D.5 ベッセル関数

D.5.1 離心近点離角のフーリエ展開

$$e \sin u = \sum_{n=1}^{\infty} b_n \sin n\ell \quad \cdots (D.27)$$

教科書はここに2がついているが、いつも自分がフーリエ・サイン級数展開するときの方法でやる。

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e \sin u \sin n\ell \, d\ell$$

$$= \frac{2}{\pi}\int_{0}^{\pi} e \sin u \sin n\ell \, d\ell \quad (\because \text{奇関数}) \quad \cdots (D.28)$$

$$= \frac{2}{\pi}\int_{0}^{\pi} e \sin u \left(-\frac{1}{n}\cos n\ell\right)' d\ell$$

$$= \frac{2}{\pi}\left\{-\left[\frac{e}{n}\sin u \cos n\ell\right]_0^{\pi} + \frac{1}{n}\int_0^{\pi} \cos n\ell \frac{d(e\sin u)}{d\ell} \cdot d\ell\right\}$$

$$= \underbrace{\left[-\frac{2e}{n\pi}\sin u \cos n\ell\right]_0^{\pi}}_{=0} + \frac{2}{n\pi}\int_0^{\pi} \cos n\ell \frac{d(e\sin u)}{d\ell} d\ell$$

$\left(\because\ \ell = \pi, 0 \text{のとき} \; u = \pi, 0\right)$

ℓとuは異なる変数だが、同じ周期で動いている

$$= \frac{2}{n\pi}\int_0^{\pi} \cos n\ell \cdot \frac{d}{d\ell}(u-\ell) \cdot d\ell$$

$$= \frac{2}{n\pi}\int_0^{\pi} \cos n\ell \left(\frac{du}{d\ell} - 1\right) \cdot d\ell$$

$$= \frac{2}{n\pi}\left(\int_0^{\pi} \cos n\ell \, du - \underbrace{\int_0^{\pi} \cos n\ell \cdot d\ell}_{=0}\right)$$

$$= \frac{2}{n\pi}\int_0^{\pi} \cos n(u - e\sin u) du \quad (\because \text{ケプラー方程式})$$

$$\cdots (D.29)$$

ここで、

$$f(x)=\frac{1}{\pi}\int_0^{\pi}\cos\theta\, du \quad , \quad \theta=nu-x\sin u \quad \cdots (D.30)$$

と定義する

(D.29)と(D.30)より、

$$b_n=\frac{1}{n}f(ne) \quad \cdots (D.31)$$

$$\frac{df(x)}{dx}=\frac{1}{\pi}\int_0^{\pi}\frac{d}{dx}(\cos\theta)\,du$$

$$=\frac{1}{\pi}\int_0^{\pi}(-\sin\theta)(-\sin u)\,du$$

$$=\frac{1}{\pi}\int_0^{\pi}\sin u\sin\theta\, du \quad \cdots (D.32)$$

$$=\frac{1}{\pi}\int_0^{\pi}(-\cos u)'\sin\theta\, du$$

$$=\frac{1}{\pi}\left\{\underbrace{\Big[-\cos u\sin\theta\Big]_{u=0}^{u=\pi}}_{=0}+\int_0^{\pi}\cos u\,\frac{d(\sin\theta)}{du}\,du\right\}$$

$$\left(\begin{array}{l}\because \theta=nu-x\sin u \text{ より} \\ \begin{array}{c|ccc} u & 0 & \cdots & \pi \\ \hline \theta & 0 & \cdots & n\pi \end{array}\end{array}\right)$$

$$=\frac{1}{\pi}\int_0^{\pi}\cos u\cdot\cos\theta\cdot\frac{d\theta}{du}\cdot du$$

$$=\frac{1}{\pi}\int_0^{\pi}(n-x\cos u)\cos u\cos\theta\, du \quad \cdots (D.33)$$

(D.32) を x で微分すると、

$$\frac{d^2 f(x)}{dx^2} = \frac{1}{\pi}\int_0^{\pi} \frac{d}{dx}\left(\sin u \sin\theta\right) du$$

$$= \frac{1}{\pi}\int_0^{\pi}\Big(\cos u \sin\theta \underbrace{\frac{du}{d\theta}}_{=0} + \sin u \cdot \cos\theta \cdot \frac{d\theta}{dx}\Big) du$$

∵ (u は離心近点離角、θ は ne なので
互いに独立)

$$= \frac{1}{\pi}\int_0^{\pi} \sin u \cos\theta \left(-\sin u\right) du$$

$$= \frac{-1}{\pi}\int_0^{\pi} \sin^2 u \cdot \cos\theta \, du \quad \cdots \text{(D.34)}$$

(D.34)(D.35) から、

$$x^2\frac{d^2 f}{dx^2} + x\frac{df}{dx} + (x^2-n^2) f$$

$$= \frac{-x^2}{\pi}\int_0^{\pi}\sin^2 u \cos\theta \, du + \frac{x}{\pi}\int_0^{\pi}(n - x\cos u)\cos u \cos\theta \, du + \frac{(x^2-n^2)}{\pi}\int_0^{\pi}\cos\theta \, du$$

$$= \int_0^{\pi}\frac{\cos\theta}{\pi}\left(-x^2\sin^2 u + nx\cos u - x^2\cos^2 u + x^2 - n^2\right) du$$

$$= \int_0^{\pi}\frac{\cos\theta}{\pi}\left(nx\cos u - n^2\right) du$$

$$= -\frac{n^2}{\pi}\int_0^{\pi}\left(1 - \frac{x}{n}\cos u\right)\cos\theta \, du \quad \cdots \text{(D.35)}$$

ここで、$u^* = u - \frac{x}{n}\sin u$ とおくと、ビブン $\left(\frac{du^*}{du} = 1 - \frac{x}{n}\cos u\right)$、(D.35) (の右辺) は、

$$(\text{右辺}) = -\frac{n^2}{\pi}\int_0^{\pi}\frac{du^*}{du}\cdot\cos(nu^*)\cdot du \quad \left(\because \begin{cases}\theta = nu - x\sin u \\ u^* = u - \frac{x}{n}\sin u\end{cases} \rightarrow \theta = nu^*\right)$$

$$= -\frac{n^2}{\pi}\int_0^{\pi}\cos(nu^*)\, du^*$$

$$= 0 \quad \cdots \text{(D.36)}$$

以上より、

$$x^2\frac{d^2f}{dx^2}+x\frac{df}{dx}+(x^2-n^2)f=0$$

$$\frac{d^2f}{dx^2}+\frac{1}{x}\frac{df}{dx}+\left(1-\frac{n^2}{x^2}\right)f=0 \quad \cdots (D.37)$$

よって、ケプラー方程式を変形していくと、

$$u-l=e\sin u$$

$$u=l+e\sin u$$

$$u=l+2\sum_{n=1}^{\infty}b_n\sin nl \quad (\because D.27)$$

$$u=l+2\sum_{n=1}^{\infty}\frac{1}{n}f(ne)\sin nl \quad (\because D.31)$$

$$u=l+2\sum_{n=1}^{\infty}\frac{1}{n}J_n(ne)\sin nl \quad \cdots (D.38)$$

D.5.2

ケプラー方程式を、$u(l,e)$であることに注意してlとeについて偏微分する

$$u-e\sin u=l \quad \cdots (D.45)$$

・lについて偏微分

$$\frac{\partial u}{\partial l}-e\cos u\frac{\partial u}{\partial l}=1$$

$$\frac{\partial u}{\partial l}=\frac{1}{1-e\cos u}$$

$$=\frac{a}{r} \quad \cdots (D.46) \quad (\because 2.59)$$

・eについて偏微分

$$\frac{\partial u}{\partial e}-\left(\sin u+e\cos u\frac{\partial u}{\partial e}\right)=0$$

$$(1-e\cos u)\frac{\partial u}{\partial e}=\sin u$$

$$\frac{\partial u}{\partial e}=\frac{a}{r}\sin u \quad \cdots (D.47) \qquad (\because 2.59)$$

(D.39)は(D.46)から求められる

$$\frac{a}{r}=\frac{\partial u}{\partial l} \quad \cdots (D.46)$$

$$=\frac{\partial}{\partial l}\left(l+2\sum_{n=1}^{\infty}\frac{1}{n}J_n(ne)\sin nl\right)$$

$$=1+2\sum_{n=1}^{\infty}\frac{1}{n}J_n(ne)\cdot n\cos nl$$

$$=1+2\sum_{n=1}^{\infty}J_n(ne)\cos nl \quad \cdots (D.39)$$

(D.43)を求めるために、$\cos u$をlで偏微分して、フーリエ展開したのちlで積分する

・lで偏微分

$$\frac{\partial}{\partial l}\cos u=-\sin u\cdot\frac{\partial u}{\partial l}$$

$$=-\sin u\cdot\frac{a}{r} \quad (\because D.46)$$

$$=-\frac{\partial u}{\partial e} \quad (\because D.47)$$

・uの lについての フーリエ展開

$$=-\frac{\partial}{\partial e}\left\{l+2\sum_{n=1}^{\infty}\frac{1}{n}J_n(ne)\sin nl\right\} \quad (\because D.38)$$

$$=-2\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{d}{de}J_n(ne)\right)\sin nl \quad \cdots (D.49)$$