2.6 ケプラー運動の平均値

ケプラー運動をしている天体の力学的量Aの時間平均

$$\langle A \rangle = \frac{1}{P}\int_0^P A\,dt \quad \cdots (2.210)$$

積分変数を t から l(平均近点離角)に変換する ($n\,dt = dl$ である)

$$\langle A \rangle = \frac{1}{2\pi}\int_0^{2\pi} A\,dl \quad \cdots (2.211)$$

当然 (2.210) と (2.211) は等しい

以下に、いくつかの力学量~~変~~についての平均を実際に求める

[例 2.6a] 動径 r の平均値

$$\left\langle \frac{r}{a} \right\rangle = \frac{1}{2\pi}\int_0^{2\pi} \frac{r}{a}\,dl$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{r}{a}\right)^2 du$$

$l = u - e\sin u \; (= n(t - t_0))$ 上で微分

$$\frac{dl}{dt} = \frac{du}{dt} - e\cos u\frac{du}{dt}$$

$$dl = \frac{r}{a}\,du \quad (\because 2.59)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} (1 - e\cos u)^2\,du \quad (\because 2.59)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \left\{1 - 2e\cos u + \frac{1}{2}e^2(1 + \cos 2u)\right\} du$$

$$= 1 + \frac{1}{2}e^2 \quad \cdots (2.212)$$

[例2.6b]

$$\left\langle \frac{a}{r} \right\rangle = \frac{1}{2\pi}\int_0^{2\pi} \frac{a}{r}\, dl = \frac{1}{2\pi}\int_0^{2\pi} \frac{a}{r}\cdot\frac{r}{a}\, du = \frac{1}{2\pi}\int_0^{2\pi} 1\, du = 1 \quad \cdots (2.213)$$

[例2.6c]

$$\left\langle \left(\frac{a}{r}\right)^3 \right\rangle = \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{a}{r}\right)^3 dl$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{a}{r}\right)^3 \cdot \frac{r^2}{a^2 n}\, df$$

$\left(\because \ \dfrac{df}{dt} = \dfrac{a^2 n \eta}{r^2} \cdots (2.74),\ \dfrac{dl}{dt} = n \cdots (\because\ l\text{の定義より}) \right.$

この2つを使うと、

$\dfrac{df}{dl} = \dfrac{df}{dt}\cdot\dfrac{dt}{dl} = \dfrac{a^2\eta}{r^2}$

$\left. \Rightarrow dl = \dfrac{r^2}{a^2\eta}\, df \right)$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{a}{r}\cdot\frac{1}{\eta}\, df$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{a}{\eta}\cdot\frac{1+e\cos f}{a\eta^2}\, df \quad (\because 2.56)$$

$$= \frac{1}{2\pi\eta^3}\int_0^{2\pi} (1+e\cos f)\, df$$

$$= \frac{1}{2\pi\eta^3}\Big[f + e\sin f \Big]_0^{2\pi}$$

$$= \frac{1}{\eta^3} \quad \cdots (2.214)$$

[例2.6d]

$$\begin{cases} \dfrac{1}{2}\dot{r}^2 - \dfrac{\mu}{r} = E & \cdots (2.16):\text{エネルギー積分} \\ E = -\dfrac{\mu}{2a} & \cdots (2.30) \end{cases}$$

この2式より、

$$v^2 = \frac{2\mu}{r} - \frac{\mu}{a}$$

と表すことができ、平均をとるときに変数となるのはrだけなので、

$$v^2 = \frac{2\mu}{a}\underbrace{\left\langle \frac{a}{r} \right\rangle}_{=1\ (\because 2.213)} - \frac{\mu}{a} = \frac{\mu}{a} \underset{(\because 2.35)}{=} (na)^2 \quad \cdots (2.215)$$

[例2.6e]

$$\langle v \rangle = \frac{1}{2\pi}\int_0^{2\pi} v\, dl$$

$$= \frac{1}{2\pi}\left[\int_0^{\pi} v\, dl + \int_{\pi}^{2\pi} v\, dl\right]$$

$$= \frac{1}{2\pi}\cdot 2\int_0^{\pi} v\, dl$$

$$= \frac{1}{\pi}\int_0^{\pi} v\, dl$$

$$= \frac{1}{\pi}\int_0^{\pi} \sqrt{v^2}\, dl$$

$$= \frac{1}{\pi}\int_0^{\pi} \sqrt{\frac{2\mu}{r} - \frac{\mu}{a}}\, dl$$ (∵エネルギー積分(2.16)と(2.30)より)

$$= \frac{1}{\pi}\int_0^{\pi} \sqrt{n^2 a^3\left(\frac{2}{r} - \frac{1}{a}\right)}\, dl$$ (∵2.35)

$$= \frac{na}{\pi}\int_0^{\pi} \sqrt{\frac{2a}{r} - 1}\cdot\frac{r}{a}\, du$$ (∵ [例2.6a]より $\frac{du}{dl} = \frac{a}{r}$)

$$= \frac{na}{\pi}\int_0^{\pi} \sqrt{2(1 - e\cos u) - (1 - e\cos u)^2}\cdot du$$ (∵(2.59)より $\frac{r}{a} = 1 - e\cos u$)

$$= \frac{na}{\pi}\int_0^{\pi} \sqrt{1 - e^2\cos^2 u}\, du$$

$\downarrow$ 第2種完全楕円積分 … 求め方よくわからない

$$= \frac{2na}{\pi}E(e)$$

$$= na\left\{1 - \frac{1}{4}e^2 - \frac{3}{64}e^4 + O(e^6)\right\} \cdots (2.216)$$

[例2.6f]

$$\left\langle \left(\frac{r}{a}\right)^2 \cos f \right\rangle = \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{r}{a}\right)^2 \cos f \, dl$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{r}{a}\right)^2 \cos f \cdot \frac{r}{a}\, du \qquad \left(\because \frac{du}{dl} = \frac{a}{r}\right)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \underset{(\because 2.59)}{(1-e\cos u)^3} \cdot \underset{(\because 2.61)}{\frac{\cos u - e}{1-e\cos u}}\, du$$

$$= \frac{1}{2\pi}\int_0^{2\pi} (1-e\cos u)^2(\cos u - e)\, du$$

$$= \frac{1}{2\pi}\int_0^{2\pi} (\cos u - e - 2e\cos^2 u + 2e^2\cos u + e^2\cos^3 u - e^3\cos^2 u)\, du$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \left\{-e + (2e^2+1)\cos u - (e^3+2e)\cos^2 u + e^2\cos^3 u\right\} du$$

$$\cos^2 u = \frac{1}{2}(\cos 2u + 1) \qquad \cos^3 u = \frac{1}{4}(\cos 3u + 3\cos u)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \left\{\left(-2e - \frac{1}{2}e^3\right) + \underbrace{\left(\frac{11}{4}e^2+1\right)\cos u - \left(\frac{1}{2}e^3+e\right)\cos 2u + \frac{3}{4}e^2\cos 3u}_{\int_0^{2\pi}\text{で積分したら全部}0}\right\} du$$

$$= -e\left(2 + \frac{e^2}{2}\right) \qquad \cdots (2.217)$$

[例2.6g]

$$\left\langle \left(\frac{a}{r}\right)^3 \cos f \right\rangle = \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{a}{r}\right)^3 \cos f \, dl$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{a}{r}\right)^3 \cos f \cdot \frac{r^2}{a^2\eta}\, df \qquad (\because 例2.6c)$$

$$= \frac{1}{2\pi\eta}\int_0^{2\pi} \frac{a}{r} \cos f \, df$$

$$= \frac{1}{2\pi\eta}\int_0^{2\pi} \underset{(\because 2.56)}{\frac{1+e\cos f}{\eta^2}} \cos f \, df$$

$$= \frac{1}{2\pi\eta^3}\int_0^{2\pi} (1+e\cos f)\cos f \, df$$

$$\vdots$$

$$= \frac{1}{2\pi\eta^3}\int_0^{2\pi} \cos f + e\underbrace{\cos^2 f}_{\left(\cos^2 f = \frac{1}{2}(\cos 2f + 1)\right)}\, df$$

$$= \frac{1}{2\pi\eta^3}\int_0^{2\pi}\left(\frac{e}{2} + \underbrace{\cos f + \frac{e}{2}\cos 2f}_{=0}\right) df$$

$$= \frac{1}{2\pi\eta^3}\cdot\frac{e}{2}\cdot 2\pi$$

$$= \frac{e}{2\eta^3} \quad \cdots (2.218)$$

[例 2.6h]

時間がかかりそうだから後でやる