4.1

P_3の慣性系での運動方程式

$$\frac{d^2\mathbf{r}}{dt^2} = -\frac{GM_1}{r_1^3}\mathbf{r}_1 - \frac{GM_2}{r_2^3}\mathbf{r}_2 = -\frac{\partial U}{\partial \mathbf{r}} \quad \cdots (4.1)$$

$$U = -\frac{GM_1}{r_1} - \frac{GM_2}{r_2} \quad \cdots (4.2)$$

(ξ, η, ζ)座標と(X, Y, Z)座標の関係は、

$$(\xi, \eta, \zeta) \underset{\ominus\text{回転}}{\overset{Z\text{軸まわりで}\ominus\text{回転}}{\rightleftarrows}} (X, Y, Z)$$ なので、

$$\begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

$$= \begin{pmatrix} X\cos\theta - Y\sin\theta \\ X\sin\theta + Y\cos\theta \\ Z \end{pmatrix} \begin{matrix} \cdots (4.3) \\ \cdots (4.4) \\ \cdots (4.5) \end{matrix}$$

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) & 0 \\ \sin(-\theta) & \cos(-\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix}$$

$$= \begin{pmatrix} \xi\cos\theta + \eta\sin\theta \\ -\xi\sin\theta + \eta\cos\theta \\ \zeta \end{pmatrix} \begin{matrix} \cdots (4.6) \\ \cdots (4.7) \\ \ \end{matrix}$$

(4.3)(4.4)より、それぞれの時間微分は、

$$\dot{\xi} = \dot{X}\cos\theta - X\dot{\theta}\sin\theta - \dot{Y}\sin\theta - Y\dot{\theta}\cos\theta$$

$$= \dot{X}\cos\theta - \dot{Y}\sin\theta - n'X\sin\theta - n'Y\cos\theta \quad \cdots (4.8)$$

← ($n't = \theta$ より $\dot{\theta} = n'$)

$$\dot{\eta} = \dot{X}\sin\theta + X\dot{\theta}\cos\theta + \dot{Y}\cos\theta - Y\dot{\theta}\sin\theta$$

$$= \dot{X}\sin\theta + \dot{Y}\cos\theta + n'X\cos\theta - n'Y\sin\theta \quad \cdots (4.9)$$

さらに(4.8)(4.9)を時間微分

$$\ddot{\xi} = \ddot{X}\cos\theta - n'\dot{X}\sin\theta - \ddot{Y}\sin\theta - n'\dot{Y}\cos\theta - n'\dot{X}\sin\theta - n'^2X\cos\theta - n'\dot{Y}\cos\theta + n'^2Y\sin\theta$$

$$= \ddot{X}\cos\theta - \ddot{Y}\sin\theta - 2n'\dot{X}\sin\theta - 2n'\dot{Y}\cos\theta - n'^2X\cos\theta + n'^2Y\sin\theta \quad \cdots (4.10)$$

$$\ddot{\eta} = \ddot{X}\sin\theta + n'\dot{X}\cos\theta + \ddot{Y}\cos\theta - n'\dot{Y}\sin\theta + n'\dot{X}\cos\theta - n'^2X\sin\theta - n'\dot{Y}\sin\theta - n'^2Y\cos\theta$$

$$= \ddot{X}\sin\theta + \ddot{Y}\cos\theta + 2n'\dot{X}\cos\theta - 2n'\dot{Y}\sin\theta - n'^2X\sin\theta - n'^2Y\cos\theta \quad \cdots (4.11)$$

$\frac{\partial U}{\partial \xi}$, $\frac{\partial U}{\partial \eta}$ を X, Y についての偏微分に書き直す

$$\frac{\partial U}{\partial \xi} = \frac{\partial U}{\partial X}\frac{\partial X}{\partial \xi} + \frac{\partial U}{\partial Y}\cdot\frac{\partial Y}{\partial \xi}$$

$$= \frac{\partial U}{\partial X}\cdot\underbrace{\cos\theta}_{(\because 4.6)} + \frac{\partial U}{\partial Y}\cdot\underbrace{(-\sin\theta)}_{(\because 4.7)}$$

$$= \frac{\partial U}{\partial X}\cos\theta - \frac{\partial U}{\partial Y}\sin\theta \quad \cdots (4.12)$$

$$\frac{\partial U}{\partial \eta} = \frac{\partial U}{\partial X}\cdot\frac{\partial X}{\partial \eta} + \frac{\partial U}{\partial Y}\cdot\frac{\partial Y}{\partial \eta}$$

$$= \frac{\partial U}{\partial X}\sin\theta + \frac{\partial U}{\partial Y}\cos\theta \quad \cdots (4.13)$$

慣性系(ξ,η,ζ)での運動方程式から回転座標系(X,Y,Z)での運動方程式を導出する　

・まずは、慣性系での運動方程式(4.1)の左辺に(4.10)(4.11)右辺に(4.12)(4.13)を代入する

$$\frac{d^2\mathbf{r}}{dt^2} = -\frac{\partial U}{\partial \mathbf{r}}$$

$$\begin{bmatrix} \dfrac{d^2\xi}{dt^2} \\ \dfrac{d^2\eta}{dt^2} \\ \dfrac{d^2\zeta}{dt^2} \end{bmatrix} = \begin{bmatrix} -\dfrac{\partial U}{\partial \xi} \\ -\dfrac{\partial U}{\partial \eta} \\ -\dfrac{\partial U}{\partial \zeta} \end{bmatrix}$$

(4.12)(4.13)代入

↓(4.10)(4.11)代入

$$\begin{cases} \ddot{X}\cos\theta - \ddot{Y}\sin\theta - 2n'\dot{X}\sin\theta - 2n'\dot{Y}\cos\theta - n'^2X\cos\theta + n'^2Y\sin\theta = -\dfrac{\partial U}{\partial X}\cos\theta + \dfrac{\partial U}{\partial Y}\sin\theta & \text{①} \\ \ddot{X}\sin\theta + \ddot{Y}\cos\theta + 2n'\dot{X}\cos\theta - 2n'\dot{Y}\sin\theta - n'^2X\sin\theta - n'^2Y\cos\theta = -\dfrac{\partial U}{\partial X}\sin\theta - \dfrac{\partial U}{\partial Y}\cos\theta & \text{②} \\ \ddot{Z} = -\dfrac{\partial U}{\partial Z} \quad (\because 4.5) & (4.16) \end{cases}$$

・①,②を用いて、

①×$\cos\theta$＋②×$\sin\theta$

$$\begin{aligned} &\ddot{X}\cos^2\theta - \ddot{Y}\sin\theta\cos\theta - 2n'\dot{X}\sin\theta\cos\theta - 2n'\dot{Y}\cos^2\theta - n'^2X\cos^2\theta + n'^2Y\sin\theta\cos\theta = -\frac{\partial U}{\partial X}\cos^2\theta + \frac{\partial U}{\partial Y}\sin\theta\cos\theta \\ +)\ &\ddot{X}\sin^2\theta + \ddot{Y}\sin\theta\cos\theta + 2n'\dot{X}\sin\theta\cos\theta - 2n'\dot{Y}\sin^2\theta - n'^2X\sin^2\theta - n'^2Y\sin\theta\cos\theta = -\frac{\partial U}{\partial X}\sin^2\theta - \frac{\partial U}{\partial Y}\sin\theta\cos\theta \\ \hline &\ddot{X} - 2n'\dot{Y} - n'^2X = -\frac{\partial U}{\partial X} \quad \cdots (4.14) \end{aligned}$$

①×$(-\sin\theta)$＋②×$\cos\theta$

$$\begin{aligned} &-\ddot{X}\sin\theta\cos\theta + \ddot{Y}\sin^2\theta + 2n'\dot{X}\sin^2\theta + 2n'\dot{Y}\sin\theta\cos\theta + n'^2X\sin\theta\cos\theta - n'^2Y\sin^2\theta = \frac{\partial U}{\partial X}\sin\theta\cos\theta - \frac{\partial U}{\partial Y}\sin^2\theta \\ +)\ &\ddot{X}\sin\theta\cos\theta + \ddot{Y}\cos^2\theta + 2n'\dot{X}\cos^2\theta - 2n'\dot{Y}\sin\theta\cos\theta - n'^2X\sin\theta\cos\theta - n'^2Y\cos^2\theta = -\frac{\partial U}{\partial X}\sin\theta\cos\theta - \frac{\partial U}{\partial Y}\cos^2\theta \\ \hline &\ddot{Y} + 2n'\dot{X} - n'^2Y = -\frac{\partial U}{\partial Y} \quad \cdots (4.15) \end{aligned}$$

$$U = -\frac{GM_1}{r_1} - \frac{GM_2}{r_2} \qquad \left(\begin{aligned} n'^2 a'^3 &= G(M_1+M_2) \\ G &= \frac{n'^2 a'^3}{M_1+M_2} \end{aligned} \right)$$

$$= -\frac{M_1}{r_1}\cdot\frac{n'^2 a'^3}{M_1+M_2} - \frac{M_2}{r_2}\cdot\frac{n'^2 a'^3}{M_1+M_2}$$

$$= -\left(\frac{M_1}{M_1+M_2}\frac{a'^3}{r_1} + \frac{M_2}{M_1+M_2}\frac{a'^3}{r_2}\right) n'^2 \quad \cdots (4.17)$$

図4.1より、

$$r_1 = \sqrt{\left(X + \frac{M_2}{M_1+M_2}a'\right)^2 + Y^2 + Z^2} \quad \cdots (4.18)$$

$$r_2 = \sqrt{\left(X - \frac{M_1}{M_1+M_2}a'\right)^2 + Y^2 + Z^2} \quad \cdots (4.19)$$

運動方程式(4.14)と(4.15)の遠心力項を右辺に移して、

$$\ddot{X} - 2n'\dot{Y} = -\frac{\partial U^*}{\partial X} \quad \left(= -\frac{\partial U}{\partial X} + n'^2 X\right) \quad \cdots (4.20)$$

$$\ddot{Y} + 2n'\dot{X} = -\frac{\partial U^*}{\partial Y} \quad \left(= -\frac{\partial U}{\partial Y} + n'^2 Y\right) \quad \cdots (4.21)$$

$$\ddot{Z} = -\frac{\partial U^*}{\partial Z} \quad \cdots (4.22)$$

(4.20)(4.21)より

$$U^* = U - \frac{1}{2}n'^2(X^2 + Y^2) \quad \cdots (4.23)$$