Geometrical visualization of the projection of a point onto a hyperplane

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1 Orthogonal Projection

Given a point $v_i \in \mathbb{R}^N$, the orthogonal projection onto a hyperplane $H = \{x \in \mathbb{R}^N | a^T x = b, \|a\| = 1\}$ can be obtained in the following way (see Figure 1): First we observe that the projection has to be along the direction of the hyperplane normal from v_i, i.e. the we will obtain a form of $v_i' = v_i - \lambda a$, whereby λ is the distance of v_i towards the hyperplane. Since the distance of the hyperplane to the origin along a is given by b, and the distance of v_i to the origin along a is given by $v_i^T a$, we obtain the distance $\lambda = v_i^T a - b$.

2 Projection along line L

Let $L = \{x \in \mathbb{R}^N | x = c + \lambda(d - c)\}$ be a line through point $c, d \in \mathbb{R}^N$. Let $H = \{x \in \mathbb{R}^N | a^T x = b\}$ be the hyperplane. The intersection of L and H is given by

$$\begin{aligned} a^T(c + \lambda(d - c)) &= b \\ \lambda a^T(d - c) &= b - a^T c \\ \lambda &= \frac{b - a^T c}{a^T(d - c)} \quad | a^T(d - c) \neq 0 \end{aligned} \tag{1}$$

Thus, the intersection of L and H is given by the point

$$x_{LH} = c + \frac{b - a^T c}{a^T(d - c)}(d - c) \tag{2}$$

for the case that $a^T(d-c) \neq 0$. If $a^T(d-c) = 0$, then the line is perpendicular to the plane normal, and thus parallel to the plane, i.e. either it does not intersect the plane, or it lies inside the hyperplane.

3 Special Case: Projection of a point $v \in \mathbb{R}^3$ along line parallel to z-axis

Let $v \in \mathbb{R}^3$ be given.

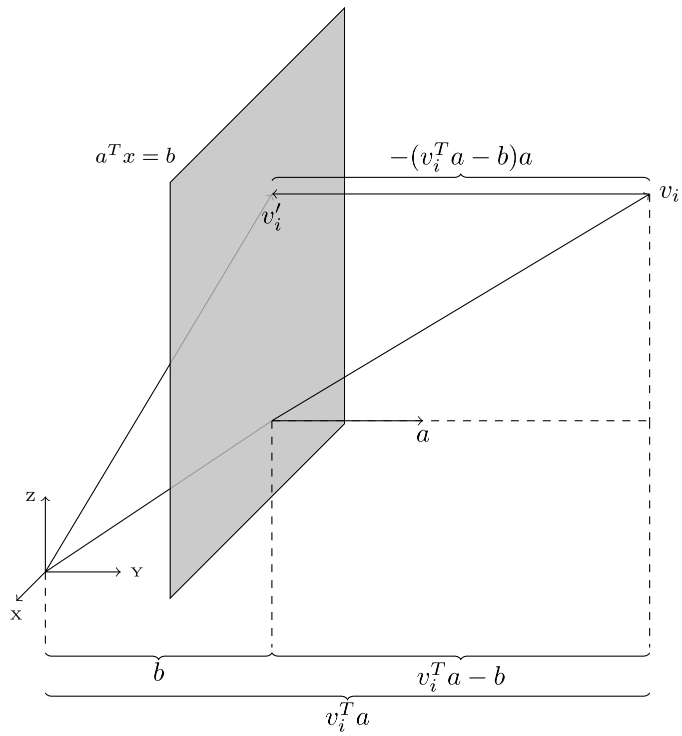

Figure 1: Geometrical visualization that the orthogonal projection of a point v_i onto a hyperplane defined by $a^T x = b$ is given by $v_i' = v_i - (v_i^T a - b)a$.

$$L = \{x \in \mathbb{R}^3 | x = v - \lambda(0,0,1)^T\}$$
$$H = \{x \in \mathbb{R}^N | a^T x = b\}$$

Assume that $v \notin H$, and $a^T(0,0,1)^T \neq 0$. Then $\lambda = \frac{b - a^T v}{a^T (0,0,1)^T}$. And the projection of v onto H along L is given by

$$v' = v - \frac{b - a^T v}{a^T (0,0,1)^T}(0,0,1)^T$$