

# Value-at-Risk Constraints, Robustness, and Aggregation

Aleksei Oskolkov

[alekseioskolkov@princeton.edu](mailto:alekseioskolkov@princeton.edu)

January 3, 2026

# this talk

**Objective:** solve macro-finance models with “financial shocks” and volatile risk premia

**Existing approaches to volatile risk premia:**

- ▶ preference shocks (risk aversion, robustness concerns, habits)
- ▶ preference/technology heterogeneity + redistribution

**This paper:** a value-at-risk constraint

- ▶ tractable portfolios with long-lived agents and time-varying risk tolerance
- ▶ interpretation through robustness concerns and model misspecification
- ▶ aggregation in general equilibrium
- ▶ clean separation of preference shocks and redistribution

# outline

Portfolio choice with value-at-risk constraints

- ▶ foundation through robustness preferences

Aggregation results: “as-if representative” agent

- ▶ interest rate, asset prices, risk premia depend on wealth distribution through single scalar
- ▶ asset prices, wealth distribution driven by “as-if representative” agent’s misspecified model
- ▶ financial shocks (constraints tightening) do not induce redistribution, output shocks do

Application:

- ▶ a model with stochastic heterogeneous risk limits
- ▶ decomposition of changes in risk premium into risk limit shocks + redistribution

# **literature**

## Related preferences and portfolio constraints:

- ▶ Danielsson, Shin, and Zigrand (2012), Adrian and Boyarchenko (2018), Hofmann, Shim, and Shin (2022), Coimbra (2020), Coimbra and Rey (2024)
- ▶ Gromb and Vayanos (2002), Gromb and Vayanos (2018), Vayanos and Vila (2021), Gourinchas, Ray, and Vayanos (2022), Greenwood, Hanson, Stein, and Sunderam (2023)
- ▶ Itskhoki and Mukhin (2021), Kekre and Lenel (2024), Kekre and Lenel (2025)

## Empirics on value-at-risk:

- ▶ Adrian and Shin (2010), Adrian and Shin (2014), Coimbra, Kim, and Rey (2022), Barbiero, Bräuning, Joaquim, and Stein (2024)

## Robustness concerns:

- ▶ Gilboa and Schmeidler (1989), Hansen and Sargent (2001)

A value-at-risk constraint

# environment

State  $x_t$  is  $d$ -dimensional, driven by a  $b$ -dimensional Brownian motion  $\{Z_t\}_{t \geq 0}$ :

$$dx_t = \mu_x(x_t)dt + \sigma_x(x_t)dZ_t$$

Risk-free instant-maturity bond pays  $r(x_t)$  and  $k$  risky assets with excess returns  $dR_t$ :

$$dR_t = \mu_R(x_t)dt + \sigma_R(x_t)dZ_t$$

Budget constraint:

$$dw_t = (r(x_t)w_t - c_t)dt + w_t \theta'_t dR_t$$

Agent's problem: given a process for  $\gamma_t \in [0, 1]$ ,

heuristic explanation

$$\max_{\{c_t, \theta_t\}_{t \geq 0}} \mathbb{E} \int_0^\infty \rho e^{-\rho t} \log(c_t) dt$$

$$\text{s.t. } \mathbb{V}_t[\theta'_t dR_t] \leq \gamma_t \mathbb{E}_t[\theta'_t dR_t]$$

(value-at-risk)

# consumption and portfolio choice

## Proposition 1

Consumption and portfolio choice are

recursive formulation

extensions

$$c^*(w_t, x_t) = \rho w_t$$

$$\theta^*(w_t, x_t) = \min\{1, \gamma_t\} \cdot [\sigma_R(x_t) \sigma_R(x_t)']^{-1} \mu_R(x_t)$$

- ▶ capping std: Danielsson, Shin, and Zigrand (2012), Adrian and Boyarchenko (2018)

$$\theta^*(w_t, x_t) = \lambda(\gamma_t, w_t, x_t) \cdot [\sigma_R(x_t) \sigma_R(x_t)']^{-1} \mu_R(x_t)$$

- ▶ recursive preferences of Kreps and Porteus (1978), Duffie and Epstein (1992)

recursive preferences

$$\theta^*(w_t, x_t) = \gamma_t \cdot [\sigma_R(x_t) \sigma_R(x_t)']^{-1} \mu_R(x_t) + f(x_t)$$

# a foundation through robustness preferences

Consider an agent who assumes there could be mistakes in model for returns (but not states)

- shocks  $dZ_t$  might be biased downwards:  $dZ_t = dB_t - \textcolor{red}{h}_t dt$ , where  $\mathbb{E}[dB_t] = 0$

$$dx_t = \mu_x(x_t)dt + \sigma_x(x_t)dB_t$$

$$dR_t = \mu_R(x_t)dt + \sigma_R(x_t)dZ_t \equiv (\mu_R(x_t) - \sigma_R(x_t)\textcolor{red}{h}_t)dt + \sigma_R(x_t)dB_t$$

Entertains alternative measures  $\mathbb{Q}$  under which  $dB_t$  is truly a standard Brownian motion:

$$\max_{\{c_t, \theta_t\}} \inf_{\mathbb{Q}} \mathbb{E}^{\mathbb{Q}} \left[ \int_0^\infty \rho e^{-\rho t} \log(c_t) dt + \int_0^\infty e^{-\rho t} \psi_t dm_t \right]$$

## Proposition 2

A robust agent with a cost parameter  $\psi_t$  has the same consumption and portfolio choice as the value-at-risk constrained agent with a multiplier  $\gamma_t = \frac{\psi_t}{\psi_t + 1}$ .

technical details

# Aggregation

# an economy with integrated markets

- agents  $i \in \{1, \dots, n\}$  identical except for individual states: multipliers  $\{\gamma_{it}\}$  and wealth  $\{w_{it}\}$
- risky assets  $j \in \{1, \dots, k\}$  in fixed supply  $\{s_j\}$  priced at  $\{p_{jt}\}$ , pay dividends  $\{y_{jt}\}$
- risk-free instant maturity bonds in zero net supply pay  $r_t$
- agents portfolio shares  $\{\theta_{ijt}\}$  translate to holdings  $h_{ijt} = \theta_{ijt} w_{it} / p_{jt}$  and  $b_{it} = (1 - \theta'_{it} 1_k) w_{it}$

Given shocks  $\{y_{jt}, \gamma_{it}\}_{t \geq 0}$ , an **equilibrium** is a set of adapted processes for prices  $\{p_{jt}, r_t\}_{t \geq 0}$  and quantities  $\{w_{it}, c_{it}, b_{it}, h_{ijt}\}_{t \geq 0}$  that solve agents' problems with prices taken as given and satisfy

$$\sum_i h_{ijt} = s_j \text{ for all } j$$

$$\sum_i b_{it} = 0$$

$$\sum_i c_{it} = \sum_j s_j y_{jt}$$

# equilibrium characterization

With  $\mathbf{y}_t = \{y_{jt}\}$ ,  $\boldsymbol{\gamma}_t = \{\gamma_{it}\}$ ,  $\mathbf{w}_t = \{w_{it}\}$ , aggregate states are  $x_t = (\mathbf{y}_t, \boldsymbol{\gamma}_t, \bar{w}_t)$ , where  $\bar{w}_t = \mathbf{w}_t$  a.s.

$$d\mathbf{y}_t = \mu_y(\mathbf{y}_t)dt + \sigma_y(\mathbf{y}_t)dZ_t$$

$$d\boldsymbol{\gamma}_t = \mu_\gamma(\boldsymbol{\gamma}_t)dt + \sigma_\gamma(\boldsymbol{\gamma}_t)dW_t$$

Characterize prices  $\mathbf{p}(x_t) = \{p_j(x_t)\}$  and  $r(x_t)$  as functions of aggregate states:

returns

$$d\mathbf{p}(x_t) = \mu_p(x_t)dt + \sigma_{p,y}(x_t)dZ_t^y + \sigma_{p,\gamma}(x_t)dZ_t^\gamma$$

# preliminaries

Total wealth is exogenous:

$$\rho \sum_i w_{it} = \sum_j s_j y_{jt}$$

Denote  $w_t = \sum_i w_{it}$  and define  $\mu_w(x_t)$  and  $\sigma_w(x_t)$  by

$$\frac{dw_t}{w_t} \equiv \mu_w(y_t)dt + \sigma_w(y_t)dZ_t^y = \frac{1}{s' y_t} [s' \mu_y(y_t)dt + s' \sigma_y(y_t)dZ_t^y]$$

Denote wealth shares by  $\nu_{it} = \frac{w_{it}}{w_t}$  and define the weighted average  $\Gamma_t$  and dispersion  $\Delta_t$

$$\Gamma_t = \sum_i \nu_{it} \gamma_{it}$$

$$\Delta_t = \sum_i \nu_{it} \gamma_{it}^2 - \left( \sum_i \nu_{it} \gamma_{it} \right)^2$$

# asset prices

## Proposition 3

The interest rate and asset prices solve

corollary

$$r(x_t) = \rho + \mu_w(\mathbf{y}_t) - \frac{|\sigma_w(\mathbf{y}_t)|^2}{\Gamma_t}$$

$$r(x_t) \mathbf{p}(x_t) = \mathbf{y}_t + \mu_p(x_t) - \frac{\sigma_{p,y}(x_t) \sigma_w(\mathbf{y}_t)'}{\Gamma_t}$$

- ▶ only loadings of prices on output shocks  $\sigma_{p,y}(x_t)$  matter: loadings  $\sigma_{p,\gamma}(x_t)$  do not enter
- ▶ shocks to  $\gamma_t$  do not change aggregate wealth
- ▶  $\Gamma_t < 1$  increases the risk premium, decreases interest rate
- ▶ regular log utility nested as  $\Gamma_t \equiv 1$

# Gorman aggregation

Wealth distribution enters asset pricing relations through one weighted average

- ▶ “as-if representative” agent with  $\gamma_t = \Gamma_t$  on all sample paths
- ▶  $\Gamma_t$  not Markov, only complete state  $x_t = (\gamma_t, y_t, v_t)$  is Markov
- ▶ non-trivial wealth distribution dynamics

# representative model

Hypothetical single agent with robustness parameter  $\psi_t = \frac{\Gamma_t}{1 - \Gamma_t}$

- ▶ has access to single asset: total stock market index with excess return  $dR_t$
- ▶ chooses an alternative probability measure  $Q_t$  out of robustness concerns
- ▶ expectation error process  $dW_t \equiv dR_t - \mathbb{E}^{Q_t}[dR_t]$

## Proposition 4

The equilibrium total stock market return  $dR_t$  and the expectation error  $dW_t$  are

$$dR_t = \frac{1}{\Gamma_t} |\sigma_w(\mathbf{y}_t)|^2 dt + \sigma_w(\mathbf{y}_t) dZ_t^y$$

$$dW_t = \underbrace{\frac{1 - \Gamma_t}{\Gamma_t} |\sigma_w(\mathbf{y}_t)|^2 dt}_{>0 \text{ for finite } \psi_t} + \sigma_w(\mathbf{y}_t) dZ_t^y$$

# stochastic discount factor

## Proposition 5

Asset prices satisfy  $\Lambda_t^{-1} p(x_t) = \mathbb{E}_t \int_t^\infty \Lambda_s^{-1} y_s ds$ , where  $\Lambda_0 = 1$  and

$$d\Lambda_t = (\rho + \mu_w(y_t))dt + \frac{dW_t}{\Gamma_t}$$

- $\Gamma_t < 1$  increases risk premia by increasing  $\mathbb{C}_t[d\Lambda_t^{-1}, dR_t]$
- drift if  $dW_t$  dampens discounting due to lower estimated benchmark returns

# dynamics of wealth distribution

Define agent  $i$ 's leverage as the total risky share of her portfolio:  $\lambda_{it} \equiv \sum_j \theta_{ijt}$

## Proposition 6

Agent  $i$ 's leverage is  $\lambda_{it} = \frac{\gamma_{it}}{\Gamma_t}$ . Her wealth share  $v_{it}$  evolves as

$$\frac{dv_{it}}{v_{it}} = (\lambda_{it} - 1)dW_t$$

- wealth share positively exposed to  $\sigma_w(\mathbf{y}_t)dZ_t^y$  iff  $\gamma_{it} > \Gamma_t$
- drift in wealth shares if expectation error  $dW_t$  has one
- no contribution of  $dZ_t^\gamma$ : shocks to constraints/robustness parameters do not redistribute

# dynamics of risk premia

## Proposition 7:

The wealth-weighted average multiplier evolves as

$$d\Gamma_t = \frac{\Delta_t}{\Gamma_t} dW_t + \nu'_t d\gamma_t$$

- ▶ two forces: direct change in preferences  $\nu'_t d\gamma_t$  and redistribution  $\frac{\Delta_t}{\Gamma_t} dW_t$
- ▶ redistribution only works if there is heterogeneity,  $\Delta_t > 0$
- ▶ redistribution suppresses risk premia after positive output shocks
- ▶ redistribution has a drift if expectation error  $dW_t$  does

Example: financial cycle

# decomposing risk premia into shocks and redistribution

Consider a simple economy:

- ▶ one Lucas tree, output evolves as  $\frac{dy_t}{y_t} = \mu dt + \sigma dZ_t^y$
- ▶ two agents: one with time-varying  $d\gamma_t = \mu_\gamma(\gamma_t)dt + \sigma_\gamma(\gamma_t)dZ_t^\gamma$ , one with a fixed  $\gamma_t = \hat{\gamma}$
- ▶ two states: exogenous multiplier  $\gamma_t$  and endogenous wealth share  $v_t$  of the first agent
- ▶ weighted average  $\Gamma(\gamma_t, v_t) = \hat{\gamma} + v_t(\gamma_t - \hat{\gamma})$

Interest rate:

$$r(\gamma_t, v_t) = \rho + \mu - \pi(\gamma_t, v_t)$$

Here the risk premium is  $\pi(\gamma_t, v_t) = \frac{\sigma^2}{\Gamma(\gamma_t, v_t)}$

## special case: only multiplier shocks

Let  $\nu_t = 1$ : the agent with a time-varying  $\gamma_t$  takes over the market

Risk premium and interest rate driven by  $\gamma_t$ , are exogenous

### Proposition 8

Let  $\kappa_r > 0$  and  $\sigma_r$  be parameters and suppose the process for  $\gamma_t$  is

$$\frac{d\gamma_t}{\gamma_t} = \left( \kappa_r + \frac{\sigma_r^2}{\sigma^4} \gamma_t^2 \right) dt + \frac{\sigma_r}{\sigma^2} \gamma_t dZ_t^\gamma$$

with a reflecting boundary at  $\gamma_t = 1$ . The process for the interest rate is that of Vasicek 1977:

$$dr_t = \kappa_r(\rho + \mu - r_t)dt + \sigma_r dZ_t^\gamma$$

with a reflecting boundary at  $r_t = \rho + \mu - \sigma^2$ .

# special case: redistribution only

Suppose  $\gamma_t > \hat{\gamma}$  and is constant

- agent with higher  $\gamma$  levers up to bet on output growth
- positive shocks make here relatively richer, suppress risk premium
- risk premium becomes a Markov process:

$$\frac{d\pi_t}{\pi_t} = \mu_\pi(\pi_t)dt + \sigma_\pi(\pi_t)dZ_t^y$$

## Proposition 9

The drift and volatility of the risk premium are

$$\mu_\pi(\pi_t) = \left( \frac{\sigma}{\bar{\pi}\underline{\pi}} \right)^2 \cdot (\sigma^2(\bar{\pi} + \underline{\pi} - \pi_t) - \bar{\pi}\underline{\pi}) \cdot (\bar{\pi} - \pi_t)(\pi_t - \underline{\pi}) < 0$$

$$\sigma_\pi(\pi_t) = -\frac{\sigma}{\bar{\pi}\underline{\pi}} \cdot (\bar{\pi} - \pi_t)(\pi_t - \underline{\pi}) < 0$$

# output shocks generate risk premium shocks

Caballero and Simsek (2020): risk premium shocks  $\rightarrow$  real shocks

- ▶ speculators with heterogeneous beliefs and risk tolerance make bets
- ▶ speculation redistributes wealth and changes aggregate risk tolerance
- ▶ natural interest rate changes
- ▶ failure to adjust policy rate is a monetary shock with real effects

Value-at-risk or robustness concerns:

- ▶ heterogeneity in  $\gamma$  is heterogeneity in chosen beliefs

# conclusion

A version of value-at-risk constraint that preserves tractable portfolios with

- ▶ long-lived agents
- ▶ time-varying risk tolerance

This risk limit comes from robustness concerns

Simple aggregation in general equilibrium

- ▶ one scalar summary of wealth distribution determines asset prices and risk premia
- ▶ forecast errors from the “representative” model determine all dynamics
- ▶ clean separation of risk premium shocks into limit shocks and redistribution

# references I

- Adrian, T. and N. Boyarchenko (2018). Liquidity policies and systemic risk. *Journal of Financial Intermediation* 35, 45–60.
- Adrian, T. and H. S. Shin (2010). Liquidity and leverage. *Journal of financial intermediation* 19(3), 418–437.
- Adrian, T. and H. S. Shin (2014). Procyclical leverage and value-at-risk. *The Review of Financial Studies* 27(2), 373–403.
- Barbiero, O., F. Bräuning, G. Joaquim, and H. Stein (2024). Dealer risk limits and currency returns. Available at SSRN.
- Blanchard, O. J. (1985). Debt, deficits, and finite horizons. *Journal of political economy* 93(2), 223–247.

## references II

- Caballero, R. J. and A. Simsek (2020). A risk-centric model of demand recessions and speculation. *The Quarterly Journal of Economics* 135(3), 1493–1566.
- Coimbra, N. (2020). Sovereigns at risk: A dynamic model of sovereign debt and banking leverage. *Journal of International Economics* 124, 103298.
- Coimbra, N., D. Kim, and H. Rey (2022). Central bank policy and the concentration of risk: Empirical estimates. *Journal of Monetary Economics* 125, 182–198.
- Coimbra, N. and H. Rey (2024). Financial cycles with heterogeneous intermediaries. *Review of Economic Studies* 91(2), 817–857.
- Danielsson, J., H. S. Shin, and J.-P. Zigrand (2012). Endogenous and systemic risk. In *Quantifying systemic risk*, pp. 73–94. University of Chicago Press.

## references III

- Duffie, D. and L. G. Epstein (1992). Stochastic differential utility. *Econometrica: Journal of the Econometric Society*, 353–394.
- Gilboa, I. and D. Schmeidler (1989). Maxmin expected utility with non-unique prior. *Journal of mathematical economics* 18(2), 141–153.
- Gourinchas, P.-O., W. Ray, and D. Vayanos (2022). A preferred-habitat model of term premia, exchange rates, and monetary policy spillovers. Technical report, National Bureau of Economic Research.
- Greenwood, R., S. Hanson, J. C. Stein, and A. Sunderam (2023). A quantity-driven theory of term premia and exchange rates. *The Quarterly Journal of Economics* 138(4), 2327–2389.

## references IV

- Gromb, D. and D. Vayanos (2002). Equilibrium and welfare in markets with financially constrained arbitrageurs. *Journal of financial Economics* 66(2-3), 361–407.
- Gromb, D. and D. Vayanos (2018). The dynamics of financially constrained arbitrage. *The Journal of Finance* 73(4), 1713–1750.
- Hansen, L. P. and T. J. Sargent (2001). Robust control and model uncertainty. *American Economic Review* 91(2), 60–66.
- Hofmann, B., I. Shim, and H. S. Shin (2022). Risk capacity, portfolio choice and exchange rates. Available at SSRN 4028446.
- Itskhoki, O. and D. Mukhin (2021). Exchange rate disconnect in general equilibrium. *Journal of Political Economy* 129(8), 2183–2232.

## references V

- Kekre, R. and M. Lenel (2024). Exchange rates, natural rates, and the price of risk. *University of Chicago, Becker Friedman Institute for Economics Working Paper* (2024-114).
- Kekre, R. and M. Lenel (2025). A model of us monetary policy and the global financial cycle.
- Kreps, D. M. and E. L. Porteus (1978). Temporal resolution of uncertainty and dynamic choice theory. *Econometrica: journal of the Econometric Society*, 185–200.
- Vayanos, D. and J.-L. Vila (2021). A preferred-habitat model of the term structure of interest rates. *Econometrica* 89(1), 77–112.
- Yaari, M. E. (1965). Uncertain lifetime, life insurance, and the theory of the consumer. *The Review of Economic Studies* 32(2), 137–150.

# heuristic explanation

Take some  $(L_t, \alpha_t)$ :

back

$$\mathbb{P}\{\theta'_t dR_t \leq -\sqrt{L_t dt}\} \leq \alpha_t$$

Equivalently,

$$\Phi\left(-\frac{\sqrt{L_t dt} + \theta'_t \mu_R(x_t) dt}{\sqrt{\theta'_t \sigma_R(x) \sigma_R(x)' \theta_t dt}}\right) \leq \alpha_t$$

Suppose  $\alpha \leq 1/2$ , in the limit  $dt \rightarrow 0$ ,

$$\theta'_t \sigma_R(x_t) \sigma_R(x_t)' \theta_t' \leq \frac{L_t}{(\Phi^{-1}(\alpha_t))^2}$$

With  $L_t = \theta'_t \mu_R(x_t)$  and  $\alpha_t = \Phi(-\sqrt{1/\gamma_t})$ ,

$$\mathbb{V}_t[\theta'_t dR_t] = \theta'_t \sigma_R(x_t) \sigma_R(x_t)' \theta_t' dt \leq \gamma_t \cdot \theta'_t \mu_R(x_t) dt = \gamma_t \mathbb{E}_t[\theta'_t dR_t]$$

# recursive problem formulation

With  $(w, x)$  as states, value  $V(w, x)$  solves

back

$$\begin{aligned}\rho V(w, x) = \max_{c, \theta} & \rho \log(c) + (r(x)w - c + w\theta' \mu_R(x))V_w(w, x) + \frac{\theta' \sigma_R(x) \sigma_R(x)' \theta}{2} V_{ww}(w, x) \\ & + \mu_x(x)' V_{x'}(w, x) + \frac{1}{2} \text{tr}[\sigma_x(x)' V_{xx'}(w, x) \sigma_x(x)] + w\theta' \sigma_R(x) \sigma_x(x)' V_{wx'}(w, x)\end{aligned}$$

$$\text{s.t. } \theta' \sigma_R(x) \sigma_R(x)' \theta \leq \gamma \cdot \theta' \mu_R(x)$$

# extensions

Simple portfolios survive with income from outside of financial markets:

- ▶ taxes (inducing stationarity)
- ▶ perpetual youth of Yaari (1965), Blanchard (1985)

Key to preserve consumption and portfolio choice: additional terms linear in own wealth

$$dw_t = (r(x_t)w_t - c_t)dt + \theta'_t dR_t - \underbrace{w_t \zeta(x_t) dt}_{\text{deterministic tax}} - \underbrace{w_t \tau(x_t)' dZ_t}_{\text{stochastic tax}}$$

Can handle any deterministic tax  $\zeta(x_t)$ , stochastic “profit” taxes  $\tau(x_t)' \propto \theta(x_t)' \sigma_R(x_t)$

result

back

# relation to recursive preferences

Take Kreps and Porteus (1978) preferences in Duffie and Epstein (1992) form, keep EIS=1:

$$V_t = \mathbb{E}_t \int_t^{\infty} \varphi(c_s, V_s) ds \quad \text{with} \quad \varphi(c, v) = \frac{\rho v(\gamma - 1)}{\gamma} \left[ \log(c) - \frac{\gamma}{\gamma - 1} \log \left( \frac{v(\gamma - 1)}{\gamma} \right) \right]$$

Value is no longer separable over  $w$  and  $x$ :

$$V(w, x) = \frac{(w\eta(x))^{1-1/\gamma}}{1 - 1/\gamma}$$

Optimal portfolio includes hedging motives if  $\gamma \neq 1$ :

$$c^*(w, x) = \rho w$$

$$\theta^*(w, x) = \gamma \cdot [\sigma_R(x)\sigma_R(x)']^{-1}\mu_R(x) + (\gamma - 1)[\sigma_R(x)\sigma_R(x)']^{-1} \underbrace{\sigma_R(x)\sigma_x(x)' \frac{\eta_{x'}(x)}{\eta(x)}}_{\text{hedging motives}}$$

back

# multiplier problem

Let  $\{Z_t\}_{t \geq 0}$  be a standard Brownian on  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ , take an adapted process  $\{h_t\}_{t \geq 0}$

- ▶ consider an adapted process  $\{M_t\}_{t \geq 0} : M_0 = 1$  and  $dM_t = -h_t M_t dZ_t$
- ▶ defines a probability measure  $\mathbb{Q} : \mathbb{E}^{\mathbb{Q}}[\xi_t] = \mathbb{E}^{\mathbb{P}}[M_t \xi_t]$  for all bounded  $\{\xi_t\}_{t \geq 0}$  and all  $t \geq 0$
- ▶  $\{B_t\}_{t \geq 0}$  with  $B_0 = 0$  and  $dB_t = dZ_t - h_t dt$  is a standard Brownian on  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{Q})$
- ▶ given an adapted process  $\{\psi_t\}_{t \geq 0}$  and  $m_t \equiv \log(M_t)$ , agent solves a multiplier problem

$$\max_{\{c_t, \theta_t\}} \inf_{\mathbb{Q}} \mathbb{E}^{\mathbb{Q}} \left[ \int_0^\infty \rho e^{-\rho t} \log(c_t) dt + \int_0^\infty e^{-\rho t} \psi_t dm_t \right]$$

solving the problem

back

# solving the multiplier problem

Log-likelihood process  $m_t$  evolves as

back

$$dm_t = -\frac{1}{2}|h_t|^2 dt - h'_t dZ_t = \frac{1}{2}|h_t|^2 dt - h'_t dB_t$$

Recursive formulation:

$$\rho V(w, x) = \max_{c, \theta} \min_h \rho \log(c) + \frac{\psi |h|^2}{2}$$

$$\begin{aligned} &+ (r(x)w - c + w\theta'(\mu_R(x) - \sigma_R(x)h))V_w(w, x) + \frac{1}{2}\theta'\sigma_R(x)\sigma_R(x)'\theta V_{ww}(w, x) \\ &+ \mu_x(x)'V_{x'}(w, x) + \frac{1}{2}\text{tr}[\sigma_x(x)'\sigma_x(x)] + w\theta'\sigma_R(x)\sigma_x(x)'V_{wx'}(w, x) \end{aligned}$$

Separability preserved:  $V(w, x) = \log(w) + \hat{\eta}(x)$  and

standard setup

$$c^*(w, x) = \rho w$$

$$\theta^*(w, x) = \frac{\psi}{\psi + 1} \cdot [\sigma_R(x)\sigma_R(x)']^{-1}\mu_R(x)$$

# relation to standard robustness setup

back

In the standard case, model for states is misspecified too:

$$dx_t = \mu_x(x_t)dt + \sigma_x(x_t)dZ_t \equiv (\mu_R(x_t) - \sigma_x(x_t)h_t)dt + \sigma_x(x_t)dB_t$$

Recursive formulation:

$$\begin{aligned} \rho V(w, x) &= \max_{c, \theta} \min_h \rho \log(c) + \frac{\psi |h|^2}{2} \\ &+ (r(x)w - c + w\theta'(\mu_R(x) - \sigma_R(x)h))V_w(w, x) + \frac{1}{2}\theta'\sigma_R(x)\sigma_R(x)'\theta V_{ww}(w, x) \\ &+ (\underbrace{\mu_x(x) - \sigma_x(x)h}_\text{new})'V_{x'}(w, x) + \frac{1}{2}\text{tr}[\sigma_x(x)'V_{xx'}(w, x)\sigma_x(x)] + w\theta'\sigma_R(x)\sigma_x(x)'V_{wx'}(w, x) \end{aligned}$$

Separability  $V(w, x) = \log(w) + \hat{\eta}(x)$  preserved but optimal  $h$  and  $\theta$  pick up  $V_{x'}(w, x)$ :

$$\theta^*(w, x) = \frac{\psi}{\psi + 1} \cdot [\sigma_R(x)\sigma_R(x)']^{-1}\mu_R(x) - \frac{1}{\psi + 1}[\sigma_R(x)\sigma_R(x)']^{-1}\sigma_R(x)\sigma_x(x)' \frac{\hat{\eta}_{x'}(x)}{\hat{\eta}(x)}$$

# stochastic taxes proportional to profits

Consider the following class of tax rates:

back

$$\tau(x_t) = \zeta(x_t)\gamma_t \cdot \sigma_R(x_t)' [\sigma_R(x_t)\sigma_R(x_t)']^{-1} \mu_R(x_t)$$

Tax payments proportional to resulting profits:

$$\tau(x_t)' dZ_t = \zeta(x_t)\theta(x_t)' \sigma_R(x_t) dZ_t = \zeta(x_t)\theta(x_t)' (dR_t - \mu_R(x_t)dt)$$

Optimal portfolio the same unless  $\zeta(x_t)$  very negative:

$$\theta(w_t, x_t) = \min\{\gamma_t, 1 + \zeta(x_t)\gamma_t\} \cdot [\sigma_R(x_t)\sigma_R(x_t)']^{-1} \mu_R(x_t)$$

# returns

Vector of excess returns:

back

$$\begin{aligned} dR_t &\equiv \mu_R(x_t)dt + \sigma_{R,y}(x_t)dZ_t + \sigma_{R,\gamma}(x_t)dW_t \\ &= D(\boldsymbol{p}_t)^{-1}(\mu_p(x_t) + \boldsymbol{y}_t - r(x_t)\boldsymbol{p}(x_t))dt + D(\boldsymbol{p}_t)^{-1}(\sigma_{p,y}(x_t)dZ_t + \sigma_{p,\gamma}(x_t)dW_t) \end{aligned}$$

# PDE for asset prices

**Corollary:** the PDE for asset prices is linear.

back

$$r(x_t)p_j(x_t) = y_{jt} + \mathcal{D}p_j(x_t) \left[ \underbrace{\mu_x(x_t) - \frac{1}{\Gamma_t} \sigma_{x,z}(x_t) \sigma_w(y_t)' }_{\text{risk adjustment}} \right] + \frac{1}{2} \text{tr}[\mathcal{H} p_j(x_t) \sigma_x(x_t) \sigma_x(x_t)']$$