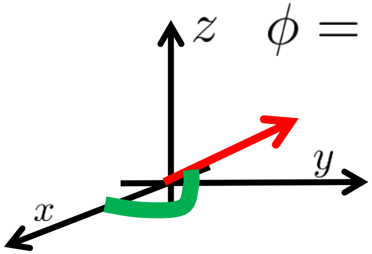
$$\phi = 0^\circ$$

Starting with the spin “vector” in the +x direction (i.e. ϴ=90° and ϴ =0) rotate the atom in two different ways to the –x direction. Is there a difference?



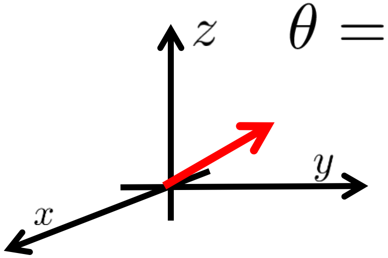
$\phi = 0^\circ$

$$= \frac{1}{\sqrt{2}}(\uparrow) + \frac{1}{\sqrt{2}}(\downarrow)$$



$\phi = 180^\circ$

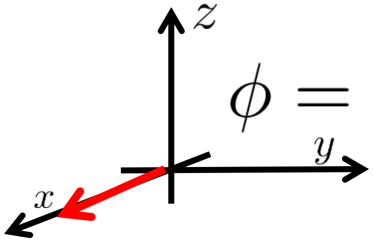
$$= i\frac{1}{\sqrt{2}}(\uparrow) - i\frac{1}{\sqrt{2}}(\downarrow)$$



$\theta = 90^\circ$ to $\theta = 270^\circ$

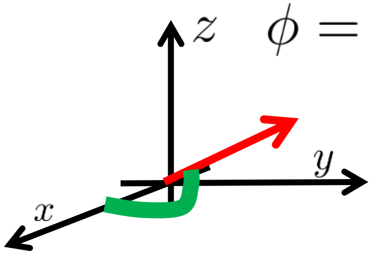
$\phi = 0^\circ$

Starting with the spin "vector" in the +x direction (i.e. Ɵ=90º and Ɵ =0) rotate the atom in two different ways to the –x direction. Is there a difference?



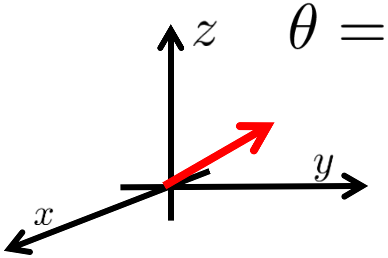
$\phi = 0^\circ$

$$= \frac{1}{\sqrt{2}}(\uparrow) + \frac{1}{\sqrt{2}}(\downarrow)$$



$\phi = 180^\circ$

$$= i\frac{1}{\sqrt{2}}(\uparrow) - i\frac{1}{\sqrt{2}}(\downarrow)$$

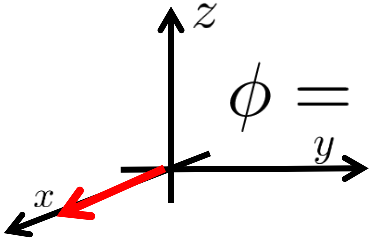


$\theta = 90^\circ$ to $\theta = 270^\circ$

$\phi = 0^\circ$

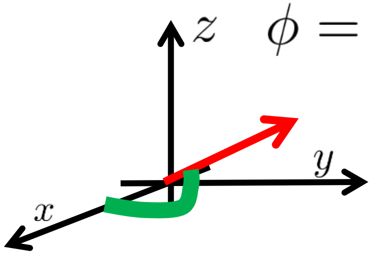
$$= \cos\frac{\theta}{2}\uparrow + \sin\frac{\theta}{2}\downarrow$$

Starting with the spin "vector" in the +x direction (i.e. ϴ=90° and ϴ =0) rotate the atom in two different ways to the –x direction. Is there a difference?



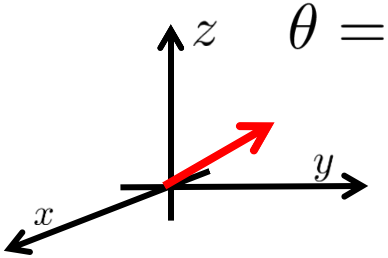
$$\phi = 0^\circ$$

$$= \frac{1}{\sqrt{2}}(\uparrow) + \frac{1}{\sqrt{2}}(\downarrow)$$



$$\phi = 180^\circ$$

$$= i\frac{1}{\sqrt{2}}(\uparrow) - i\frac{1}{\sqrt{2}}(\downarrow)$$



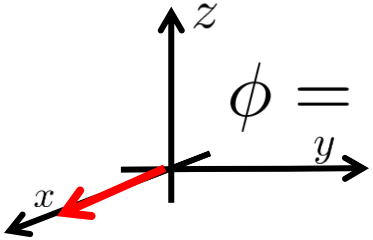
$\theta = 90^\circ$ to $\theta = 270^\circ$

$$\phi = 0^\circ$$

$$\cos\left(\frac{270^\circ}{2}\right) = -\frac{1}{\sqrt{2}}$$

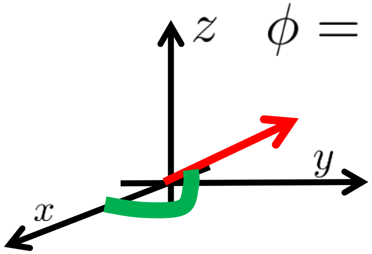
$$\sin\left(\frac{270^\circ}{2}\right) = +\frac{1}{\sqrt{2}}$$

Starting with the spin "vector" in the +x direction (i.e. ϴ=90° and ϴ =0) rotate the atom in two different ways to the –x direction. Is there a difference?



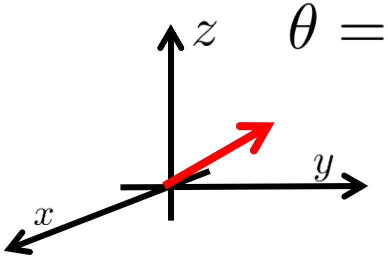
$\phi = 0^\circ$

$$= \frac{1}{\sqrt{2}}(\uparrow) + \frac{1}{\sqrt{2}}(\downarrow)$$



$\phi = 180^\circ$

$$= i\frac{1}{\sqrt{2}}(\uparrow) - i\frac{1}{\sqrt{2}}(\downarrow)$$



$\theta = 90^\circ$ to $\theta = 270^\circ$

$\phi = 0^\circ$

$$= -\frac{1}{\sqrt{2}}(\uparrow) + \frac{1}{\sqrt{2}}(\downarrow)$$

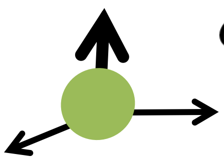
Weird!

- Rotating one way = -i (Rotating the other way)

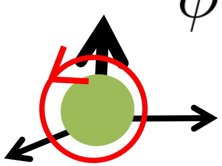
- In rotating the atom from one spin direction to another, the final state of the atom **depends on how the atom was rotated into that final spin direction!**

Spin ½ particles -Fermions

- Atoms, electrons, protons neutrons

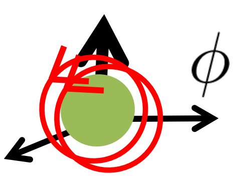


$\phi = 0^\circ$



$\phi = 360^\circ$

$$-1$$



$\phi = 720^\circ$

$$+1$$

Spin 0

- Sphere

Spin 1 - Bosons

- Rotate 360°

Coin Toss

- What is the chance of rolling a 2?
- What is the chance of rolling a 2 and then rolling a 5?
- What is the chance of rolling two die at the same time and getting a 2 and 5?

- Put two quantum particles in a box. Let's say they are bosons

- The probability amplitude that boson 1 is in state n at x_a is $\psi_n(x_a)$

- The probability amplitude that boson 2 is in state m at x_b is $\psi_m(x_b)$

What is the chance of finding one at x_a and one at x_b ?

- They are indistinguishable so

$$\psi_n(x_a)\psi_m(x_b) + \psi_n(x_b)\psi_m(x_a)$$

What is the chance of them being in the same quantum state?

$$\psi_{total} = \psi_n(x_a)\psi_n(x_b) + \psi_n(x_b)\psi_n(x_a)$$

$$\psi_{total} = 2\psi_n(x_a)\psi_n(x_b)$$

$$P = |\psi_{total}|^2 = 4\psi_n^2(x_a)\psi_n^2(x_b)$$

This is double what we would expect classically!

- Put two quantum particles in a box. Let's say they are fermions

- The probability amplitude that fermion 1 is in state n at x_a is $\psi_n(x_a)$

- The probability amplitude that fermion 2 is in state m at x_b is $\psi_m(x_b)$

What is the chance of finding one at x_a and one at x_b ?

- Fermions are ANTI-SYMMETRIC!

$$\psi_{total} = \psi_n(x_a)\psi_m(x_b) - \psi_n(x_b)\psi_m(x_a)$$

What is the chance of them being in the same quantum state?

$$\psi_{total} = \psi_n(x_a)\psi_n(x_b) - \psi_n(x_b)\psi_n(x_a)$$

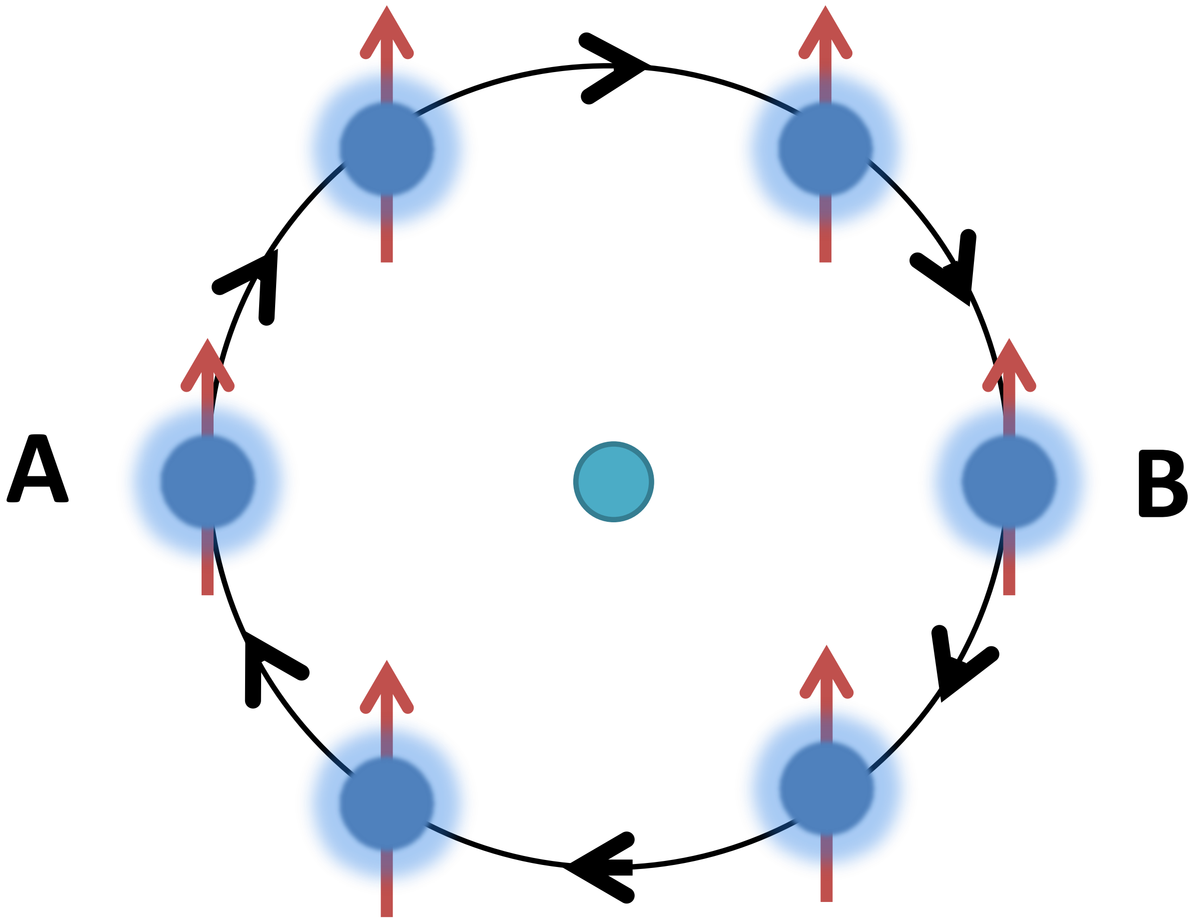
$$\psi_{total} = 0$$

Two fermions can't exist in the same state!

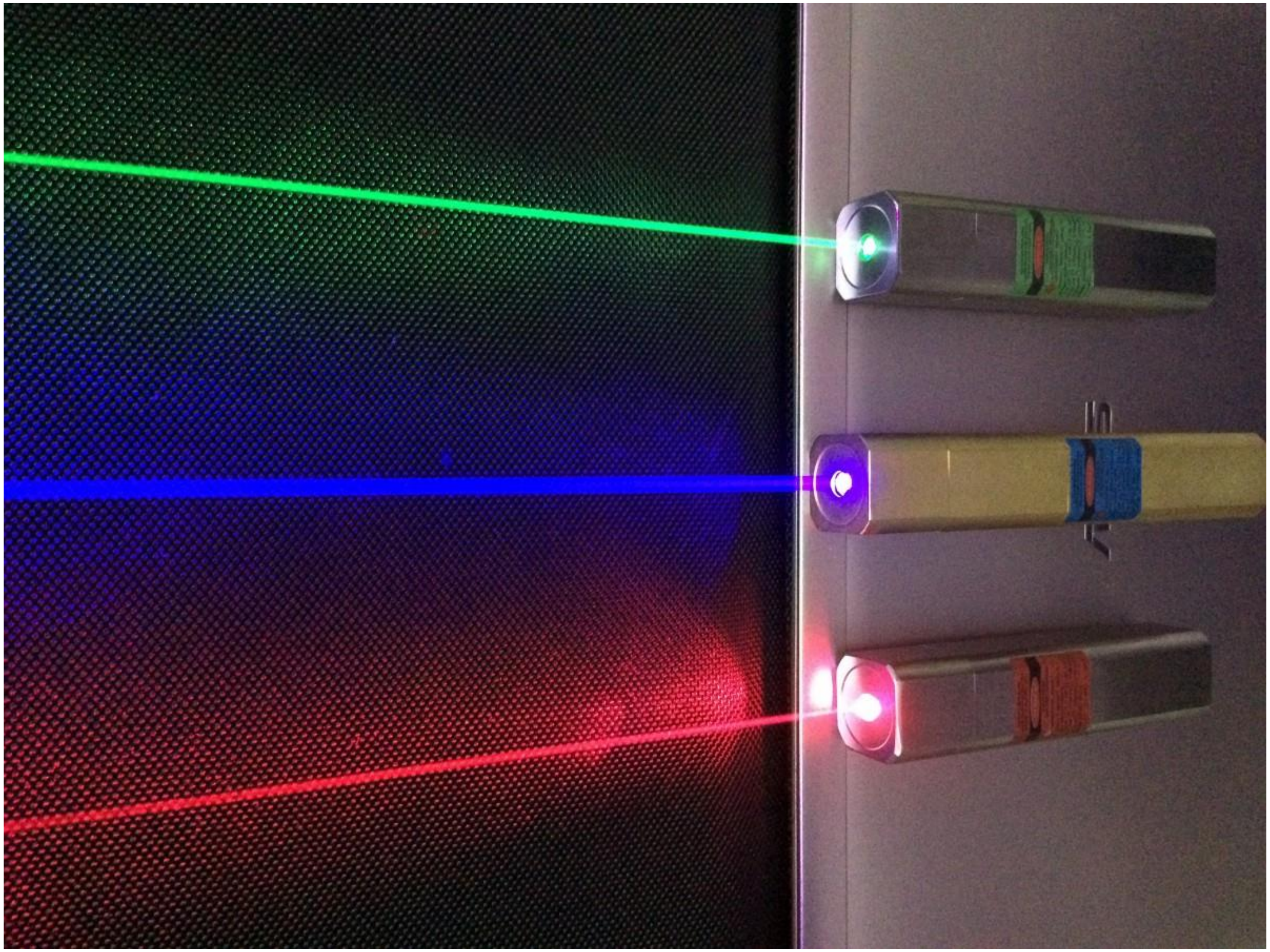
Pauli's Exclusion Principle

- The mysterious minus sign that cropped up just through our math approach shows us something pretty amazing!

- This is why we can't walk through walls!


A
B

- FERMIONS: “Two is crowd”
- BOSONS: “The more the merrier”



Stimulated Emission in a Mirrored Laser Cavity

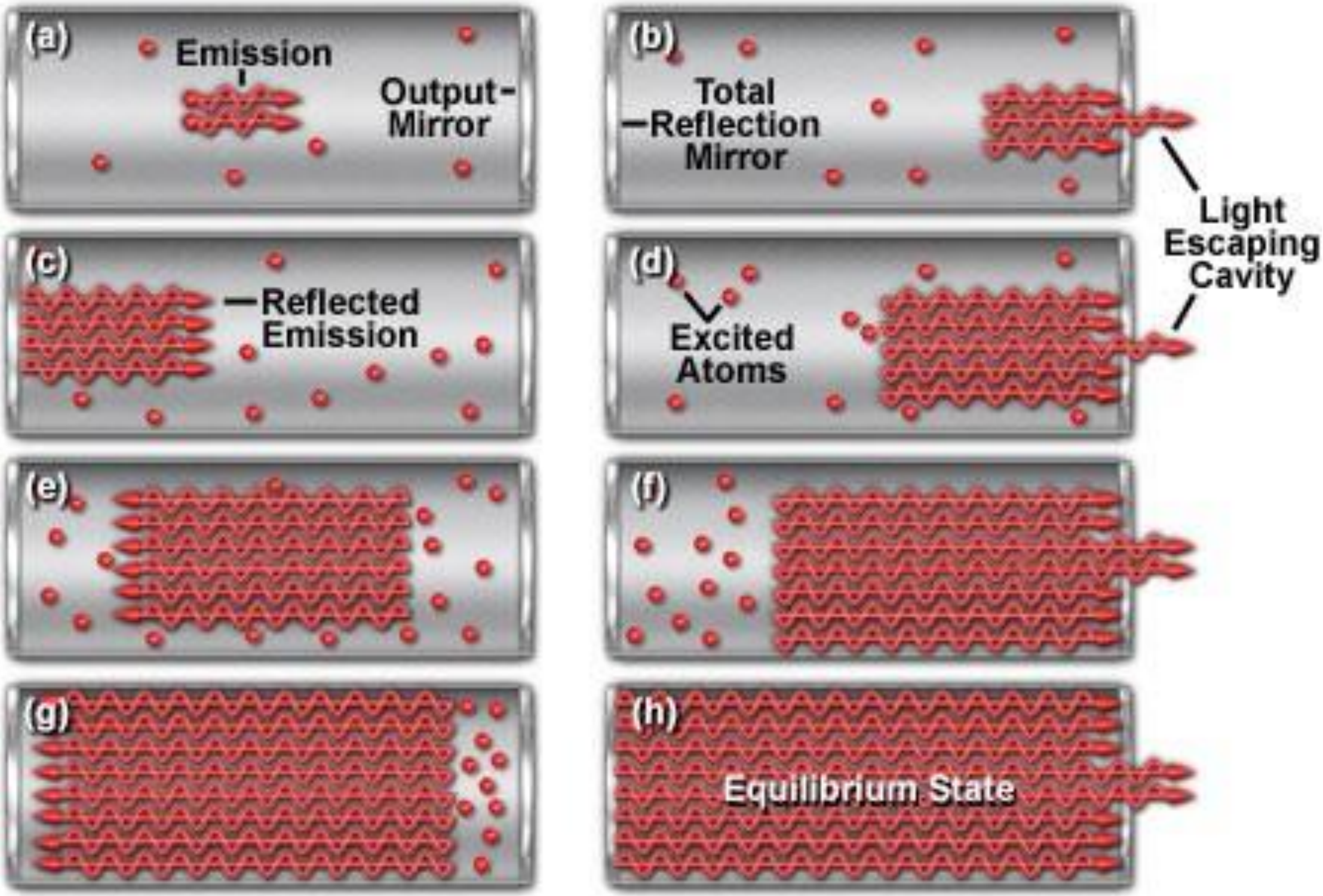
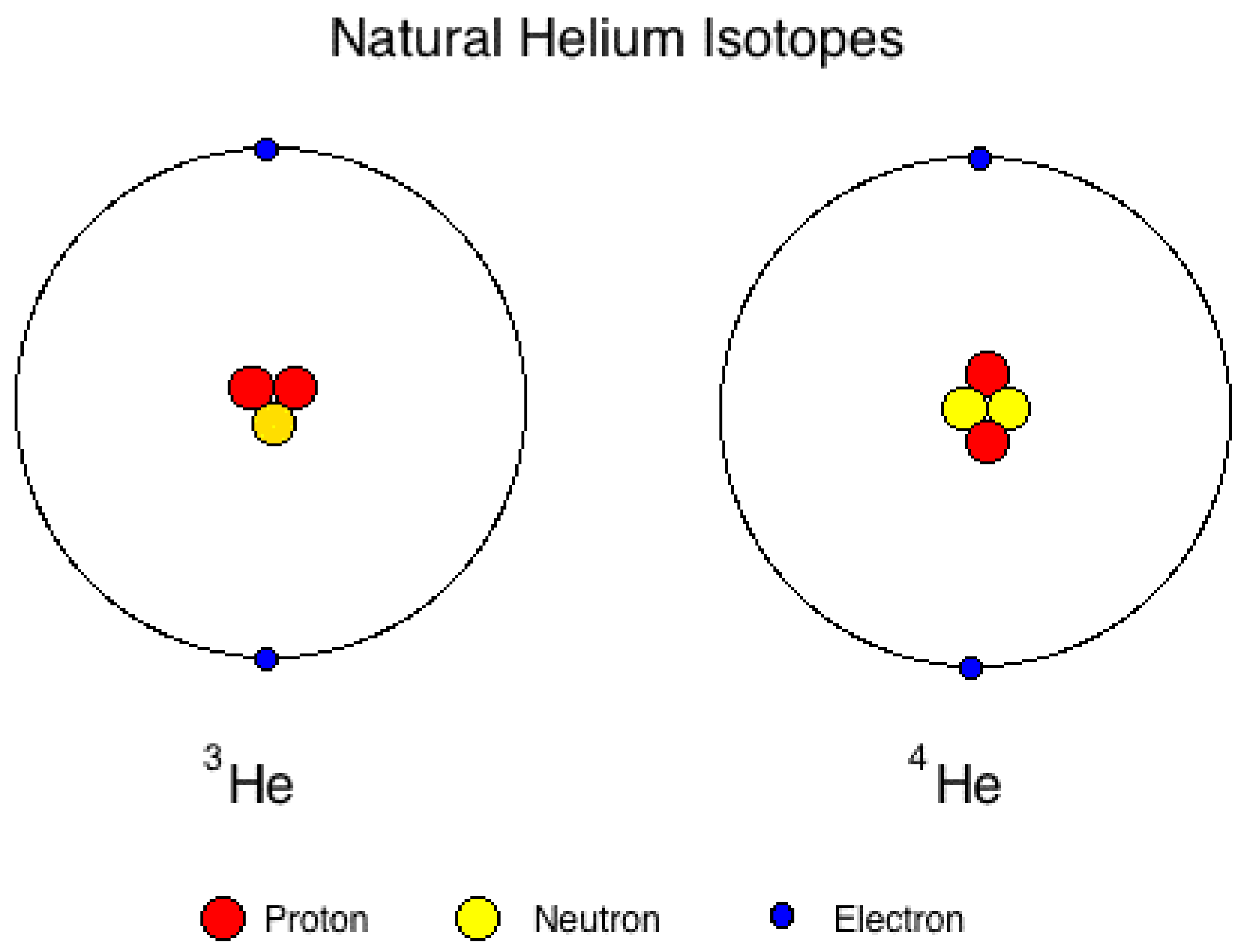


Figure 1


Natural Helium Isotopes
3He
4He
Proton
Neutron
Electron

[redacted]