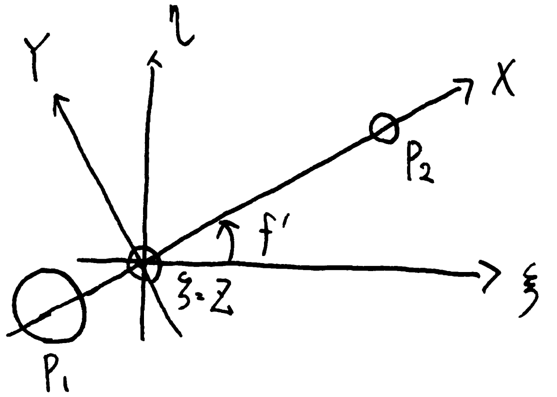
4.7 楕円制限3体問題

◦ まずは、円制限3体問題のときと同じようにして、回転座標系(X, Y, Z)でのP_3の運動方程式を求める



この節の計算は
$a'=1,\ G=1,\ m_1+m_2=1,\ n'=1$
で規格化されているので注意

◦ 慣性座標系(ξ, η, ζ)と回転座標系(X, Y, Z)の変換式は、

$$\begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} = \begin{bmatrix} \cos f' & -\sin f' & 0 \\ \sin f' & \cos f' & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} X\cos f' - Y\sin f' \\ X\sin f' + Y\cos f' \\ Z \end{bmatrix} \quad \cdots (4.114)$$

◦ ここで、さらに新しい座標系$(\hat{X}, \hat{Y}, \hat{Z})$を導入する

$$X = r'\hat{X}_{(f')}, \quad Y = r'\hat{Y}_{(f')}, \quad Z = r'\hat{Z}_{(f')} \quad \cdots ①$$

この座標系では長さの単位をP_1, P_2の距離r'にしてある。
つまり、(X, Y, Z)座標系では楕円運動していたP_1, P_2が$(\hat{X}, \hat{Y}, \hat{Z})$座標系では円運動しているようにみえる。もちろん$r'$は変数。
また、独立変数をtからf'へ変換してある。

○ $(\hat{x}, \hat{y}, \hat{z})$ と (ξ, η, ζ) の変換式は、
(4.114) へ ① を代入して求められる

$$\begin{cases} \xi = r'(\hat{x}\cos f' - \hat{y}\sin f') \\ \eta = r'(\hat{x}\sin f' + \hat{y}\cos f') \\ \zeta = r'\hat{z} \end{cases} \quad \cdots ②$$

違う表し方をすれば、

$$\begin{cases} \hat{x} = \dfrac{1}{r'}(\xi\cos f' + \eta\sin f') \\ \hat{y} = \dfrac{1}{r'}(-\xi\sin f' + \eta\cos f') \\ \hat{z} = \dfrac{1}{r'}\zeta \end{cases} \quad \cdots ③$$

○ 慣性系での運動方程式は、(4.1) より

$$\frac{d^2\mathbf{r}}{dt^2} = -\frac{\partial U}{\partial \mathbf{r}}$$

$$\Rightarrow \begin{cases} \ddot{\xi} = -\dfrac{\partial U}{\partial \xi} \\ \ddot{\eta} = -\dfrac{\partial U}{\partial \eta} \\ \ddot{\zeta} = -\dfrac{\partial U}{\partial \zeta} \end{cases} \quad \cdots ④$$

と求められているので、この④と②,③を利用して、慣性系 (ξ, η, ζ) の運動方程式を変換して $(\hat{x}, \hat{y}, \hat{z})$ 系での運動方程式を導出する。

・まず、(ξ,η,ζ)系では独立変数はtであったが、$(\hat{X},\hat{Y},\hat{Z})$系では独立変数を$f'$としているので、$\hat{X},\hat{Y},\hat{Z}$の時間微分と$f'$微分の関係式を求める。

$$\frac{d\hat{X}}{dt}=\frac{df'}{dt}\cdot\frac{d\hat{X}}{df'}=\dot{f}'\frac{d\hat{X}}{df'} \qquad \frac{d\hat{Y}}{dt}=\dot{f}'\frac{d\hat{Y}}{df'} \qquad \frac{d\hat{Z}}{dt}=\dot{f}'\frac{d\hat{Z}}{df'}$$

$$\frac{d^2\hat{X}}{dt^2}=\frac{d}{dt}\left(\dot{f}'\frac{d\hat{X}}{df'}\right)=\ddot{f}'\frac{d\hat{X}}{df'}+\dot{f}'\frac{df'}{dt}\frac{d}{df'}\left(\frac{d\hat{X}}{df'}\right)=\ddot{f}'\frac{d\hat{X}}{df'}+\dot{f}'^2\frac{d^2\hat{X}}{df'^2}$$

$$\frac{d^2\hat{Y}}{dt^2}=\ddot{f}'\frac{d\hat{Y}}{df'}+\dot{f}'^2\frac{d^2\hat{Y}}{df'^2}$$

$$\frac{d^2\hat{Z}}{dt^2}=\ddot{f}'\frac{d\hat{Z}}{df'}+\dot{f}'^2\frac{d^2\hat{Z}}{df'^2}$$

…⑤

$\hat{X},\hat{Y},\hat{Z}$の式に変形する

・④の左辺を計算する　(∵②)

$$\dot{\xi}=\dot{r}'(\hat{X}\cos f'-\hat{Y}\sin f')+r'(\dot{\hat{X}}\cos f'-\hat{X}\dot{f}'\sin f'-\dot{\hat{Y}}\sin f'-\hat{Y}\dot{f}'\cos f')$$
$$=\dot{r}'(\hat{X}\cos f'-\hat{Y}\sin f')+r'\{(\dot{\hat{X}}-\dot{f}'\hat{Y})\cos f'-(\dot{\hat{Y}}+\dot{f}'\hat{X})\sin f'\}$$

$$\begin{aligned}\ddot{\xi}=&\ddot{r}'(\hat{X}\cos f'-\hat{Y}\sin f')+\dot{r}'(\dot{\hat{X}}\cos f'-\dot{f}'\hat{X}\sin f'-\dot{\hat{Y}}\sin f'-\dot{f}'\hat{Y}\cos f')\\&+\dot{r}'\{(\dot{\hat{X}}-\dot{f}'\hat{Y})\cos f'-(\dot{\hat{Y}}+\dot{f}'\hat{X})\sin f'\}\\&+r'\{(\ddot{\hat{X}}-\ddot{f}'\hat{Y}-\dot{f}'\dot{\hat{Y}})\cos f'+(-\dot{\hat{X}}+\dot{f}'\hat{Y})\dot{f}'\sin f'\\&\quad-(\ddot{\hat{Y}}+\ddot{f}'\hat{X}+\dot{f}'\dot{\hat{X}})\sin f'-(\dot{\hat{Y}}+\dot{f}'\hat{X})\dot{f}'\cos f'\}\end{aligned}$$

↓ つづく ＋ ⑤代入

$\vdots$

$$\ddot{\xi} = \ddot{r}'(\hat{X}\cos f' - \hat{Y}\sin f') + 2\dot{r}'\left\{\left(\dot{f}'\frac{d\hat{X}}{df'} - \dot{f}'\hat{Y}\right)\cos f' - \left(\dot{f}'\frac{d\hat{Y}}{df'} + \dot{f}'\hat{X}\right)\sin f'\right\}$$

$$+ r'\left\{\left(\ddot{f}'\frac{d\hat{X}}{df'} + \dot{f}'^2\frac{d^2\hat{X}}{df'^2} - 2\dot{f}'^2\frac{d\hat{Y}}{df'} - \ddot{f}'\hat{Y} - \dot{f}'^2\hat{X}\right)\cos f'\right.$$

$$\left. + \left(-\ddot{f}'\frac{d\hat{Y}}{df'} - \dot{f}'^2\frac{d^2\hat{Y}}{df'^2} - 2\dot{f}'^2\frac{d\hat{X}}{df'} + \dot{f}'^2\hat{Y} - \ddot{f}'\hat{X}\right)\sin f'\right\}$$

$$= \ddot{r}'(\hat{X}\cos f' - \hat{Y}\sin f')$$

$$+ (2\dot{r}'\dot{f}' + r'\ddot{f}')\left\{\left(\frac{d\hat{X}}{df'} - \hat{Y}\right)\cos f' - \left(\frac{d\hat{Y}}{df'} + \hat{X}\right)\sin f'\right\}$$

$$+ r'\dot{f}'^2\left\{\left(\frac{d^2\hat{X}}{df'^2} - 2\frac{d\hat{Y}}{df'} - \hat{X}\right)\cos f' - \left(\frac{d^2\hat{Y}}{df'^2} + 2\frac{d\hat{X}}{df'} - \hat{Y}\right)\sin f'\right\} \quad \cdots ⑥$$

$$\dot{\eta} = \dot{r}'(\hat{X}\sin f' + \hat{Y}\cos f') + r'(\dot{\hat{X}}\sin f' + \hat{X}\dot{f}'\cos f' + \dot{\hat{Y}}\cos f' - \hat{Y}\dot{f}'\sin f')$$

$$= \dot{r}'(\hat{X}\sin f' + \hat{Y}\cos f') + r'\left\{(\dot{\hat{Y}} + \hat{X}\dot{f}')\cos f' + (\dot{\hat{X}} - \hat{Y}\dot{f}')\sin f'\right\}$$

$$\ddot{\eta} = \ddot{r}'(\hat{X}\sin f' + \hat{Y}\cos f') + \dot{r}'(\dot{\hat{X}}\sin f' + \dot{f}'\hat{X}\cos f' + \dot{\hat{Y}}\cos f' - \dot{f}'\hat{Y}\sin f')$$

$$+ \dot{r}'\left\{(\dot{\hat{Y}} + \hat{X}\dot{f}')\cos f' + (\dot{\hat{X}} - \hat{Y}\dot{f}')\sin f'\right\}$$

$$+ r'\left\{(\ddot{\hat{Y}} + \dot{f}'\dot{\hat{X}} + \ddot{f}'\hat{X})\cos f' - (\dot{\hat{Y}} + \hat{X}\dot{f}')\dot{f}'\sin f'\right.$$

$$\left. + (\ddot{\hat{X}} - \ddot{f}'\hat{Y} - \dot{f}'\dot{\hat{Y}})\sin f' + (\dot{\hat{X}} - \hat{Y}\dot{f}')\dot{f}'\cos f'\right\}$$

$\downarrow$ ⑤を代入

$$= \ddot{r}'(\hat{X}\sin f' + \hat{Y}\cos f') + 2\dot{r}'\left\{\left(\dot{f}'\frac{d\hat{Y}}{df'} + \dot{f}'\hat{X}\right)\cos f' + \left(\dot{f}'\frac{d\hat{X}}{df'} - \dot{f}'\hat{Y}\right)\sin f'\right\}$$

$$+ r'\left\{\left(\ddot{f}'\frac{d\hat{Y}}{df'} + \dot{f}'^2\frac{d^2\hat{Y}}{df'^2} + 2\dot{f}'^2\frac{d\hat{X}}{df'} + \ddot{f}'\hat{X} - \dot{f}'^2\hat{Y}\right)\cos f'\right.$$

$$\left. + \left(\ddot{f}'\frac{d\hat{X}}{df'} + \dot{f}'^2\frac{d^2\hat{X}}{df'^2} - 2\dot{f}'^2\frac{d\hat{Y}}{df'} - \ddot{f}'\hat{Y} - \dot{f}'^2\hat{X}\right)\sin f'\right\}$$

$$= \ddot{r}'(\hat{X}\sin f' + \hat{Y}\cos f')$$

$$+ (2\dot{r}'\dot{f}' + r'\ddot{f}')\left\{\left(\frac{d\hat{X}}{df'} - \hat{Y}\right)\sin f' + \left(\frac{d\hat{Y}}{df'} + \hat{X}\right)\cos f'\right\}$$

$$+ r'\dot{f}'^2\left\{\left(\frac{d^2\hat{X}}{df'^2} - 2\frac{d\hat{Y}}{df'} - \hat{X}\right)\sin f' + \left(\frac{d^2\hat{Y}}{df'^2} + 2\frac{d\hat{X}}{df'} - \hat{Y}\right)\cos f'\right\} \quad \cdots ⑦$$

$$\dot{\zeta} = \dot{r}'\tilde{Z} + r'\dot{\tilde{Z}}$$

$$\ddot{\zeta} = \ddot{r}'\tilde{Z} + \dot{r}'\dot{\tilde{Z}} + \dot{r}'\dot{\tilde{Z}} + r'\ddot{\tilde{Z}}$$

↓⑤代入

$$= \ddot{r}'\tilde{Z} + (2\dot{r}'\dot{f}' + r'\ddot{f}')\frac{d\tilde{Z}}{df'} + r'\dot{f}'^2\frac{d^2\tilde{Z}}{df'^2} \quad \cdots ⑧$$

◦ ここで、⑥,⑦,⑧をもう少し整理する

$h' = r'^2\dot{f}' = \sqrt{1-e^2} = \text{const}$ より

$$\frac{dh'}{dt} = 0$$

$$2r'\dot{r}'\dot{f}' + r'^2\ddot{f}' = 0$$

$$r'(2\dot{r}'\dot{f}' + r'\ddot{f}') = 0$$

$$\Rightarrow 2\dot{r}'\dot{f}' + r'\ddot{f}' = 0$$

$$\dot{f}'^2 = \frac{h'^2}{r'^4}$$

これらを⑥,⑦,⑧へ代入する

$$\ddot{\xi} = \ddot{r}'(\tilde{X}\cos f' - \tilde{Y}\sin f') + \frac{h'^2}{r'^3}\left\{\left(\frac{d^2\tilde{X}}{df'^2} - 2\frac{d\tilde{Y}}{df'} - \tilde{X}\right)\cos f' - \left(\frac{d^2\tilde{Y}}{df'^2} + 2\frac{d\tilde{X}}{df'} - \tilde{Y}\right)\sin f'\right\} \quad \cdots ⑨$$

$$\ddot{\eta} = \ddot{r}'(\tilde{X}\sin f' + \tilde{Y}\cos f') + \frac{h'^2}{r'^3}\left\{\left(\frac{d^2\tilde{X}}{df'^2} - 2\frac{d\tilde{Y}}{df'} - \tilde{X}\right)\sin f' + \left(\frac{d^2\tilde{Y}}{df'^2} + 2\frac{d\tilde{X}}{df'} - \tilde{Y}\right)\cos f'\right\} \quad \cdots ⑩$$

$$\ddot{\zeta} = \ddot{r}'\tilde{Z} + \frac{h'^2}{r'^3}\frac{d^2\tilde{Z}}{df'^2} \quad \cdots ⑪$$

○④の右辺を計算する (∵③)

(x̂, ŷ, ẑ)の偏微分に書き換える

$$\frac{\partial U}{\partial \xi} = \frac{\partial \hat{x}}{\partial \xi}\cdot\frac{\partial U}{\partial \hat{x}} + \frac{\partial \hat{y}}{\partial \xi}\cdot\frac{\partial U}{\partial \hat{y}}$$

$$= \left(\frac{1}{r'}\cos f'\right)\frac{\partial U}{\partial \hat{x}} + \left(-\frac{1}{r'}\sin f'\right)\frac{\partial U}{\partial \hat{y}}$$

$$= \frac{1}{r'}\left(\cos f'\frac{\partial U}{\partial \hat{x}} - \sin f'\frac{\partial U}{\partial \hat{y}}\right) \quad \cdots ⑫$$

$$\frac{\partial U}{\partial \eta} = \frac{\partial \hat{x}}{\partial \eta}\cdot\frac{\partial U}{\partial \hat{x}} + \frac{\partial \hat{y}}{\partial \eta}\cdot\frac{\partial U}{\partial \hat{y}}$$

$$= \left(\frac{1}{r'}\sin f'\right)\frac{\partial U}{\partial \hat{x}} + \left(\frac{1}{r'}\cos f'\right)\frac{\partial U}{\partial \hat{y}}$$

$$= \frac{1}{r'}\left(\sin f'\frac{\partial U}{\partial \hat{x}} + \cos f'\frac{\partial U}{\partial \hat{y}}\right) \quad \cdots ⑬$$

$$\frac{\partial U}{\partial \zeta} = \frac{\partial \hat{z}}{\partial \zeta}\cdot\frac{\partial U}{\partial \hat{z}}$$

$$= \frac{1}{r'}\frac{\partial U}{\partial \hat{z}} \quad \cdots ⑭$$

○④へ⑨~⑪と⑫~⑭を代入すると.

$$\ddot{r}'(\hat{x}\cos f' - \hat{y}\sin f') + \frac{h'^2}{r'^3}\left\{\left(\frac{d^2\hat{x}}{df'^2} - 2\frac{d\hat{y}}{df'} - \hat{x}\right)\cos f' - \left(\frac{d^2\hat{y}}{df'^2} + 2\frac{d\hat{x}}{df'} - \hat{y}\right)\sin f'\right\}$$
$$+ \frac{1}{r'}\left(\frac{\partial U}{\partial \hat{x}}\cos f' - \frac{\partial U}{\partial \hat{y}}\sin f'\right) = 0 \quad \cdots ⑮$$

$$\ddot{r}'(\hat{x}\sin f' + \hat{y}\cos f') + \frac{h'^2}{r'^3}\left\{\left(\frac{d^2\hat{x}}{df'^2} - 2\frac{d\hat{y}}{df'} - \hat{x}\right)\sin f' + \left(\frac{d^2\hat{y}}{df'^2} + 2\frac{d\hat{x}}{df'} - \hat{y}\right)\cos f'\right\}$$
$$+ \frac{1}{r'}\left(\frac{\partial U}{\partial \hat{x}}\sin f' + \frac{\partial U}{\partial \hat{y}}\cos f'\right) = 0 \quad \cdots ⑯$$

$$\ddot{r}'\hat{z} + \frac{h'^2}{r'^3}\frac{d^2\hat{z}}{df'^2} + \frac{1}{r'}\frac{\partial U}{\partial \hat{z}} = 0 \quad \cdots ⑰$$

- ⑮, ⑯, ⑰を整理する
  - ⑮ $\times \cos f'$ + ⑯ $\times \sin f'$

$$\ddot{r}'\tilde{X} + \frac{h'^2}{r'^3}\left(\frac{d^2\tilde{X}}{df'^2} - 2\frac{d\tilde{Y}}{df'} - \tilde{X}\right) + \frac{1}{r'}\frac{\partial U}{\partial \tilde{X}} = 0$$

$$\downarrow \times \frac{r'^3}{h'^2}$$

$$\frac{d^2\tilde{X}}{df'^2} - 2\frac{d\tilde{Y}}{df'} + \left(\frac{r'^3\ddot{r}'}{h'^2} - 1\right)\tilde{X} + \frac{r'^2}{h'^2}\frac{\partial U}{\partial \tilde{X}} = 0 \quad \cdots ⑱$$

  - ⑮ $\times (-\sin f')$ + ⑯ $\times \cos f'$

$$\ddot{r}'\tilde{Y} + \frac{h'^2}{r'^3}\left(\frac{d^2\tilde{Y}}{df'^2} + 2\frac{d\tilde{X}}{df'} - \tilde{Y}\right) + \frac{1}{r'}\frac{\partial U}{\partial \tilde{Y}} = 0$$

$$\downarrow \times \frac{r'^3}{h'^2}$$

$$\frac{d^2\tilde{Y}}{df'^2} + 2\frac{d\tilde{X}}{df'} + \left(\frac{r'^3\ddot{r}'}{h'^2} - 1\right)\tilde{Y} + \frac{r'^2}{h'^2}\frac{\partial U}{\partial \tilde{Y}} = 0 \quad \cdots ⑲$$

  - ⑰ $\times \frac{r'^3}{h'^2}$

$$\frac{d^2\tilde{Z}}{df'^2} + \frac{r'^3\ddot{r}'}{h'^2}\tilde{Z} + \frac{r'^2}{h'^2}\frac{\partial U}{\partial \tilde{Z}} = 0 \quad \cdots ⑳$$

- ここで、$\tilde{U}$ という $(\tilde{X}, \tilde{Y}, \tilde{Z})$ 系でのポテンシャルのような関数を導入して、⑱, ⑲, ⑳を以下のように書き換える

$$\frac{d^2\tilde{X}}{df'^2} - 2\frac{d\tilde{Y}}{df'} + \frac{\partial \tilde{U}}{\partial \tilde{X}} = 0 \quad \cdots (4.115)$$

$$\frac{d^2\tilde{Y}}{df'^2} + 2\frac{d\tilde{X}}{df'} + \frac{\partial \tilde{U}}{\partial \tilde{Y}} = 0 \quad \cdots (4.116)$$

$$\frac{d^2\tilde{Z}}{df'^2} + \frac{\partial \tilde{U}}{\partial \tilde{Z}} = 0 \quad \cdots (4.117)$$

・⑱、⑲、⑳と(4.115)、(4.116)、(4.117)を比較して$\tilde{U}$の具体的な形を求める

$$\tilde{U}=\frac{r'^2}{h'^2}U+\frac{1}{2}\left(\frac{r'^3\ddot{r}'}{h'^2}-1\right)\hat{X}^2+\frac{1}{2}\left(\frac{r'^3\ddot{r}'}{h'^2}-1\right)\tilde{Y}^2+\frac{1}{2}\frac{r'^3\ddot{r}'}{h'^2}\hat{Z}^2$$

rについての運動eq（P.30 問題2.3）より

$$\ddot{r}'-r'\dot{f}'^2=-\frac{1}{r'^2} \qquad \text{（規格化済み）}$$

$$\Rightarrow \ddot{r}'=\frac{h'^2}{r'^3}-\frac{1}{r'^2}$$

$$\rightarrow r'^3\ddot{r}'=h'^2-r'$$

これを代入

$$=\frac{1}{2}\left(\frac{h'^2-r'}{h'^2}-1\right)\hat{X}^2+\frac{1}{2}\left(\frac{h'^2-r'}{h'^2}-1\right)\tilde{Y}^2+\frac{1}{2}\frac{h'^2-r'}{h'^2}\tilde{Z}^2+\frac{r'^2}{h'^2}U$$

$$=\frac{1}{2}\left(-\frac{r'}{h'^2}\right)(\hat{X}^2+\hat{Y}^2)+\frac{1}{2}\left(1-\frac{r'}{h'^2}\right)\tilde{Z}^2+\frac{r'^2}{h'^2}U$$

(2.56)(2.69)より

$$\frac{r'}{h'^2}=\frac{1-e^2}{1+e'\cos f'}\cdot\frac{1}{1-e^2}=\frac{1}{1+e'\cos f'}$$

これを代入

$$=-\frac{1}{2}\frac{1}{1+e'\cos f'}(\hat{X}^2+\hat{Y}^2)+\frac{1}{2}\frac{e'\cos f'}{1+e'\cos f'}\tilde{Z}^2+\frac{r'}{1+e'\cos f'}U$$

$$=-\frac{1}{1+e'\cos f'}\left\{\frac{1}{2}(\hat{X}^2+\hat{Y}^2-e'\cos f'\hat{Z}^2)-r'U\right\} \quad \cdots (4.118)$$

楕円制限3体問題の平衡解は、

$$\frac{\partial \hat{U}}{\partial \hat{X}}=0,\ \frac{\partial \hat{U}}{\partial \hat{Y}}=0,\ \frac{\partial \hat{U}}{\partial \hat{Z}}=0 \quad \cdots (4.119)$$

より求められる。

これは、円制限3体問題の平衡解を求めるための条件式

$$\frac{\partial U^*}{\partial X}=0,\ \frac{\partial U^*}{\partial Y}=0,\ \frac{\partial U^*}{\partial Z}=0 \quad \cdots ㉑$$

と同じなので、円制限3体問題の平衡解を求める際に使った(4.35)〜(4.37)は楕円制限3体問題の平衡解を求めるときも、そのままの形で使える。

。ここで、(4.119)と㉑が同じであることを示しておく。

$$\frac{\partial U^*}{\partial X}=0 \quad より、$$

$$\Rightarrow \frac{\partial U}{\partial X}-X=0 \quad \cdots ㉒$$

$(\because U^*=U-\frac{1}{2}n'^2(X^2+Y^2)$ から $n'=1$ で規格化$)$

$$\frac{\partial \hat{U}}{\partial \hat{X}}=0 \quad より$$

$$\Rightarrow \frac{\partial \hat{U}}{\partial X}\cdot\frac{\partial X}{\partial \hat{X}}=0$$

$(\because X=r'\hat{X})$

$$\Rightarrow r'\cdot\frac{\partial \hat{U}}{\partial X}=0$$

$(\because 4.118)$

$$\Rightarrow -\frac{r'^2}{1+e'\cos f'}\left\{X-\frac{\partial U}{\partial X}\right\}=0$$

$$\Rightarrow X-\frac{\partial U}{\partial X}=0 \quad \cdots ㉓$$

㉒㉓より、$\frac{\partial U^*}{\partial X}=0$ と $\frac{\partial \hat{U}}{\partial \hat{X}}=0$ は同じであることがわかる

$\frac{\partial U^*}{\partial Y}=0$ と $\frac{\partial \hat{U}}{\partial \hat{Y}}=0$ が同じ式を導き出すことも同様にして求められる

$\frac{\partial \hat{U}}{\partial \hat{Z}}=0$ についてはふつうに計算すれば、

$$\frac{\partial \hat{U}}{\partial \hat{Z}}=\frac{\partial \hat{U}}{\partial Z}$$

$$=-\frac{1}{1+e'\cos f'}\left\{-e'\cos f'\cdot Z-r'\cdot\frac{\partial U}{\partial Z}\right\}$$

$$=-\frac{1}{1+e'\cos f'}\left\{-e'\cos f'\cdot Z-r'\left[\frac{a'^3n'^2}{m_1+m_2}\left(\frac{m_1}{r_1^3}+\frac{m_2}{r_2^3}\right)Z\right]\right\}$$

(∵ノート 4.4-② ⑦式の展開途中)

$$=\frac{1}{1+e'\cos f'}\left\{\left[e'\cos f'+\frac{a'^3n'^2}{m_1+m_2}\left(\frac{m_1}{r_1^3}+\frac{m_2}{r_2^3}\right)\right]Z\right\}=0$$

$$\Rightarrow Z=0$$

よって、$\frac{\partial U^*}{\partial Z}=0$ のときと同じ式を導きだしている。

以上より、楕円制限3体問題における平衡解を求める式は、円制限3体問題における平衡解を求める式(4.35)〜(4.36)と全く同じであることが示せた。

楕円制限3体問題の平衡点の安定性

平衡解は円制限3体問題のときと同じ（座標は変換してあるが）であったが、平衡点周りの微小運動の安定性については議論が異なる。

今回は例として L_4 周りの安定性について議論していく。

- まずは、L_4 からの微小変位を $\hat{x}, \hat{y}, \hat{z}$ とおくと、P_3 の位置は

  $$\hat{X} = \hat{X}_4 + \hat{x}, \quad \hat{Y} = \hat{Y}_4 + \hat{y}, \quad \hat{Z} = \hat{Z}_4 + \hat{z}$$

  と表すことができる。

- これらを楕円制限3体問題の運動方程式 (4.115)〜(4.118) へ代入すると、

$$\begin{cases} \dfrac{d^2\hat{x}}{df'^2} - 2\dfrac{d\hat{y}}{df'} + \dfrac{\partial \hat{U}}{\partial \hat{X}} = 0 & \cdots ㉔ \\ \dfrac{d^2\hat{y}}{df'^2} + 2\dfrac{d\hat{x}}{df'} + \dfrac{\partial \hat{U}}{\partial \hat{Y}} = 0 & \cdots ㉕ \\ \dfrac{d^2\hat{z}}{df'^2} + \dfrac{\partial \hat{U}}{\partial \hat{Z}} = 0 & \cdots ㉖ \end{cases}$$

- $\dfrac{\partial \hat{U}}{\partial \hat{X}}, \dfrac{\partial \hat{U}}{\partial \hat{Y}}, \dfrac{\partial \hat{U}}{\partial \hat{Z}}$ を L_4 周りでテイラー展開する

$$\frac{\partial \hat{U}}{\partial \hat{X}} = \underbrace{\left.\frac{\partial \hat{U}}{\partial \hat{X}}\right|_{L_4}}_{0} + \left.\frac{\partial}{\partial \hat{X}}\left(\frac{\partial \hat{U}}{\partial \hat{X}}\right)\right|_{L_4}\hat{x} + \left.\frac{\partial}{\partial \hat{Y}}\left(\frac{\partial \hat{U}}{\partial \hat{X}}\right)\right|_{L_4}\hat{y} + \left.\frac{\partial}{\partial \hat{Z}}\left(\frac{\partial \hat{U}}{\partial \hat{X}}\right)\right|_{L_4}\tilde{z}$$

$$= \left.\frac{\partial^2 \hat{U}}{\partial \hat{X}^2}\right|_{L_4}\tilde{x} + \left.\frac{\partial^2 \hat{U}}{\partial \hat{X}\partial \hat{Y}}\right|_{L_4}\tilde{y} + \left.\frac{\partial^2 \hat{U}}{\partial \hat{X}\partial \hat{Z}}\right|_{L_4}\tilde{z} \quad \cdots ㉗$$

同様にして、

$$\frac{\partial \hat{U}}{\partial \hat{Y}} = \left.\frac{\partial^2 \hat{U}}{\partial \hat{X} \partial \hat{Y}}\right|_{L4} \hat{x} + \left.\frac{\partial^2 \hat{U}}{\partial \hat{Y}^2}\right|_{L4} \hat{y} + \left.\frac{\partial^2 \hat{U}}{\partial \hat{Y} \partial \hat{Z}}\right|_{L4} \hat{z} \quad \cdots ㉘$$

$$\frac{\partial \hat{U}}{\partial \hat{Z}} = \left.\frac{\partial^2 \hat{U}}{\partial \hat{X} \partial \hat{Z}}\right|_{L4} \hat{x} + \left.\frac{\partial^2 \hat{U}}{\partial \hat{Y} \partial \hat{Z}}\right|_{L4} \hat{y} + \left.\frac{\partial^2 \hat{U}}{\partial \hat{Z}^2}\right|_{L4} \hat{z} \quad \cdots ㉙$$

㉗~㉙へ $\underbrace{\left.\frac{\partial^2 \hat{U}}{\partial \hat{X} \partial \hat{Z}}\right|_{L4}, \left.\frac{\partial^2 \hat{U}}{\partial \hat{Y} \partial \hat{Z}}\right|_{L4} = 0}_{}$ (*) を代入して整理すると、

$$\frac{\partial \hat{U}}{\partial \hat{X}} = \left.\frac{\partial^2 \hat{U}}{\partial \hat{X}^2}\right|_{L4} \hat{x} + \left.\frac{\partial^2 \hat{U}}{\partial \hat{X} \partial \hat{Y}}\right|_{L4} \hat{y} \quad \cdots ㉚$$

$$\frac{\partial \hat{U}}{\partial \hat{Y}} = \left.\frac{\partial^2 \hat{U}}{\partial \hat{X} \partial \hat{Y}}\right|_{L4} \hat{x} + \left.\frac{\partial^2 \hat{U}}{\partial \hat{Y}^2}\right|_{L4} \hat{y} \quad \cdots ㉛$$

$$\frac{\partial \hat{U}}{\partial \hat{Z}} = \left.\frac{\partial^2 \hat{U}}{\partial \hat{Z}^2}\right|_{L4} \hat{z} \quad \cdots ㉜$$

(*)部分が成り立つことを示しておく

$$\hat{U} = -\frac{1}{1+e'\cos f'}\left\{\frac{1}{2}\left(\hat{X}^2+\hat{Y}^2-e'\cos f'\cdot \hat{Z}^2\right)+\frac{1-\nu}{\hat{r}_1}+\frac{\nu}{\hat{r}_2}\right\} \cdots (4.118)$$

より、

$$\frac{\partial \hat{U}}{\partial \hat{X}} = -\frac{1}{1+e'\cos f'}\left\{\hat{X}-(1-\nu)\hat{r}_1^{-2}\cdot\frac{\partial \hat{r}_1}{\partial \hat{X}}-\nu\hat{r}_2^{-2}\cdot\frac{\partial \hat{r}_2}{\partial \hat{X}}\right\}$$

$$\frac{\partial \hat{U}}{\partial \hat{X}\partial \hat{Z}} = \frac{\partial}{\partial \hat{Z}}\left(\frac{\partial \hat{U}}{\partial \hat{X}}\right)$$

$$= -\frac{1}{1+e'\cos f'}\left\{2(1-\nu)\hat{r}_1^{-3}\frac{\partial \hat{r}_1}{\partial \hat{Z}}\cdot\frac{\partial \hat{r}_1}{\partial \hat{X}}-(1-\nu)\hat{r}_1^{-2}\frac{\partial \hat{r}_1}{\partial \hat{X}\partial \hat{Z}}\right.$$

$$\left.+2\nu\hat{r}_2^{-3}\frac{\partial \hat{r}_2}{\partial \hat{Z}}\cdot\frac{\partial \hat{r}_2}{\partial \hat{X}}-\nu\hat{r}_2^{-2}\frac{\partial r_2}{\partial \hat{X}\partial \hat{Z}}\right\}$$

以上より、

$$\left.\frac{\partial^2 \hat{U}}{\partial\hat{X}\partial\hat{Z}}\right|_{\substack{\hat{X}=\hat{X}_4=\frac{1}{2}-\nu \\ \hat{Y}=\hat{Y}_4=\frac{\sqrt{3}}{2} \\ \hat{Z}=\hat{Z}_4=0}} = 0$$

$$\because \left.\frac{\partial\hat{r}_1}{\partial\hat{Z}}\right|_{L4} = \left.\frac{\partial\hat{r}_2}{\partial\hat{Z}}\right|_{L4} = \left.\frac{\partial^2\hat{r}_1}{\partial\hat{X}\partial\hat{Z}}\right|_{L4} = \left.\frac{\partial^2\hat{r}_2}{\partial\hat{X}\partial\hat{Z}}\right|_{L4} = 0$$

$\hat{r}_1 = \hat{r}_1^{\frac{1}{2}}$ とおくと、

(∵ 4.18、4.19)

$$\frac{\partial\hat{r}_1}{\partial\hat{Z}} = \frac{1}{2}\hat{r}_1^{-\frac{1}{2}}\cdot 2\hat{Z} = \hat{r}_1^{-\frac{1}{2}}\cdot\hat{Z}$$ つまり、 $$\left.\frac{\partial\hat{r}_1}{\partial\hat{Z}}\right|_{\hat{Z}=0} = 0$$

$$\frac{\partial^2\hat{r}_1}{\partial\hat{X}\partial\hat{Z}} = \frac{\partial^2\hat{r}_1}{\partial\hat{Z}\partial\hat{X}} = \frac{\partial}{\partial\hat{X}}\frac{\partial\hat{r}_1}{\partial\hat{Z}}$$

$$= \hat{Z}\left(-\frac{1}{2}\hat{r}_1^{-\frac{3}{2}}\cdot 2(\hat{X}+\nu\hat{\alpha}')\right)$$ つまり、 $$\left.\frac{\partial^2\hat{r}_1}{\partial\hat{X}\partial\hat{Z}}\right|_{\hat{Z}=0} = 0$$

$\left.\frac{\partial^2\hat{U}}{\partial\hat{Y}\partial\hat{Z}}\right|_{L4}$ についても同様に計算すれば 0 となる

・次に、㉚~㉜の右辺各項の具体的な値を計算していく。

$\left.\frac{\partial^2\hat{U}}{\partial\hat{X}^2}\right|_{L4}$ について

$$\frac{\partial^2\hat{U}}{\partial\hat{X}^2} = \frac{\partial}{\partial\hat{X}}\left[-\frac{1}{1+e'\cos f'}\left\{\hat{X} - r'\frac{\partial U}{\partial\hat{X}}\right\}\right]$$

$$= -\frac{1}{1+e'\cos f'}\left(1 - r'\frac{\partial^2 U}{\partial\hat{X}^2}\right)$$

$$= -\frac{1}{1+e'\cos f'}\left(1 - r'^3\frac{\partial^2 U}{\partial X^2}\right) \quad \cdots ㉝$$

$\left.\dfrac{\partial^2 U}{\partial X^2}\right|_{L_4}$ について先に計算する

$$\frac{\partial^2 U}{\partial X^2} = -3(1-\nu)\frac{(X+a'\nu)^2}{r_1^5} - 3\nu\frac{\{X-a'(1-\nu)\}^2}{r_2^5} + \frac{1-\nu}{r_1^3} + \frac{\nu}{r_2^3} \quad (\because 1\text{-}1\text{-}4.6\text{-⑦参照})$$

L_4の値を代入する前に$\hat{X}_4, \hat{Y}_4, \hat{Z}_4$の値を慣性系$(X, Y, Z)$での値へ直す

$$\begin{cases} \hat{X}_4 = \frac{1}{2} - \nu \\ \hat{Y}_4 = \frac{\sqrt{3}}{2} \\ \hat{Z}_4 = 0 \\ \hat{r}_1 = 1 \\ \hat{r}_2 = 1 \\ \hat{a}' = 1 \end{cases} \quad \begin{pmatrix} X = r'\hat{X} \\ Y = r'\hat{Y} \text{ より} \\ \Rightarrow \end{pmatrix} \begin{cases} X_4 = r'\left(\frac{1}{2} - \nu\right) \\ Y_4 = r' \cdot \frac{\sqrt{3}}{2} \\ Z_4 = 0 \\ r_1 = r' \\ r_2 = r' \\ a' = r' \end{cases} \quad \cdots ㉞$$

$$\left.\frac{\partial^2 U}{\partial X^2}\right|_{\substack{X=X_4 \\ Y=Y_4 \\ Z=Z_4}} = -3(1-\nu)\frac{\left\{r'\left(\frac{1}{2}-\nu+\nu\right)\right\}^2}{r'^5} - 3\nu\frac{\left\{r'\left(\frac{1}{2}-\nu-1+\nu\right)\right\}^2}{r'^5} + \frac{1-\nu}{r'^3} + \frac{\nu}{r'^3}$$

$$= \frac{1}{4r'^3} \quad \cdots ㉟$$

この㉟を㉝へ戻すと、

$$\left.\frac{\partial^2 \hat{U}}{\partial \hat{X}^2}\right|_{L_4} = -\frac{1}{1+e'\cos f'} \cdot \frac{3}{4} \quad \cdots ㊱$$

$\left.\dfrac{\partial^2 \hat{U}}{\partial \hat{X}\partial \hat{Y}}\right|_{L4}$ について

$$\frac{\partial^2 \hat{U}}{\partial \hat{X}\partial \hat{Y}} = \frac{\partial}{\partial \hat{Y}}\left[-\frac{1}{1+e'\cos f'}\left\{\hat{X} - r'\frac{\partial U}{\partial \hat{X}}\right\}\right]$$

$$= -\frac{1}{1+e'\cos f'}\left\{-r'^2\frac{\partial}{\partial \hat{Y}}\left(\frac{\partial U}{\partial X}\right)\right\}$$

$$= \frac{r'^3}{1+e'\cos f'}\cdot\frac{\partial^2 U}{\partial X\partial Y} \quad \cdots ㊲$$

$$\frac{\partial^2 U}{\partial X\partial Y} = -3(1-\nu)\frac{(X+\nu a')}{r_1^5}Y - 3\nu\frac{\{X-(1-\nu)a'\}}{r_2^5}Y \quad (\because 1\text{-}1\text{-}4\text{-}6\text{-}⑰ 参照)$$

$$\left.\frac{\partial^2 U}{\partial X\partial Y}\right|_{\substack{X=X_4\\ Y=Y_4\\ Z=Z_4}} (\because ㉞) = -3(1-\nu)\frac{\{r'(\frac{1}{2}-\nu)+r'\nu\}\frac{\sqrt{3}}{2}r'}{r'^5} - 3\nu\frac{\{r'(\frac{1}{2}-\nu)-(1-\nu)r'\}\frac{\sqrt{3}}{2}r'}{r'^5}$$

$$= \frac{3\sqrt{3}}{4r'^3}(2\nu-1) \quad \cdots ㊳$$

㊳を㊲へ戻すと、

$$\left.\frac{\partial^2 \hat{U}}{\partial \hat{X}\partial \hat{Y}}\right|_{L4} = \frac{1}{1+e'\cos f'}\cdot\frac{3\sqrt{3}}{4}(2\nu-1) \quad \cdots ㊴$$

$\left.\dfrac{\partial^2 \hat{U}}{\partial \hat{Y}^2}\right|_{L4}$ について

($\hat{U}$は$\hat{X}$と$\hat{Y}$を入れ替えても同じ形になるので、$\left.\dfrac{\partial^2 \hat{U}}{\partial \hat{X}^2}\right|_{L4}$の計算を参考にすると簡単に計算ができる。)

$$\frac{\partial^2 \hat{U}}{\partial \hat{Y}^2} = -\frac{1}{1+e'\cos f'}\left(1 - r'^3\frac{\partial^2 U}{\partial Y^2}\right) \quad \cdots ㊵$$

$$\frac{\partial^2 U}{\partial Y^2} = -3(1-\nu)\frac{Y^2}{r_1^5} - 3\nu\frac{Y^2}{r_2^5} + \frac{1-\nu}{r_1^3} + \frac{\nu}{r_1^3} \quad (\because ノート 4.6-⑦)$$

$$\left.\frac{\partial^2 U}{\partial Y^2}\right|_{L4} \overset{(\because ㉞)}{=} -3(1-\nu)\cdot\frac{\frac{3}{4}r'^2}{r'^5} - 3\nu\frac{\frac{3}{4}r'^2}{r'^5} + \frac{1-\nu}{r'^3} + \frac{\nu}{r'^3}$$

$$= -\frac{5}{4r'^3} \quad \cdots ㊶$$

㊶を㊵へ代入

$$\left.\frac{\partial^2 \hat{U}}{\partial \hat{Y}^2}\right|_{L4} = -\frac{1}{1+e'\cos f'}\cdot\frac{9}{4} \quad \cdots ㊷$$

$\left.\dfrac{\partial^2 \hat{U}}{\partial \hat{Z}^2}\right|_{L4}$ について

$$\frac{\partial^2 \hat{U}}{\partial \hat{Z}^2} = \frac{\partial}{\partial \hat{Z}}\left[-\frac{1}{1+e'\cos f'}\left(-e'\cos f'\cdot\hat{Z} - r'\frac{\partial U}{\partial \hat{Z}}\right)\right]$$

$$= \frac{1}{1+e'\cos f'}\left(e'\cos f' + r'\frac{\partial^2 U}{\partial \hat{Z}^2}\right)$$

$$= \frac{1}{1+e'\cos f'}\left(e'\cos f' + r'^3\underset{\sim\sim}{\frac{\partial^2 U}{\partial Z^2}}\right)$$

$$\frac{\partial^2 U}{\partial Z^2} = \frac{1-\nu}{r_1^3} + \frac{\nu}{r_2^3} \quad (\because 4.85を参考にして)$$

$$\left.\frac{\partial^2 U}{\partial Z^2}\right|_{L4} \underset{㉞}{=} \frac{1}{r'^3}$$

$$\left.\frac{\partial^2 \hat{U}}{\partial \hat{Z}^2}\right|_{L4} = \frac{1}{1+e'\cos f'}\left(e'\cos f' + 1\right) = 1 \quad \cdots ㊸$$

・㉚.㉛.㉜へ㊱.㊴.㊷.㊸を代入する

$$\frac{\partial \tilde{U}}{\partial \tilde{x}} = -\frac{1}{1+e'\cos f'}\left\{\frac{3}{4}\tilde{x} + \frac{3\sqrt{3}}{4}(1-2\nu)\tilde{y}\right\} \quad \cdots ㊹$$

$$\frac{\partial \tilde{U}}{\partial \tilde{y}} = -\frac{1}{1+e'\cos f'}\left\{\frac{3\sqrt{3}}{4}(1-2\nu)\tilde{x} + \frac{9}{4}\tilde{y}\right\} \quad \cdots ㊺$$

$$\frac{\partial \tilde{U}}{\partial \tilde{z}} = \tilde{z} \quad \cdots ㊻$$

・この㊹.㊺.㊻を㉔.㉕.㉖へ代入する

$$\frac{d^2\tilde{x}}{df'^2} - 2\frac{d\tilde{y}}{df'} - \frac{1}{1+e'\cos f'}\left\{\frac{3}{4}\tilde{x} + \frac{3\sqrt{3}}{4}(1-2\nu)\tilde{y}\right\} = 0 \quad \cdots (4.120)$$

$$\frac{d^2\tilde{y}}{df'^2} + 2\frac{d\tilde{x}}{df'} - \frac{1}{1+e'\cos f'}\left\{\frac{3\sqrt{3}}{4}(1-2\nu)\tilde{x} + \frac{9}{4}\tilde{y}\right\} = 0 \quad \cdots (4.121)$$

$$\frac{d^2\tilde{z}}{df'^2} + \tilde{z} = 0 \quad \cdots (4.122)$$