A ラグランジュ括弧式の評価

A.1 ラグランジュ括弧式の初等的評価

◦ 慣性座標系における質点の座標 x, y, z と
軌道面を基準面,近点方向を新たなx軸とする座標 $x^*, y^*, (z^*=0)$
との関係は以下の様に書ける。(2.8.2項参照)

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A \begin{pmatrix} x^* \\ y^* \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = A \begin{pmatrix} \dot{x}^* \\ \dot{y}^* \\ 0 \end{pmatrix} \quad \cdots (A.1)$$

$$A = R_3(-\Omega) R_1(-I) R_3(-\omega) = (A_{ij}) \quad \cdots (A.2)$$

この関係を

$$\mathbb{r} = A\mathbb{r}^*, \quad \dot{\mathbb{r}} = A\dot{\mathbb{r}}^* \quad \cdots (A.3)$$

と書く。

◦ ラグランジュ括弧式を展開していく

$$[c_l, c_j] = \sum_{i=1}^{3} \frac{\partial(x_i, \dot{x}_i)}{\partial(c_l, c_j)}$$

$$= \sum_{i=1}^{3} \left\{ \left(\frac{\partial x_i}{\partial c_l}, \frac{\partial \dot{x}_i}{\partial c_j} \right) - \left(\frac{\partial x_i}{\partial c_j}, \frac{\partial \dot{x}_i}{\partial c_j} \right) \right\}$$

$$= \left(\frac{\partial \mathbb{r}}{\partial c_l}, \frac{\partial \dot{\mathbb{r}}}{\partial c_j} \right) - \left(\frac{\partial \mathbb{r}}{\partial c_j}, \frac{\partial \dot{\mathbb{r}}}{\partial c_l} \right) \quad \cdots (A.4)$$

◦ ラグランジュ括弧式に具体的な値を入れて評価する

$c_\ell = \sigma$, $c_j = a$ として (A.3)を(A.4)へ代入する

$$
\begin{aligned}
[\sigma, a] &= \left(\frac{\partial \mathbf{r}}{\partial \sigma}, \frac{\partial \mathbf{r}}{\partial a}\right) - \left(\frac{\partial \mathbf{r}}{\partial a}, \frac{\partial \mathbf{r}}{\partial \sigma}\right) \\
&= \left(\frac{\partial}{\partial \sigma} A\mathbf{r}^*, \frac{\partial}{\partial a} A\mathbf{r}^*\right) - \left(\frac{\partial}{\partial a} A\mathbf{r}^*, \frac{\partial}{\partial \sigma} A\mathbf{r}^*\right) \qquad \cdots (A.5) \\
&= \left(A\frac{\partial \mathbf{r}^*}{\partial \sigma}, A\frac{\partial \mathbf{r}^*}{\partial a}\right) - \left(A\frac{\partial \mathbf{r}^*}{\partial a}, A\frac{\partial \mathbf{r}^*}{\partial \sigma}\right)
\end{aligned}
$$

(∵ Aはσとaを含まないので)

$$
= \left(A\frac{\partial \mathbf{r}^*}{\partial \sigma}\right)^t A\frac{\partial \mathbf{r}^*}{\partial a} - \left(A\frac{\partial \mathbf{r}^*}{\partial a}\right)^t A\frac{\partial \mathbf{r}^*}{\partial \sigma}
$$

$\left(\because (\mathbf{a}, \mathbf{b}) = \mathbf{a}^t \mathbf{b}\right.$ cf. 線形代数入門 P.61 $\left.\right)$

$$
= \left(\frac{\partial \mathbf{r}^*}{\partial \sigma}\right)^t \underbrace{A^t A}_{= E} \frac{\partial \mathbf{r}^*}{\partial a} - \left(\frac{\partial \mathbf{r}^*}{\partial a}\right)^t \underbrace{A^t A}_{= E} \frac{\partial \mathbf{r}^*}{\partial \sigma} \qquad \cdots (A.6)
$$

定理

n次の回転行列は直交行列

$M^t = M^{-1}$

$$
= \left(\frac{\partial \mathbf{r}^*}{\partial \sigma}\right)^t \frac{\partial \mathbf{r}^*}{\partial a} - \left(\frac{\partial \mathbf{r}^*}{\partial a}\right)^t \frac{\partial \mathbf{r}^*}{\partial \sigma}
$$

$$
= \begin{pmatrix} \frac{\partial x^*}{\partial \sigma} \\ \frac{\partial y^*}{\partial \sigma} \\ 0 \end{pmatrix} \left(\frac{\partial x^*}{\partial a}, \frac{\partial y^*}{\partial a}, 0\right) - \begin{pmatrix} \frac{\partial x^*}{\partial a} \\ \frac{\partial y^*}{\partial a} \\ 0 \end{pmatrix} \left(\frac{\partial x^*}{\partial \sigma}, \frac{\partial y^*}{\partial \sigma}, 0\right)
$$

$$
= \frac{\partial x^*}{\partial \sigma}\frac{\partial x^*}{\partial a} + \frac{\partial y^*}{\partial \sigma}\frac{\partial y^*}{\partial a} - \frac{\partial x^*}{\partial a}\cdot\frac{\partial x^*}{\partial \sigma} - \frac{\partial y^*}{\partial a}\cdot\frac{\partial y^*}{\partial \sigma}
$$

$\vdots$

$\cdots (A.7)$

$$\frac{\partial x^*}{\partial \sigma} = \frac{\dot{x}^*}{n},\quad \frac{\partial y^*}{\partial \sigma} = \frac{\dot{y}^*}{n},\quad \frac{\partial \dot{x}^*}{\partial \sigma} = \frac{\ddot{x}^*}{n} = -\frac{\mu}{nr^3}x^*,\quad \frac{\partial \dot{y}^*}{\partial \sigma} = \frac{\ddot{y}^*}{n} = -\frac{\mu}{nr^3}y^*$$

であるから.

確認　$l = nt + \sigma$ …(2.241) を使う　通常 x^* を l の関数として $x^*(l)$ と表されることが多い.

$$\left(\frac{\partial x^*}{\partial t}\right)_{n,\sigma} = \left(\frac{\partial l}{\partial t}\right)_{n,\sigma}\frac{dx^*}{dl} = n\frac{dx^*}{dl}$$

$$\left(\frac{\partial x^*}{\partial \sigma}\right)_{n,t} = \left(\frac{\partial l}{\partial \sigma}\right)_{n,t}\frac{dx^*}{dl} = \frac{dx^*}{dl}$$

$\xRightarrow{2式を比較して}$ $\left(\frac{\partial x^*}{\partial \sigma}\right)_{n,t} = \dfrac{\left(\frac{\partial x^*}{\partial t}\right)_{n,\sigma}}{n} = \dfrac{\dot{x}^*}{n}$

同様にして,

$$\frac{\partial y^*}{\partial \sigma} = \frac{\dot{y}^*}{n},\quad \frac{\partial \dot{x}^*}{\partial \sigma} = \frac{\ddot{x}^*}{n},\quad \frac{\partial \dot{y}^*}{\partial \sigma} = \frac{\ddot{y}^*}{n}$$

も導ける

また, 運動方程式 (2.8) より

$$\ddot{\mathbf{r}}^* = -\mu\frac{\mathbf{r}^*}{r^3} \Rightarrow \begin{cases} \ddot{x}^* = -\mu\dfrac{x^*}{r^3} \\ \ddot{y}^* = -\mu\dfrac{y^*}{r^3} \end{cases}$$

これらを利用すると,

$$\frac{\partial \dot{x}^*}{\partial \sigma} = \frac{\ddot{x}}{n} = -\frac{\mu}{nr^3}x^*$$

$$\frac{\partial \dot{y}^*}{\partial \sigma} = \frac{\ddot{y}}{n} = -\frac{\mu}{nr^3}y^*$$

⋮ A.7からの(つづき)

⋮

$$[\sigma, a] = \frac{1}{n}\left\{\dot{x}^*\frac{\partial \dot{x}^*}{\partial a} + \dot{y}^*\frac{\partial \dot{y}^*}{\partial a} + \frac{\mu}{r^3}\left(x^*\frac{\partial x^*}{\partial a} + y^*\frac{\partial y^*}{\partial a}\right)\right\}$$

$$= \frac{1}{n}\left\{\frac{1}{2}\frac{\partial}{\partial a}\dot{x}^{*2} + \frac{1}{2}\frac{\partial}{\partial a}\dot{y}^{*2} + \frac{\mu}{r^3}\left(\frac{1}{2}\frac{\partial}{\partial a}x^{*2} + \frac{1}{2}\frac{\partial}{\partial a}y^{*2}\right)\right\}$$

⋮

⋮

$$[\sigma, a] = \frac{1}{n}\left\{\frac{1}{2}\frac{\partial}{\partial a}v^2 + \frac{\mu}{2r^3}\cdot\frac{\partial}{\partial a}r^2\right\}$$

$$= \frac{1}{n}\left(\frac{1}{2}\frac{\partial}{\partial a}v^2 + \frac{\mu}{r^3}r\frac{\partial r}{\partial a}\right)$$

$$= \frac{1}{n}\frac{\partial}{\partial a}\left(\underbrace{\frac{1}{2}v^2 - \frac{\mu}{r}}_{E}\right) \quad \cdots (A.8)$$

$$= \frac{1}{n}\frac{\partial E}{\partial a} \quad \cdots ①$$

$$= \frac{1}{n}\frac{\partial}{\partial a}\left(-\frac{\mu}{2a}\right) \qquad (\because 2.68)$$

$$= \frac{\mu}{2na^2}$$

$$= \frac{1}{2}na \quad \cdots (A.9) \qquad (\because \text{ケプラーの第3法則 } \mu = n^2a^3 \ (2.35))$$

・同様にして、$[\sigma, e]$についても計算する

$[\sigma, a]$のときと全く同じように展開していけばよい。（∵Aはσとeを含まない）

$$[\sigma, e] = \frac{1}{n}\frac{\partial E}{\partial e} \qquad (\because ①)$$

$$= 0$$

・次に$[\sigma, \Omega]$を求める

$$[\sigma, \Omega] = \left(\frac{\partial}{\partial \sigma} A\boldsymbol{r}^*, \frac{\partial}{\partial \Omega} A\boldsymbol{r}^*\right) - \left(\frac{\partial}{\partial \Omega} A\boldsymbol{r}^*, \frac{\partial}{\partial \sigma} A\boldsymbol{r}^*\right)$$

(∵ Aはσには依らないがΩには依る
$\boldsymbol{r}^*$はΩには依らないがσには依る)

$$= \left(A\frac{\partial \boldsymbol{r}^*}{\partial \sigma}, \frac{\partial A}{\partial \Omega}\boldsymbol{r}^*\right) - \left(\frac{\partial A}{\partial \Omega}\boldsymbol{r}^*, A\frac{\partial \boldsymbol{r}^*}{\partial \sigma}\right)$$

$$= \left(A\frac{\partial \boldsymbol{r}^*}{\partial \sigma}\right)^t \left(\frac{\partial A}{\partial \Omega}\right)\boldsymbol{r}^* - \left(\frac{\partial A}{\partial \Omega}\boldsymbol{r}^*\right)^t A\frac{\partial \boldsymbol{r}^*}{\partial \sigma} \qquad (∵ (A, B) = A^t B)$$

$$= \left(\frac{\partial \boldsymbol{r}^*}{\partial \sigma}\right)^t \underbrace{A^t \left(\frac{\partial A}{\partial \Omega}\right)}_{B_1}\boldsymbol{r}^* - \boldsymbol{r}^{*t} \underbrace{\left(\frac{\partial A}{\partial \Omega}\right)^t A}_{B_1^t}\frac{\partial \boldsymbol{r}^*}{\partial \sigma}$$

$$= \left(\frac{\partial \boldsymbol{r}^*}{\partial \sigma}\right)^t B_1 \boldsymbol{r}^* - \boldsymbol{r}^{*t} B_1^t \frac{\partial \boldsymbol{r}^*}{\partial \sigma}$$

(∵ ノートA1-3 確認 を参考にして、

$$\frac{\partial \boldsymbol{r}^*}{\partial \sigma} = \frac{\dot{\boldsymbol{r}}^*}{n}, \quad \frac{\partial \dot{\boldsymbol{r}}^*}{\partial \sigma} = -\mu \frac{\boldsymbol{r}^*}{r^3}$$)

$$= \left(\frac{\dot{\boldsymbol{r}}^*}{n}\right)^t B_1 \boldsymbol{r}^* - \boldsymbol{r}^{*t} B_1^t \left(-\mu \frac{\boldsymbol{r}^*}{r^3}\right)$$

$$= \frac{1}{n}\left(\dot{\boldsymbol{r}}^{*t} B_1 \boldsymbol{r}^* + \frac{\mu}{r^3}\boldsymbol{r}^{*t} B_1^t \boldsymbol{r}^*\right) \quad \cdots (A.11)$$

ここで B_1について計算しておく.

$$B_1 = A^t \frac{\partial A}{\partial \Omega} \qquad \left(\because \overset{P.60}{A} = R_3(-\Omega)R_1(-I)R_3(-\omega)\right)$$

R_1, R_3の形もP.60

$$= R_3(\omega)R_1(I)R_3(\Omega)\frac{\partial}{\partial \Omega}\Big(R_3(-\Omega)R_1(-I)R_3(-\omega)\Big)$$

$$= R_3(\omega)R_1(I)R_3(\Omega)\frac{\partial R_3(-\Omega)}{\partial \Omega}R_1(-I)R_3(-\omega)$$

$$= \begin{pmatrix} \cos\omega & \sin\omega & 0 \\ -\sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos I & \sin I \\ 0 & -\sin I & \cos I \end{pmatrix}\begin{pmatrix} \cos\Omega & \sin\Omega & 0 \\ -\sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} -\sin\Omega & -\cos\Omega & 0 \\ \cos\Omega & -\sin\Omega & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos I & -\sin I \\ 0 & \sin I & \cos I \end{pmatrix}\begin{pmatrix} \cos\omega & -\sin\omega & 0 \\ \sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\omega & \sin\omega\cos I & \sin\omega\sin I \\ -\sin\omega & \cos\omega\cos I & \cos\omega\sin I \\ 0 & -\sin I & \cos I \end{pmatrix}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \cos\omega & -\sin\omega & 0 \\ \cos I\sin\omega & \cos I\cos\omega & -\sin I \\ \sin I\sin\omega & \sin I\cos\omega & \cos I \end{pmatrix}$$

$$= \begin{pmatrix} \sin\omega\cos I & -\cos\omega & 0 \\ \cos\omega\cos I & \sin\omega & 0 \\ -\sin I & 0 & 0 \end{pmatrix}\begin{pmatrix} \cos\omega & -\sin\omega & 0 \\ \cos I\sin\omega & \cos I\cos\omega & -\sin I \\ \sin I\sin\omega & \sin I\cos\omega & \cos I \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -\cos I & \sin I\cos\omega \\ \cos I & 0 & -\sin I\sin\omega \\ -\sin I\cos\omega & \sin I\sin\omega & 0 \end{pmatrix} \quad \cdots (A.12)$$

この(A.12)を(A.11)へ代入

$$[\sigma,\Omega]=\frac{1}{n}\left\{(\dot{x}^*,\dot{y}^*,0)\begin{pmatrix}0 & -\cos I & \sin I\cos\omega\\ \cos I & 0 & -\sin I\sin\omega\\ -\sin I\cos\omega & \sin I\sin\omega & 0\end{pmatrix}\begin{pmatrix}\dot{x}^*\\ \dot{y}^*\\ 0\end{pmatrix}\cdots\right.$$

$$\left.+\frac{\mu}{r^3}(x^*,y^*,0)\begin{pmatrix}0 & \cos I & -\sin I\cos\omega\\ -\cos I & 0 & \sin I\sin\omega\\ \sin I\cos\omega & -\sin I\sin\omega & 0\end{pmatrix}\begin{pmatrix}x^*\\ y^*\\ 0\end{pmatrix}\right\}$$

$$=\frac{1}{n}\left\{(\dot{x}^*,\dot{y}^*,0)\begin{pmatrix}-\dot{y}^*\cos I\\ \dot{x}^*\cos I\\ 0\end{pmatrix}+\frac{\mu}{r^3}(x^*,y^*,0)\begin{pmatrix}y^*\cos I\\ -x^*\cos I\\ x^*\sin I\cos\omega-y^*\sin I\sin\omega\end{pmatrix}\right\}$$

$$=0 \qquad \cdots(A.13)$$

・次に$[\Omega,a]$を求める

$$[\Omega,a]=\left(\frac{\partial}{\partial\Omega}A\mathbb{r}^*,\frac{\partial}{\partial a}A\mathbb{r}^*\right)-\left(\frac{\partial}{\partial a}A\mathbb{r}^*,\frac{\partial}{\partial\Omega}A\mathbb{r}^*\right)$$

(∵ Aはaに依らないがΩに依る
$\mathbb{r}^*$はΩ 〃 a 〃)

$$=\left(\frac{\partial A}{\partial\Omega}\mathbb{r}^*,A\frac{\partial\mathbb{r}^*}{\partial a}\right)-\left(A\frac{\partial\mathbb{r}^*}{\partial a},\frac{\partial A}{\partial\Omega}\mathbb{r}^*\right)$$

$$=\left(\frac{\partial A}{\partial\Omega}\mathbb{r}^*\right)^t A\frac{\partial\mathbb{r}^*}{\partial a}-\left(A\frac{\partial\mathbb{r}^*}{\partial a}\right)^t\frac{\partial A}{\partial\Omega}\mathbb{r}^*$$

$$=\mathbb{r}^{*t}\underbrace{\left(\frac{\partial A}{\partial\Omega}\right)^t A}_{=B_1^t}\frac{\partial\mathbb{r}^*}{\partial a}-\left(\frac{\partial\mathbb{r}^*}{\partial a}\right)^t\underbrace{A^t\frac{\partial A}{\partial\Omega}}_{=B_1}\mathbb{r}^*$$

⋮

$$[\Omega, a] \doteq \mathbb{r}^{*t} B_1^{\,t} \frac{\partial \mathbb{r}^*}{\partial a} - \left(\frac{\partial \mathbb{r}^*}{\partial a}\right)^t B_1 \dot{\mathbb{r}}^*$$

$$= (x^*, y^*, 0)\begin{pmatrix} 0 & \cos I & -\sin I \cos\omega \\ -\cos I & 0 & \sin I \sin\omega \\ \sin I \cos\omega & -\sin I \sin\omega & 0 \end{pmatrix}\begin{pmatrix} \frac{\partial \dot{x}^*}{\partial a} \\ \frac{\partial \dot{y}^*}{\partial a} \\ 0 \end{pmatrix}$$

$$- \left(\frac{\partial x^*}{\partial a}, \frac{\partial y^*}{\partial a}, 0\right)\begin{pmatrix} 0 & -\cos I & \sin I \cos\omega \\ \cos I & 0 & -\sin I \sin\omega \\ -\sin I \cos\omega & \sin I \sin\omega & 0 \end{pmatrix}\begin{pmatrix} \dot{x}^* \\ \dot{y}^* \\ 0 \end{pmatrix}$$

$$= (x^*, y^*, 0)\begin{pmatrix} \frac{\partial \dot{y}^*}{\partial a}\cos I \\ -\frac{\partial \dot{x}^*}{\partial a}\cos I \\ \frac{\partial \dot{y}^*}{\partial a}\sin I \sin\omega \end{pmatrix} - \left(\frac{\partial x^*}{\partial a}, \frac{\partial y^*}{\partial a}, 0\right)\begin{pmatrix} -\dot{y}^*\cos I \\ \dot{x}^*\cos I \\ \underbrace{\qquad\qquad}_{\uparrow} \end{pmatrix}$$

どうせ消えるから省略

$$= x^*\frac{\partial \dot{y}^*}{\partial a}\cos I - y^*\frac{\partial \dot{x}^*}{\partial a}\cos I + \dot{y}^*\frac{\partial x^*}{\partial a}\cos I - \dot{x}^*\frac{\partial y^*}{\partial a}\cos I$$

$$= \cos I\left(\frac{\partial x^*}{\partial a}\dot{y}^* + x^*\frac{\partial \dot{y}^*}{\partial a} - \frac{\partial y^*}{\partial a}\dot{x}^* - y^*\frac{\partial \dot{x}^*}{\partial a}\right)$$

$$= \cos I \frac{\partial}{\partial a}\left(\underbrace{x^*\dot{y}^* - y^*\dot{x}^*}_{\substack{\parallel \\ h\,:\,\text{角運動量}}}\right) \quad \cdots (A.15)$$

$$= \cos I \frac{\partial}{\partial a}\sqrt{\mu a(1-e^2)} \qquad (\because 2.69)$$

$$= \cos I\left\{\frac{1}{2}\left[\mu a(1-e^2)\right]^{-\frac{1}{2}} \cdot \mu(1-e^2)\right\}$$

$$= \frac{\cos I}{2}\sqrt{\frac{\mu(1-e^2)}{a}} \qquad \left(\because \begin{array}{l}\text{ケプラーの第3法則 (2.66)} \\ n^2a^3=\mu\end{array} \quad + \quad \eta=\sqrt{1-e^2}\right)$$

$$= \frac{\cos I}{2}\sqrt{\frac{n^2a^3\eta^2}{a}}$$

$$= \frac{1}{2}na\eta\cos I \quad \cdots (A.16)$$