2.4 2体問題の諸公式

2.4.1 楕円運動

(d) 時間微分を含んだ関係式

$$(1-e\cos u)\frac{du}{dt}=n \quad \cdots ⓐ$$

この式を変形する

$$\frac{du}{dt}=\frac{n}{1-e\cos u}$$

$$=\frac{an}{r} \quad \left(\because \begin{smallmatrix}1.13\\ \text{又は}\\ 2.59\end{smallmatrix} \quad r=a(1-e\cos u) \Rightarrow 1-e\cos u=\frac{r}{a}\right)$$

$$\cdots(2.72)$$

r と f の時間変化の式 (1.18) と (1.19) へ (2.72) を代入する

(1.18) - $$\frac{dr}{dt}=ae\sin u\frac{du}{dt}$$

$$=ae\sin u\cdot\frac{an}{r}$$

$$=ae\cdot\frac{r\sin f}{a\sqrt{1-e^2}}\cdot\frac{an}{r} \quad (\because 1.12)$$

$$=\frac{aen\sin f}{\sqrt{1-e^2}}$$

$$=\frac{aen}{\eta}\sin f \quad \cdots(2.73) \qquad (\because \eta=\sqrt{1-e^2})$$

(1.19) - $$\frac{df}{dt}=\frac{a\sqrt{1-e^2}}{r}\frac{du}{dt}$$

$$=\frac{a^2n\eta}{r^2} \quad (\because 2.72,\ \eta=\sqrt{1-e^2})$$

$$\cdots(2.74)$$

次に(x^*, y^*)系での速度を求める

$$\frac{dx^*}{dt} = \frac{d}{dt}\left\{a(\cos u - e)\right\} \quad (\because 2.57)$$
$$= -a\sin u \cdot \frac{du}{dt}$$
$$= -a\sin u \cdot \frac{an}{r} \quad (\because 2.72)$$
$$= -\frac{a^2 n \sin u}{r} \quad \cdots (2.75)$$

$$\frac{dy^*}{dt} = \frac{d}{dt}\left\{a\sqrt{1-e^2}\sin u\right\} \quad (\because 2.58)$$
$$= a\sqrt{1-e^2}\cos u \cdot \frac{du}{dt}$$
$$= \frac{a^2 n \eta \cos u}{r} \quad \cdots (2.76)$$

(2.75), (2.76)を真近点離角fを用いて表現すると、

$$\frac{dx^*}{dt} = -\frac{a^2 n}{r} \cdot \frac{r\sin f}{a\sqrt{1-e^2}} \quad (\because 1.12)$$
$$= -\frac{an}{\eta}\sin f \quad \cdots (2.77)$$

$$\frac{dy^*}{dt} = a^2 n \eta \cdot \underbrace{\frac{1+e\cos f}{a\eta^2}}_{(\because 1.6)} \cdot \underbrace{\frac{\cos f + e}{1+e\cos f}}_{(\because 2.61)}$$
$$= \frac{an}{\eta}(\cos f + e) \quad \cdots (2.78)$$

$$v = \sqrt{\left(\frac{dx^*}{dt}\right)^2 + \left(\frac{dy^*}{dt}\right)^2}$$

$$= \sqrt{\left(\frac{an}{\eta}\right)^2 \sin^2 f + \left(\frac{an}{\eta}\right)^2 \left(\cos^2 f + 2e\cos f + e^2\right)} \qquad (\because 2.77,\ 2.78)$$

$$= \frac{an}{\eta}\sqrt{1 + 2e\cos f + e^2} \quad \cdots (2.79)$$

$$= \frac{an}{\eta}\sqrt{1 + 2e\cdot \underbrace{\frac{\cos u - e}{1 - e\cos u}}_{(\because 2.63)} + e^2}$$

$$= \frac{an}{\eta}\sqrt{\frac{(1+e^2)(1-e\cos u) + 2e(\cos u - e)}{1 - e\cos u}}$$

$$= \frac{an}{\sqrt{1-e^2}}\sqrt{\frac{(1-e^2) + (1-e^2)e\cos u}{1 - e\cos u}}$$

$$= an\sqrt{\frac{1 + e\cos u}{1 - e\cos u}} \quad \cdots (2.80)$$

$$= an\sqrt{1 + e\cos u}\cdot\sqrt{\frac{a}{r}} \qquad (\because 2.59)$$

$$= \sqrt{\frac{\mu}{r}}\sqrt{1 + e\cos u} \quad \cdots (2.81) \qquad (\because 2.66)$$

~~(2.81)~~ (2.79)~(2.81) はエネルギー積分 ~~(2.79)~~ (2.82) からも導きだせる

$$\frac{1}{2}v^2 - \frac{\mu}{r} = -\frac{\mu}{2a}$$

$$v = \sqrt{\frac{2\mu}{r} - \frac{\mu}{a}}$$

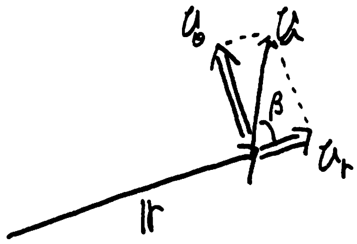
$$= \sqrt{\frac{2\mu}{r} - \frac{\mu(1 - e\cos u)}{r}}$$

$$= \sqrt{\frac{\mu}{r}}\sqrt{1 + e\cos u} \quad \cdots (2.81)$$

速さの極座標表示は、

$$v_r = \frac{dr}{dt} \underset{(\because 2.73)}{=} \frac{aen}{\eta}\sin f \quad \cdots (2.83)$$

$$v_\theta = r\dot{\theta} = r\dot{f} \underset{(\because 2.74)}{=} r\cdot\frac{a^2 n\eta}{r^2} \underset{(\because 2.56)}{=} a^2 n\eta\cdot\frac{1+e\cos f}{a\eta^2} = \frac{an}{\eta}(1+e\cos f) \quad \cdots (2.84)$$



$$\tan\beta = \frac{v_\theta}{v_r} \underset{(\because 2.83,\ 2.84)}{=} \frac{an(1+e\cos f)}{\eta}\cdot\frac{\eta}{aen\sin f} = \frac{1+e\cos f}{e\sin f} \quad \cdots (2.86)$$

また、$\cos\beta$, $\sin\beta$についてもだしておくと、

$$\begin{cases} \cos^2\beta + \sin^2\beta = 1 & \cdots ① \\ \dfrac{\sin\beta}{\cos\beta} = \dfrac{1+e\cos f}{e\sin f} & \cdots (2.86) \quad \cdots ② \end{cases}$$

①、②を用いて、

$$\cos\beta = \frac{e\sin f}{\sqrt{1+2e\cos f+e^2}} \quad \cdots (2.87)$$

$$\sin\beta = \frac{1+e\cos f}{\sqrt{1+2e\cos f+e^2}} \quad \cdots (2.88)$$

(2.86) で $\tan\beta$ を f の関数として表したが、これを u の関数に変形する

$$\tan\beta = \frac{1 + e\cdot\dfrac{\cos u - e}{1 - e\cos u}}{e\cdot\dfrac{\sqrt{1-e^2}\sin u}{1 - e\cos u}} = \frac{\sqrt{1-e^2}}{e\sin u} \quad \cdots (2.89)$$

$\vdots$
$(\because 2.61,\ 2.62)$

角運動量 h と β の関係は、

$$h = r^2\dot{\theta} = r^2\dot{f}$$

$$= r v_\theta$$

$$= r v \sin\beta \quad (\because 2.85) \quad \cdots (2.90)$$

e) r, f の u, a, e についての偏微分

極座標 r, f と離心近点離角の関係

$$\begin{cases} r\cos f = a(\cos u - e) & \cdots (2.91) \\ r\sin f = a\eta\sin u & \cdots (2.92) \end{cases} \quad (\because 1.11,\ 1.12)$$

この2式を u について偏微分する

$$\begin{cases} \dfrac{\partial r}{\partial u}\cos f - r\sin f\dfrac{\partial f}{\partial u} = -a\sin u & \cdots (2.93) \\ \dfrac{\partial r}{\partial u}\sin f + r\cos f\dfrac{\partial f}{\partial u} = a\eta\cos u & \cdots (2.94) \end{cases}$$

(2.93)と(2.94)から $\frac{\partial r}{\partial u}, \frac{\partial f}{\partial u}$ について解く

・(2.93)×cos f + (2.94)×sin f

$$\frac{\partial r}{\partial u}\cos^2 f - r\sin f\cos f\frac{\partial f}{\partial u} = -a\sin u\cos f$$

$$+)\ \frac{\partial r}{\partial u}\sin^2 f + r\sin f\cos f\frac{\partial f}{\partial u} = a\eta\cos u\sin f$$

$$\frac{\partial r}{\partial u}(\sin^2 f + \cos^2 f) = -a\sin u\cos f + a\eta\cos u\sin f$$

$$\frac{\partial r}{\partial u} = a\left\{-\cos f\underbrace{\left(\frac{\eta\sin f}{1+e\cos f}\right)}_{(\because 2.62)} + \eta\sin f\underbrace{\left(\frac{\cos f+e}{1+e\cos f}\right)}_{(\because 2.61)}\right\}$$

$$\frac{\partial r}{\partial u} = \frac{ae\eta\sin f}{1+e\cos f}$$

$$\frac{\partial r}{\partial u} = \frac{r}{a\eta^2}\cdot ae\eta\sin f \quad (\because 2.56)$$

$$\frac{\partial r}{\partial u} = \frac{e}{\eta}r\sin f \quad \cdots (2.95)$$

・(2.93)×(−sin f) + (2.94)×cos f

$$-\frac{\partial r}{\partial u}\sin f\cos f + r\sin^2 f\frac{\partial f}{\partial u} = a\sin u\sin f$$

$$+)\ \frac{\partial r}{\partial u}\sin f\cos f + r\cos^2 f\frac{\partial f}{\partial u} = a\eta\cos u\cos f$$

$$r\frac{\partial f}{\partial u}(\sin^2 f + \cos^2 f) = a(\sin u\sin f + \eta\cos u\cos f)$$

$$= a\left\{\underbrace{\left(\frac{\eta\sin f}{1+e\cos f}\right)}_{(\because 2.62)}\sin f + \eta\underbrace{\left(\frac{\cos f+e}{1+e\cos f}\right)}_{(\because 2.61)}\cos f\right\}$$

$$= \frac{a}{1+e\cos f}(\eta\sin^2 f + \eta\cos^2 f + \eta e\cos f)$$

$$= \frac{a\eta}{1+e\cos f}(1+e\cos f)$$

$$\frac{\partial f}{\partial u} = \frac{a\eta}{r} \quad \cdots (2.96)$$

r, fをuの関数として陽に表現した式(1.13), (1.17)からも
(2.95), (2.96)は求められる

$$\begin{cases} r = a(1 - e\cos u) & \cdots (1.13) \\ \tan\frac{f}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{u}{2} & \cdots (1.17) \end{cases}$$

$$\frac{\partial r}{\partial u} = ae\sin u$$

$$= ae\frac{\eta\sin f}{1+e\cos f} \quad (\because 2.62)$$

$$= \frac{e}{\eta}r\sin f \quad (\because 2.56)$$

r, f ~~[struck]~~ の a についての偏微分（独立変数 a, e, t）

$$\frac{\partial r}{\partial a} = \frac{\partial}{\partial a}\left\{a(1-e\cos u)\right\} \qquad (\because 2.59)$$

$$= (1-e\cos u) + ae\sin u\frac{\partial u}{\partial a}$$

$$= \underbrace{\frac{r}{a}}_{(\because 2.59)} + ae\cdot\underbrace{\frac{\eta\sin f}{1+e\cos f}}_{(\because 2.62)}\frac{\partial u}{\partial a}$$

$$= \frac{r}{a} + \frac{r\eta e}{\eta^2}\sin f\frac{\partial u}{\partial a} \qquad (\because 2.56)$$

$$= \frac{r}{a} + \frac{e}{\eta}r\sin f\frac{\partial u}{\partial a} \qquad \cdots (2.97)$$

$\frac{\partial r}{\partial a} = \frac{\partial r}{\partial u}\cdot\frac{\partial u}{\partial a}$ は×

$\frac{\partial r}{\partial u}$ は(2.95)、$\frac{\partial u}{\partial a}$ は(2.99)で求めているが、共に偏微分したときの条件が違う。（どの文字を定数としたか）よって上の式にはあてはめられない。

$\frac{\partial f}{\partial a}$ について求めるため、まずは(2.91)と(2.92)の両辺を a で偏微分する
独立変数は a, e, t とする

$$\begin{cases}\frac{\partial r}{\partial a}\cos f - r\sin f\frac{\partial f}{\partial a} = \cos u - e - a\sin u\frac{\partial u}{\partial a} & \cdots ① \\ \frac{\partial r}{\partial a}\sin f + r\cos f\frac{\partial f}{\partial a} = \eta\sin u + a\eta\cos u\frac{\partial u}{\partial a} & \cdots ②\end{cases}$$

①×(−sin f) + ②×cos f

$$-\frac{\partial r}{\partial a}\sin f\cos f + r\sin^2 f\frac{\partial f}{\partial a} = -\sin f\cos u + e\sin f + a\sin f\sin u\frac{\partial u}{\partial a}$$

$$+)\ \frac{\partial r}{\partial a}\sin f\cos f + r\cos^2 f\frac{\partial f}{\partial a} = \eta\sin u\cos f + a\eta\cos u\cos f\frac{\partial u}{\partial a}$$

$$r\frac{\partial f}{\partial a} = \eta\sin u\cos f - \cos u\sin f + e\sin f + a\frac{\partial u}{\partial a}(\sin f\sin u + \eta\cos f\cos u)$$

⋮

⋮

$$r\frac{\partial f}{\partial a} = \eta\frac{\eta\sin f\cos f}{1+e\cos f} - \frac{\sin f\cos f + e\sin f}{1+e\cos f} + \frac{e\sin f + e^2\sin f\cos f}{1+e\cos f}$$

$$+ a\frac{\partial u}{\partial a}\left(\frac{\eta\sin^2 f}{1+e\cos f} + \frac{\eta\cos^2 f + e\eta\cos f}{1+e\cos f}\right) \quad (\because 2.61, 2.62)$$

$$= \frac{1}{1+e\cos f}\left\{\sin f\cos f(\eta^2-1+e^2)\right\} + a\frac{\partial u}{\partial a}\cdot\frac{1}{1+e\cos f}\left\{\eta + e\eta\cos f\right\}$$

$$= a\eta\frac{\partial u}{\partial a}$$

$$\frac{\partial f}{\partial a} = \frac{a\eta}{r}\frac{\partial u}{\partial a} \quad \cdots (2.98)$$

ケプラーの方程式の両辺をaについて偏微分すると、
$(u - e\sin u = n(t-t_0))$ ($\because$ 2.65 など)

$$\frac{\partial u}{\partial a} - e\cos u\frac{\partial u}{\partial a} = \frac{\partial n}{\partial a}(t-t_0)$$

$$\frac{\partial u}{\partial a} = \frac{1}{1-e\cos u}\cdot\frac{\partial n}{\partial a}(t-t_0)$$

$$= \frac{a}{r}\frac{\partial n}{\partial a}(t-t_0) \quad \cdots (2.99)$$

教科書は $t_0=0$ のときを計算している

~~$\frac{\partial u}{\partial a} = \frac{\partial n}{\partial a}$~~

でも普通はt_0=0で考えるよね。
だから教科書通りの式でいいと思う。

ケプラーの第3法則 ($n^2a^3=\mu=G(m_1+m_2)$) -(2.66)
をaで偏微分すると、

$$2n\cdot\frac{\partial n}{\partial a}\cdot a^3+n^2\cdot 3a^2=0$$

$$\frac{\partial n}{\partial a}=-\frac{3n}{2a}\quad\cdots(2.100)$$

これを用いて、(2.100)を(2.99)に代入して、

$$\frac{\partial u}{\partial a}=\frac{a}{r}\left(-\frac{3n}{2a}\right)(t-t_0)$$

$$=-\frac{3}{2r}n(t-t_0)$$

$$=-\frac{3}{2}\frac{l}{r}\quad(\because 1.31)\quad\cdots(2.101)$$

同様に、ケプラー方程式の両辺を次はeで偏微分する

$$\frac{\partial u}{\partial e}-\sin u-e\cos u\frac{\partial u}{\partial e}=0$$

← (∵ケプラーの第3法則より、nはeに依存しないことがわかっている)

$$(1-e\cos u)\frac{\partial u}{\partial e}=\sin u$$

$$\frac{\partial u}{\partial e}=\frac{\sin u}{1-e\cos u}$$

$$=\frac{a}{r}\cdot\frac{\eta\sin f}{1+e\cos f}\quad(\because 2.59,\ 2.62)$$

$$=\frac{a}{r}\cdot\frac{r}{a\eta^2}\eta\sin f\quad(\because 2.56)$$

$$=\frac{\sin f}{\eta}\quad\cdots(2.102)$$

$r=a(1-e\cos u)$ … (2.59) について e で偏微分する

$$\frac{\partial r}{\partial e}=a\left(-\cos u+e\sin u\frac{\partial u}{\partial e}\right)$$

$$=a\Bigg(-\underbrace{\frac{\cos f+e}{1+e\cos f}}_{(\because 2.61)}+e\cdot\underbrace{\frac{\eta\sin f}{1+e\cos f}}_{(\because 2.62)}\cdot\underbrace{\frac{\sin f}{\eta}}_{(\because 2.102)}\Bigg)$$

$$=\frac{a}{1+e\cos f}\left(-\cos f-e+e-e\cos^2 f\right)$$

$$=\frac{a}{1+e\cos f}(1+e\cos f)(-\cos f)$$

$$=-a\cos f \quad \cdots (2.103)$$

$\frac{\partial f}{\partial e}$ を求めるため、(2.91) を e で偏微分する ((2.92) でも同じことはできるはず)

だってないけど

$$\frac{\partial r}{\partial e}\cos f-r\sin f\frac{\partial f}{\partial e}=a\left(-\sin u\cdot\frac{\partial u}{\partial e}-1\right)$$

$$\underbrace{-a\cos f}_{(\because 2.103)}\cdot\cos f-r\sin f\cdot\frac{\partial f}{\partial e}=a\Bigg(-\underbrace{\frac{\eta\sin f}{1+e\cos f}}_{(\because 2.62)}\cdot\underbrace{\frac{\sin f}{\eta}}_{(\because 2.102)}-1\Bigg)$$

$$-a\cos^2 f-r\sin f\frac{\partial f}{\partial e}=a\left(-\frac{\sin^2 f}{1+e\cos f}-1\right)$$

$$r\sin f\frac{\partial f}{\partial e}=a\left(\frac{\sin^2 f}{1+e\cos f}+1-\cos^2 f\right)$$

$$\frac{\partial f}{\partial e}=\frac{a}{r}\left(\frac{\sin f}{1+e\cos f}+\sin f\right)$$

$$=\frac{a\sin f}{r}\Bigg(\underbrace{\frac{r}{a\eta^2}}_{(\because 2.56)}+1\Bigg)$$

$$=\left(\frac{a}{r}+\frac{1}{\eta^2}\right)\sin f \quad \cdots (2.104)$$