MATH 454 McGill University
Assignment 1 Fall 2020

This assignment is due on Friday, September 18 at 23:00pm. Please submit your answers on Crowdmark (which you may access via the navigation bar of the course webpage on myCourses).

Questions 2 and 5 are from Stein and Shakarchi's book (Chapter 1).

Question 1. [20 marks] Let S be the set of all closed cubes in $\mathbb{R}^d$, S' be the set of all open cubes in $\mathbb{R}^d$ and S'' be the set of all rectangles R such that $(a_1, b_1) \times ... \times (a_d, b_d) \subseteq R \subseteq [a_1, b_1] \times ... \times [a_d, b_d]$ for some $a_1 < b_1, \ldots, a_d < b_d \in \mathbb{R}^d$. For every set $A \subseteq \mathbb{R}^d$, let $m_*(A)$, $m'_*(A)$ and $m''_*(A)$ be defined as

$$m_*(A) = \inf \sum_{k=1}^{\infty} \mathrm{vol}(Q_k), \quad m'_*(A) = \inf \sum_{k=1}^{\infty} \mathrm{vol}(Q'_k) \qquad \text{and} \quad m''_*(A) = \inf \sum_{k=1}^{\infty} \mathrm{vol}(R_k),$$

where the infima are taken over all countable coverings of A by $Q_k \in S$, $Q'_k \in S'$ and $R_k \in S''$, respectively. Prove that $m''_*(A) = m'_*(A) = m_*(A)$.

Hint: Prove that $m''_*(A) \leq m'_*(A) \leq m_*(A) \leq m''_*(A)$. You may use the following facts on rectangles without proofs:

(1) *For every rectangle R and every $\varepsilon > 0$, there exists an open rectangle R_ε such that $R \subset R_\varepsilon$ and $\mathrm{vol}(R_\varepsilon) \leq \mathrm{vol}(R) + \varepsilon$ (it suffices to slightly extend the sides of R). Similarly, for every cube Q and every $\varepsilon > 0$, there exists an open cube Q_ε such that $Q \subset Q_\varepsilon$ and $\mathrm{vol}(Q_\varepsilon) \leq \mathrm{vol}(Q) + \varepsilon$.*

(2) *Every open rectangle R can be written as $R = \bigcup_{k=1}^{\infty} Q_k$ for some cubes $Q_k \in S$ which interiors are disjoint (we actually proved this fact in class for all open sets).*

(3) *If a sequence of disjoint rectangles $(R_k)_{k\in\mathbb{N}}$ is such that $\bigcup_{k=1}^{\infty} R_k \subset R$ for some rectangle R, then $\sum_{k=1}^{\infty} \mathrm{vol}(R_k) \leq \mathrm{vol}(R)$ (we also discussed this fact in class).*

Question 2. [20 marks] For every set $A \subseteq \mathbb{R}$, let $m^*_J(A)$ be defined as

$$m^*_J(A) = \inf \sum_{k=1}^{n} \ell(I_k),$$

where the infimum is taken over all *finite* coverings of A by intervals I_k, $\ell(I_k)$ stands for the length of I_k if I_k is bounded and $\ell(I_k) = \infty$ if I_k is unbounded. The function m^*_J is called the *exterior Jordan measure.*

(1) Prove that $m^*_J(A) = m^*_J(\overline{A})$ for all sets $A \subseteq \mathbb{R}$, where $\overline{A}$ denotes the closure of A.

(2) Give an example of a countable set A such that $m^*_J(A) = 1$. Compare with $m_*(A)$.

Question 3. [20 marks] For every $A \subseteq \mathbb{R}^d$, $\delta = (\delta_1, \ldots, \delta_d) \in (0, \infty)^d$ and $y = (y_1, \ldots, y_d) \in \mathbb{R}^d$, let $A_{\delta,y}$ be the set defined as

$$A_{\delta,y} = \{(\delta_1 x_1 + y_1, \ldots, \delta_d x_d + y_d) : x = (x_1, \ldots, x_d) \in A\}.$$

(1) Prove that $m_*(A_{\delta,y}) = \delta_1 \cdots \delta_k \, m_*(A)$.

(2) Prove that A is measurable if and only if $A_{\delta,y}$ is measurable.

Please turn over

Question 4. [10 marks] Given $\varepsilon > 0$, provide an example of an open set Ω that is dense in $\mathbb{R}$ and such that $m(\Omega) < \varepsilon$.

Question 5. [10 marks] Let $A, B \subseteq \mathbb{R}^d$ be such that A and B are measurable and $m(A) = m(B) < \infty$. Prove that every set E such that $A \subseteq E \subseteq B$ is measurable.

Question 6. [20 marks] Prove that a set $A \subseteq \mathbb{R}^d$ is measurable if and only if for every set $B \subseteq \mathbb{R}^d$ (not necessarily measurable), we have

$$m_*(B) = m_*(B \cap A) + m_*(B \backslash A).$$

Hint: It may be helpful to use the fact (proven in class) that for every set $E \subseteq \mathbb{R}^d$, we have

$$m_*(E) = \inf \{m(\mathcal{O}) : \mathcal{O} \text{ open, } E \subseteq \mathcal{O}\}.$$