2.7

2.7.1

$$u = l + e\sin u \quad \cdots (2.228)$$ より、

$$u_0 = l$$
$$u_1 = l + e\sin u_0$$
$$u_2 = l + e\sin u_1$$
$$\vdots$$
$$u_{n+1} = l + e\sin u_n \quad \cdots (2.229)$$

○近似の精度を調べる

$$u_n = u + \epsilon_n \quad \cdots (2.230)$$

これを(2.229)へ代入する

$$u + \epsilon_{n+1} = l + e\sin(u + \epsilon_n) \quad \cdots (2.231)$$

$$\left(\begin{aligned} &f(\epsilon) = \sin(u+\epsilon_n) \text{ とおいてマクローリン展開} \\ &f'(\epsilon) = \cos(u+\epsilon_n) \\ &f(\epsilon) = f(0) + f'(0)\epsilon_n + O(\epsilon_n^2) \\ &\quad\;\; = \sin u + (\cos u)\epsilon_n + O(\epsilon_n^2) \end{aligned}\right)$$

$$u + \epsilon_{n+1} = l + e\sin u + (e\cos u)\epsilon_n + O(\epsilon^2) \quad \cdots (2.232)$$

ここで、$u = l + e\sin u$ を左辺に代入

$$l + e\sin u + \epsilon_{n+1} = l + e\sin u + (e\cos u)\epsilon_n + O(\epsilon^2)$$

$$\epsilon_{n+1} = (e\cos u)\epsilon_n \quad \cdots (2.233)$$

2.7.2

$$y - f(x_0) = f'(x_0)(x - x_0)$$

$$y = f'(x_0)(x - x_0) + f(x_0)$$

$y=0$ となる x の値を x_1 とおくと、

$$f'(x_0)(x_1 - x_0) + f(x_0) = 0$$

$$\Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

このようにして、順次 $x_2, x_3, \cdots$ と求めていけば、

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \cdots (2.234)$$

根を x^*、第n近似での誤差を ϵ_n とすると、

$$x_n = x^* + \epsilon_n \quad \cdots (2.235)$$

$f(x)$ を x^* 周りでテイラー展開

$$f(x_n) = f(x^*) + \epsilon_n f'(x^*) + \frac{\epsilon_n^2}{2} f''(x^*) + O(\epsilon_n^3) \quad \cdots (2.236)$$

$$= \epsilon_n f'(x^*) + \frac{\epsilon_n^2}{2} f''(x^*) + O(\epsilon_n^3) \quad \cdots (2.237) \quad (\because f(x^*) = 0)$$

これより、

$$f'(x_n) = f'(x^*) + \epsilon_n f''(x^*) + O(\epsilon_n^2) \quad \cdots (2.238)$$

(2.235), (2.237), (2.238) を (2.234) へ代入

$$x^* + \epsilon_{n+1} = x^* + \epsilon_n - \frac{\epsilon_n f'(x^*) + \frac{\epsilon_n^2}{2} f''(x^*)}{f'(x^*) + \epsilon_n f''(x^*)}$$

$$\epsilon_{n+1} = \epsilon_n - \epsilon_n \frac{1 + \frac{\epsilon_n}{2}\frac{f''}{f'}}{1 + \epsilon_n \frac{f''}{f'}}$$

$$\vdots$$

$$
\begin{aligned}
\epsilon_{n+1} &= \epsilon_n - \epsilon_n\left(1+\frac{\epsilon_n}{2}\frac{f''}{f'}\right)\left(1-\epsilon_n\frac{f''}{f'}+O(\epsilon_n^2)\right) \quad \left(\because \frac{1}{1+x}=1-x+x^2+x^3\cdots\right)\\
&= \epsilon_n - \epsilon_n\left(1-\epsilon_n\frac{f''}{f'}+\frac{\epsilon_n}{2}\frac{f''}{f'}+O(\epsilon_n^2)\right)\\
&= \epsilon_n\left(1-1+\frac{\epsilon_n f''}{2f'}+O(\epsilon_n^2)\right)\\
&= \frac{\epsilon_n^2 f''}{2f'} \quad \cdots (2.239)
\end{aligned}
$$

ケプラー方程式に(2.239)を適用すると、

$$
\begin{cases}
f(u) = u - e\sin u - l\\
f'(u) = 1 - e\cos u\\
f''(u) = e\sin u
\end{cases}
$$

$$
\begin{aligned}
\epsilon_{n+1} &= \frac{f''(u^*)}{2f'(u^*)}\epsilon_n^2\\
&= \frac{e\sin u^*}{2(1-e\cos u^*)}\epsilon_n^2 \quad \cdots (2.240)
\end{aligned}
$$