Fourier Optics

Appendix C

Fourier Integrals

Source: “A. Nussbaum and R. A. Phillips, “Contemporary Optics for Scientists and Engineers” is the well-known Xerox system, using electrostatic image formation from a selenium photon detector. Although these newer systems are the basis of commercial photocopy uses, they are not regarded as detectors in the conventional sense.

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APPENDIX A

FOURIER INTEGRALS

A-1 Fourier Series

The square pulse shown in Fig. A-1 can be defined by the expression

$$\begin{aligned} y &= 1 \quad \text{when} \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \\ y &= 0 \quad \text{when} \quad \frac{\pi}{2} < |x| \leq \pi \end{aligned} \tag{A-1}$$

It is sometimes convenient, however, to have an analytic equivalent of Eq. (A-1), and this may be obtained if y is expressed in terms of the sine and

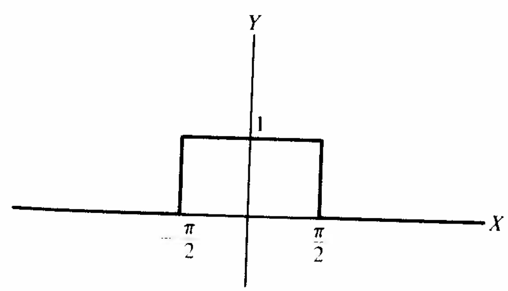

Figure A-1

cosine functions as

$$y = a_0 + a_1 \cos x + a_2 \cos 2x + \cdots + b_1 \sin x + b_2 \sin 2x + \cdots \tag{A-2}$$

or

$$y = \sum_{n=0}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{A-3}$$

Such an expression is known as a *Fourier series*.

To determine the coefficients a_n and b_n, we can multiply Eq. (A-3) by $\cos mx$ or $\sin mx$, where m is a positive integer, and integrate over the interval $(-\pi, \pi)$. For example, to determine a_m,

$$\int_{-\pi}^{\pi} y \cos mx \, dx = \int_{-\pi}^{\pi} \sum (a_n \cos nx + b_n \sin nx) \cos mx \, dx$$

To evaluate this expression, we use the definite integrals

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \begin{cases} \pi & \text{if} \quad m = n \\ 0 & \text{if} \quad m \neq n \end{cases}$$

$$\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0$$

The latter may easily be demonstrated by means of the identity

$$\sin mx \cos nx = \tfrac{1}{2}[\sin (m+n)x + \sin (m-n)x]$$

As a result, we see that multiplying y by $\cos m_x$ and integrating from $-\pi$ to π has the effect of picking out the coefficient a_m, since only this term is nonzero when the definite integral is evaluated:

$$\int_{-\pi}^{\pi} y \cos mx \, dx = \int_{-\pi}^{\pi} \cos^2 mx \, dx + 0 + 0 + \cdots = \pi a_m$$

This may be rewritten as

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} y \cos mx \, dx \qquad (m \neq 0) \tag{A-4}$$

Similarly

$$b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} y \sin mx \, dx \tag{A-5}$$

When $m = 0$, then $b_m = 0$, but $\int_{-\pi}^{\pi} y \, dx = \int_{-\pi}^{\pi} a_0 \, dx = 2\pi a_0$ or $a_0 = (1/2\pi) \int_{-\pi}^{\pi} y \, dx$. If we replace a_0 in Eq. (A-2) by a new constant $a_0/2$, then Eq. (A-4) becomes

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} y \cos nx \, dx \qquad (n = 0, 1, 2, \ldots) \tag{A-6}$$

and in the same way

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} y \sin nx \, dx \qquad (n = 0, 1, 2, \ldots) \tag{A-7}$$

For the example of Fig. A-1, we find that

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} y \, dx = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} (1) \, dx = 1$$

$$a_1 = \frac{1}{\pi} \int_{-\pi}^{\pi} y \cos x \, dx = \frac{1}{\pi} (\sin x) \Big|_{-\pi/2}^{\pi/2} = \frac{2}{\pi}$$

$$b_1 = \frac{1}{\pi} \int_{-\pi}^{\pi} y \sin x \, dx = -\frac{1}{\pi} \cos x \Big|_{-\pi/2}^{\pi/2} = 0$$

$$a_2 = \frac{1}{\pi} \int_{-\pi}^{\pi} y \cos 2x \, dx = \frac{1}{2\pi} (\sin 2x) \Big|_{-\pi/2}^{\pi/2} = 0$$

$$b_2 = 0$$

$$a_3 = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \cos 3x \, dx = \frac{1}{3\pi} \sin 3x \Big|_{-\pi/2}^{\pi/2} = -\frac{2}{3\pi}$$

etc.

so that

$$\begin{aligned} y &= \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \cdots + b_1 \sin x + \cdots \\ &= \frac{1}{2} + \frac{2}{\pi} \cos x - \frac{2}{3\pi} \cos 3x + \cdots \\ &= \frac{1}{2} + \frac{2}{\pi} \left[\cos x - \frac{1}{3} \cos 3x + \frac{1}{5} \cos 5x - \cdots \right] \end{aligned} \tag{A-8}$$

The term in square brackets can be pictured as shown in Fig. A-2. Figure A-2(a) at the top is $\cos x$, and this curve is also repeated at the bottom. To this, we add $(-1/3) \cos 3x$ of Fig. A-2(b) to obtain Fig. A-2(d). Then we combine the curve in Fig. A-2(d) with the curve for $(1/5) \cos 5x$ of curve (c) in the figure to obtain the curve in Fig. A-2(e), and so on. The resulting square pulse has a height of $2(\pi/4)$, for at $x = 0$

$$\cos x - \frac{1}{3} \cos 3x + \frac{1}{5} \cos 5x - \cdots = 1 - \frac{1}{3} + \frac{1}{5} - \cdots = \frac{\pi}{4}$$

as may be demonstrated if we numerically sum the series for the first few terms. Taking $2/\pi$ times this result and shifting the entire square pulse up a distance $y = 0.5$ then gives Fig. A-1, as it should.

The function expressed by Eq. (A-1) and shown in Fig. A-1 is a single square pulse which is nonzero only from $-\pi/2$ to $\pi/2$. The Fourier series expression in Eq. (A-8), however, does not vanish outside this interval, because $\cos x$, $\cos 3x$, ... are periodic functions which repeat themselves

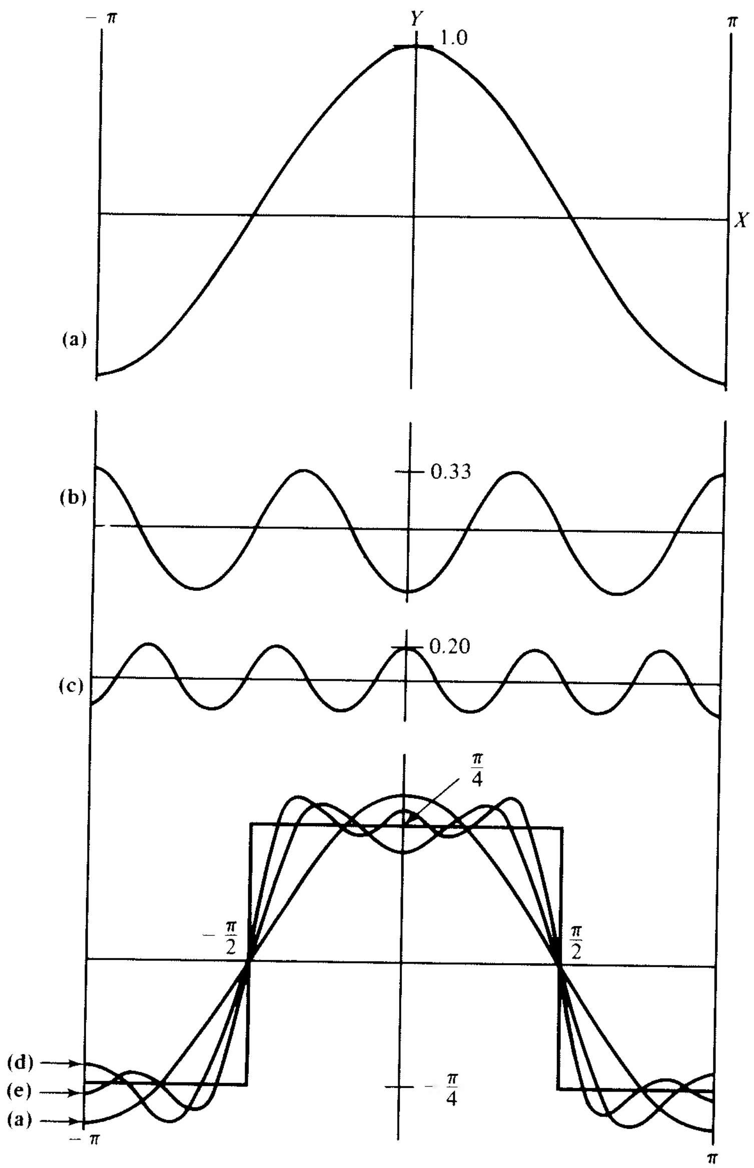

Figure A-2

to the right and to the left of the vertical boundaries of Fig. A-2. Hence, Eq. (A-8) describes the function of Eq. (A-1) only in the interval $(-\pi, \pi)$. However, it *is* an analytic approximation to the periodic function

$$f(x) = \begin{cases} 1 & \text{when } |x| \leq \dfrac{\pi}{2} \\ 0 & \text{when } \pi > |x| > \dfrac{\pi}{2} \end{cases}$$

$$f(x \pm 2\pi) = f(x)$$

and this train of square waves is shown in Fig. A-3.

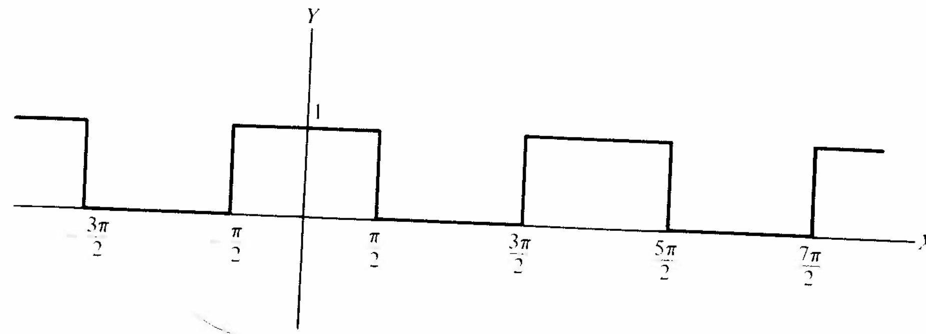

Figure A-3

Equation (A-3) as applied to an arbitrary periodic function $f(x)$ is then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{A-9}$$

It is convenient to change x to a new variable z, defined as

$$z = kx = \frac{2\pi x}{\lambda}$$

where k is the propagation constant of Eq. (5-2). The Fourier series of Eq. (A-9) is then

$$f(z) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nz + b_n \sin nz) \tag{A-10}$$

or

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nkx + b_n \sin nkx) \tag{A-11}$$

where the coefficients, by Eqs. (A-6) and (A-7), become

$$\left.\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(z) \cos nz \, dz = \frac{2}{\lambda} \int_{-\lambda/2}^{\lambda/2} f(x) \cos nkx \, dx \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(z) \sin nz \, dz = \frac{2}{\lambda} \int_{-\lambda/2}^{\lambda/2} f(x) \sin nkx \, dx \end{aligned}\right\} \tag{A-12}$$

The series for $f(x)$ just given applies to a function whose wavelength is 2π units. For a function of arbitrary wavelength, which we shall designate as $2L$, the substitution $\lambda = 2L$ converts (Eq. (A-11) to)

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) \tag{A-13}$$

where

$$\left.\begin{aligned} a_n &= \frac{1}{L}\int_{-L/2}^{L/2} f(x)\cos\frac{n\pi x}{L}\,dx \\ b_n &= \frac{1}{L}\int_{-L/2}^{L/2} f(x)\sin\frac{n\pi x}{L}\,dx \end{aligned}\right\} \tag{A-14}$$

Using

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$$

$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

we convert Eq. (A-11) into the complex form

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inkx} \tag{A-15}$$

where

$$c_n = \frac{a_n - ib_n}{2} \qquad (n = 1, 2, \ldots)$$

$$c_0 = \frac{a_0}{2} \qquad (n = 0)$$

$$c_n = \frac{a_n + ib_n}{2} \qquad (n = -1, -2, \ldots)$$

Since

$$\int_{-\pi}^{\pi} e^{i(m-n)x}\,dx = \begin{cases} 2\pi & \text{if} \quad m = n \\ 0 & \text{if} \quad m \neq n \end{cases} \tag{A-16}$$

then

$$c_n = \frac{1}{\lambda}\int_{-\lambda/2}^{\lambda/2} f(x)e^{-inkx}\,dx \tag{A-16a}$$

A-2 The Fourier Integral and the Fourier Transform

We shall replace the variable x in Eqs. (A-11) and (A-12) by a new variable y (since the symbol assigned to the variable of integration in a definite integral is arbitrary), and the coefficients are then

$$\left.\begin{aligned} a_n &= \frac{2}{\lambda}\int_{-\lambda/2}^{\lambda/2} f(y)\cos nky\,dy \\ b_n &= \frac{2}{\lambda}\int_{-\lambda/2}^{\lambda/2} f(y)\sin nky\,dy \end{aligned}\right\} \tag{A-17}$$

Substituting Eq. (A-17) into Eq. (A-16) gives

$$f(x) = \frac{1}{\lambda}\int_{-\lambda/2}^{\lambda/2} f(y)\,dy + \sum_{n=1}^{\infty}\frac{2}{\lambda}\int_{-\lambda/2}^{\lambda/2} f(y)\cos\{nk(x-y)\}\,dy \tag{A-18}$$

where we have used the well-known identity for $\cos(a - b)$.

Now let

$$k_n = \frac{2\pi n}{\lambda} = nk$$

so that

$$k_n - k_{n-1} = \Delta k_n = \frac{2\pi n}{\lambda} - \frac{2\pi(n-1)}{\lambda} = \frac{2\pi}{\lambda}$$

If we let λ approach infinity, then the second term on the right of Eq. (A-18) becomes

$$\frac{1}{\pi}\sum \Delta k_n \int_{-\lambda/2}^{\lambda/2} f(y)\cos\{k_n(x-y)\}\,dy = \frac{1}{\pi}\int_0^{\infty} dk \int_{-\infty}^{\infty} f(y)\cos\{k(x-y)\}\,dy$$

where the summation from $n = 1$ to $n = \infty$ is assumed to be approximately equal to the integral over the continuous variable w from 0 to ∞. We further assume that the integral in the first term on the right of Eq. (A-18) has a finite value so that the entire term vanishes as $\lambda \to \infty$, and we have

$$f(x) = \frac{1}{\pi}\int_0^{\infty} dk \int_{-\infty}^{\infty} f(y)\cos\{k(x-y)\}\,dy \tag{A-19}$$

This is the *Fourier integral theorem.*

If we write this as

$$\begin{aligned} f(x) &= \frac{1}{\pi}\int_0^{\infty}\cos(kx)\,dk\int_{-\infty}^{\infty} f(y)\cos(ky)\,dy \\ &\quad + \frac{1}{\pi}\int_0^{\infty}\sin(kx)\int_{-\infty}^{\infty} f(y)\sin(ky)\,dy \end{aligned} \tag{A-20}$$

we may look on the new form as an extension of Eq. (A-3) to an interval $(-\infty, \infty)$, where the term $\int_{-\infty}^{\infty} f(y)\sin(ky)\,dy$ may be regarded as the coefficient $a(k)$ and $\int_{-\infty}^{\infty} f(y)\cos(ky)\,dy$ is $b(k)$.

Using

$$\cos\{k(x-y)\} = \frac{e^{ik(x-y)} + e^{-ik(x-y)}}{2}$$

we convert Eq. (A-19) into

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(y)\left\{\int_0^{\infty}\left(\frac{e^{ik(x-y)} + e^{-ik(x-y)}}{2}\right)dk\right\}dy$$

The inner integral in this expression may be written

$$\int_0^{\infty}(e^{ik(x-y)} + e^{-ik(x-y)})\,dk = \int_0^{\infty}\frac{e^{ik(x-y)}}{2}\,dk + \int^{0}\frac{e^{ik(x-y)}}{2}\,dk$$

from which we obtain

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(y)\left\{\int_{-\infty}^{\infty} e^{ik(x-y)}\,dk\right\}dy$$

or

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} f(y)e^{ik(x-y)}\,dk\right\}dy \tag{A-21}$$

This is the *complex form* of the Fourier integral theorem.

We next define the *Fourier transform* $\mathfrak{F}[f(x)] = F(k)$ of the function $f(x)$ as

$$\mathfrak{F}[f(x)] = \int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx = F(k) \tag{A-22}$$

Comparing Eqs. (A-21) and (A-22) shows that the *inverse Fourier transform* will be

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} f(y)e^{-iky}\,dy\right\}e^{ikx}\,dk$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)e^{ikx}\,dk = \mathfrak{F}^{-1}[F(k)] = \mathfrak{F}^{-1}\mathfrak{F}[f(x)] = f(x) \tag{A-23}$$

We define a *pair of Fourier transforms* $f(x)$ and $F(k)$ as two functions which satisfy both Eqs. (A-22) and (A-23).

Let us identify x in Eq. (A-22) or (A-23) as one coordinate of an arbitrary point whose position vector is $\mathbf{r}$ and k as the x component of the propagation vector $\mathbf{k}$ for an arbitrary wave described by the Fourier integral theorem. Then the three-dimensional analogue of Eq. (A-22) is

$$F(\mathbf{k}) = \mathfrak{F}[f(\mathbf{r})] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\mathbf{r})e^{-i\mathbf{k}\cdot\mathbf{r}}\,dx\,dy\,dz \tag{A-24}$$

and the inverse is

$$f(r) = \mathfrak{F}^{-1}[F(k)] = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}}\,dk_x\,dk_y\,dk_z \tag{A-25}$$

The linear differential equation of the form

$$A\frac{d^2f(t)}{dt^2} + B\frac{df(t)}{dt} + Cf(t) = g(t) \tag{A-26}$$

or

$$A\ddot{f}(t) + B\dot{f}(t) + Cf(t) = g(t) \tag{A-27}$$

governs the behavior of damped oscillating systems. For example, if we apply a potential $v(t) = g(t)$ to a resistor, an inductor, and a capacitor in series, then the resulting current $i(t) = f(t)$ is obtained by solving Eq. (A-26). In fact, we call $g(t)$ the *excitation* or *forcing function* and $i(t)$ is the *response function.*

To solve Eq. (A-26) we shall take the Fourier transform of both sides. First, we replace x in Eq. (A-22) by the time t, so that k is replaced by the

angular frequency ω of the applied potential. Equation (A-22) becomes

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt \tag{A-28}$$

and Eq. (A-23) is now

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}\,d\omega \tag{A-29}$$

Next, we integrate by parts to obtain

$$\int_{-\infty}^{\infty}\frac{df}{dt}e^{-i\omega t}\,dt = fe^{-i\omega t}\Big|_{-\infty}^{\infty} + i\omega\int_{-\infty}^{\infty} fe^{-i\omega t}\,dt \tag{A-30}$$

where

$$v = e^{-i\omega t}, \qquad dv = \frac{df}{dt}\,dt$$

Since the magnitude of $e^{-i\omega t}$ is unity, the integrated term will vanish if we require $f(t)$ to vanish when $|t|$ is large. We arrange for the current $i(t) = f(t)$ to start at $t = 0$, so that it has a value of zero for $t < 0$, and we know that it decays in a damped circuit; thus, these conditions are satisfied. Hence

$$\mathfrak{F}\left[\frac{d}{dt}f(t)\right] = i\omega\mathfrak{F}[f(t)] \tag{A-31}$$

and similarly, we may show that

$$\mathfrak{F}\left[\frac{d^2}{dt^2}f(t)\right] = (i\omega)^2\mathfrak{F}[f(t)] \tag{A-32}$$

Then Eq. (A-27) becomes

$$\{(i\omega)^2A + (i\omega)B + C\}F(\omega) = G(\omega) \tag{A-33}$$

where

$$G(\omega) = \mathfrak{F}[g(t)] \tag{A-34}$$

or

$$F(\omega) = \frac{G(\omega)}{(i\omega)^2A + (i\omega)B + C} \tag{A-35a}$$

The polynomial in $(i\omega)$ of the form

$$H(\omega) = \frac{1}{(i\omega)^2A + (i\omega)B + C} \tag{A-36}$$

is called the *transfer function.* Since Eq. (A-35a) becomes

$$F(\omega) = G(\omega)H(\omega) \tag{A-35b}$$

where $F(\omega)$ and $G(\omega)$ are the transforms of the current and voltage, respectively, then $H(\omega)$ must be the admittance $Y(\omega)$ (i.e., the reciprocal of the impedance) of the series circuit expressed as a function of ω.

To convert back to time as a variable, we use the definitions

$$h(t) = \mathfrak{F}^{-1}[H(\omega)]$$

$$f(t) = \mathfrak{F}^{-1}[F(\omega)]$$

so that

$$F(\omega) = G(\omega)H(\omega) = \int_{-\infty}^{\infty} g(\alpha)e^{-i\omega\alpha}\,d\alpha \int_{-\infty}^{\infty} h(\beta)e^{-i\omega\beta}\,d\beta$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\alpha)h(\beta)e^{-i\omega(\alpha+\beta)}\,d\alpha\,d\beta$$

Let

$$\alpha + \beta = t$$

and integrate first over β, holding α constant. This gives

$$F(\omega) = \int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} g(\beta)h(t-\alpha)e^{-i\omega t}\,dt\right\}d\alpha$$

Then we reverse the order of integration to obtain

$$F(\omega) = \int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} g(\alpha)h(t-\alpha)\,d\alpha\right\}e^{-i\omega t}\,dt$$

By Eq. (A-29) and Eq. (A-37), this result is equivalent to

$$f(t) = \mathfrak{F}^{-1}[F(\omega)] = \int_{-\infty}^{\infty} g(\alpha)h(t-\alpha)\,d\alpha \tag{A-38a}$$

which is known as the *convolution theorem*, and we say that *the response of the circuit is the convolution of the excitation with the transfer function*. This is written as

$$f(t) = g(t) * h(t) \tag{A-38b}$$

and we realize that the Fourier transform of the convolution of two functions is the product of the individual transforms, using Eq. (A-35b).

As a concrete example, consider a resistance of 4 ohms in series with an inductance of 2 henries. An exponential voltage specified by the relation

$$v(t) = \begin{Bmatrix} 0 & \text{for} \quad t \leqq 0 \\ 10e^{-t} & \text{for} \quad t \geqq 0 \end{Bmatrix} \tag{A-39}$$

is applied to this circuit. The differential equation to be solved is then

$$v = L\frac{di}{dt} + iR \tag{A-40}$$

Since there is a response only for $t \geqq 0$, we write the equation as

$$2\frac{di}{dt} + 4i = 10e^{-t}$$

and integrate only from $t = 0$ to $t = \infty$, obtaining the Fourier transform of $v(t)$ as

$$V(\omega) = \int_0^{\infty} (10e^{-t})e^{-i\omega t}\,dt = 10\left.\frac{e^{-(1+i\omega)t}}{-(1+i\omega)}\right|_0^{\infty}$$

$$= 10\left(\frac{1}{1+i\omega}\right)$$

Combining this with Eq. (A-31) and Eq. (A-40) gives

$$[(i\omega)2 + 4]I(\omega) = 10\left(\frac{1}{1+i\omega}\right)$$

or

$$I(\omega) = 5\frac{1}{(2+i\omega)(1+\omega)}$$

To find the inverse transform, we can write the function on the right as

$$\frac{1}{(2+i\omega)(1+i\omega)} = \frac{1}{1+i\omega} - \frac{1}{2+i\omega}$$

But we have just seen that the Fourier transform of the function given by Eq. (A-39) is e^{-t}, and we similarly can show that the transform of the function

$$f(t) = \begin{cases} 0 & \text{for} \quad t \leqq 0 \\ Ae^{-at} & \text{for} \quad t \geqq 0 \end{cases}$$

is

$$F(\omega) = \frac{A}{a+i\omega}$$

Using

$$\mathfrak{F}^{-1}[\mathfrak{F}[f(t)]] = f(t)$$

then gives

$$\mathfrak{F}^{-1}\left[\frac{1}{a+i\omega}\right] = e^{-at}$$

Hence

$$i(t) = \mathfrak{F}^{-1}[I(\omega)]$$

$$= 5\mathfrak{F}^{-1}\left[\frac{1}{1+i\omega} - \frac{1}{2+i\omega}\right]$$

$$= \begin{cases} 0 & t \leqq 0 \\ 5(e^{-t} - e^{-2t}) & t \geqq 0 \end{cases}$$

To summarize, we have obtained the following results:

	Time Domain	Frequency Domain
Excitation	$v(t) = \begin{cases} 0 & t \leqq 0 \\ 10e^{-t} & t \geqq 0 \end{cases}$	$V(\omega) = \dfrac{10}{1+i\omega}$
Response	$i(t) = \begin{cases} 0 & t \leqq 0 \\ 5(e^{-t} - e^{-2t}) & t \geqq 0 \end{cases}$	$I(\omega) = \dfrac{5}{(2+i\omega)(1+i\omega)}$
Transfer Function	$y(t) = \dfrac{1}{2}e^{-2t}$	$Y(\omega) = \dfrac{1}{2(2+i\omega)}$

$$i(t) = v(t) * y(t)$$

$$I(\omega) = V(\omega)Y(\omega)$$

A-3 The Meaning of Convolution Processes

The convolution operation is of such practical importance in optics and engineering that it is worthwhile to examine its properties and its physical significance. A very clear discussion of these points has been given by Healy[(A-1)] and the material which follows is extracted from his article.

Transformers of a certain type are delivered by two companies, which will be called A and B, in lots of three and four, respectively. In a lot from company A, there will be zero, one, two, or three defective transformers. The probability that each of these numbers of defects, represented by $a(a = 0, 1, 2, 3)$, will occur is assumed to be

a	$P(a)$
0	0.4
1	0.3
2	0.2
3	0.1

Similarly, the probability of $b(b = 0, 1, 2, 3, 4)$ occurrences is taken as

b	$P(b)$
0	0.3
1	0.2
2	0.2
3	0.2
4	0.1

in a shipment from company B. The problem is to find the probabilities of the total number of defects in two shipments, one from each company. This sum or total number of defects will be indicated as $c(c = 0, 1, 2, 3, 4, 5, 6, 7)$.

The probability that $c = 0$ is the probability that $a = 0$ and $b = 0$; that is, there are no defects in either shipment. This is written as

$$P(c = 0) = P(a = 0, b = 0)$$

If the events $a = 0$ and $b = 0$ are independent of each other, which we assume here and which is necessary if the solution is to be a convolution, then this probability reduces to the product

$$\begin{aligned} P(c = 0) &= P(a = 0) \times P(b = 0) \\ &= 0.4 \times 0.3 \\ &= 0.12 \end{aligned}$$

The probability that we have a total of one defect is the probability that the shipment from A has one defect and the shipment from B has none, or

vice versa. That is,

$$\begin{aligned} P(c = 1) &= P[(a = 1, b = 0) \text{ or } (a = 0, b = 1)] \\ &= P(a = 1, b = 0) + P(a = 0, b = 1) \\ &= P(a = 1)P(b = 0) + P(a = 0)P(b = 1) \\ &= (0.3 \times 0.3) + (0.4 \times 0.2) \\ &= 0.17 \end{aligned}$$

Similarly,

$$\begin{aligned} P(c = 2) &= P[(a = 2, b = 0), (a = 1, b = 1), \text{ or } (a = 0, b = 2)] \\ &= P(a = 2)P(b = 0) + P(a = 1)P(b = 1) + P(a = 0)P(b = 2) \\ &= (0.2 \times 0.3) + (0.3 \times 0.2) + (0.4 \times 0.2) \\ &= 0.20 \end{aligned}$$

Continuing in this way, we can obtain the probabilities for all eight possible values of c. The result is given as

c	$P(c)$
0	0.12
1	0.17
2	0.20
3	0.21
4	0.16
5	0.09
6	0.04
7	0.01

Although it is possible to solve this problem in the manner just described, it is highly desirable to find a shortcut to determine the probabilities of the values of c. Notice how the entries for a and b are used to find those for c. The first entry for c is the product of the first entries for a and b. The second entry for c is entry 1 of b times entry 2 of a plus entry 2 of a times entry 1 of b. The third entry of c is the sum of cross terms 1 and 3, 2 and 2, 3 and 1 of a and b.

We can systematize the process of finding the entries for c in the following way. Write the probabilities for a and b as sequences A and B

$$\begin{aligned} A &= [0.4, 0.3, 0.2, 0.1] \\ B &= [0.3, 0.2, 0.2, 0.2, 0.1] \end{aligned}$$

Reverse the order of one of the sequences, say B. Call the reversed sequence B_{inv}, where

$$B_{\text{inv}} = [0.1, 0.2, 0.2, 0.2, 0.3]$$

Position sequence A and the inverted sequence B_{inv} so that the first righthand term of B_{inv} is under the first left-hand term of A

$$\begin{array}{l} \qquad\quad [0.4, 0.3, 0.2, 0.1] \\ [0.1, 0.2, 0.2, 0.2, 0.3] \end{array}$$

The probability of zero defects is the product of the overlapping numbers 0.4 and 0.3. Now shift the inverted sequence one position to the right:

$$\begin{array}{l} \qquad\quad [0.4, 0.3, 0.2, 0.1] \\ [0.1, 0.2, 0.2, 0.2, 0.3] \end{array}$$

The probability of *one* defect is the sum of the overlapping products 0.2×0.4 and 0.3×0.3. The remaining terms in c are obtained by shifting the inverted sequence one step at a time to the right, and for each step summing the overlap products.

The process of inverting a sequence, sliding it one step at a time to the right, and summing the overlap products is called *discrete convolution* or *serial multiplication*.

The asterisk (*) is generally used to indicate convolution. Thus, we write

$$C = A * B \tag{A-41}$$

or

$$\begin{aligned} C &= [0.4, 0.3, 0.2, 0.1] * [0.3, 0.2, 0.2, 0.2, 0.1] \\ &= [0.12, 0.17, 0.20, 0.21, 0.16, 0.09, 0.04, 0.01] \end{aligned}$$

where the last term in C is the first product, the next-to-last term is the sum of the second and third products, and so on.

Using this example as a guide, we see that the convolution of two sequences A and B, where

$$A = [a_0, a_1, \ldots, a_i, \ldots]$$
$$B = [b_0, b_1, \ldots, b_i, \ldots]$$

to obtain a sequence

$$C = [c_1, c_2, \ldots, c_i, \ldots]$$

has the following properties:

(a) The convolution process is commutative. That is

$$A * B = B * A \tag{A-42}$$

(b) The ith term of C is given by the relation

$$c_i = \sum_{j=0}^{i} a_j b_{i-j} \tag{A-43}$$

(c) The number of terms n_C in C is

$$n_C = n_A + n_B - 1 \tag{A-44}$$

where n_A and n_B are the number of terms in A and B, respectively.

(d) The product of the sum of the elements of A and the sum of the elements in B is the sum of the elements in C, or

$$\left(\sum_{i=0}^{n_A-1} a_i\right)\left(\sum_{j=0}^{n_B-1} b_j\right) = \sum_{k=0}^{n_A+n_B-1} c_k \tag{A-45}$$

Returning to Eq. (A-38a),

$$f(t) = \int_{-\infty}^{\infty} g(\alpha)h(t-\alpha)\,d\alpha \tag{A-38a}$$

where $f(t)$ is a *continuous convolution*. To see why, let $g(t)$ and $h(t)$ be two arbitrary functions whose appearance is sketched in Fig. A-4(a). Let α be some value of t at which we shall evaluate $h(t)$, so that $h(-\alpha)$ represents what happens when we take the function $f(t)$ and fold or convolve it about the vertical axis to obtain its mirror image. Then we slide it back to the right somewhat (Fig. A-4(b)) by changing the argument to $(-\alpha + t)$, so that we end up with $h(t-\alpha)$ as shown. We multiply this function by $g(t)$ evaluated at the same value of α to obtain the shaded area in Fig. A-4(c), and then we "add" all these products—i.e., integrate over all possible values of α—to obtain the convolution in Fig. A-4(d). This process is essentially what we did in the discrete case: one of the two sets of variables is reversed, the overlaping values are computed, and then the overlaps are totaled. In fact, the dummy variable of integration α is analogous to the dummy summation index j. We have a further parallel which comes from letting

$$t - \alpha = \beta$$

Then

$$f(t) = g(t) * h(t) = -\int_{+\infty}^{-\infty} g(t-\beta)h(\beta)\,d\beta = h(t) * g(t) \tag{A-46}$$

which is the analogue of Eq. (A-42). We can find the relation which corresponds to Eq. (A-45) by integrating Eq. (A-38a), obtaining

$$\begin{aligned} \int_{-\infty}^{\infty} f(t)\,dt &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\alpha)h(t-\alpha)\,d\alpha\,dt \\ &= \int_{-\infty}^{\infty} g(\alpha)\left\{\int_{-\infty}^{\infty} h(t-\alpha)\,dt\right\} d\alpha \\ &= \int_{-\infty}^{\infty} g(t)\left\{\int_{-\infty}^{\infty} h(t)\,dt\right\} dt \\ &= \int_{-\infty}^{\infty} g(t)\,dt \int_{-\infty}^{\infty} h(t)\,dt \end{aligned} \tag{A-47}$$

where we have used the fact that $h(t-\alpha)$ and $h(t)$ take on the same set of values if we are integrating from $-\infty$ to ∞. Equation (A-47) states that the product of the areas under the curves for two functions is equal to the area produced by this convolution.

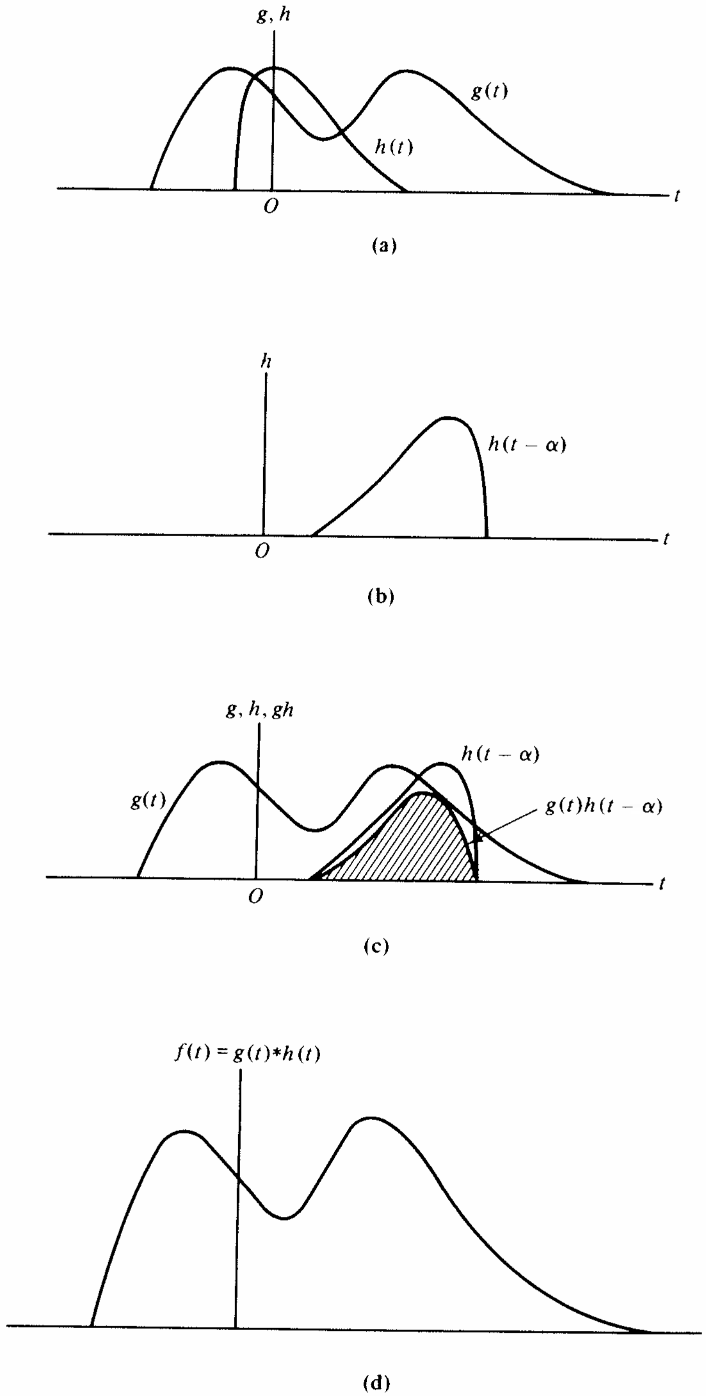

Figure A-4

A-4 The Delta Function

The function $f(t)$ defined by

$$f(t) = \frac{1}{\sqrt{2\pi a^2}} e^{-t^2/2a^2} \tag{A-48}$$

is called the *Gaussian function*. The constant $1/\sqrt{2\pi a^2}$ makes the area under the curve in the range $-\infty < x < \infty$ have a value of unity, as may be verified by a table of definite integrals. The quantity a in the exponent is a measure of the width of the symmetric Gaussian curve. Figure A-5 shows these curves

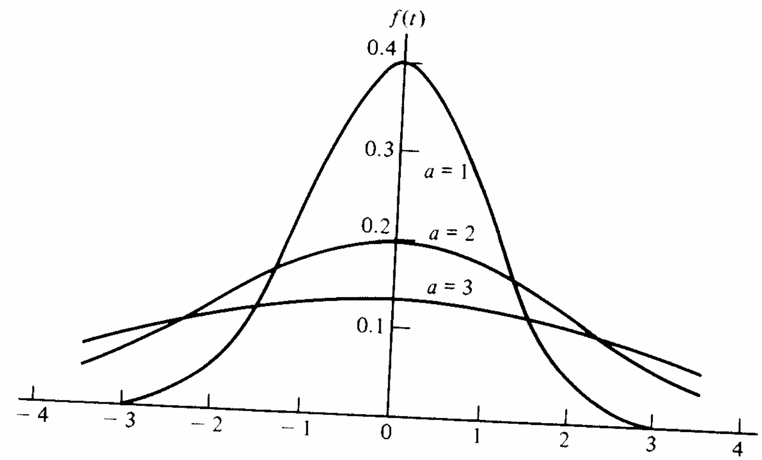

Figure A-5

for $a = 1$, 2, and 3. If we consider the width $2w$ as being specified by the values of $f(t)$ which are $e^{-1} = 0.37$ times the value at $t = 0$, then Eq. (A-48) indicates that

$$w = \pm\sqrt{2}a \tag{A-49}$$

The Fourier transform of the Gaussian function is

$$F(\omega) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi a^2}} e^{-t^2/2a^2} e^{-i\omega t}\, dt = e^{-2\pi^2 a^2 \omega^2} \tag{A-50}$$

This function is also Gaussian, as shown in Fig. A-6.

Let us consider the effect on $f(t)$ and $F(\omega)$ if a is permitted to become indefinitely small. As Fig. A-6 indicates, the curve for $f(t)$ will become higher and narrower. If we let $a \longrightarrow 0$, we have, in fact, a width of zero and a height of infinity, but the area under the curve remains equal to unity. Under these conditions, we call $f(t)$ the *Dirac delta function* $\delta(t)$, and specify it by the

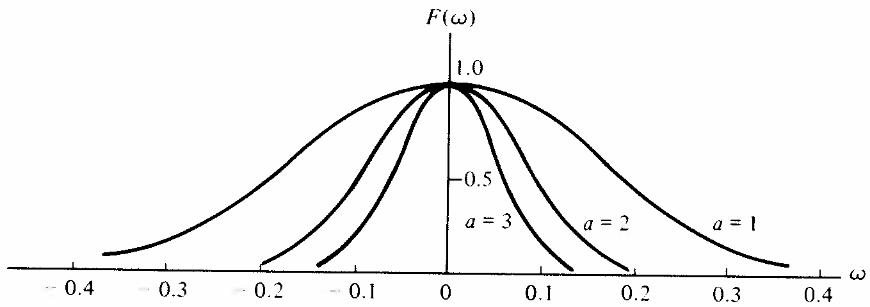

Figure A-6

relations

$$\delta(t) = 0 \qquad t \neq 0 \tag{A-51}$$

$$\delta(t) = \infty \qquad t = 0 \tag{A-52}$$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1 \tag{A-53}$$

It is not really correct to call $\delta(t)$ a function—it is properly known as a *distribution*[A-2]—but for our purposes here we need not worry about mathematical rigor. Taking the same limit in Eq. (A-50) shows that

$$F(\omega) = 1 \tag{A-54}$$

That is, we have obtained a Gaussian curve of infinite width. Figure A-7(a) shows the way in which the delta function might be represented and Fig. A-7(b) is its Fourier transform.

One of the useful properties of the delta function is obtained if we consider

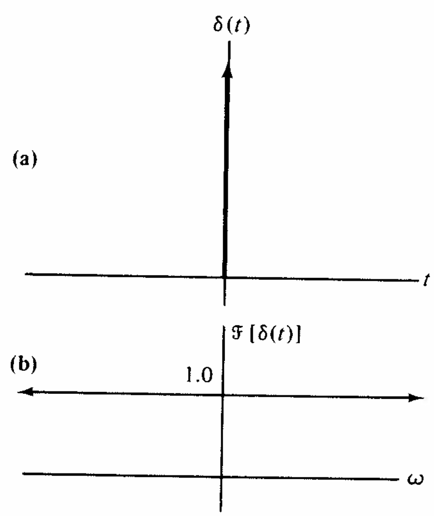

Figure A-7

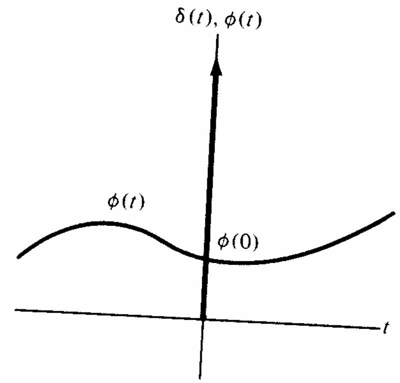

Figure A-8

its effect on an arbitrary function $\varphi(t)$. Let $\varphi(t)$ have the general behavior shown in Fig. A-8 and consider the integral

$$I = \int_{-\infty}^{\infty} \varphi(t)\delta(t)\, dt \tag{A-55}$$

Since the product $\varphi(t)\delta(t)$ vanishes for $t \neq 0$, the only value of $\varphi(t)$ which has any effect on the value of the integral I is $\varphi(0)$. But $\varphi(0)$ is a constant and can be placed outside the integral. Hence

$$I = \varphi(0) \int_{-\infty}^{\infty} \delta(t)\, dt = \varphi(0) \tag{A-56}$$

We can generalize this result by shifting the origin of coordinates to $t = a$. Then Eq. (A-53) becomes

$$\int_{-\infty}^{\infty} \delta(t - a)\, d(t - a) = \int_{-\infty}^{\infty} \delta(t - a)\, dt = 1 \tag{A-57}$$

and the arguments just given lead to the relation

$$\int_{-\infty}^{\infty} \varphi(t)\delta(t - a)\, dt = \varphi(a) \tag{A-58}$$

This equation expresses the *sampling property* of the delta function. We can visualize it as shown in Fig. A-9; placing $\delta(t)$ at $t = a$ picks out the corresponding value $\varphi(a)$ of $\varphi(t)$ as we integrate from $t = -\infty$ to $t = \infty$.

The sampling property may also be obtained from Fourier transforms involving the delta function. We have seen from Eq. (A-50) and Eq. (A-54) that

$$F(\omega) = \mathfrak{F}[\delta(t)] = 1 \tag{A-59}$$

and the inverse of this is

$$\mathfrak{F}^{-1}[1] = \delta(t) \tag{A-60}$$

Using Eq. (A-29), we see that Eq. (A-60) is equivalent to the relation

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t}\, d\omega = \delta(t) \tag{A-61}$$

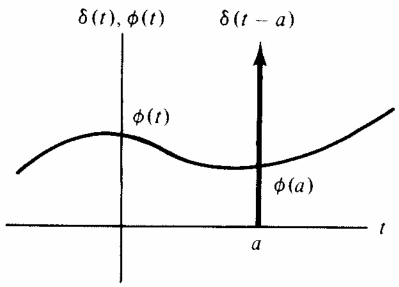

Figure A-9

or replacing t by $t - t'$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(t-t')}\,d\omega = \delta(t-t') \tag{A-62}$$

Returning to Eq. (A-21), let $x = t$, $y = t'$, and $k = \omega$ to obtain

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t')e^{i\omega(t-t')}\,d\omega\,dt' \tag{A-63}$$

By Eq. (A-62)

$$f(t) = \int_{-\infty}^{\infty} f(t')\delta(t-t')\,dt' \tag{A-64}$$

which is equivalent to Eq. (A-58), since

$$\delta(t-t') = \delta(t'-t) \tag{A-65}$$

This result follows from the fact that the area defined by the delta function $\delta(x)$ is unity for any value of x.

If we rewrite Eq. (A-16) as

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(m-n)x}\,dx = \delta_{mn} \tag{A-66}$$

where

$$\delta_{mn} = \begin{cases} 1 & \text{if} \quad m = n \\ 0 & \text{if} \quad m \neq n \end{cases}$$

is the Kronecker delta, then Eq. (A-66)—which involves the integers m and n—is analogous to the continuous relation of Eq. (A-62) and explains the source of the name given to $\delta(t)$.

The use of the delta function in the convolution process leads to interesting results. Since the sampling property as illustrated in Fig. A-9 simply picks out the value of $\varphi(t)$ at $t = a$, it follows that the convolution, $\varphi(t) * \delta(t-a)$ will first reverse the delta function (which leaves it unaltered) and then reproduce each of the values of φ in a new position. That is, $\varphi(t)$ is converted into $\varphi(t-a)$ as shown in Fig. A-10. We express this analytically, from Eq. (A-46), as

$$\varphi(t) * \delta(t-a) = \delta(t-a) * \varphi(t) = \int_{-\infty}^{\infty} \delta(\alpha - a)\varphi(t-\alpha)\,d\alpha \tag{A-67}$$

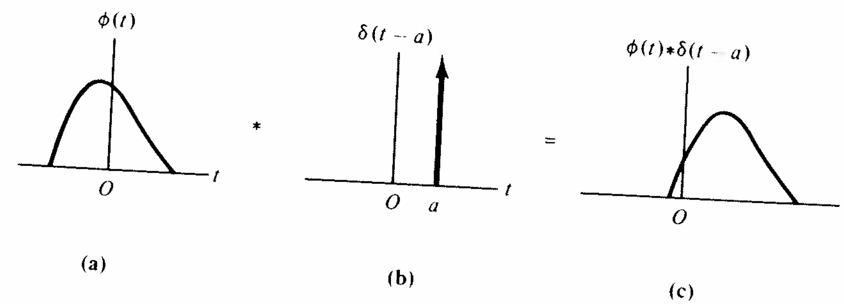

Figure A-10

The sampling property will reduce the integral to the value of φ for which $\alpha = a$. Hence Eq. (A-67) becomes

$$\varphi(t) * \delta(t-a) = \varphi(t-a) \tag{A-68}$$

as we have already deduced.

Consider finally a series of evenly spaced delta functions, known as a *Dirac comb* (Fig. A-11a). This function is expressible as

$$\text{comb}\,(t) = \sum_{n=-\infty}^{\infty} \delta(t - na) \tag{A-69}$$

The convolution of $\varphi(t)$ and the comb function then repeats $\varphi(t)$ at equally spaced intervals of $\pm a$, $\pm 2a$, . . . , as shown in Fig. A-11b.

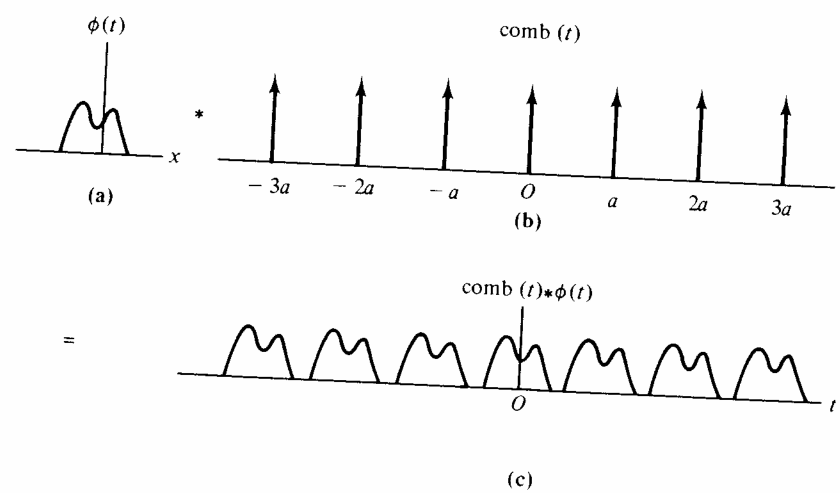

Figure A-11

A-5 The Fourier Spectrum

A periodic function can be expanded in a Fourier series of trigonometric functions as specified by Eq. (A-11) or in a series of complex exponential functions as given by Eq. (A-12). To express the series, we need merely know the set of coefficients a_n, b_n or the set c_n and the value of k. As an example, consider a generalization (Fig. A-12) of the pulse train in Fig. A-3. Let

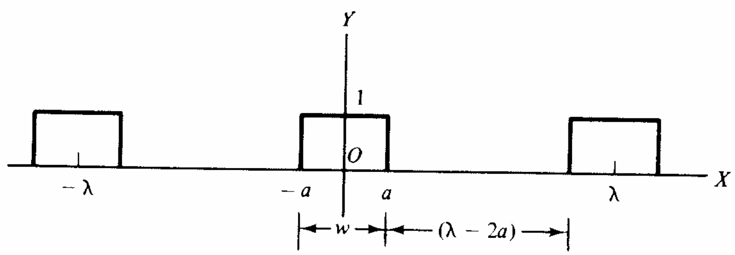

Figure A-12

the pulses have a width $2a$ and a spacing $(\lambda - 2a)$. To determine the c_n, we use Eq. (A-14) and obtain

$$c_n = \frac{1}{\lambda}\int_{-\lambda/2}^{\lambda/2} f(x)e^{-inkx}\,dx$$

$$= \frac{1}{\lambda}\int_{-a}^{a} (1)e^{-inkx}\,dx = \left.\frac{e^{-inkx}}{\lambda(-ink)}\right|_{-a}^{a}$$

$$= \frac{2\sin nka}{\lambda nk} = \frac{\sin nka}{\pi n} = \frac{2a}{\lambda}\frac{\sin nka}{nka} = \frac{\omega}{\lambda}\operatorname{sinc}(nka) \qquad \text{(A-70)}$$

This result does not hold for $n = 0$, since c_0 has the form $0/0$. However, using a Taylor series for $\sin nka$, we find that

$$c_0 = \lim_{n\to 0}\frac{nka + \frac{1}{3}!(nka)^3 + \cdots}{\pi n} = \frac{ka}{\pi} = \frac{\omega}{\lambda}$$

We now wish to find the dependence of c_n on nk. To do this, we must specify a and λ numerically. For example, if $\lambda = 6$ and $a = 1$ so that

$$a = \frac{\lambda}{6}$$

then

$$c_0 = \frac{ka}{\pi} = \frac{2a}{\lambda} = \frac{1}{3}$$

$$c_1 = \frac{\sin(\pi/3)}{\pi} = \frac{\sqrt{3}}{2\pi}$$

$$c_2 = \frac{\sin(2\pi/3)}{2\pi} = \frac{\sqrt{3}}{4\pi}$$

$$c_3 = \frac{\sin(3\pi/3)}{2\pi} = 0$$

and so on, producing the graph in Fig. A-13(a). This dependence of c_n on nk is called the *Fourier spectrum* and, as stated above, provides a compact summary of information such as that in Figs. A-2(a), (b), and (c). That is, it gives the amplitudes and wavelengths of the constituent parts of the periodic function. Note that the ordinates are labeled with the values of n.

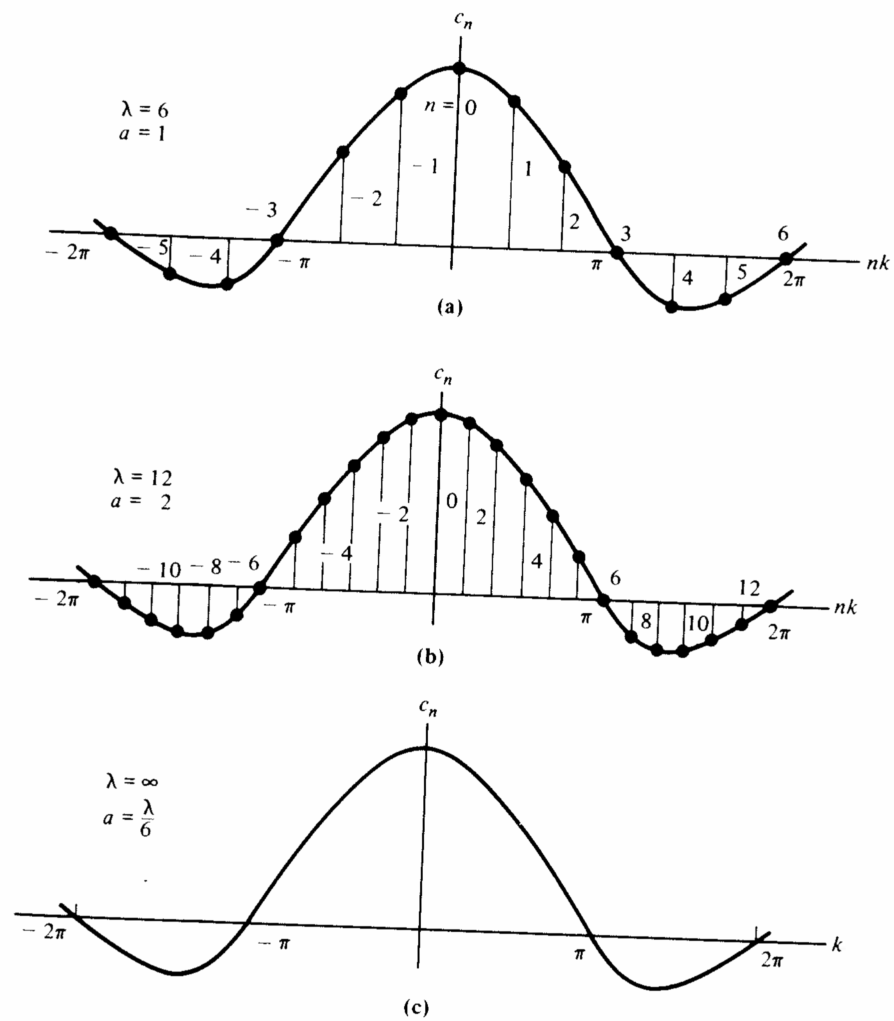

Figure A-13

If the pulse train has its wavelength doubled by letting

$$\lambda = 12, \qquad a = 2$$

then the ratio a/λ is unchanged; therefore c_0 is unchanged. A condition that

$$\sin nka = 0$$

is that

$$nka = \pi$$

In the case $\lambda = 6$, this is equivalent to $n = 3$, and when $\lambda = 12$, this gives $n = 6$. Hence, we obtain the spectrum shown in Fig. A-13(b), which has the same shape as the curve above it, but twice as many points.

It then is clear that as λ increases, we approach the curve of Fig. A-13(c) as a limit. That is, we have the Fourier spectrum for the function

$$f(x) = 1 \quad \text{for} \quad -\infty \leqq x \leqq \infty \tag{A-71}$$

This constant function contains all values of k in its spectrum, rather than a comparatively small number of discrete values. Also, we cannot talk about a Fourier series for this function; instead, we have a Fourier integral. By Eq. (A-23), we may express an arbitrary function as

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)e^{ikx}\,dk$$

where $F(k)$ may be interpreted as the infinite set of Fourier coefficients. It is, however, the Fourier transform of $f(x)$. Hence, the limit of the Fourier spectrum as λ becomes infinite is the Fourier transform. In fact, for the function of Eq. (A-71), we have

$$F(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} (1)e^{-ikx}\,dx$$

We can evaluate this integral by simply letting the limits be very large, rather than infinity. Then

$$F(k) = \frac{\sqrt{2}\,\sin kx}{\sqrt{\pi}\,k}$$

This is the function shown in Fig. A-13(c), thus verifying the statement that the Fourier transform is the limiting spectrum.

References

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