

Value-at-Risk Constraints, Robustness, and Aggregation

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January 3, 2026

this talk

Objective: solve macro-finance models with “financial shocks” and volatile risk premia

Existing approaches to volatile risk premia:

- ▶ preference shocks (risk aversion, robustness concerns, habits)
- ▶ preference/technology heterogeneity + redistribution

This paper: a value-at-risk constraint

- ▶ tractable portfolios with long-lived agents and time-varying risk tolerance
- ▶ interpretation through robustness concerns and model misspecification
- ▶ aggregation in general equilibrium
- ▶ clean separation of preference shocks and redistribution

outline

Portfolio choice with value-at-risk constraints

- ▶ foundation through robustness preferences

Aggregation results: “as-if representative” agent

- ▶ interest rate, asset prices, risk premia depend on wealth distribution through single scalar
- ▶ asset prices, wealth distribution driven by “as-if representative” agent’s misspecified model
- ▶ financial shocks (constraints tightening) do not induce redistribution, output shocks do

Application:

- ▶ a model with stochastic heterogeneous risk limits
- ▶ decomposition of changes in risk premium into risk limit shocks + redistribution

literature

Related preferences and portfolio constraints:

- ▶ Danielsson, Shin, and Zigrand (2012), Adrian and Boyarchenko (2018), Hofmann, Shim, and Shin (2022), Coimbra (2020), Coimbra and Rey (2024)
- ▶ Gromb and Vayanos (2002), Gromb and Vayanos (2018), Vayanos and Vila (2021), Gourinchas, Ray, and Vayanos (2022), Greenwood, Hanson, Stein, and Sunderam (2023)
- ▶ Itskhoki and Mukhin (2021), Kekre and Lenel (2024), Kekre and Lenel (2025)

Empirics on value-at-risk:

- ▶ Adrian and Shin (2010), Adrian and Shin (2014), Coimbra, Kim, and Rey (2022), Barbiero, Bräuning, Joaquim, and Stein (2024)

Robustness concerns:

- ▶ Gilboa and Schmeidler (1989), Hansen and Sargent (2001)

A value-at-risk constraint

environment

State x_t is d -dimensional, driven by a b -dimensional Brownian motion $\{Z_t\}_{t \geq 0}$:

$$dx_t = \mu_x(x_t)dt + \sigma_x(x_t)dZ_t$$

Risk-free instant-maturity bond pays $r(x_t)$ and k risky assets with excess returns dR_t :

$$dR_t = \mu_R(x_t)dt + \sigma_R(x_t)dZ_t$$

Budget constraint:

$$dw_t = (r(x_t)w_t - c_t)dt + w_t\theta'_t dR_t$$

Agent's problem: given a process for $\gamma_t \in [0, 1]$,

heuristic explanation

$$\max_{\{c_t, \theta_t\}_{t \geq 0}} \mathbb{E} \int_0^\infty \rho e^{-\rho t} \log(c_t) dt$$

$$\text{s.t. } \mathbb{V}_t[\theta'_t dR_t] \leq \gamma_t \mathbb{E}_t[\theta'_t dR_t]$$

(value-at-risk)

consumption and portfolio choice

Proposition 1

Consumption and portfolio choice are

recursive formulation

extensions

$$c^*(w_t, x_t) = \rho w_t$$

$$\theta^*(w_t, x_t) = \min\{1, \gamma_t\} \cdot [\sigma_R(x_t)\sigma_R(x_t)']^{-1} \mu_R(x_t)$$

- capping **std**: Danielsson, Shin, and Zigrand (2012), Adrian and Boyarchenko (2018)

$$\theta^*(w_t, x_t) = \lambda(\gamma_t, w_t, x_t) \cdot [\sigma_R(x_t)\sigma_R(x_t)']^{-1} \mu_R(x_t)$$

- recursive preferences of Kreps and Porteus (1978), Duffie and Epstein (1992)

recursive preferences

$$\theta^*(w_t, x_t) = \gamma_t \cdot [\sigma_R(x_t)\sigma_R(x_t)']^{-1} \mu_R(x_t) + f(x_t)$$

a foundation through robustness preferences

Consider an agent who assumes there could be mistakes in model for returns (but not states)

- shocks dZ_t might be biased downwards: $dZ_t = dB_t - h_t dt$, where $\mathbb{E}[dB_t] = 0$

$$dx_t = \mu_x(x_t)dt + \sigma_x(x_t)dB_t$$

$$dR_t = \mu_R(x_t)dt + \sigma_R(x_t)dZ_t \equiv (\mu_R(x_t) - \sigma_R(x_t)h_t)dt + \sigma_R(x_t)dB_t$$

Entertains alternative measures \mathbb{Q} under which dB_t is truly a standard Brownian motion:

$$\max_{\{c_t, \theta_t\}} \inf_{\mathbb{Q}} \mathbb{E}^{\mathbb{Q}} \left[\int_0^\infty \rho e^{-\rho t} \log(c_t) dt + \int_0^\infty e^{-\rho t} \psi_t dm_t \right]$$

Proposition 2

A robust agent with a cost parameter ψ_t has the same consumption and portfolio choice as the value-at-risk constrained agent with a multiplier $\gamma_t = \frac{\psi_t}{\psi_t + 1}$.

technical details

Aggregation

an economy with integrated markets

- ▶ agents $i \in \{1, \dots, n\}$ identical except for individual states: multipliers $\{\gamma_{it}\}$ and wealth $\{w_{it}\}$
- ▶ risky assets $j \in \{1, \dots, k\}$ in fixed supply $\{s_j\}$ priced at $\{p_{jt}\}$, pay dividends $\{y_{jt}\}$
- ▶ risk-free instant maturity bonds in zero net supply pay r_t
- ▶ agents portfolio shares $\{\theta_{ijt}\}$ translate to holdings $h_{ijt} = \theta_{ijt}w_{it}/p_{jt}$ and $b_{it} = (1 - \theta'_{it}1_k)w_{it}$

Given shocks $\{y_{jt}, \gamma_{it}\}_{t \geq 0}$, an **equilibrium** is a set of adapted processes for prices $\{p_{jt}, r_t\}_{t \geq 0}$ and quantities $\{w_{it}, c_{it}, b_{it}, h_{ijt}\}_{t \geq 0}$ that solve agents' problems with prices taken as given and satisfy

$$\sum_i h_{ijt} = s_j \text{ for all } j$$

$$\sum_i b_{it} = 0$$

$$\sum_i c_{it} = \sum_j s_j y_{jt}$$

equilibrium characterization

With $\mathbf{y}_t = \{y_{jt}\}$, $\gamma_t = \{\gamma_{it}\}$, $\mathbf{w}_t = \{w_{it}\}$, aggregate states are $x_t = (\mathbf{y}_t, \gamma_t, \bar{\mathbf{w}}_t)$, where $\bar{\mathbf{w}}_t = \mathbf{w}_t$ a.s.

$$d\mathbf{y}_t = \mu_y(\mathbf{y}_t)dt + \sigma_y(\mathbf{y}_t)dZ_t$$

$$d\gamma_t = \mu_\gamma(\gamma_t)dt + \sigma_\gamma(\gamma_t)dW_t$$

Characterize prices $\mathbf{p}(x_t) = \{p_j(x_t)\}$ and $r(x_t)$ as functions of aggregate states:

returns

$$d\mathbf{p}(x_t) = \mu_p(x_t)dt + \sigma_{p,y}(x_t)dZ_t^y + \sigma_{p,\gamma}(x_t)dZ_t^\gamma$$

preliminaries

Total wealth is exogenous:

$$\rho \sum_i w_{it} = \sum_j s_j y_{jt}$$

Denote $w_t = \sum_i w_{it}$ and define $\mu_w(x_t)$ and $\sigma_w(x_t)$ by

$$\frac{dw_t}{w_t} \equiv \mu_w(y_t)dt + \sigma_w(y_t)dZ_t^y = \frac{1}{s'y_t} [s'\mu_y(y_t)dt + s'\sigma_y(y_t)dZ_t^y]$$

Denote wealth shares by $v_{it} = \frac{w_{it}}{w_t}$ and define the weighted average Γ_t and dispersion Δ_t

$$\begin{aligned}\Gamma_t &= \sum_i v_{it} \gamma_{it} \\ \Delta_t &= \sum_i v_{it} \gamma_{it}^2 - \left(\sum_i v_{it} \gamma_{it} \right)^2\end{aligned}$$

asset prices

Proposition 3

The interest rate and asset prices solve

corollary

$$r(x_t) = \rho + \mu_w(\mathbf{y}_t) - \frac{|\sigma_w(\mathbf{y}_t)|^2}{\Gamma_t}$$
$$r(x_t)\mathbf{p}(x_t) = \mathbf{y}_t + \mu_p(x_t) - \frac{\sigma_{p,y}(x_t)\sigma_w(\mathbf{y}_t)'}{\Gamma_t}$$

- ▶ only loadings of prices on output shocks $\sigma_{p,y}(x_t)$ matter: loadings $\sigma_{p,\gamma}(x_t)$ do not enter
- ▶ shocks to γ_t do not change aggregate wealth
- ▶ $\Gamma_t < 1$ increases the risk premium, decreases interest rate
- ▶ regular log utility nested as $\Gamma_t \equiv 1$

Gorman aggregation

Wealth distribution enters asset pricing relations through one weighted average

- ▶ “as-if representative” agent with $\gamma_t = \Gamma_t$ on all sample paths
- ▶ Γ_t not Markov, only complete state $x_t = (\gamma_t, \mathbf{y}_t, \mathbf{v}_t)$ is Markov
- ▶ non-trivial wealth distribution dynamics

representative model

Hypothetical single agent with robustness parameter $\psi_t = \frac{\Gamma_t}{1 - \Gamma_t}$

- ▶ has access to single asset: total stock market index with excess return dR_t
- ▶ chooses an alternative probability measure \mathbb{Q}_t out of robustness concerns
- ▶ expectation error process $dW_t \equiv dR_t - \mathbb{E}^{\mathbb{Q}_t}[dR_t]$

Proposition 4

The equilibrium total stock market return dR_t and the expectation error dW_t are

$$\begin{aligned}dR_t &= \frac{1}{\Gamma_t} |\sigma_w(\mathbf{y}_t)|^2 dt + \sigma_w(\mathbf{y}_t) dZ_t^y \\dW_t &= \underbrace{\frac{1 - \Gamma_t}{\Gamma_t} |\sigma_w(\mathbf{y}_t)|^2 dt}_{>0 \text{ for finite } \psi_t} + \sigma_w(\mathbf{y}_t) dZ_t^y\end{aligned}$$

stochastic discount factor

Proposition 5

Asset prices satisfy $\Lambda_t^{-1} p(x_t) = \mathbb{E}_t \int_t^\infty \Lambda_s^{-1} y_s ds$, where $\Lambda_0 = 1$ and

$$d\Lambda_t = (\rho + \mu_w(\mathbf{y}_t))dt + \frac{dW_t}{\Gamma_t}$$

- ▶ $\Gamma_t < 1$ increases risk premia by increasing $\mathbb{C}_t[d\Lambda_t^{-1}, dR_t]$
- ▶ drift if dW_t dampens discounting due to lower estimated benchmark returns

dynamics of wealth distribution

Define agent i 's leverage as the total risky share of her portfolio: $\lambda_{it} \equiv \sum_j \theta_{ijt}$

Proposition 6

Agent i 's leverage is $\lambda_{it} = \frac{\gamma_{it}}{\Gamma_t}$. Her wealth share v_{it} evolves as

$$\frac{dv_{it}}{v_{it}} = (\lambda_{it} - 1)dW_t$$

- ▶ wealth share positively exposed to $\sigma_w(\mathbf{y}_t)dZ_t^y$ iff $\gamma_{it} > \Gamma_t$
- ▶ drift in wealth shares if expectation error dW_t has one
- ▶ no contribution of dZ_t^γ : shocks to constraints/robustness parameters do not redistribute

dynamics of risk premia

Proposition 7:

The wealth-weighted average multiplier evolves as

$$d\Gamma_t = \frac{\Delta_t}{\Gamma_t} dW_t + \nu'_t d\gamma_t$$

- ▶ two forces: direct change in preferences $\nu'_t d\gamma_t$ and redistribution $\frac{\Delta_t}{\Gamma_t} dW_t$
- ▶ redistribution only works if there is heterogeneity, $\Delta_t > 0$
- ▶ redistribution suppresses risk premia after positive output shocks
- ▶ redistribution has a drift if expectation error dW_t does

Example: financial cycle

decomposing risk premia into shocks and redistribution

Consider a simple economy:

- ▶ one Lucas tree, output evolves as $\frac{dy_t}{y_t} = \mu dt + \sigma dZ_t^y$
- ▶ two agents: one with time-varying $d\gamma_t = \mu_\gamma(\gamma_t)dt + \sigma_\gamma(\gamma_t)dZ_t^\gamma$, one with a fixed $\gamma_t = \hat{\gamma}$
- ▶ two states: exogenous multiplier γ_t and endogenous wealth share v_t of the first agent
- ▶ weighted average $\Gamma(\gamma_t, v_t) = \hat{\gamma} + v_t(\gamma_t - \hat{\gamma})$

Interest rate:

$$r(\gamma_t, v_t) = \rho + \mu - \pi(\gamma_t, v_t)$$

Here the risk premium is $\pi(\gamma_t, v_t) = \frac{\sigma^2}{\Gamma(\gamma_t, v_t)}$

special case: only multiplier shocks

Let $v_t = 1$: the agent with a time-varying γ_t takes over the market

Risk premium and interest rate driven by γ_t , are exogenous

Proposition 8

Let $\kappa_r > 0$ and σ_r be parameters and suppose the process for γ_t is

$$\frac{d\gamma_t}{\gamma_t} = \left(\kappa_r + \frac{\sigma_r^2}{\sigma^4} \gamma_t^2 \right) dt + \frac{\sigma_r}{\sigma^2} \gamma_t dZ_t^\gamma$$

with a reflecting boundary at $\gamma_t = 1$. The process for the interest rate is that of Vasicek 1977:

$$dr_t = \kappa_r(\rho + \mu - r_t)dt + \sigma_r dZ_t^\gamma$$

with a reflecting boundary at $r_t = \rho + \mu - \sigma^2$.

special case: redistribution only

Suppose $\gamma_t > \hat{\gamma}$ and is constant

- ▶ agent with higher γ levers up to bet on output growth
- ▶ positive shocks make here relatively richer, suppress risk premium
- ▶ risk premium becomes a Markov process:

$$\frac{d\pi_t}{\pi_t} = \mu_\pi(\pi_t)dt + \sigma_\pi(\pi_t)dZ_t^y$$

Proposition 9

The drift and volatility of the risk premium are

$$\mu_\pi(\pi_t) = \left(\frac{\sigma}{\bar{\pi}\underline{\pi}} \right)^2 \cdot (\sigma^2(\bar{\pi} + \underline{\pi} - \pi_t) - \bar{\pi}\underline{\pi}) \cdot (\bar{\pi} - \pi_t)(\pi_t - \underline{\pi}) < 0$$

$$\sigma_\pi(\pi_t) = -\frac{\sigma}{\bar{\pi}\underline{\pi}} \cdot (\bar{\pi} - \pi_t)(\pi_t - \underline{\pi}) < 0$$

output shocks generate risk premium shocks

Caballero and Simsek (2020): risk premium shocks \longrightarrow real shocks

- ▶ speculators with heterogeneous beliefs and risk tolerance make bets
- ▶ speculation redistributes wealth and changes aggregate risk tolerance
- ▶ natural interest rate changes
- ▶ failure to adjust policy rate is a monetary shock with real effects

Value-at-risk or robustness concerns:

- ▶ heterogeneity in γ is heterogeneity in chosen beliefs

conclusion

A version of value-at-risk constraint that preserves tractable portfolios with

- ▶ long-lived agents
- ▶ time-varying risk tolerance

This risk limit comes from robustness concerns

Simple aggregation in general equilibrium

- ▶ one scalar summary of wealth distribution determines asset prices and risk premia
- ▶ forecast errors from the “representative” model determine all dynamics
- ▶ clean separation of risk premium shocks into limit shocks and redistribution

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heuristic explanation

Take some (L_t, α_t) :

back

$$\mathbb{P}\{\theta'_t dR_t \leq -\sqrt{L_t dt}\} \leq \alpha_t$$

Equivalently,

$$\Phi\left(-\frac{\sqrt{L_t dt} + \theta'_t \mu_R(x_t) dt}{\sqrt{\theta'_t \sigma_R(x) \sigma_R(x)' \theta_t dt}}\right) \leq \alpha_t$$

Suppose $\alpha \leq 1/2$, in the limit $dt \rightarrow 0$,

$$\theta'_t \sigma_R(x_t) \sigma_R(x_t) \theta'_t \leq \frac{L_t}{(\Phi^{-1}(\alpha_t))^2}$$

With $L_t = \theta'_t \mu_R(x_t)$ and $\alpha_t = \Phi(-\sqrt{1/\gamma_t})$,

$$\mathbb{V}_t[\theta'_t dR_t] = \theta'_t \sigma_R(x_t) \sigma_R(x_t) \theta'_t dt \leq \gamma_t \cdot \theta'_t \mu_R(x_t) dt = \gamma_t \mathbb{E}_t[\theta'_t dR_t]$$

recursive problem formulation

With (w, x) as states, value $V(w, x)$ solves

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$$\begin{aligned}\rho V(w, x) = \max_{c, \theta} & \rho \log(c) + (r(x)w - c + w\theta' \mu_R(x)) V_w(w, x) + \frac{\theta' \sigma_R(x) \sigma_R(x)' \theta}{2} V_{ww}(w, x) \\ & + \mu_x(x)' V_{x'}(w, x) + \frac{1}{2} \text{tr}[\sigma_x(x)' V_{xx'}(w, x) \sigma_x(x)] + w\theta' \sigma_R(x) \sigma_x(x)' V_{wx'}(w, x)\end{aligned}$$

$$\text{s.t. } \theta' \sigma_R(x) \sigma_R(x)' \theta \leq \gamma \cdot \theta' \mu_R(x)$$

extensions

Simple portfolios survive with income from outside of financial markets:

- ▶ taxes (inducing stationarity)
- ▶ perpetual youth of Yaari (1965), Blanchard (1985)

Key to preserve consumption and portfolio choice: additional terms linear in own wealth

$$dw_t = (r(x_t)w_t - c_t)dt + \theta'_t dR_t - \underbrace{w_t \zeta(x_t)dt}_{\text{deterministic tax}} - \underbrace{w_t \tau(x_t)' dZ_t}_{\text{stochastic tax}}$$

Can handle any deterministic tax $\zeta(x_t)$, stochastic “profit” taxes $\tau(x_t)' \propto \theta(x_t)' \sigma_R(x_t)$

[result](#)[back](#)

relation to recursive preferences

Take Kreps and Porteus (1978) preferences in Duffie and Epstein (1992) form, keep EIS=1:

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$$V_t = \mathbb{E}_t \int_t^\infty \varphi(c_s, V_s) ds \quad \text{with} \quad \varphi(c, v) = \frac{\rho v(\gamma - 1)}{\gamma} \left[\log(c) - \frac{\gamma}{\gamma - 1} \log \left(\frac{v(\gamma - 1)}{\gamma} \right) \right]$$

Value is no longer separable over w and x :

$$V(w, x) = \frac{(w\eta(x))^{1-1/\gamma}}{1-1/\gamma}$$

Optimal portfolio includes hedging motives if $\gamma \neq 1$:

$$c^*(w, x) = \rho w$$

$$\theta^*(w, x) = \gamma \cdot [\sigma_R(x)\sigma_R(x)']^{-1} \mu_R(x) + (\gamma - 1) \underbrace{[\sigma_R(x)\sigma_R(x)']^{-1} \sigma_R(x)\sigma_x(x)' \frac{\eta_{x'}(x)}{\eta(x)}}_{\text{hedging motives}}$$

multiplier problem

Let $\{Z_t\}_{t \geq 0}$ be a standard Brownian on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$, take an adapted process $\{h_t\}_{t \geq 0}$

- ▶ consider an adapted process $\{M_t\}_{t \geq 0} : M_0 = 1$ and $dM_t = -h_t M_t dZ_t$
- ▶ defines a probability measure $\mathbb{Q} : \mathbb{E}^{\mathbb{Q}}[\xi_t] = \mathbb{E}^{\mathbb{P}}[M_t \xi_t]$ for all bounded $\{\xi_t\}_{t \geq 0}$ and all $t \geq 0$
- ▶ $\{B_t\}_{t \geq 0}$ with $B_0 = 0$ and $dB_t = dZ_t - h_t dt$ is a standard Brownian on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{Q})$
- ▶ given an adapted process $\{\psi_t\}_{t \geq 0}$ and $m_t \equiv \log(M_t)$, agent solves a multiplier problem

$$\max_{\{c_t, \theta_t\}} \inf_{\mathbb{Q}} \mathbb{E}^{\mathbb{Q}} \left[\int_0^\infty \rho e^{-\rho t} \log(c_t) dt + \int_0^\infty e^{-\rho t} \psi_t dm_t \right]$$

solving the problem

back

solving the multiplier problem

Log-likelihood process m_t evolves as

back

$$dm_t = -\frac{1}{2}|h_t|^2 dt - h_t' dZ_t = \frac{1}{2}|h_t|^2 dt - h_t' dB_t$$

Recursive formulation:

$$\begin{aligned} \rho V(w, x) = & \max_{c, \theta} \min_h \rho \log(c) + \frac{\psi |h|^2}{2} \\ & + (r(x)w - c + w\theta'(\mu_R(x) - \sigma_R(x)h))V_w(w, x) + \frac{1}{2}\theta'\sigma_R(x)\sigma_R(x)'\theta V_{ww}(w, x) \\ & + \mu_x(x)'V_{x'}(w, x) + \frac{1}{2}\text{tr}[\sigma_x(x)'V_{xx'}(w, x)\sigma_x(x)] + w\theta'\sigma_R(x)\sigma_x(x)'V_{wx'}(w, x) \end{aligned}$$

Separability preserved: $V(w, x) = \log(w) + \hat{\eta}(x)$ and

standard setup

$$c^*(w, x) = \rho w$$

$$\theta^*(w, x) = \frac{\psi}{\psi + 1} \cdot [\sigma_R(x)\sigma_R(x)']^{-1}\mu_R(x)$$

relation to standard robustness setup

In the standard case, model for states is misspecified too:

back

$$dx_t = \mu_x(x_t)dt + \sigma_x(x_t)dZ_t \equiv (\mu_R(x_t) - \sigma_x(x_t)h_t)dt + \sigma_x(x_t)dB_t$$

Recursive formulation:

$$\begin{aligned} \rho V(w, x) = & \max_{c, \theta} \min_h \rho \log(c) + \frac{\psi |h|^2}{2} \\ & + (r(x)w - c + w\theta'(\mu_R(x) - \sigma_R(x)h))V_w(w, x) + \frac{1}{2}\theta'\sigma_R(x)\sigma_R(x)'\theta V_{ww}(w, x) \\ & + (\underbrace{\mu_x(x) - \sigma_x(x)h}_{\text{new}})'V_{x'}(w, x) + \frac{1}{2}\text{tr}[\sigma_x(x)'\underbrace{V_{xx'}}_{\text{new}}(w, x)\sigma_x(x)] + w\theta'\sigma_R(x)\sigma_x(x)'\underbrace{V_{wx'}}_{\text{new}}(w, x) \end{aligned}$$

Separability $V(w, x) = \log(w) + \hat{\eta}(x)$ preserved but optimal h and θ pick up $V_{x'}(w, x)$:

$$\theta^*(w, x) = \frac{\psi}{\psi + 1} \cdot [\sigma_R(x)\sigma_R(x)']^{-1}\mu_R(x) - \frac{1}{\psi + 1}[\sigma_R(x)\sigma_R(x)']^{-1}\sigma_R(x)\sigma_x(x)'\frac{\hat{\eta}_{x'}(x)}{\hat{\eta}(x)}$$

stochastic taxes proportional to profits

Consider the following class of tax rates:

back

$$\tau(x_t) = \zeta(x_t)\gamma_t \cdot \sigma_R(x_t)'[\sigma_R(x_t)\sigma_R(x_t)']^{-1}\mu_R(x_t)$$

Tax payments proportional to resulting profits:

$$\tau(x_t)'dZ_t = \zeta(x_t)\theta(x_t)'\sigma_R(x_t)dZ_t = \zeta(x_t)\theta(x_t)'(dR_t - \mu_R(x_t)dt)$$

Optimal portfolio the same unless $\zeta(x_t)$ very negative:

$$\theta(w_t, x_t) = \min\{\gamma_t, 1 + \zeta(x_t)\gamma_t\} \cdot [\sigma_R(x_t)\sigma_R(x_t)']^{-1}\mu_R(x_t)$$

returns

Vector of excess returns:

back

$$\begin{aligned}dR_t &\equiv \mu_R(x_t)dt + \sigma_{R,y}(x_t)dZ_t + \sigma_{R,\gamma}(x_t)dW_t \\&= D(\mathbf{p}_t)^{-1}(\mu_p(x_t) + \mathbf{y}_t - r(x_t)\mathbf{p}(x_t))dt + D(\mathbf{p}_t)^{-1}(\sigma_{p,y}(x_t)dZ_t + \sigma_{p,\gamma}(x_t)dW_t)\end{aligned}$$

PDE for asset prices

Corollary: the PDE for asset prices is linear.

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$$r(x_t)p_j(x_t) = y_{jt} + \mathcal{D}p_j(x_t) \left[\underbrace{\mu_x(x_t) - \frac{1}{\Gamma_t} \sigma_{x,z}(x_t) \sigma_w(\mathbf{y}_t)'}_{\text{risk adjustment}} \right] + \frac{1}{2} \text{tr}[\mathcal{H}p_j(x_t) \sigma_x(x_t) \sigma_x(x_t)']$$