Chapter 5: Histograms

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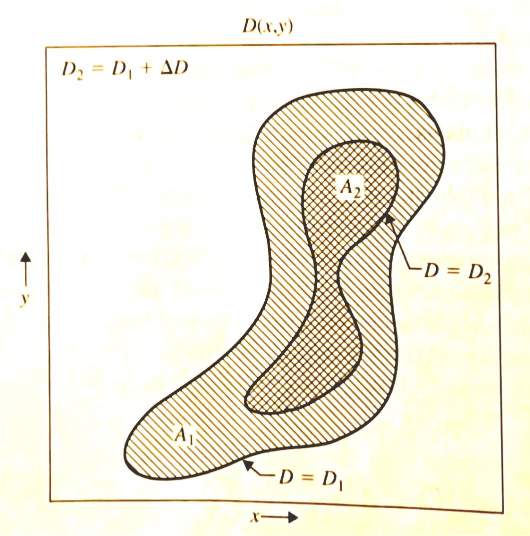
Outline

Area Function

Area Function

The area contained by all contours of gray level D is called the area function.

Area Function Contours

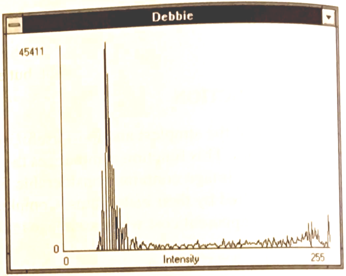


Histogram

Gray Level Histogram

A function showing, for each gray level in an image, how many pixels have that gray level.

Histogram Example



Histogram Definition

The histogram is the negative of the derivative of the area function.

$$H(D) = \lim_{\Delta D \to 0} \frac{A(D) - A(D + \Delta D)}{\Delta D} = -\frac{d}{dD} A(D) \tag{1}$$

For discrete images, ΔD is set to 1.

$$H(D) = A(D) - A(D + 1) \tag{2}$$

Integral of the Histogram

The integral of the histogram is the area function.

$$\int_{D}^{\infty} H(P)dP = -A(P)|_{D}^{\infty} = A(D) \tag{3}$$

If we set D=0, we obtain the area of the image.

$$\int_{0}^{\infty} H(P)dP = \text{area of image} \tag{4}$$

Sum of the Discrete Histogram

For discrete images, assuming 256 gray levels:

$$\sum_{D=0}^{255} H(D) = NR * NC = \text{number of pixels in image} \tag{5}$$

where NR and NC are the number of rows and columns in the image, respectively.

Connection to Probability

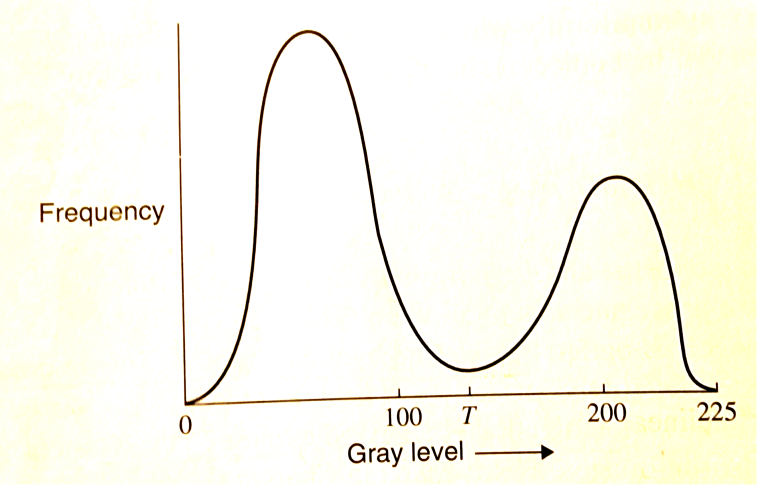
When normalized by the number of pixels in an image

- $H(D)$ is the same as the probability density function (PDF)
- $-A(D)$ is the same as the cumulative distribution function (CDF)

Uses of the Histogram

- Quality Control
  - The histogram can give a simple visual indication of whether an image uses all available gray levels.
  - Can help determine whether an image is being clipped at D=0 or D=255
- Boundary Threshold Selection
  - Can help select the gray level to use as a threshold for a boundary between two objects in, for example, a bimodal image

Bimodal Histogram



Identifying Objects

If an image contains a single gray object on a contrasting background, surrounded by a contour of intensity D_1, then

$$\int_{D_1}^{\infty} H(D)dD = \text{area of object} \tag{6}$$

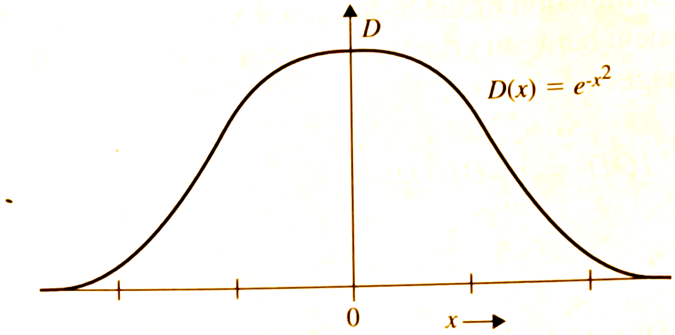
Integrated Optical Density

The histogram can be used to find the integrated optical density of an image:

$$\begin{aligned} \mathrm{IOD} &= \sum_{i=1}^{NR} \sum_{j=1}^{NC} D(i,j) \\ &= \sum_{k=0}^{255} kH(k) \end{aligned} \tag{7}$$

- The IOD is equivalent to the first moment (mean) in probability theory.
- The mean interior gray level of an object is its IOD divided by its area (which is the zero-order moment).

Example: 1D Gaussian Image



Area Function of 1D Gaussian Image

A 1D image represented as a Gaussian

$$D(x) = e^{-x^2}, \qquad -\infty \leq x \leq \infty \tag{8}$$

can be inverted to give the area function for the right half of the image

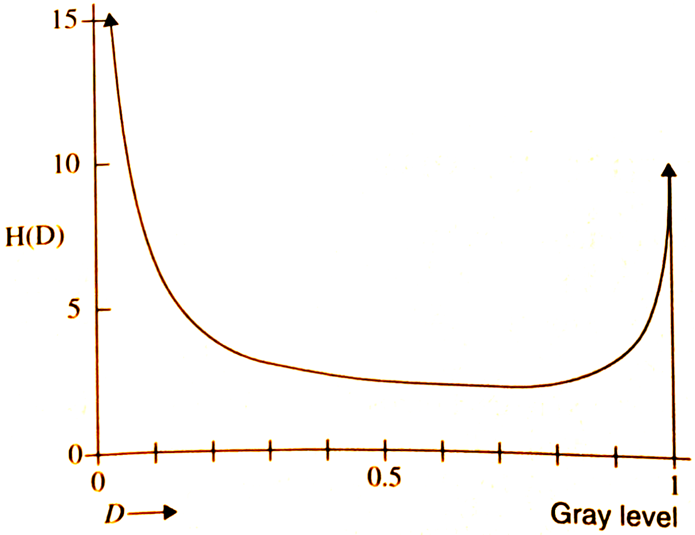
$$A_{\frac{1}{2}}(D) = x(D) = \sqrt{-ln(D)}, \qquad x \geq 0 \tag{9}$$

Histogram of 1D Gaussian Image

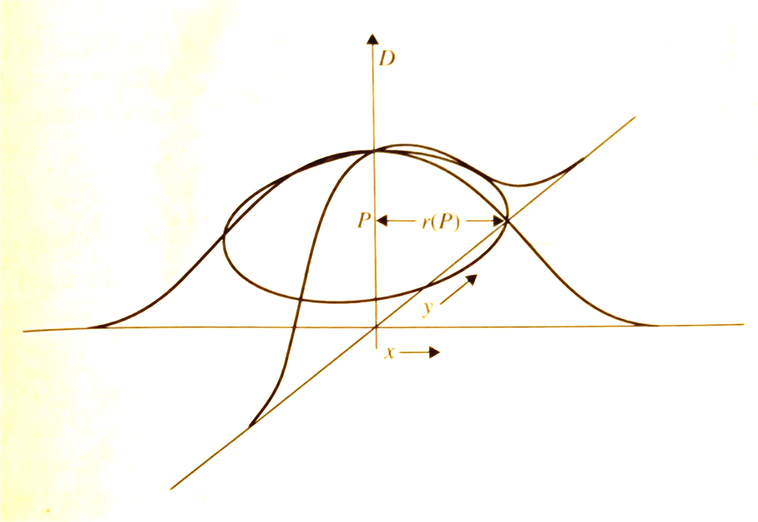
The histogram is then given by

$$H(D) = -\frac{d}{dD}[2\sqrt{-ln(D)}] = \frac{1}{D\sqrt{-ln(D)}} \tag{10}$$

Histogram of 1D Gaussian Image (Plot)



Example: 2D Gaussian Image



2D Gaussian Image

A 2D Gaussian can be represented in polar coordinates as

$$D(r,\theta) = e^{-r^2}, \qquad 0 \leq r \leq \infty, \quad 0 \leq \theta \leq 2\pi \tag{11}$$

A contour of constant gray level P is a circle of radius

$$r(P) = \sqrt{-ln(P)} \tag{12}$$

Histogram of 2D Gaussian Image

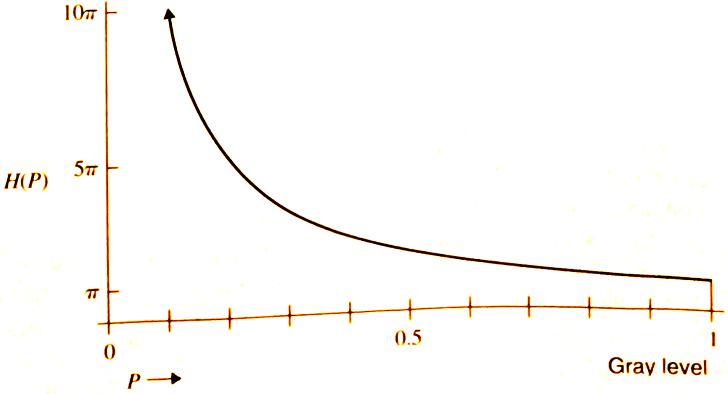
Such a contour encloses an area

$$A(P) = \pi[r(P)]^2 = -\pi ln(P) \tag{13}$$

Which can be differentiated to give the histogram

$$H(P) = -\frac{d}{dP}A(P) = \frac{\pi}{P} \tag{14}$$

Histogram of 2D Gaussian Image (Plot)



Summary of Important Points

1. The histogram is the negative of the derivative of the threshold area function.
2. The histogram shows how many pixels occur at each gray level.
3. Inspection of the histogram helps with segmentation and quality control.
4. The area of an object can be computed from the histogram of its image.
5. The histogram of an image of a specified functional form can be derived with the aid of the area function.