D.4

ωの定義(D.20)より、

$$\frac{1+\omega}{1-\omega} = \frac{1+\frac{e}{1+\sqrt{1-e^2}}}{1-\frac{e}{1+\sqrt{1-e^2}}}$$

$$= \frac{1+\sqrt{1-e^2}+e}{1+\sqrt{1-e^2}-e} = \frac{\sqrt{1+e}\sqrt{1+e}+\sqrt{1+e}\sqrt{1-e}}{\sqrt{1-e}\sqrt{1-e}+\sqrt{1+e}\sqrt{1-e}}$$

$$= \frac{\sqrt{1+e}(\sqrt{1+e}+\sqrt{1-e})}{\sqrt{1-e}(\sqrt{1-e}+\sqrt{1+e})} = \sqrt{\frac{1+e}{1-e}} \quad \cdots (D.22)$$

三角関数の指数関数表示 $\sin x = \frac{1}{2i}(e^{ix}-e^{-ix})$, $\cos x = \frac{1}{2}(e^{ix}+e^{-ix})$
を使って(D.21)を書き換えると、

$$\tan\frac{f}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{u}{2} \quad \cdots (D.21)$$

$$\frac{\sin\frac{f}{2}}{\cos\frac{f}{2}} = \sqrt{\frac{1+e}{1-e}}\,\frac{\sin\frac{u}{2}}{\cos\frac{u}{2}}$$

$$\frac{1}{i}\cdot\frac{e^{\frac{if}{2}}-e^{-\frac{if}{2}}}{e^{\frac{if}{2}}+e^{-\frac{if}{2}}} = \underbrace{\frac{1+\omega}{1-\omega}}_{(\because D.22)}\cdot\frac{1}{i}\cdot\frac{e^{\frac{iu}{2}}-e^{-\frac{iu}{2}}}{e^{\frac{iu}{2}}+e^{-\frac{iu}{2}}} \quad \cdots (D.23)$$

$$\frac{e^{if}-1}{e^{if}+1} = \frac{1+\omega}{1-\omega}\,\frac{1-e^{-iu}}{1+e^{-iu}}$$

$$(1-\omega)(1+e^{-iu})(e^{if}-1) = (1+\omega)(1-e^{-iu})(e^{if}+1)$$

$$\{(1-\omega)(1+e^{-iu})-(1+\omega)(1-e^{-iu})\}e^{if} = (1+\omega)(1-e^{-iu})+(1-\omega)(1+e^{-iu})$$

$$e^{if} = \frac{1-\omega e^{-iu}}{e^{-iu}-\omega} = e^{iu}\frac{1-\omega e^{-iu}}{1-\omega e^{iu}} \quad \cdots (D.24)$$

(D.24)の両辺の自然対数をとる

$$\log e^{if} = \log\left(e^{iu}\frac{1-\omega e^{-iu}}{1-\omega e^{iu}}\right)$$

$$\log e^{if} = \log e^{iu} + \log(1-\omega e^{-iu}) - \log(1-\omega e^{iu})$$

$$if = iu + \underbrace{\log(1-\omega e^{-iu})}_{f(\omega)} - \underbrace{\log(1-\omega e^{iu})}_{g(\omega)}$$

f(ω)とg(ω)をマクローリン展開

$$f(\omega) = \log(1-\omega e^{-iu})$$

$$f'(\omega) = -e^{-iu}(1-\omega e^{-iu})^{-1}$$

$$f''(\omega) = -e^{-2iu}(1-\omega e^{-iu})^{-2}$$

$$f(\omega) = f(0) + f'(0)\omega + \frac{1}{2!}f''(0)\omega^2 + \cdots$$
$$= -\omega e^{-iu} - \frac{\omega^2}{2}e^{-2iu} + \cdots$$

$$g(\omega) = \log(1-\omega e^{iu})$$

$$g'(\omega) = -e^{iu}(1-\omega e^{iu})^{-1}$$

$$g''(\omega) = -e^{2iu}(1-\omega e^{iu})^{-2}$$

$$g(\omega) = g(0) + g'(0)\omega + \frac{1}{2!}g''(0)\omega^2 \cdots$$
$$= -\omega e^{iu} - \frac{\omega^2}{2}e^{2iu} + \cdots$$

$$if = iu - \left(\omega e^{-iu} + \frac{\omega^2}{2}e^{-2iu} + \cdots\right) + \left(\omega e^{iu} + \frac{\omega^2}{2}e^{2iu} + \cdots\right) \quad \cdots (D.25)$$

$$f = u + \omega\frac{e^{iu}-e^{-iu}}{i} + \frac{\omega^2}{2}\,\frac{e^{2iu}-e^{-2iu}}{i} + \cdots$$

$$= u + 2\omega\sin u + \frac{\omega^2}{2}\cdot 2\sin(2u) + \cdots$$

$$= u + 2\sum_{k=1}^{\infty}\frac{\omega^k}{k}\sin(ku) \quad \cdots (D.19)$$