問題2.1

$\mathbf{r}_1'$ と $\mathbf{r}_2'$ の関係は、

$$\begin{cases} m_1 r_1' = m_2 r_2' \\ \dfrac{\mathbf{r}_1'}{r_1'} = \dfrac{\mathbf{r}_2'}{r_2'} \end{cases} \text{より} \Rightarrow \mathbf{r}_2' = -\frac{m_1}{m_2}\mathbf{r}_1' \quad \cdots ①$$

となる

$$KE = \frac{1}{2}m_1\dot{\mathbf{r}}_1^2 + \frac{1}{2}m_2\dot{\mathbf{r}}_2^2$$

$$= \frac{1}{2}m_1(\dot{\mathbf{r}}_c + \dot{\mathbf{r}}_1')^2 + \frac{1}{2}m_2(\dot{\mathbf{r}}_c + \dot{\mathbf{r}}_2')^2$$

$$= \frac{1}{2}(m_1+m_2)\dot{\mathbf{r}}_c^2 (\underbrace{m_1\dot{\mathbf{r}}_c\dot{\mathbf{r}}_1' + m_2\dot{\mathbf{r}}_c\dot{\mathbf{r}}_2'}_{=0\ (\because ①)}) + \frac{1}{2}m_1\dot{\mathbf{r}}_1'^2 + \frac{1}{2}m_2\dot{\mathbf{r}}_2'^2$$

$$= \frac{1}{2}M\dot{\mathbf{r}}_c^2 + \frac{1}{2}\sum_{i=1}^{2} m_i\dot{\mathbf{r}}_i'^2 \quad \cdots (2.10)$$

$$\mathbf{h} = \mathbf{r}_1 \times m_1\dot{\mathbf{r}}_1 + \mathbf{r}_2 \times m_2\dot{\mathbf{r}}_2$$

$$= (\mathbf{r}_c + \mathbf{r}_1') \times m_1(\dot{\mathbf{r}}_c + \dot{\mathbf{r}}_1') + (\mathbf{r}_c + \mathbf{r}_2') \times m_2(\dot{\mathbf{r}}_c + \dot{\mathbf{r}}_2')$$

$$= m_1(\mathbf{r}_c \times \dot{\mathbf{r}}_c + \mathbf{r}_c \times \dot{\mathbf{r}}_1' + \mathbf{r}_1' \times \dot{\mathbf{r}}_c + \mathbf{r}_1' \times \dot{\mathbf{r}}_1')$$

$$+ m_2(\mathbf{r}_c \times \dot{\mathbf{r}}_c + \mathbf{r}_c \times \dot{\mathbf{r}}_2' + \mathbf{r}_2' \times \dot{\mathbf{r}}_c + \mathbf{r}_2' \times \dot{\mathbf{r}}_2')$$

$$= (m_1+m_2)(\mathbf{r}_c \times \dot{\mathbf{r}}_c) + \{\underbrace{m_1(\mathbf{r}_c \times \dot{\mathbf{r}}_1' + \mathbf{r}_1' \times \dot{\mathbf{r}}_c) + m_2(\mathbf{r}_c \times \dot{\mathbf{r}}_2' + \mathbf{r}_2' \times \dot{\mathbf{r}}_c)}_{=0\ (\because ①)}\} + m_1\mathbf{r}_1' \times \dot{\mathbf{r}}_1' + m_2\mathbf{r}_2' \times \dot{\mathbf{r}}_2'$$

$$= M\mathbf{r}_c \times \dot{\mathbf{r}}_c + \sum_{i=1}^{2} m_i\mathbf{r}_i' \times \dot{\mathbf{r}}_i' \quad \cdots (2.11)$$

(2.10), (2.11) の第2項を相対座標 $\mathbb{r}$ を用いて書き直す

$$\begin{cases} \mathbb{r}_1' = \mathbb{r}_1 - \mathbb{r}_c & \cdots ② \\ \mathbb{r}_2' = \mathbb{r}_2 - \mathbb{r}_c & \cdots ③ \\ \mathbb{r}_c = \dfrac{m_1\mathbb{r}_1 + m_2\mathbb{r}_2}{m_1 + m_2} & \cdots ④ \quad (2.3) \end{cases}$$

を用いる

$$\begin{aligned}
\frac{1}{2}\sum_{i=1}^{2} m_i \dot{\mathbb{r}}_i'^2 &= \frac{1}{2}m_1\dot{\mathbb{r}}_1'^2 + \frac{1}{2}m_2\dot{\mathbb{r}}_2'^2 \\
&= \frac{1}{2}m_1(\dot{\mathbb{r}}_1 - \dot{\mathbb{r}}_c)^2 + \frac{1}{2}m_2(\dot{\mathbb{r}}_2 - \dot{\mathbb{r}}_c)^2 \qquad (\because ②, ③) \\
&= \frac{1}{2}m_1\dot{\mathbb{r}}_1^2 + \frac{1}{2}m_2\dot{\mathbb{r}}_2^2 - m_1\dot{\mathbb{r}}_1\dot{\mathbb{r}}_c - m_2\dot{\mathbb{r}}_2\dot{\mathbb{r}}_c + \frac{1}{2}(m_1+m_2)\dot{\mathbb{r}}_c^2 \\
&= \frac{1}{2}m_1\dot{\mathbb{r}}_1^2 + \frac{1}{2}m_2\dot{\mathbb{r}}_2^2 - (m_1\dot{\mathbb{r}}_1 + m_2\dot{\mathbb{r}}_2)\cdot\frac{m_1\dot{\mathbb{r}}_1 + m_2\dot{\mathbb{r}}_2}{m_1+m_2} + \frac{1}{2}(m_1+m_2)\left(\frac{m_1\dot{\mathbb{r}}_1 + m_2\dot{\mathbb{r}}_2}{m_1+m_2}\right)^2 \\
&= \frac{1}{2}m_1\dot{\mathbb{r}}_1^2 + \frac{1}{2}m_2\dot{\mathbb{r}}_2^2 - \frac{(m_1\dot{\mathbb{r}}_1 + m_2\dot{\mathbb{r}}_2)^2}{m_1+m_2} + \frac{1}{2}\frac{(m_1\dot{\mathbb{r}}_1 + m_2\dot{\mathbb{r}}_2)^2}{m_1+m_2} \qquad (\because ④) \\
&= \frac{1}{2}\left(m_1\dot{\mathbb{r}}_1^2 + m_2\dot{\mathbb{r}}_2^2\right) - \frac{1}{2}\frac{(m_1\dot{\mathbb{r}}_1 + m_2\dot{\mathbb{r}}_2)^2}{m_1+m_2} \\
&= \frac{1}{2(m_1+m_2)}\left\{m_1^2\dot{\mathbb{r}}_1^2 + m_1m_2\dot{\mathbb{r}}_2^2 + m_1m_2\dot{\mathbb{r}}_1^2 + m_2^2\dot{\mathbb{r}}_2^2 - \left(m_1^2\dot{\mathbb{r}}_1^2 + m_2^2\dot{\mathbb{r}}_2^2 + 2m_1m_2\dot{\mathbb{r}}_1\dot{\mathbb{r}}_2\right)\right\} \\
&= \frac{m_1m_2}{2M}\left\{\dot{\mathbb{r}}_2^2 - 2\dot{\mathbb{r}}_1\dot{\mathbb{r}}_2 + \dot{\mathbb{r}}_1^2\right\} \\
&= \frac{m_1m_2}{2M}\left(\dot{\mathbb{r}}_2 - \dot{\mathbb{r}}_1\right)^2 \\
&= \frac{m_1m_2}{2M}\dot{\mathbb{r}}^2
\end{aligned}$$

$$\sum_{i=1}^{2} m_i \mathbf{r}_i' \times \dot{\mathbf{r}}_i'$$

$$= m_1 \mathbf{r}_1' \times \dot{\mathbf{r}}_1' + m_2 \mathbf{r}_2' \times \dot{\mathbf{r}}_2'$$

$$= m_1 (\mathbf{r}_1 - \mathbf{r}_c) \times (\dot{\mathbf{r}}_1 - \dot{\mathbf{r}}_c) + m_2 (\mathbf{r}_2 - \mathbf{r}_c) \times (\dot{\mathbf{r}}_2 - \dot{\mathbf{r}}_c) \qquad (\because ②, ③)$$

$$= m_1 \left(\mathbf{r}_1 - \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2}\right) \times \left(\dot{\mathbf{r}}_1 - \frac{m_1 \dot{\mathbf{r}}_1 + m_2 \dot{\mathbf{r}}_2}{m_1 + m_2}\right) + m_2 \left(\mathbf{r}_2 - \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2}\right) \times \left(\dot{\mathbf{r}}_2 - \frac{m_1 \dot{\mathbf{r}}_1 + m_2 \dot{\mathbf{r}}_2}{m_1 + m_2}\right) \quad (\because ④)$$

$$= m_1 \left(\frac{m_2 \mathbf{r}_1 - m_2 \mathbf{r}_2}{m_1 + m_2}\right) \times \left(\frac{m_2 \dot{\mathbf{r}}_1 - m_2 \dot{\mathbf{r}}_2}{m_1 + m_2}\right) + m_2 \left(\frac{m_1 \mathbf{r}_2 - m_1 \mathbf{r}_1}{m_1 + m_2}\right) \times \left(\frac{m_1 \dot{\mathbf{r}}_2 - m_1 \dot{\mathbf{r}}_1}{m_1 + m_2}\right)$$

$$= \frac{m_1 m_2^2}{(m_1 + m_2)^2} (\mathbf{r}_1 - \mathbf{r}_2) \times (\dot{\mathbf{r}}_1 - \dot{\mathbf{r}}_2) + \frac{m_2 m_1^2}{(m_1 + m_2)^2} (\mathbf{r}_2 - \mathbf{r}_1) \times (\dot{\mathbf{r}}_2 - \dot{\mathbf{r}}_1)$$

$$= \frac{m_1 m_2 (m_1 + m_2)}{(m_1 + m_2)^2} (\mathbf{r}_2 - \mathbf{r}_1) \times (\dot{\mathbf{r}}_2 - \dot{\mathbf{r}}_1)$$

$$= \frac{m_1 m_2}{M} \mathbf{r} \times \dot{\mathbf{r}} \qquad \cdots \quad (2.13)$$