Math Primer Solutions 1:
BINOMIAL SERIES

1. Expand $(1+x)^7$

$$(1+x)^7 = \binom{7}{0}x^0 + \binom{7}{1}x^1 + \binom{7}{2}x^2 + \binom{7}{3}x^3 + \binom{7}{4}x^4 + \binom{7}{5}x^5 + \binom{7}{6}x^6 + \binom{7}{7}x^7$$
$$= 1 + 7x + 21x^2 + 35x^3 + 35x^4 + 21x^5 + 7x^6 + x^7$$

2a. Write out 5 terms in the expansion of $\sqrt{1.2}$

$$\sqrt{1.2} = (1+0.2)^{\frac{1}{2}}$$
$$= 1 + \frac{1}{2}(0.2) - \frac{1}{8}(0.2)^2 + \frac{1}{16}(0.2)^3 - .\frac{5}{128}(0.2)^4 ..$$
$$= 1 + 0.1 - 0.005 + 0.0005 - 0.0000625...$$
$$= 1.0954375...$$

2b. Quickly approximate $\sqrt{1.2}$

$$\sqrt{1.2} \approx 1 + \frac{1}{2}(0.2) = 1.1$$

2c. Quickly approximate $\sqrt{5}$

$$\sqrt{5} = \sqrt{4+1}$$
$$= (4+1)^{\frac{1}{2}}$$
$$= 4^{\frac{1}{2}}\left(1+\frac{1}{4}\right)^{\frac{1}{2}}$$
$$\approx 2\left(1+\frac{1}{2}\frac{1}{4}\right)$$
$$= 2 + \frac{1}{4}$$
$$= 2.25$$

3. Einstein discovered that for an object of ***rest mass*** m, moving with ***relativistic momentum*** p, the relationship between m, p and its ***total relativistic energy***, E, is: $E^2 = m^2c^4 + p^2c^2$.

a. Take the positive square root and factor out mc^2 from within the radical sign.

$$
\begin{aligned}
E &= \sqrt{m^2c^4 + p^2c^2} \\
&= \sqrt{m^2c^4\left(1 + \frac{p^2c^2}{m^2c^4}\right)} \\
&= mc^2\sqrt{1 + \frac{p^2}{m^2c^2}}
\end{aligned}
$$

b. Given that the relativistic definition of momentum is $p = \frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}}$, insert this into question 2a's equation and simplify until you obtain a single term with the denominator as $\sqrt{1 - \frac{v^2}{c^2}}$.

$$
E = mc^2\sqrt{1 + \frac{\left(\frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}}\right)^2}{m^2c^2}} = mc^2\sqrt{1 + \frac{m^2v^2}{m^2c^2\left(1 - \frac{v^2}{c^2}\right)}} = mc^2\sqrt{1 + \frac{v^2}{c^2 - v^2}}
$$

$$
= mc^2\sqrt{\frac{c^2 - v^2 + v^2}{c^2 - v^2}} = mc^2\sqrt{\frac{c^2}{c^2 - v^2}} = mc^2\sqrt{\frac{c^2}{c^2} \cdot \frac{1}{1 - \frac{v^2}{c^2}}} = mc^2\sqrt{\frac{1}{1 - \frac{v^2}{c^2}}} = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}}
$$

c. Write this quotient as a product and apply the binomial approximation to the radical term.

$$mc^2 \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = mc^2\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}$$

$$= mc^2\left(1-\left(-\frac{1}{2}\left(\frac{v^2}{c^2}\right)\right)\right)$$

$$= mc^2\left(1+\frac{1}{2}\frac{v^2}{c^2}\right)$$

d. Multiply through.

$$mc^2\left(1+\frac{1}{2}\frac{v^2}{c^2}\right) = mc^2 + \frac{1}{2}mv^2$$

e. What does this new equation tell us?

For slowly moving objects, Einstein's general relationship $E^2 = m^2c^4 + p^2c^2$ reduces, approximately, to:

$$E \approx mc^2 + \frac{1}{2}mv^2 \qquad .$$
$$= \text{"rest energy"} + \text{"kinetic energy"}$$