Value-at-Risk Constraints, Robustness, and Aggregation

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this talk

Objective: solve macro-finance models with “financial shocks” and volatile risk premia

This paper: a portfolio constraint that allows for

- fully dynamic model: long-lived agents, endogenous interest rates
- simple “myopic” portfolios with time-varying risk tolerance
- simple aggregation in general equilibrium

Existing approaches:

- preference shocks (risk aversion, robustness concerns, habits)
- preference, technology heterogeneity + redistribution

Improvements: closed-form solutions + aggregation + reduced demands for state space

outline

Portfolio constraint and portfolio choice

- foundation through robustness preferences

Aggregation results

- interest rate, asset prices, risk premia depend on wealth distribution through single scalar
- a notion of representative agent
- financial (constraint) shocks do not induce redistribution, output shocks do

Applications:

- risk premium dynamics with output shocks
- bond-stock correlation

literature

Related preferences and portfolio constraints:

- Danielsson, Shin, and Zigrand (2012), Adrian and Boyarchenko (2018), Hofmann, Shim, and Shin (2022), Coimbra (2020), Coimbra and Rey (2024)
- Gromb and Vayanos (2002), Gromb and Vayanos (2018), Vayanos and Vila (2021), Gourinchas, Ray, and Vayanos (2022), Greenwood, Hanson, Stein, and Sunderam (2023)

Empirics on value-at-risk:

- Adrian and Shin (2010), Adrian and Shin (2014), Coimbra, Kim, and Rey (2022), Barbiero, Bräuning, Joaquim, and Stein (2024)

Robustness concerns:

- Gilboa and Schmeidler (1989), Hansen and Sargent (2001)

Value-at-risk in mathematical finance:

- Sentana (2001), Yiu (2004), Alexander and Baptista (2003), and many others

A value-at-risk constraint

environment

State x_t is d-dimensional, driven by a b-dimensional Brownian motion $\{Z_t\}_{t\geq 0}$:

$$dx_t = \mu_x(x_t)dt + \sigma_x(x_t)dZ_t$$

Risk-free instant-maturity bond pays $r(x_t)$ and k risky assets with excess returns dR_t:

$$dR_t = \mu_R(x_t)dt + \sigma_R(x_t)dZ_t$$

Budget constraint:

$$dw_t = (r(x_t)w_t - c_t)dt + w_t\theta_t' dR_t$$

Agent's problem: given a process for $\gamma_t \in [0, 1]$,

heuristic explanation

$$\max_{\{c_t,\theta_t\}_{t\geq 0}} \mathbb{E}\int_0^\infty \rho e^{-\rho t}\log(c_t)dt$$

$$\text{s.t. } \mathbb{V}_t[\theta_t' dR_t] \leq \gamma_t \mathbb{E}_t[\theta_t' dR_t] \qquad \text{(value-at-risk)}$$

a foundation through robustness preferences

Result: same consumption and portfolio choice with a version of robust preferences technical details

Take an "alternative" Brownian motion $\{B_t\}_{t\geq 0} : B_0 = Z_0$ and $dB_t = dZ_t - h_t dt$

Agent entertains alternative models under which dB_t is truly a standard Brownian motion

Assumes the following processes for excess returns and states:

$$dR_t = \mu_R(x_t)dt + \sigma_R(x_t)dZ_t \equiv (\mu_R(x_t) - \sigma_R(x_t)h_t)dt + \sigma_R(x_t)dB_t$$

$$dx_t = \mu_x(x_t)dt + \underbrace{\sigma_x(x_t)dB_t}_{\text{no mistake}}$$

Willingness to entertain pessimistic scenarios: parameter $\psi_t \mapsto$ risk-tolerance γ_t

consumption and portfolio choice

Result: value separable over states w_t and x_t: $V(w_t, x_t) = \log(w_t) + \eta(x_t)$ with

recursive formulation

$$c^*(w_t, x_t) = \rho w_t$$

$$\theta^*(w_t, x_t) = \min\{1, \gamma_t\} \cdot [\sigma_R(x_t)\sigma_R(x_t)']^{-1}\mu_R(x_t)$$

- capping **std**: Danielsson, Shin, and Zigrand (2012), Adrian and Boyarchenko (2018)

$$\theta^*(w_t, x_t) = \lambda(\gamma_t, w_t, x_t) \cdot [\sigma_R(x_t)\sigma_R(x_t)']^{-1}\mu_R(x_t)$$

- myopic agents: Vayanos and Vila (2021)
- recursive preferences of Kreps and Porteus (1978), Duffie and Epstein (1992)

recursive preferences

$$\theta^*(w_t, x_t) = \gamma_t \cdot [\sigma_R(x_t)\sigma_R(x_t)']^{-1}\mu_R(x_t) + f(x_t)$$

extensions

Simple portfolios survive with income from outside of financial markets:

- taxes (inducing stationarity)
- perpetual youth of Yaari (1965), Blanchard (1985)

Key to preserve consumption and portfolio choice: additional terms linear in own wealth

$$dw_t = (r(x_t)w_t - c_t)dt + \theta_t' dR_t - \underbrace{w_t \varsigma(x_t) dt}_{\text{deterministic tax}} - \underbrace{w_t \tau(x_t)' dZ_t}_{\text{stochastic tax}}$$

Can handle any deterministic tax $\varsigma(x_t)$, stochastic "profit" taxes $\tau(x_t)' \propto \theta(x_t)' \sigma_R(x_t)$ result

Aggregation

an economy with integrated markets

- agents $i \in \{1, ...n\}$ identical except for individual states: multipliers $\{\gamma_{it}\}$ and wealth $\{w_{it}\}$
- risky assets $j \in \{1, ...k\}$ in fixed supply $\{s_j\}$ priced at $\{p_{jt}\}$, pay dividends $\{y_{jt}\}$
- risk-free instant maturity bonds in zero net supply pay r_t
- agents portfolio shares $\{\theta_{ijt}\}$ translate to holdings $h_{ijt} = \theta_{ijt} w_{it} / p_{jt}$ and $b_{it} = (1 - \theta'_{it} 1_k) w_{it}$

Given shocks $\{y_{jt}, \gamma_{it}\}_{t \geq 0}$, an **equilibrium** is a set of adapted processes for prices $\{p_{jt}, r_t\}_{t \geq 0}$ and quantities $\{w_{it}, c_{it}, b_{it}, h_{ijt}\}_{t \geq 0}$ that solve agents' problems with prices taken as given and satisfy

$$\sum_i h_{ijt} = s_j \text{ for all } j$$

$$\sum_i b_{it} = 0$$

$$\sum_i c_{it} = \sum_j s_j y_{jt}$$

equilibrium characterization

With $\boldsymbol{y}_t = \{y_{jt}\}$, $\gamma_t = \{\gamma_{it}\}$, $\boldsymbol{w}_t = \{w_{it}\}$, aggregate states are $x_t = (\boldsymbol{y}_t, \boldsymbol{\gamma}_t, \overline{\boldsymbol{w}}_t)$, where $\overline{\boldsymbol{w}}_t = \boldsymbol{w}_t$ a.s.

$$d\boldsymbol{y}_t = \mu_y(\boldsymbol{y}_t)dt + \sigma_y(\boldsymbol{y}_t)dZ_t$$

$$d\boldsymbol{\gamma}_t = \mu_\gamma(\boldsymbol{\gamma}_t)dt + \sigma_\gamma(\boldsymbol{\gamma}_t)dW_t$$

Characterize prices $\boldsymbol{p}(x_t) = \{p_j(x_t)\}$ and $r(x_t)$ as functions of aggregate states: returns

$$d\boldsymbol{p}(x_t) = \mu_p(x_t)dt + \sigma_{p,y}(x_t)dZ_t + \sigma_{p,\gamma}(x_t)dW_t$$

preliminaries

Total wealth is exogenous:

$$\rho \sum_i w_{it} = \sum_j s_j y_{jt}$$

Denote $w_t = \sum_i w_{it}$ and define $\mu_w(x_t)$ and $\sigma_w(x_t)$ by

$$\frac{dw_t}{w_t} \equiv \mu_w(y_t)dt + \sigma_w(y_t)dZ_t = \frac{1}{\boldsymbol{s}'\boldsymbol{y}_t}[\boldsymbol{s}'\mu_w(\boldsymbol{y}_t)dt + \boldsymbol{s}'\sigma_y(\boldsymbol{y}_t)dZ_t]$$

Denote wealth shares by $\nu_{it} = \dfrac{w_{it}}{w_t}$ and define the weighted average Γ_t and dispersion Δ_t

$$\Gamma_t = \sum_i \nu_{it}\gamma_{it}$$

$$\Delta_t = \sum_i \nu_{it}\gamma_{it}^2 - \left(\sum_i \nu_{it}\gamma_{it}\right)^2$$

asset prices

Proposition 1

The interest rate and asset prices solve

corollary

$$r(x_t) = \rho + \mu_w(\boldsymbol{y}_t) - \frac{|\sigma_w(\boldsymbol{y}_t)|^2}{\Gamma_t}$$

$$r(x_t)\boldsymbol{p}(x_t) = \boldsymbol{y}_t + \mu_p(x_t) - \frac{\sigma_{p,y}(x_t)\sigma_w(\boldsymbol{y}_t)'}{\Gamma_t}$$

Proposition 2

Asset prices satisfy $\Lambda_t \boldsymbol{p}(x_t) = \mathbb{E}_t \int_t^\infty \Lambda_s \boldsymbol{y}_s ds$, where $\Lambda_0 = 1$ and

$$d\log(\Lambda_t) = -(\rho + \mu_w(\boldsymbol{y}_t))dt - \frac{1}{\Gamma_t} \cdot \left[\frac{1-\Gamma_t}{\Gamma_t}|\sigma_w(\boldsymbol{y}_t)|^2 dt + \sigma_w(\boldsymbol{y}_t)dZ_t\right]$$

leverage and risk tolerance

Proposition 3

Agent i's leverage $\lambda_{it} \equiv \sum_j \theta_{ijt}$ is $\lambda_{it} = \frac{\gamma_{it}}{\Gamma_t}$. Her wealth share ν_{it} evolves as

$$\frac{d\nu_{it}}{\nu_{it}} = (\lambda_{it} - 1) \cdot \left[\frac{1 - \Gamma_t}{\Gamma_t} |\sigma_w(\boldsymbol{y}_t)|^2 dt + \sigma_w(\boldsymbol{y}_t) dZ_t + \mathbf{0}' dW_t \right]$$

Proposition 4:

The wealth-weighted average multiplier evolves as

$$d\Gamma_t = \frac{\Delta_t}{\Gamma_t} \cdot \left[\frac{1 - \Gamma_t}{\Gamma_t} |\sigma_w(\boldsymbol{y}_t)|^2 dt + \sigma_w(\boldsymbol{y}_t) dZ_t + \mathbf{0}' dW_t \right] + \boldsymbol{\nu}_t' d\boldsymbol{\gamma}_t$$

tax-funded debt

Can introduce long-term debt (perpetuity with coupon τ) funded by wealth taxes

Crucial: taxes assessed in proportion to wealth to keep budget constraints linear in wealth

Proposition 5

Asset prices, including the price of the perpetuity, satisfy the same equation as in Proposition 1, while the interest rate is shifted up:

$$r(x_t)\boldsymbol{p}(x_t) = \mu_p(x_t) + \boldsymbol{y}_t - \frac{\sigma_{p,y}(x_t)\sigma_w(\boldsymbol{y}_t)'}{\Gamma_t}$$

$$r(x_t) = \rho + \mu_w(\boldsymbol{y}_t) + \frac{\rho\tau}{w_t} - \frac{|\sigma_w(\boldsymbol{y}_t)|^2}{\Gamma_t}$$

Example: integrated markets

risk premia driven by output shocks

Caballero and Simsek (2020): risk premium shocks $\longrightarrow$ real shocks

- speculators with heterogenous beliefs and risk tolerance make bets
- speculation redistributes wealth and changes aggregate risk tolerance
- natural interest rate changes
- failure to adjust policy rate is a monetary shock with real effects

Value-at-risk: no speculation needed, just productivity shocks, closed-form solutions

- two agents with different value-at-risk multipliers + one tree + risk-free debt
- low output $\longrightarrow$ high risk premium $\longrightarrow$ low interest rate
- closely related: He and Krishnamurthy (2012)

two agents, one tree

Lucas tree with $\frac{dy_t}{y_t} = \mu dt + \sigma dZ_t$ in unit supply, two agents with fixed multipliers $\overline{\gamma}$ and $\underline{\gamma}$

- total wealth is $p_t = w_t = \rho^{-1} y_t$, growth and volatility $\mu_w = \mu$ and $\sigma_w = \sigma$
- wealth shares $\overline{\nu}_t$ and $\underline{\nu}_t = 1 - \overline{\nu}_t$
- weighted average $\Gamma_t = \underline{\gamma} + \overline{\nu}_t(\overline{\gamma} - \underline{\gamma})$ determines interest rate and risk premium:

$$r_t = \rho + \mu - \underbrace{\frac{\sigma^2}{\underline{\gamma} + \overline{\nu}_t(\overline{\gamma} - \underline{\gamma})}}_{\text{risk premium}} \equiv \rho + \mu - x_t$$

- more risk-tolerant agent borrows from less risk-tolerant

$$\overline{\lambda}_t = \frac{\overline{\gamma}}{\Gamma_t} > 1 > \frac{\underline{\gamma}}{\Gamma_t} = \underline{\lambda}_t$$

wealth shares and risk premium

More risk-tolerant agent's wealth share:

$$\frac{d\overline{v}_t}{\overline{v}_t} = \underbrace{\frac{(1-\overline{v}_t)(\overline{\gamma}-\underline{\gamma})}{\underline{\gamma}+\overline{v}_t(\overline{\gamma}-\underline{\gamma})}}_{\text{excess leverage} > 0} \cdot \left[\underbrace{\frac{1-\underline{\gamma}-\overline{v}_t(\overline{\gamma}-\underline{\gamma})}{\underline{\gamma}+\overline{v}_t(\overline{\gamma}-\underline{\gamma})}}_{\text{risk compensation} > 0} \sigma^2 dt + \sigma dZ_t \right]$$

Risk premium $x_t \in \left[\frac{\sigma^2}{\overline{\gamma}}, \frac{\sigma^2}{\underline{\gamma}}\right]$:

$$\frac{dx_t}{x_t} = \frac{(\overline{\gamma}x_t - \sigma^2)(\sigma^2 - \underline{\gamma}x_t)}{\sigma^6} \cdot \underbrace{x_t(\sigma^2(\overline{\gamma}+\underline{\gamma}-1) - \overline{\gamma}\underline{\gamma}x_t)}_{<0} dt - \frac{(\overline{\gamma}x_t - \sigma^2)(\sigma^2 - \underline{\gamma}x_t)}{\sigma^3} dZ_t$$

stationary economy

Can impose wealth taxes to make the economy stationary

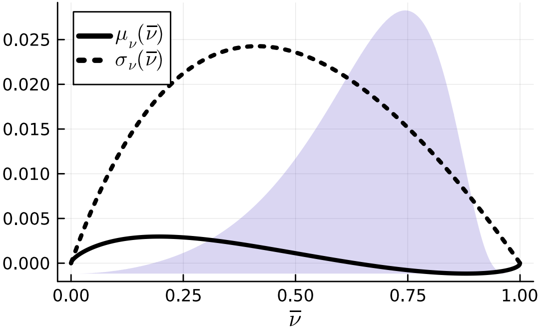

Figure: drift and volatility of the more risk-tolerant agent's wealth share $\overline{\nu}_t$, stationary distribution.

conclusion

A version of value-at-risk constraint that preserves tractable portfolios with

- long-lived agents
- time-varying risk tolerance

Robustness interpretation

Simple aggregation in general equilibrium

- potential for studying segmented markets

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heuristic explanation

Take some (L_t, α_t): back

$$\mathbb{P}\{\theta_t' dR_t \leq -\sqrt{L_t dt}\} \leq \alpha_t$$

Equivalently,

$$\Phi\left(-\frac{\sqrt{L_t dt} + \theta_t' \mu_R(x_t) dt}{\sqrt{\theta_t' \sigma_R(x)\sigma_R(x)' \theta_t dt}}\right) \leq \alpha_t$$

Suppose $\alpha \leq 1/2$, in the limit $dt \longrightarrow 0$,

$$\theta_t' \sigma_R(x_t)\sigma_R(x_t)\theta_t' \leq \frac{L_t}{(\Phi^{-1}(\alpha_t))^2}$$

With $L_t = \theta_t' \mu_R(x_t)$ and $\alpha_t = \Phi(-\sqrt{1/\gamma_t})$,

$$\mathbb{V}_t[\theta_t' dR_t] = \theta_t' \sigma_R(x_t)\sigma_R(x_t)\theta_t' dt \leq \gamma_t \cdot \theta_t' \mu_R(x_t) dt = \gamma_t \mathbb{E}_t[\theta_t' dR_t]$$

recursive problem formulation

With (w, x) as states, value $V(w, x)$ solves

back

$$\rho V(w,x) = \max_{c,\theta} \rho \log(c) + (r(x)w - c + w\theta' \mu_R(x)) V_w(w,x) + \frac{\theta' \sigma_R(x) \sigma_R(x)' \theta}{2} V_{ww}(w,x)$$
$$+ \mu_x(x)' V_{x'}(w,x) + \frac{1}{2} \mathrm{tr}[\sigma_x(x)' V_{xx'}(w,x) \sigma_x(x)] + w\theta' \sigma_R(x) \sigma_x(x)' V_{wx'}(w,x)$$

s.t. $\theta' \sigma_R(x) \sigma_R(x)' \theta \leq \gamma \cdot \theta' \mu_R(x)$

relation to recursive preferences

Take Kreps and Porteus (1978) preferences in Duffie and Epstein (1992) form, keep EIS=1: back

$$V_t = \mathbb{E}_t \int_t^\infty \varphi(c_s, V_s) ds \quad \text{with} \quad \varphi(c,v) = \frac{\rho v(\gamma-1)}{\gamma}\left[\log(c) - \frac{\gamma}{\gamma-1}\log\left(\frac{v(\gamma-1)}{\gamma}\right)\right]$$

Value is no longer separable over w and x:

$$V(w,x) = \frac{(w\eta(x))^{1-1/\gamma}}{1-1/\gamma}$$

Optimal portfolio includes hedging motives if $\gamma \neq 1$:

$$c^*(w,x) = \rho w$$

$$\theta^*(w,x) = \gamma \cdot [\sigma_R(x)\sigma_R(x)']^{-1}\mu_R(x) + (\gamma-1)[\sigma_R(x)\sigma_R(x)']^{-1}\underbrace{\sigma_R(x)\sigma_x(x)'\frac{\eta_{x'}(x)}{\eta(x)}}_{\text{hedging motives}}$$

multiplier problem

Let $\{Z_t\}_{t\geq 0}$ be a standard Brownian on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, take an adapted process $\{h_t\}_{t\geq 0}$

- consider an adapted process $\{M_t\}_{t\geq 0} : M_0 = 1$ and $dM_t = -h_t M_t dZ_t$
- defines a probability measure $\mathbb{Q} : \mathbb{E}^{\mathbb{Q}}[\xi_t] = \mathbb{E}^{\mathbb{P}}[M_t \xi_t]$ for all bounded $\{\xi_t\}_{t\geq 0}$ and all $t \geq 0$
- $\{B_t\}_{t\geq 0}$ with $B_0 = 0$ and $dB_t = dZ_t - h_t dt$ is a standard Brownian on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{Q})$
- given an adapted process $\{\psi_t\}_{t\geq 0}$ and $m_t \equiv \log(M_t)$, agent solves a multiplier problem

$$\max_{\{c_t, \theta_t\}} \inf_{\mathbb{Q}} \mathbb{E}^{\mathbb{Q}} \left[\int_0^\infty \rho e^{-\rho t} \log(c_t) dt + \int_0^\infty e^{-\rho t} \psi_t dm_t \right]$$

solving the problem back

solving the multiplier problem

Log-likelihood process m_t evolves as

back

$$dm_t = -\frac{1}{2}|h_t|^2 dt - h_t' dZ_t = \frac{1}{2}|h_t|^2 dt - h_t' dB_t$$

Recursive formulation:

$$\begin{aligned}\rho V(w,x) = \max_{c,\theta} \min_{h} \; & \rho \log(c) + \frac{\psi |h|^2}{2} \\ & + (r(x)w - c + w\theta'(\mu_R(x) - \sigma_R(x)h))V_w(w,x) + \frac{1}{2}\theta'\sigma_R(x)\sigma_R(x)'\theta V_{ww}(w,x) \\ & + \mu_x(x)'V_{x'}(w,x) + \frac{1}{2}\mathrm{tr}[\sigma_x(x)'V_{xx'}(w,x)\sigma_x(x)] + w\theta'\sigma_R(x)\sigma_x(x)'V_{wx'}(w,x)\end{aligned}$$

Separability preserved: $V(w,x) = \log(w) + \hat{\eta}(x)$ and

standard setup

$$c^*(w,x) = \rho w$$

$$\theta^*(w,x) = \frac{\psi}{\psi + 1} \cdot [\sigma_R(x)\sigma_R(x)']^{-1}\mu_R(x)$$

relation to standard robustness setup

In the standard case, model for states is misspecified too:

back

$$dx_t = \mu_x(x_t)dt + \sigma_x(x_t)dZ_t \equiv (\mu_R(x_t) - \sigma_x(x_t)h_t)dt + \sigma_x(x_t)dB_t$$

Recursive formulation:

$$\begin{aligned}\rho V(w,x) = \max_{c,\theta}\ \min_{h}\ \rho\log(c) + \frac{\psi|h|^2}{2} \\ + (r(x)w - c + w\theta'(\mu_R(x) - \sigma_R(x)h))V_w(w,x) + \frac{1}{2}\theta'\sigma_R(x)\sigma_R(x)'\theta V_{ww}(w,x) \\ + (\mu_x(x) - \underbrace{\sigma_x(x)h}_{\text{new}})'V_{x'}(w,x) + \frac{1}{2}\mathrm{tr}[\sigma_x(x)'V_{xx'}(w,x)\sigma_x(x)] + w\theta'\sigma_R(x)\sigma_x(x)'V_{wx'}(w,x)\end{aligned}$$

Separability $V(w,x) = \log(w) + \hat{\eta}(x)$ preserved but optimal h and θ pick up $V_{x'}(w,x)$:

$$\theta^*(w,x) = \frac{\psi}{\psi+1}\cdot[\sigma_R(x)\sigma_R(x)']^{-1}\mu_R(x) - \frac{1}{\psi+1}[\sigma_R(x)\sigma_R(x)']^{-1}\sigma_R(x)\sigma_x(x)'\frac{\hat{\eta}_{x'}(x)}{\hat{\eta}(x)}$$

stochastic taxes proportional to profits

Consider the following class of tax rates:

back

$$\tau(x_t) = \zeta(x_t)\gamma_t \cdot \sigma_R(x_t)'[\sigma_R(x_t)\sigma_R(x_t)']^{-1}\mu_R(x_t)$$

Tax payments proportional to resulting profits:

$$\tau(x_t)'dZ_t = \zeta(x_t)\theta(x_t)'\sigma_R(x_t)dZ_t = \zeta(x_t)\theta(x_t)'(dR_t - \mu_R(x_t)dt)$$

Optimal portfolio the same unless $\zeta(x_t)$ very negative:

$$\theta(w_t, x_t) = \min\{\gamma_t, 1 + \zeta(x_t)\gamma_t\} \cdot [\sigma_R(x_t)\sigma_R(x_t)']^{-1}\mu_R(x_t)$$

returns

Vector of excess returns:

back

$$dR_t \equiv \mu_R(x_t)dt + \sigma_{R,y}(x_t)dZ_t + \sigma_{R,\gamma}(x_t)dW_t$$
$$= D(\boldsymbol{p}_t)^{-1}(\mu_p(x_t) + \boldsymbol{y}_t - r(x_t)\boldsymbol{p}(x_t))dt + D(\boldsymbol{p}_t)^{-1}(\sigma_{p,y}(x_t)dZ_t + \sigma_{p,\gamma}(x_t)dW_t)$$

PDE for asset prices

Corollary: the PDE for asset prices is linear.

back

$$r(x_t)p_j(x_t) = y_{jt} + \mathcal{D}p_j(x_t)\left[\mu_x(x_t) \underbrace{-\frac{1}{\Gamma_t}\sigma_{x,z}(x_t)\sigma_w(\boldsymbol{y}_t)'}_{\text{risk adjustment}}\right] + \frac{1}{2}\mathrm{tr}[\mathcal{H}p_j(x_t)\sigma_x(x_t)\sigma_x(x_t)']$$