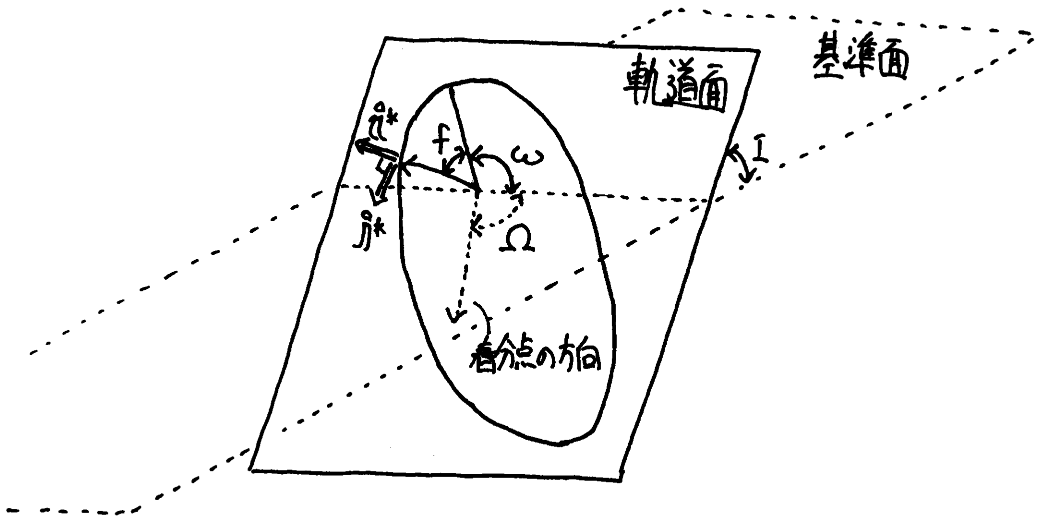
B ガウスの惑星方程式の導出

B.1 ガウスの惑星方程式の初等的導出

動径方向の単位ベクトルを $\mathbb{i}^*$, 軌道面内で動径に垂直な方向の単位ベクトル $\mathbb{j}^*$, 軌道面に垂直な方向のベクトル $\mathbb{k}$ とする。



慣性系 (x, y, z) から座標系 $(\mathbb{i}^*, \mathbb{j}^*, \mathbb{k})$ への変換行列 T は 2.8.2より

$$T = R_3(-\Omega)R_1(-I)R_3(-f-\omega) \quad \cdots \quad (B.1)$$

$$\left[\begin{pmatrix} \mathbb{i}^* \\ \mathbb{j}^* \\ \mathbb{k} \end{pmatrix} = T \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right]$$

である。

ここで、

$$\phi = f + \omega$$

と略記することにする。

2.8.2より、B.1を展開すると、

$$T=\begin{pmatrix}\cos\Omega & -\sin\Omega & 0\\ \sin\Omega & \cos\Omega & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & \cos I & -\sin I\\ 0 & \sin I & \cos I\end{pmatrix}\begin{pmatrix}\cos\phi & -\sin\phi & 0\\ \sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$=\begin{pmatrix}\cos\Omega & -\sin\Omega\cos I & \sin\Omega\sin I\\ \sin\Omega & \cos\Omega\cos I & -\cos\Omega\sin I\\ 0 & \sin I & \cos I\end{pmatrix}\begin{pmatrix}\cos\phi & -\sin\phi & 0\\ \sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{pmatrix}$$

← 後の計算で都合がいいので、この形にしておく

である。

単位ベクトル i^* の慣性系での表示は、

$$i^*=T\begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$=\begin{pmatrix}\cos\Omega & -\sin\Omega\cos I & \sin\Omega\sin I\\ \sin\Omega & \cos\Omega\cos I & -\cos\Omega\sin I\\ 0 & \sin I & \cos I\end{pmatrix}\begin{pmatrix}\cos\phi\\ \sin\phi\\ 0\end{pmatrix}$$

$$=\begin{pmatrix}\cos\Omega\cos\phi-\sin\Omega\cos I\sin\phi\\ \sin\Omega\cos\phi+\cos\Omega\cos I\sin\phi\\ \sin I\sin\phi\end{pmatrix}\qquad\cdots(B.2)$$

となる。同様にして j^*, k についても、

$$j^*=T\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}\quad\sim\quad\end{pmatrix}\begin{pmatrix}-\sin\phi\\ \cos\phi\\ 0\end{pmatrix}=\begin{pmatrix}-\cos\Omega\sin\phi-\cos I\sin\Omega\cos\phi\\ -\sin\Omega\sin\phi+\cos I\cos\Omega\cos\phi\\ \sin I\cos\phi\end{pmatrix}\qquad\cdots(B.3)$$

$$k=T\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}\quad\sim\quad\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}\sin I\sin\Omega\\ -\sin I\cos\Omega\\ \cos I\end{pmatrix}\qquad\cdots(B.4)$$

摂動力$\mathbb{X}$の$\mathbb{i}^*, \mathbb{j}^*, \mathbb{k}$成分をR, S, Wとすると、

$$\mathbb{X} = R\,\mathbb{i}^* + S\,\mathbb{j}^* + W\mathbb{k} \qquad \cdots (B.5)$$

となる。動径ベクトル$\mathbb{r}$は

$$\mathbb{r} = r\,\mathbb{i}^* \qquad \cdots (B.6)$$

である。

(5.97)は摂動力$\mathbb{X}$と動径ベクトル$\mathbb{r}$を軌道要素で偏微分したものの内積である。この内積表現に(B.5)と(B.6)を代入すると、

$$\begin{aligned}\sum_{j=1}^{6}[c_i, c_j]\frac{dc_j}{dt} &= \left(\mathbb{X}, \frac{\partial \mathbb{r}}{\partial c_\ell}\right) \\ &= \left(R\,\mathbb{i}^* + S\,\mathbb{j}^* + W\mathbb{k},\ \frac{\partial r}{\partial c_\ell}\mathbb{i}^* + r\frac{\partial \mathbb{i}^*}{\partial c_\ell}\right) \\ &= R\frac{\partial r}{\partial c_\ell} + r\underset{(a)}{\left(R\,\mathbb{i}^* + S\,\mathbb{j}^* + W\mathbb{k},\ \frac{\partial \mathbb{i}^*}{\partial c_\ell}\right)} \qquad \cdots (B.7)\end{aligned}$$

この(B.7)を$\ell = 4$の場合について評価する。(B.2)より、

$$\frac{\partial \mathbb{i}^*}{\partial \sigma} = \begin{pmatrix} -\cos\Omega\sin\phi\dfrac{\partial\phi}{\partial\sigma} - \cos I\sin\Omega\cos\phi\dfrac{\partial\phi}{\partial\sigma} \\ -\sin\Omega\sin\phi\dfrac{\partial\phi}{\partial\sigma} + \cos I\cos\Omega\cos\phi\dfrac{\partial\phi}{\partial\sigma} \\ \sin I\cos\phi\dfrac{\partial\phi}{\partial\sigma} \end{pmatrix}$$

$$= \mathbb{j}^*\frac{\partial\phi}{\partial\sigma} \qquad (\because B.3)$$

$$\cdots (B.8)$$

σとのラグランジュ括弧式は$[\sigma, a]$のほかは0となることがA.1より
わかっているので、

$$\sum_{j=1}^{6}[\sigma, c_j]\frac{dc_j}{dt} = [\sigma, a]\frac{da}{dt} \quad -①$$

また、(B.7), (B.8)より

$$\sum_{j=1}^{6}[\sigma, c_j]\frac{dc_j}{dt} = R\frac{\partial r}{\partial \sigma} + r\left(R\,\mathbf{i}^* + S\,\mathbf{j}^* + W\mathbf{k},\ \mathbf{j}^*\frac{\partial \phi}{\partial \sigma}\right) \quad -②$$

①,②まとめて、

$$[\sigma, a]\frac{da}{dt} = R\frac{\partial r}{\partial \sigma} + rS\frac{\partial \phi}{\partial \sigma} \quad \cdots (B.9)$$

となる。

(A.25): $[\sigma, a] = \frac{na}{2}$, (2.73): $\frac{\partial r}{\partial \sigma} = \frac{ae}{\eta}\sin f$, (2.74): $r\frac{\partial \phi}{\partial \sigma} = \frac{a^2\eta}{r}$ を
(B.9)へ代入すると軌道長半径 aについての方程式となる。

$$\frac{na}{2}\frac{da}{dt} = R\frac{ae}{\eta}\sin f + S\frac{a^2\eta}{r}$$

$$\therefore \frac{da}{dt} = \frac{2}{na}\left(\frac{ae\sin f}{\eta}R + \frac{a^2\eta}{r}S\right) \quad \cdots (B.10)$$

(2.73)から$\frac{\partial r}{\partial \sigma} = \frac{ae}{\eta}\sin f$の導出!

まずは、$l = nt + \sigma$より

$$\left(\frac{\partial l}{\partial t}\right)_{a,\sigma} = n \quad \cdots (i)$$

また、

$$\left(\frac{\partial l}{\partial \sigma}\right)_a = 1 \underset{(\because i)}{=} \frac{1}{n}\left(\frac{\partial l}{\partial t}\right)_{a,\sigma} \quad \cdots (ii)$$

よって、

$$\left(\frac{\partial r}{\partial \sigma}\right)_a = \left(\frac{\partial r}{\partial l}\right)_{a,e,I,\Omega,\omega}\left(\frac{\partial l}{\partial \sigma}\right)_a$$

$$= \left(\frac{\partial r}{\partial l}\right)_{a,e,I,\Omega,\omega}\underbrace{\frac{1}{n}\left(\frac{\partial l}{\partial t}\right)_{a,\sigma}}_{(\because ii)}$$

$$= \frac{1}{n}\left(\frac{\partial r}{\partial t}\right)_{a,e,I,\Omega,\omega}$$

$$= \frac{1}{n}\cdot\frac{aen}{\eta}\sin f$$

$$= \frac{ae}{\eta}\sin f \quad ■$$

(2.74)から$\frac{\partial \phi}{\partial \sigma} = \frac{a^2\eta}{r^2}$の導出!

$\phi = f + \omega$, $l = nt + \sigma$などから、
文字同士の関係は以下の様

$$\begin{array}{ccc} & \phi & \\ f & & \omega \\ l & & \\ \sigma & & \end{array}$$

$$\left(\frac{\partial \phi}{\partial \sigma}\right)_{a,e,I,\Omega,\omega}$$

$$= \left(\frac{\partial \phi}{\partial f}\right)_\omega\left(\frac{\partial f}{\partial l}\right)_a\left(\frac{\partial l}{\partial \sigma}\right)_a$$

$$= \left(\frac{\partial \phi}{\partial f}\right)_\omega\left(\frac{\partial f}{\partial l}\right)_a\underbrace{\frac{1}{n}\left(\frac{\partial l}{\partial t}\right)}_{(\because ii)}$$

$$= \left(\frac{\partial \phi}{\partial f}\right)_\omega\left(\frac{\partial f}{\partial t}\right)_{a,\sigma}\frac{1}{n}$$

$$= 1\cdot\frac{a^2n\eta}{r^2}\cdot\frac{1}{n}$$

$$= \frac{a^2\eta}{r^2} \quad ■$$

○ $l=5$の場合 (ω)

(B.2)をωで偏微分すると.

$$\frac{\partial \mathbf{i}^*}{\partial \omega} = \begin{pmatrix} -\cos\Omega \sin\phi \frac{\partial \phi}{\partial \omega} - \cos I \sin\Omega \cos\phi \frac{\partial \phi}{\partial \omega} \\ -\sin\Omega \sin\phi \frac{\partial \phi}{\partial \omega} + \cos I \cos\Omega \cos\phi \frac{\partial \phi}{\partial \omega} \\ \sin I \cos\phi \frac{\partial \phi}{\partial \omega} \end{pmatrix}$$

$$= \mathbf{j}^* \quad \left(\because \phi = f + \omega \rightarrow \frac{\partial \phi}{\partial \omega} = 1\right)$$

$\cdots$(B.11)

rと近点方向は関係ないので

$$\frac{\partial r}{\partial \omega} = 0$$

であるから.

(B.7)より

$$\sum_{j=1}^{6} [\omega, c_j] \frac{dc_j}{dt} = r(R\mathbf{i}^* + S\mathbf{j}^* + W\mathbf{k}, \mathbf{j}^*)$$

$$= rS \quad \cdots ③$$

また. P.224より

$$\sum_{j=1}^{6} [\omega, c_j] \frac{dc_j}{dt} = [\omega, a]\frac{da}{dt} + [\omega, e]\frac{de}{dt} \quad \cdots ④$$

③, ④より.

$$[\omega, a]\frac{da}{dt} + [\omega, e]\frac{de}{dt} = rS \quad \cdots \text{(B.12)}$$

(B.12)へ(A.26),(A.27),(B.10)を代入する

$$\frac{1}{2}na\eta\cdot\frac{2}{na}\left(\frac{ae\sin f}{\eta}R+\frac{a^2\eta}{r}S\right)-\frac{na^2e}{\eta}\cdot\frac{de}{dt}=rS$$

$$\frac{na^2e}{\eta}\frac{de}{dt}=\eta\left(\frac{ae\sin f}{\eta}R+\frac{a^2\eta}{r}S\right)-rS$$

$$\frac{de}{dt}=\frac{\eta}{na^2e}\left\{ae\sin f\cdot R+\left(\frac{a^2\eta^2}{r}-r\right)S\right\}$$

$$=\frac{\eta}{na}\left\{R\sin f+S\left(\frac{a\eta^2}{re}-\frac{r}{ae}\right)\right\}$$

$$\downarrow$$

$$\left(\begin{aligned}&\frac{a\eta^2}{e}\cdot\underbrace{\frac{1+e\cos f}{a\eta^2}}_{(\because 2.56)}-\frac{1}{ae}\cdot\underbrace{a(1-e\cos u)}_{(\because 2.59)}\\&=\frac{1}{e}+\cos f-\frac{1}{e}+\cos u\\&=\cos f+\cos u\end{aligned}\right)$$

$$=\frac{\eta}{na}\left\{R\sin f+S(\cos f+\cos u)\right\}\quad\cdots(B.13)$$

◦ $l=6$ の場合 (Ω)

(B.7)より

$$\sum_{j=1}^{6}[\Omega, c_j]\frac{dc_j}{dt} = r\left(R\mathbf{i}^* + S\mathbf{j}^* + W\mathbf{k}, \frac{\partial \mathbf{i}^*}{\partial \Omega}\right) \quad \cdots ⑤ \quad (\because r と \Omega は関係ない)$$

⑤の右辺については, $\left(\mathbf{i}^*, \frac{\partial \mathbf{i}^*}{\partial \Omega}\right), \left(\mathbf{j}^*, \frac{\partial \mathbf{i}^*}{\partial \Omega}\right), \left(\mathbf{k}, \frac{\partial \mathbf{i}^*}{\partial \Omega}\right)$ について別々に求めていく。

・まずは, $\left(\mathbf{i}^*, \frac{\partial \mathbf{i}^*}{\partial \Omega}\right)$ について

$(\mathbf{i}^*, \mathbf{i}^*) = 1$ を Ω で偏微分する

$$\frac{\partial \mathbf{i}^*}{\partial \Omega}\mathbf{i}^* + \mathbf{i}^*\frac{\partial \mathbf{i}^*}{\partial \Omega} = 0$$

$$2\left(\mathbf{i}^*, \frac{\partial \mathbf{i}^*}{\partial \Omega}\right) = 0$$

$$\therefore \left(\mathbf{i}^*, \frac{\partial \mathbf{i}^*}{\partial \Omega}\right) = 0 \quad \cdots (B.15)$$

これは簡単に計算できたが, 残り2つは地道に計算しないといけない。

(B.3)より,

$$\frac{\partial \mathbf{i}^*}{\partial \Omega} = \begin{pmatrix} -\sin\Omega\cos\phi - \cos I\cos\Omega\sin\phi \\ \cos\Omega\cos\phi - \cos I\sin\Omega\sin\phi \\ 0 \end{pmatrix} \quad \cdots ⑥$$

となるので.

・$\left(\hat{h}^*, \dfrac{\partial \hat{e}^*}{\partial \Omega}\right)$ について.

$$
\left(\hat{h}^*, \frac{\partial \hat{e}^*}{\partial \Omega}\right) = \left(-\cos\Omega\sin\phi - \cos I\sin\Omega\cos\phi,\ -\sin\Omega\sin\phi + \cos I\cos\Omega\cos\phi,\ \sin I\cos\phi\right)
\begin{pmatrix} -\sin\Omega\cos\phi - \cos I\cos\Omega\sin\phi \\ \cos\Omega\cos\phi - \cos I\sin\Omega\sin\phi \\ 0 \end{pmatrix}
$$

$$
\begin{aligned}
&= (-\cos\Omega\sin\phi - \cos I\sin\Omega\cos\phi)(-\sin\Omega\cos\phi - \cos I\cos\Omega\sin\phi) \\
&\quad + (-\sin\Omega\sin\phi + \cos I\cos\Omega\cos\phi)(\cos\Omega\cos\phi - \cos I\sin\Omega\sin\phi) \\
&= \sin\Omega\cos\Omega\sin\phi\cos\phi + \cos I\cos^2\Omega\sin^2\phi \\
&\quad + \cos I\sin^2\Omega\cos^2\phi + \cos^2 I\sin\Omega\cos\Omega\sin\phi\cos\phi \\
&\quad - \sin\Omega\cos\Omega\sin\phi\cos\phi + \cos I\sin^2\Omega\sin^2\phi \\
&\quad + \cos I\cos^2\Omega\cos^2\phi - \cos^2 I\sin\Omega\cos\Omega\sin\phi\cos\phi \\
&= \cos I\sin^2\phi(\sin^2\Omega + \cos^2\Omega) + \cos I\cos^2\phi(\sin^2\Omega + \cos^2\Omega) \\
&= \cos I(\sin^2\phi + \cos^2\phi) \\
&= \cos I \qquad \cdots (B.16)
\end{aligned}
$$

・$\left(\mathbb{k}, \frac{\partial \mathbb{i}^*}{\partial \Omega}\right)$ について.

$$\left(\mathbb{k}, \frac{\partial \mathbb{i}^*}{\partial \Omega}\right) = (\sin I \sin \Omega, -\sin I \cos \Omega, \cos I)\begin{pmatrix} -\sin\Omega\cos\phi - \cos I \cos\Omega \sin\phi \\ \cos\Omega\cos\phi - \cos I \sin\Omega \sin\phi \\ 0 \end{pmatrix}$$

$$= -\sin I \sin^2\Omega \cos\phi - \sin I \cos I \sin\Omega\cos\Omega\sin\phi$$
$$- \sin I \cos^2\Omega\cos\phi + \sin I \cos I \sin\Omega \cos\Omega \sin\phi$$
$$= -\sin I \cos\phi (\sin^2\Omega + \cos^2\Omega)$$
$$= -\sin I \cos\phi \quad \cdots (B.17)$$

また、P.224より、

$$\sum_{j=1}^{6} [\Omega, c_j]\frac{dc_j}{dt} = [\Omega, a]\frac{da}{dt} + [\Omega, e]\frac{de}{dt} + [\Omega, I]\frac{dI}{dt} \quad \cdots ⑦$$

以上、⑤、⑦より、

$$[\Omega, a]\frac{da}{dt} + [\Omega, e]\frac{de}{dt} + [\Omega, I]\frac{dI}{dt} = r\left(R\mathbb{i}^* + S\mathbb{j}^* + W\mathbb{k}, \frac{\partial \mathbb{i}^*}{\partial \Omega}\right) \quad \cdots (B.14)$$

となり、この左辺へ (A.28)~(A.30) と (B.10), (B.13) を代入し、
右辺へ (B.15)~(B.17) を代入すると、

$$\frac{1}{2}na\eta\cos I \cdot \frac{2}{na}\left(\frac{ae\sin f}{\eta}R + \frac{a^2\eta}{r}S\right)$$
$$-\frac{na^2e\cos I}{\eta}\cdot\frac{\eta}{na}\left\{R\sin f + S(\cos f + \cos u)\right\}$$
$$-na^2\eta\sin I \cdot \frac{dI}{dt}$$
$$= r(S\cos I - W\sin I\cos\phi)$$

$$\eta \cos I \left(\frac{ae \sin f}{\eta} R + \frac{a^2 \eta}{r} S \right) - ae \cos I \left\{ R \sin f + S(\cos f + \cos u) \right\}$$

$$- n a^2 \eta \sin I \frac{dI}{dt} = r(S \cos I - W \sin I \cos \phi)$$

$$n a^2 \eta \sin I \frac{dI}{dt} = (ae \sin f \cos I - ae \cos I \sin f) R$$

$$+ \left(\frac{a^2 \eta^2}{r} \cos I - ae \cos I \cos f - ae \cos I \cos u - r \cos I \right) S$$

$$+ r \sin I \cos \phi \cdot W$$

$$= \left\{ a(1 + e \cos f) \cos I - ae \cos I \cos f - ae \cos I \cos u - a(1 - e \cos u) \cos I \right\} S$$

$$+ r \sin I \cos \phi \cdot W$$

$$\frac{dI}{dt} = \frac{r \sin I \cos \phi}{n a^2 \eta \sin I} W$$

$$= \frac{r \cos \phi}{n a^2 \eta} W \quad \cdots (B.18)$$

・$l=3$ の場合（I）

(B.7)より、

$$\sum_{j=1}^{6}[I, c_j]\frac{dc_j}{dt} = r\left(R\,\boldsymbol{i}^* + S\,\boldsymbol{j}^* + W\boldsymbol{k},\ \frac{\partial \boldsymbol{i}^*}{\partial I}\right) \quad \cdots ⑧$$

(∵ r と I は関係な…)

P.224より、

$$\sum_{j=1}^{6}[I, c_j]\frac{dc_j}{dt} = [I, \Omega]\frac{d\Omega}{dt} \quad \cdots ⑨$$

⑧, ⑨より、

$$[I, \Omega]\frac{d\Omega}{dt} = r\left(R\,\boldsymbol{i}^* + S\,\boldsymbol{j}^* + W\boldsymbol{k},\ \frac{\partial \boldsymbol{i}^*}{\partial I}\right) \quad \cdots (B.19)$$

(B.2)より、

$$\frac{\partial \boldsymbol{i}^*}{\partial I} = \begin{pmatrix} \sin I \sin\Omega \sin\phi \\ -\sin I \cos\Omega \sin\phi \\ \cos I \sin\phi \end{pmatrix} = \sin\phi \cdot \boldsymbol{k} \quad \cdots (B.20)$$

であるので、(B.19)へ(B.20)と(A.30)を代入すると、

$$na^2\eta \sin I \frac{d\Omega}{dt} = rW\sin\phi$$

$$\therefore \frac{d\Omega}{dt} = \frac{r\sin\phi}{na^2\eta\sin I}W \quad \cdots (B.21)$$

・$l=2$の場合（e）

(B.7)より、

$$\sum_{j=1}^{6}[e, c_j]\frac{dc_j}{dt} = R\frac{\partial r}{\partial e} + r\left(R\boldsymbol{i}^* + S\boldsymbol{j}^* + W\boldsymbol{k}, \frac{\partial \boldsymbol{i}^*}{\partial e}\right) \quad \cdots (10)$$

P.224より

$$\sum_{j=1}^{6}[e, c_j]\frac{dc_j}{dt} = [e, \omega]\frac{d\omega}{dt} + [e, \Omega]\frac{d\Omega}{dt} \quad \cdots (11)$$

(10), (11)より、

$$[e, \omega]\frac{d\omega}{dt} + [e, \Omega]\frac{d\Omega}{dt} = R\frac{\partial r}{\partial e} + r\left(R\boldsymbol{i}^* + S\boldsymbol{j}^* + W\boldsymbol{k}, \frac{\partial \boldsymbol{i}^*}{\partial e}\right) \quad \cdots (B.22)$$

(B.2)より、

$$\frac{\partial \boldsymbol{i}^*}{\partial e} = \frac{\partial \boldsymbol{i}^*}{\partial \phi}\frac{\partial \phi}{\partial f}\frac{\partial f}{\partial e}$$

$$= \begin{pmatrix} -\cos\Omega\sin\phi - \cos I\sin\Omega\cos\phi \\ -\sin\Omega\sin\phi + \cos I\cos\Omega\cos\phi \\ \sin I\cos\phi \end{pmatrix} \cdot 1 \cdot \frac{\partial f}{\partial e}$$

$$= \boldsymbol{j}^*\frac{\partial f}{\partial e} \quad \cdots (B.23)$$

(B.22)へ(A.27), (A.29), (2.103), (2.104)を代入して、

$$\frac{na^2e}{\eta}\cdot\frac{d\omega}{dt} + \frac{na^2e\cos I}{\eta}\frac{d\Omega}{dt}$$

$$= -Ra\cos f + rS\left(\frac{a}{r} + \frac{1}{\eta^2}\right)\sin f$$

⋮

$\vdots$

$$\frac{d\omega}{dt} = -\cos I \frac{d\Omega}{dt} + \frac{\eta}{na^2e}\left[-Ra\cos f + S\left(\frac{a}{r} + \frac{1}{\eta^2}\right)\sin f\right]$$

$$= -\cos I \frac{d\Omega}{dt} + \frac{\eta}{nae}\left[-R\cos f + S\left(1 + \frac{r}{a\eta^2}\right)\sin f\right] \quad \cdots (B.24)$$

∘ $l=1$ の場合 (a)

(B.7) より.

$$\sum_{j=1}^{6}[a, c_j]\frac{dc_j}{dt} = R\frac{\partial r}{\partial a} + r\left(R\hat{i}^* + S\hat{j}^* + W\hat{k}, \frac{\partial \hat{i}^*}{\partial a}\right) \quad \cdots ⑫$$

P.224 より.

$$\sum_{j=1}^{6}[a, c_j]\frac{dc_j}{dt} = [a, a]\frac{da}{dt} + [a, \omega]\frac{d\omega}{dt} + [a, \Omega]\frac{d\Omega}{dt} \quad \cdots ⑬$$

⑫, ⑬ より.

$$[a, a]\frac{da}{dt} + [a, \omega]\frac{d\omega}{dt} + [a, \Omega]\frac{d\Omega}{dt} = R\frac{\partial r}{\partial a} + r\left(R\hat{i}^* + S\hat{j}^* + W\hat{k}, \frac{\partial \hat{i}^*}{\partial a}\right)$$

$$\cdots (B.25)$$

(B.2) より.

$$\frac{\partial \hat{i}^*}{\partial a} = \frac{\partial \hat{i}^*}{\partial \phi}\frac{\partial \phi}{\partial f}\frac{\partial f}{\partial a}$$

$$= \begin{pmatrix} -\cos\Omega\sin\phi - \cos I\sin\Omega\cos\phi \\ -\sin\Omega\sin\phi + \cos I\cos\Omega\cos\phi \\ \sin I\cos\phi \end{pmatrix} \cdot 1 \cdot \frac{\partial f}{\partial a}$$

$\vdots$

$$\vdots$$

$$\frac{\partial \hat{i}^*}{\partial a} = \hat{j}^* \frac{\partial f}{\partial a} \qquad \cdots (B.26)$$

であるから、

$$(B.25)\text{の右辺} = R\frac{\partial r}{\partial a} + rS\frac{\partial f}{\partial a} \qquad \cdots ⑭$$

となる。

$$\frac{\partial r}{\partial a} = \frac{r}{a} + \frac{e}{\eta} r \sin f \frac{\partial u}{\partial a} \qquad \cdots (2.97)$$

$$= \frac{r}{a} + \frac{e}{\eta} r \sin f \cdot \frac{a}{r}\frac{dn}{da} t \qquad (\because 2.99)$$

$$= \frac{r}{a} + \frac{ae}{\eta} \sin f \frac{dn}{da} t \qquad \cdots ⑮$$

$$\frac{\partial f}{\partial a} = \frac{a\eta}{r}\frac{\partial u}{\partial a} \qquad \cdots (2.98)$$

$$= \frac{a\eta}{r} \cdot \frac{a}{r}\frac{dn}{da} t \qquad (\because 2.99)$$

$$= \frac{a^2\eta}{r^2}\frac{dn}{da} t \qquad \cdots ⑯$$

⑭へ⑮, ⑯を代入すると、

$$(B.25)\text{の右辺} = R\left(\frac{r}{a} + \frac{ae}{\eta}\sin f \frac{dn}{da} t\right) + rS \cdot \frac{a^2\eta}{r^2}\frac{dn}{da} t$$

$$= R\frac{r}{a} + t\frac{dn}{da}\left(\frac{ae}{\eta} R \sin f + \frac{a^2\eta}{r} S\right)$$

$$= R\frac{r}{a} + t\frac{dn}{da} \cdot \frac{na}{2}\frac{da}{dt} \qquad (\because B.10)$$

$$= R\frac{r}{a} + \frac{na}{2} t \frac{dn}{dt} \qquad \cdots ⑰$$

(B.25)へ (A.25)と⑰を代入する

$$-\frac{na}{2}\frac{da}{dt}+[a,\omega]\frac{d\omega}{dt}+[a,\Omega]\frac{d\Omega}{dt}=R\frac{r}{a}+\frac{na}{2}t\frac{dn}{dt}$$

$$\frac{na}{2}\frac{da}{dt}+\frac{na}{2}t\frac{dn}{dt}=[a,\omega]\frac{d\omega}{dt}+[a,\Omega]\frac{d\Omega}{dt}-\frac{r}{a}R$$

$$\therefore \underbrace{\frac{da}{dt}+t\frac{dn}{dt}}_{=\frac{da^I}{dt}\ (\because 5.133)}=\frac{2}{na}\left(\underbrace{[a,\omega]}_{=-\frac{na\eta}{2}\ (\because A.26)}\underbrace{\frac{d\omega}{dt}}_{=(B.24)}+\underbrace{[a,\Omega]}_{=-\frac{na\eta}{2}\cos I\ (\because A.28)}\underbrace{\frac{d\Omega}{dt}}_{=}-\frac{r}{a}R\right)\cdots ⑱$$

(B.21)だけど、これは後で代入しなくて…(計算の都合上)

$\Downarrow$

$$\frac{da^I}{dt}=-\eta\frac{d\omega}{dt}-\eta\cos I\frac{d\Omega}{dt}$$

$$=\eta\cos I\frac{d\Omega}{dt}-\frac{\eta^2}{nae}\left\{-R\cos f+S\left(1+\frac{r}{a\eta^2}\right)\sin f\right\}-\eta\cos I\frac{d\Omega}{dt}-\frac{2r}{na^2}R$$

$$=\left(\frac{\eta^2}{nae}\cos f-\frac{2r}{na^2}\right)R-\frac{\eta^2}{nae}\left(1+\frac{r}{a\eta^2}\right)S\sin f$$

$$=\frac{1}{na}\left(-\frac{2r}{a}+\frac{\eta^2}{e}\cos f\right)R-\frac{\eta^2}{nae}\left(1+\frac{r}{p}\right)S\sin f\quad\cdots(B.27)$$

B.2 2体問題の保存量を用いてガウスの惑星方程式の導出

摂動があるときの角運動量ベクトル(2.14)は、

$$\mathbf{h} = \mathbf{r} \times \dot{\mathbf{r}} = h \underbrace{\mathbf{k}}_{\text{軌道面に垂直な単位ベクトル}} \qquad \cdots (B.34)$$

である。

摂動があるときは、h, $\mathbf{k}$ が時間の関数である。

しかし、接触軌道要素を用いて、

$$h = \sqrt{\mu a (1-e^2)} \qquad \cdots ⑲$$

と表すことができ、$\mathbf{k}$ に関しては、(B.4)より

$$\mathbf{k} = T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin I \sin \Omega \\ -\sin I \cos \Omega \\ \cos I \end{pmatrix} \qquad \cdots (B.4)$$

である。

角運動量ベクトル $\mathbf{h}$ を時間微分する

$$\begin{aligned}
\frac{d\mathbf{h}}{dt} &= \frac{d}{dt}(\mathbf{r} \times \dot{\mathbf{r}}) \\
&= \underbrace{\dot{\mathbf{r}} \times \dot{\mathbf{r}}}_{= 0} + \mathbf{r} \times \ddot{\mathbf{r}} \\
&= \mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2} \\
&= \mathbf{r} \times \left(-\frac{\mu}{r^3}\mathbf{r} + \mathbf{X} \right) \qquad (\because 5.89) \\
&= \mathbf{r} \times \mathbf{X} \qquad \cdots (B.35)
\end{aligned}$$

$$\frac{d\mathbb{h}}{dt} = \frac{d}{dt}(h\mathbb{k})$$
$$= \frac{dh}{dt}\mathbb{k} + h\frac{d\mathbb{k}}{dt} \quad \cdots (B.36)$$

摂動力 (B.5) $\mathbb{X} = R\mathbb{i}^* + S\mathbb{j}^* + W\mathbb{k}$ を (B.35) へ代入する

$$\frac{d\mathbb{h}}{dt} = r\mathbb{i}^* \times (R\mathbb{i}^* + S\mathbb{j}^* + W\mathbb{k})$$
$$= rS\mathbb{k} - rW\mathbb{j}^* \quad \cdots (B.37)$$

(B.36) と (B.37) を比較して、

$$\frac{dh}{dt} = rS \quad \cdots (B.38)$$
$$\frac{d\mathbb{k}}{dt} = -\frac{r}{h}W\mathbb{j}^* \quad \cdots (B.39)$$

が得られる。

この (B.39) へ $\mathbb{k}$ と $\mathbb{j}^*$ の具体的表現 (B.4) と (B.3) を代入する

$$\begin{pmatrix} \cos I \sin\Omega \frac{dI}{dt} + \sin I \cos\Omega \frac{d\Omega}{dt} \\ -\cos I \cos\Omega \frac{dI}{dt} + \sin I \sin\Omega \frac{d\Omega}{dt} \\ -\sin I \frac{dI}{dt} \end{pmatrix} = -\frac{r}{h}W \begin{pmatrix} -\cos\Omega \sin\phi - \cos I \sin\Omega \cos\phi \\ -\sin\Omega \sin\phi + \cos I \cos\Omega \cos\phi \\ \sin I \cos\phi \end{pmatrix} \quad \begin{matrix} \cdots (B.40) \\ \cdots (B.41) \\ \cdots (B.42) \end{matrix}$$

(B.40)〜(B.42)からIとΩについての方程式を導出する

・(B.42)より、

$$\frac{dI}{dt}=\frac{r}{h}W\cos\phi \qquad \cdots(\mathrm{B}.43)$$

・(B.40)×cosΩ + (B.41)×sinΩ

$$\cos I\sin\Omega\cos\Omega\frac{dI}{dt}+\sin I\cos^2\Omega\frac{d\Omega}{dt}=\frac{r}{h}W(\cos^2\Omega\sin\phi+\cos I\sin\Omega\cos\Omega\cos\phi)$$

$$+)\ -\cos I\sin\Omega\cos\Omega\frac{dI}{dt}+\sin I\sin^2\Omega\frac{d\Omega}{dt}=\frac{r}{h}W(\sin^2\Omega\sin\phi-\cos I\sin\Omega\cos\Omega\cos\phi)$$

$$\sin I\frac{d\Omega}{dt}=\frac{r}{h}W\sin\phi$$

$$\therefore\ \frac{d\Omega}{dt}=\frac{r}{h\sin I}W\sin\phi \qquad \cdots(\mathrm{B}.44)$$

2体問題に特有な離心積分は、(2.52)より、

$$\frac{d\boldsymbol{r}}{dt}=\frac{\mu}{h^2}h\boldsymbol{k}\times(e\hat{\boldsymbol{i}}+\hat{\boldsymbol{i}}^*)$$

$$=\frac{\mu}{h}(e\boldsymbol{k}\times\hat{\boldsymbol{i}}+\boldsymbol{k}\times\hat{\boldsymbol{i}}^*)$$

$$=\frac{\mu}{h}(\boldsymbol{k}\times\hat{\boldsymbol{i}}^*+e\boldsymbol{j}) \qquad \cdots(\mathrm{B}.45)$$

(B.45)の両辺を時間微分する

$$-\frac{\mu}{r^3}\mathbb{r}+\mathbb{X} = \mu\frac{d}{dt}(h^{-1})\left(\mathbb{k}\times\hat{\mathbb{i}}^*+e\hat{\mathbb{j}}\right)+\frac{\mu}{h}\left(\frac{d\mathbb{k}}{dt}\times\hat{\mathbb{i}}^*+\mathbb{k}\times\frac{d\hat{\mathbb{i}}^*}{dt}+\frac{de}{dt}\hat{\mathbb{j}}+e\frac{d\hat{\mathbb{j}}}{dt}\right)$$

$(\because 5.89)$

$$\therefore -\frac{\mu}{r^3}\mathbb{r}+\mathbb{X} = -\frac{\mu}{h^2}\frac{dh}{dt}\left(\mathbb{k}\times\hat{\mathbb{i}}^*+e\hat{\mathbb{j}}\right)+\frac{\mu}{h}\left(\frac{d\mathbb{k}}{dt}\times\hat{\mathbb{i}}^*+\mathbb{k}\times\frac{d\hat{\mathbb{i}}^*}{dt}+\frac{de}{dt}\hat{\mathbb{j}}+e\frac{d\hat{\mathbb{j}}}{dt}\right) \quad \cdots (B.46)$$

$\hat{\mathbb{i}}^*$の時間微分は(2.44)より、

$$\frac{d\hat{\mathbb{i}}^*}{dt} = \frac{d}{dt}\left(\frac{\mathbb{r}}{r}\right)$$

$$= \frac{(\mathbb{r}\times\dot{\mathbb{r}})\times\mathbb{r}}{r^3}$$

$$= \frac{\mathbb{h}\times\mathbb{r}}{r^3}$$

$$= \frac{h\,\mathbb{k}\times r\hat{\mathbb{i}}^*}{r^3} \quad (\because B.34,\ B.6)$$

$$= \frac{h\,\mathbb{k}\times\hat{\mathbb{i}}^*}{r^2}$$

$$= \frac{h\hat{\mathbb{j}}^*}{r^2} \quad \cdots (B.47)$$

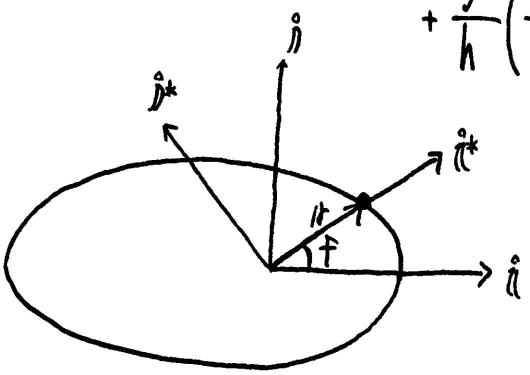
(B.46)へ(B.47),(B.38),(B.39)を代入すると、

$$-\frac{\mu}{r^3}\underbrace{r\mathbf{i}^*}_{(\because B.6)} + \mathbf{X} = -\frac{\mu}{h^2}\underbrace{rS}_{(B.38)}(\mathbf{k}\times\mathbf{i}^* + e\mathbf{j}) + \frac{\mu}{h}\Big(\underbrace{-\frac{r}{h}W\mathbf{j}^*}_{(\because B.39)}\times\mathbf{i}^* + \mathbf{k}\times\underbrace{\frac{h\mathbf{j}^*}{r^2}}_{(\because B.47)} + \frac{de}{dt}\mathbf{j} + e\frac{d\mathbf{j}}{dt}\Big)$$

$$\therefore \mathbf{X} = -\frac{\mu}{h^2}rS(\mathbf{k}\times\mathbf{i}^* + e\mathbf{j}) + \frac{\mu}{h}\Big(\frac{r}{h}W\mathbf{k} + \frac{de}{dt}\mathbf{j} + e\frac{d\mathbf{j}}{dt}\Big) \quad \cdots (B.48)$$

(B.48)の両辺とjの内積をとる

$$(R\mathbf{i}^* + S\mathbf{j}^* + W\mathbf{k})\cdot\mathbf{j} = -\frac{\mu}{h^2}rS(\mathbf{j}^*\cdot\mathbf{j} + e\underbrace{\mathbf{j}\cdot\mathbf{j}}_{=1})$$
$$+\frac{\mu}{h}\Big(\frac{r}{h}W\underbrace{\mathbf{k}\cdot\mathbf{j}}_{=0} + \frac{de}{dt}\underbrace{\mathbf{j}\cdot\mathbf{j}}_{=1} + e\underbrace{\frac{d\mathbf{j}}{dt}\cdot\mathbf{j}}_{=0}\Big)$$



(左図より $\mathbf{i}^*\cdot\mathbf{j} = 1\cdot1\cdot\cos(\frac{\pi}{2}-f) = \sin f$
$\mathbf{j}^*\cdot\mathbf{j} = 1\cdot1\cdot\cos f = \cos f$
$(\mathbf{k}\cdot\mathbf{j} = 0)$)

$$R\sin f + S\cos f = -\frac{\mu}{h^2}rS(\cos f + e) + \frac{\mu}{h}\frac{de}{dt} \quad \cdots (B.49)$$

$$\Rightarrow \frac{de}{dt} = \frac{h}{\mu}\Big\{R\sin f + S\cos f + \frac{\mu}{h^2}rS(\cos f + e)\Big\}$$

$\Big($ $\frac{h}{\mu} = \frac{\sqrt{\mu a\eta^2}}{\mu}$ (∵P.230) $= \frac{\sqrt{n^2a^4\eta^2}}{n^2a^3} = \frac{\eta}{na}$

$\frac{\mu}{h^2}r = \frac{\mu}{\mu a\eta^2}\cdot\frac{a\eta^2}{1+e\cos f}$ (∵P.230) (∵2.56) $\therefore = \frac{1}{1+e\cos f}$

$\cos f + e = \cos u(1+e\cos f)$ (∵2.61) $\Big)$

$$\therefore \frac{de}{dt} = \frac{\eta}{na}\Big\{R\sin f + S(\cos f + \cos u)\Big\} \quad \cdots (B.50)$$

$h=\sqrt{\mu a(1-e^2)}$ を時間微分する

$$\frac{dh}{dt}=\frac{1}{2}\left\{\mu a(1-e^2)\right\}^{-\frac{1}{2}}\left[\frac{d}{da}\left\{\mu a(1-e^2)\right\}+\frac{d}{de}\left\{\mu a(1-e^2)\right\}\right]$$

$$=\frac{1}{2h}\left[\mu(1-e^2)\frac{da}{dt}-2\mu ae\frac{de}{dt}\right]$$

$$=\frac{1}{2h}\left(\mu\eta^2\frac{da}{dt}-2\mu ae\frac{de}{dt}\right)$$

この式をaについての方程式の形に変形する

$$\mu\eta^2\frac{da}{dt}=2h\frac{dh}{dt}+2\mu ae\frac{de}{dt}$$

$$\frac{da}{dt}=\frac{2h}{\mu\eta^2}\underbrace{\frac{dh}{dt}}_{(\because B.38)}+\frac{2ae}{\eta^2}\underbrace{\frac{de}{dt}}_{(\because B.50)}$$

$$=\frac{2h}{\mu\eta^2}rS+\frac{2ae}{\eta^2}\cdot\frac{\eta}{na}\left\{R\sin f+S(\cos f+\cos u)\right\}$$

$$\left(\frac{2na^2\eta}{n^2a^3\eta^2}=\frac{2}{na\eta}\right)$$

$$=\frac{2}{n\eta}\left\{Re\sin f+S\left(e\cos f+e\cos u+\frac{r}{a}\right)\right\}$$

fにそろえる

$$e\cos f+e\cdot\underbrace{\frac{\cos f+e}{1+e\cos f}}_{(\because 2.61)}+\underbrace{\frac{1-e^2}{1+e\cos f}}_{(\because 2.56)}$$

$$=\frac{e\cos f+e^2\cos^2 f+e\cos f+e^2+1-e^2}{1+e\cos f}$$

$(\because p=a\eta^2)$

$$=\frac{e^2\cos^2 f+2e\cos f+1}{1+e\cos f}=1+e\cos f=\underbrace{\frac{a\eta^2}{r}}_{(\because 2.56)}=\frac{p}{r}$$

⋮

⋮

$$\frac{da}{dt} = \frac{2}{n\eta}\left(Re\sin f + \frac{p}{r}S\right) \quad \cdots (B.51)$$

(B.48)より j の時間微分を求める

$$\mathbb{X} = -\frac{\mu}{h^2} rS(k \times i^* + ej) + \frac{\mu}{h}\left(\frac{r}{h}Wk + \frac{de}{dt}j + e\frac{dj}{dt}\right) \quad \cdots (B.48)$$

$$\Rightarrow e\frac{dj}{dt} = \frac{h}{\mu}(Ri^* + Sj^* + Wk) + \frac{1}{h}rS(j^* + ej) - \left(\frac{r}{h}Wk + \frac{de}{dt}j\right)$$

$$= \underbrace{\frac{h}{\mu}R}_{\text{i}}\, i^* + \underbrace{\left(\frac{h}{\mu} + \frac{r}{h}\right)S}_{\text{ii}}\, j^* + \underbrace{\left(\frac{1}{h}rSe - \frac{de}{dt}\right)}_{\text{iii}} j + \underbrace{\left(\frac{h}{\mu} - \frac{r}{h}\right)W}_{\text{iv}}\, k$$

$$\text{i} = \frac{\sqrt{n^2a^4\eta^2}}{n^2a^3}R = \frac{na^2\eta}{n^2a^3}R = \frac{\eta}{na}R$$

$$\text{ii} = \left(\frac{h}{\mu} + \frac{r}{h}\right)S = \left(\frac{\eta}{na} + \frac{r}{na^2\eta}\right)S = \frac{\eta}{na}\left(1 + \frac{r}{a\eta^2}\right)S = \frac{\eta}{na}\left(1 + \frac{r}{p}\right)S$$

$$\text{iii} = \frac{1}{h}rSe - \frac{\eta}{na}\left\{R\sin f + S(\cos f + \cos u)\right\} \quad (\because B.50)$$

$$= \frac{1}{na^2\eta}\cdot \underbrace{\frac{a\eta^2}{1+e\cos f}}_{(\because 2.56)}\cdot Se - \frac{\eta}{na}\left\{ \sim \right\}$$

$$= \frac{\eta}{na}\left[\left(\underbrace{\frac{e}{1+e\cos f} - \cos f - \underbrace{\cos u}_{(\because 2.61)}}_{\downarrow\ f\text{に統一}}\right)S - R\sin f\right]$$

$$\left(\begin{aligned}
&\frac{e}{1+e\cos f} - \cos f - \frac{\cos f + e}{1+e\cos f} \\
&= \frac{1}{1+e\cos f}\left(e - \cos f - e\cos^2 f - \cos f - e\right) = \frac{-\cos f(1+1+e\cos f)}{1+e\cos f} \\
&= -\cos f\left(\frac{1}{1+e\cos f} + 1\right) = -\left(1 + \frac{r}{p}\right)\cos f
\end{aligned}\right)$$

⋮

$$
\begin{cases}
\text{(iii)} = \dfrac{\eta}{na}\left\{-\left(1+\dfrac{r}{p}\right)\cos f\, S - \quad R\sin f\right\} \\
\text{(iv)} = \dfrac{\eta}{na}\left(1-\dfrac{r}{p}\right)S \qquad (\because \text{(ii)})
\end{cases}
$$

⋮

$$e\frac{d\hat{\jmath}}{dt} = \frac{\eta}{na}R\hat{\imath}^* + \frac{\eta}{na}\left(1+\frac{r}{p}\right)S\hat{\jmath}^* + \frac{\eta}{na}\left\{-\left(1+\frac{r}{p}\right)\cos f\cdot S - R\sin f\right\}\hat{\jmath} + \frac{\eta}{na}\left(1-\frac{r}{p}\right)S\mathbb{k}$$

$\hat{\imath}^*, \hat{\jmath}^*$を$\hat{\imath}, \hat{\jmath}$へ変換するには、B-20ページの図からもわかるように、

$$\begin{pmatrix}\hat{\imath}^* \\ \hat{\jmath}^*\end{pmatrix} = \begin{pmatrix}\cos f & \sin f \\ -\sin f & \cos f\end{pmatrix}\begin{pmatrix}\hat{\imath} \\ \hat{\jmath}\end{pmatrix} = \begin{pmatrix}\cos f\,\hat{\imath} + \sin f\,\hat{\jmath} \\ -\sin f\,\hat{\imath} + \cos f\,\hat{\jmath}\end{pmatrix}$$

と回転させればよい。

よって、

$$e\frac{d\hat{\jmath}}{dt} = \frac{\eta}{na}R(\cos f\cdot\hat{\imath} + \sin f\cdot\hat{\jmath}) + \frac{\eta}{na}\left(1+\frac{r}{p}\right)S(-\sin f\cdot\hat{\imath} + \cos f\cdot\hat{\jmath})$$
$$+ \frac{\eta}{na}\left\{-\left(1+\frac{r}{p}\right)\cos f\cdot S - \sin f\cdot R\right\}\hat{\jmath} + \frac{\eta}{na}\left(1-\frac{r}{p}\right)S\mathbb{k}$$

$$\therefore \frac{d\hat{\jmath}}{dt} = \frac{\eta}{nae}\left[\left(1-\frac{r}{p}\right)W\mathbb{k} + \left\{R\cos f - S\left(1+\frac{r}{p}\right)\sin f\right\}\hat{\imath}\right] \quad \cdots (B.52)$$

$\hat{i}, \hat{j}$は $f=0$ のときの $\hat{i}^*, \hat{j}^*$ であるため、(B.2), (B.3)より、

$$\hat{i} = T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}\cos\Omega\cos\omega - \cos I\sin\Omega\sin\omega\\ \sin\Omega\cos\omega + \cos I\cos\Omega\sin\omega\\ \sin I\sin\omega\end{pmatrix} \quad \cdots (B.53)$$

$$\hat{j} = T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}-\cos\Omega\sin\omega - \cos I\sin\Omega\cos\omega\\ -\sin\Omega\sin\omega + \cos I\cos\Omega\cos\omega\\ \sin I\cos\omega\end{pmatrix} \quad \cdots (B.54)$$

(B.52)へ(B.53), (B.54), (B.4)を代入したものの第3成分からωについての方程式が得られる。

$$\frac{d}{dt}\begin{pmatrix}\sim\\\sim\\\sin I\cos\omega\end{pmatrix} = \frac{\eta}{nae}\left[\left(1-\frac{r}{p}\right)W\begin{pmatrix}\sim\\\sim\\\cos I\end{pmatrix} + \left\{R\cos f - S\left(1+\frac{r}{p}\right)\sin f\right\}\begin{pmatrix}\sim\\\sim\\\sin I\sin\omega\end{pmatrix}\right]$$

↓第3成分のみ取り出し

$$\cos I\cos\omega\frac{dI}{dt} - \sin I\sin\omega\frac{d\omega}{dt}$$

$$= \frac{\eta}{nae}\left[\left(1-\frac{r}{p}\right)W\cos I + \left\{R\cos f - S\left(1+\frac{r}{p}\right)\sin f\right\}\sin I\sin\omega\right]$$

$$\frac{d\omega}{dt} = \frac{\cos I\cos\omega}{\sin I\sin\omega}\cdot\frac{r}{h}W\cos\phi - \frac{\eta}{nae}\left[\left(1-\frac{r}{p}\right)W\cdot\frac{\cos I}{\sin I\sin\omega} + R\cos f - S\left(1+\frac{r}{p}\right)\sin f\right]$$

$$= \frac{\cos I}{\sin I\sin\omega}\cdot\frac{r}{na^2\eta}W\Big[\underbrace{\cos\omega\cos\phi - \frac{a\eta^2}{re}\left(1-\frac{r}{p}\right)}_{(\text{ア})}\Big]$$

$$- \frac{\eta}{nae}\left\{R\cos f - S\left(1+\frac{r}{p}\right)\sin f\right\}$$

⋮

$$
\begin{aligned}
③ &= \cos\omega\cos\phi - \frac{1+e\cos f}{a\eta^2}\cdot\frac{a\eta^2}{e}\left(1-\frac{r}{p}\right) \\
&= \cos\omega\cos\phi - \frac{1+e\cos f}{e}\left(1-\frac{r}{p}\right) \\
&\quad\downarrow (\because 2.56) \\
&= \cos\omega\cos\phi - \frac{p}{er}\left(1-\frac{r}{p}\right) \\
&= \cos\omega\cos\phi - \frac{1}{e}\left(\frac{p}{r}-1\right) \\
&\quad\downarrow (\because 2.56) \\
&= \cos\omega\cos\phi - \frac{1}{e}\left(e\cos f\right) \\
&= \cos\omega\cos\phi - \cos(\phi-\omega) \quad (\because \phi = f+\omega) \\
&= \cos\omega\cos\phi - \cos\phi\cos\omega - \sin\phi\sin\omega \\
&= -\sin\phi\sin\omega
\end{aligned}
$$

$\vdots$

$$
\begin{aligned}
\frac{d\omega}{dt} &= -\frac{\cos I\sin\phi\sin\omega}{\sin I\sin\omega}\cdot\frac{r}{na^2\eta}W - \frac{\eta}{nae}\left\{R\cos f - S\left(1+\frac{r}{p}\right)\sin f\right\} \\
&= -\frac{r\sin\phi}{na^2\eta}W\cot I + \frac{\eta}{nae}\left\{-R\cos f + S\left(1+\frac{r}{p}\right)\sin f\right\} \quad \cdots (B.55)
\end{aligned}
$$

が導かれる。

エネルギー積分 $\frac{1}{2}v^2 - \frac{\mu}{r} = -\frac{\mu}{2a}$ よりaについての方程式を導く

まずは、エネルギー積分の両辺の時間微分をとる

$$u \cdot \frac{du}{dt} - \mu(-r^{-2})\frac{dr}{dt} = -\frac{\mu}{2}(-a^{-2})\frac{da}{dt}$$

$$u \cdot \frac{du}{dt} + \frac{\mu}{r^2}\frac{dr}{dt} = \frac{\mu}{2a^2}\frac{da}{dt} \quad \cdots (B.56)$$

(B.56)の左辺第1項は、

$$u \cdot \frac{du}{dt} = u \cdot \left(-\frac{\mu}{r^3}\mathbb{r} + \mathbb{X}\right)$$

(∵ 5.89)

$$= -\frac{\mu}{r^3} u \cdot \mathbb{r} + u \cdot \mathbb{X}$$

$$= -\frac{\mu}{r^3} \cdot \frac{\mu}{h}\underbrace{\left\{(\hat{j}^* + e\hat{j}) \cdot r\hat{i}^*\right\}} + u \cdot \mathbb{X}$$

⋮

$$\left(\begin{aligned} &\cdot \hat{j}^* \cdot \hat{i}^* = 0 \\ &\cdot \hat{j} \cdot \hat{i}^* = 1 \cdot 1 \cdot \cos\left(\frac{\pi}{2} - f\right) \\ &\qquad = \sin f \end{aligned}\right)$$

$$= -\frac{n^2a^3}{r^3} \cdot \frac{n^2a^3}{na^2\eta} \cdot er\sin f + u \cdot \mathbb{X}$$

$$= -\frac{\mu an}{r^2\eta} e\sin f + \underset{\sim}{u \cdot \mathbb{X}} \quad \cdots (B.57)$$

また、(B.5)、(B.45)より、

$$u \cdot \mathbb{X} = \frac{\mu}{h}(\hat{j}^* + e\hat{j}) \cdot (R\hat{i}^* + S\hat{j}^* + W\mathbb{k})$$

⋮

$$\vdots$$

$$\boldsymbol{u}\cdot\boldsymbol{x} = \frac{\mu}{h}\Big(R\underbrace{\hat{j}^*\cdot\hat{i}^*}_{0} + S\underbrace{\hat{j}^*\cdot\hat{j}^*}_{1} + W\underbrace{\hat{j}^*\cdot\hat{k}}_{0} + eR\underbrace{\hat{j}\cdot\hat{i}^*}_{\cos(\frac{\pi}{2}-f)=\sin f} + eS\underbrace{\hat{j}\cdot\hat{j}^*}_{\cos f\ (\because 図B.1)} + eW\underbrace{\hat{j}\cdot\hat{k}}_{0}\Big)$$

$$\vdots$$

$$= \frac{\mu}{h}\left(S + eR\sin f + eS\cos f\right)$$

$$= \frac{\mu}{h}\left\{Re\sin f + (1+e\cos f)S\right\}$$

$$= \frac{\mu}{na^2\eta}\left[Re\sin f + \overset{\downarrow(\because 2.56)}{\frac{p}{r}}S\right]$$

$$= \frac{\mu}{2a^2}\frac{da}{dt} \quad \cdots (B.58)$$

(B.57), (B.58)を(B.56)へ代入すると、

$$-\frac{\mu an}{r^2\eta}e\sin f + \frac{\mu}{2a^2}\frac{da}{dt} + \frac{\mu}{r^2}\frac{dr}{dt} = \frac{\mu}{2a^2}\frac{da}{dt}$$

$$\Rightarrow \frac{dr}{dt} = \frac{an}{\eta}e\sin f$$

$$\therefore \frac{dr}{dt} = \frac{\mu}{h}e\sin f \quad \cdots (B.59)$$

摂動を受けている動径rに含まれている軌道要素はa, e, αだけなのでrの時間微分は、

$$\frac{dr}{dt} = \frac{\partial r}{\partial t} + \frac{\partial r}{\partial a}\frac{da}{dt} + \frac{\partial r}{\partial e}\frac{de}{dt} + \frac{\partial r}{\partial \alpha}\frac{d\alpha}{dt} \quad \cdots (B.60)$$

したがって、

$$\underbrace{\frac{\partial r}{\partial a}}_{\substack{(2.97)\\+\\(2.99)}}\frac{da}{dt}+\underbrace{\frac{\partial r}{\partial e}}_{(2.103)}\frac{de}{dt}+\underbrace{\frac{\partial r}{\partial \sigma}}_{(1\text{-}1,B\text{-}4)}\frac{d\sigma}{dt}=0 \qquad \cdots(B.61)$$

$$\left(\frac{r}{a}+\frac{ae}{\eta}\sin f\frac{dn}{da}t\right)\frac{da}{dt}+\left(-a\cos f\right)\frac{de}{dt}+\left(\frac{ae}{\eta}\sin f\right)\frac{d\sigma}{dt}=0$$

$$\left(\frac{r}{a}+\frac{ae}{\eta}t\sin f\frac{dn}{da}\right)\frac{da}{dt}-a\cos f\frac{de}{dt}+\frac{ae}{\eta}\sin f\frac{d\sigma}{dt}=0 \qquad \cdots(B.62)$$

$$\frac{r}{a}\underbrace{\frac{da}{dt}}_{(B.51)}-a\cos f\underbrace{\frac{de}{dt}}_{(B.50)}+\frac{ae}{\eta}\sin f\underbrace{\left(\frac{d\sigma}{dt}+t\frac{dn}{dt}\right)}_{\substack{\parallel\\ \frac{d\sigma^*}{dt}\ (\because 5.3.2項)}}=0 \qquad \cdots(B.63)$$

$$\Downarrow$$

$$\frac{ae}{\eta}\sin f\frac{d\sigma^*}{dt}=-\frac{r}{a}\cdot\frac{2}{n\eta}\left(Re\sin f+\frac{p}{r}S\right)+a\cos f\cdot\frac{1}{na}\left\{R\sin f+S(\cos f+\cos u)\right\}$$

$$=\underbrace{\left(-\frac{2re}{an\eta}\sin f+\frac{\eta\cos f}{n}\sin f\right)}_{①}R$$

$$+\underbrace{\left\{-\frac{2\eta}{n}+\frac{\eta}{n}\cos f(\cos f+\cos u)\right\}}_{②}S$$

$$①=\frac{ae}{\eta}\sin f\left(-\frac{2r}{a^2n}+\frac{\eta^2}{aen}\cos f\right)$$

$$=\frac{ae}{\eta}\sin f\cdot\frac{1}{na}\left(-\frac{2r}{a}+\frac{\eta^2}{e}\cos f\right)$$

$$⑰ = \frac{\eta}{n}\left\{-2 + \cos f\left(\cos f + \underbrace{\frac{\cos f + e}{1+e\cos f}}_{(\because 2.61)}\right)\right\}$$

$$= \frac{\eta}{n}\left\{-1 + \frac{-1 - e\cos f + \cos^2 f + e\cos^3 f + \cos^2 f + e\cos f}{1+e\cos f}\right\}$$

$$= \frac{\eta}{n}\left\{-1 + \frac{-1 + 2\cos^2 f + e\cos^3 f}{1+e\cos f}\right\}$$

$$= \frac{\eta}{n}\left\{-1 + \frac{\cos^2 f(1+e\cos f) + \cos^2 f - 1}{1+e\cos f}\right\}$$

$$= \frac{\eta}{n}\left\{-1 + \cos^2 f + \frac{\cos^2 f - 1}{1+e\cos f}\right\}$$

$$= -\frac{\eta}{n}(1-\cos^2 f)\left(1 + \frac{1}{1+e\cos f}\right)$$

$$= -\frac{\eta}{n}\sin^2 f\left(1 + \frac{r}{p}\right)$$

$\vdots$

$$\therefore \frac{da^I}{dt} = \frac{1}{na}\left(-\frac{2r}{a} + \frac{\eta^2}{e}\cos f\right)R - \frac{\eta^2}{nae}\left(1 + \frac{r}{p}\right)S\sin f \quad \cdots(B.64)$$