Value-at-Risk Constraints, Robustness, and Aggregation*

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Abstract

I build a tractable general equilibrium framework with investors operating under stochastic value-at-risk constraints. Constraints can be regulatory or self-imposed, originating in robustness concerns: intermediaries choose alternative models of asset returns to proof their decisions against model misspecification. Economies with heterogeneous value-at-risk limits admit a form of aggregation. Asset prices and interest rates depend on the representative risk limit, which is the wealth-weighted average of individual risk limits. Wealth distribution dynamics depend on a single stochastic process: the forecast error in the representative misspecified model of the total market return chosen by robust agents. The framework nests both forces driving time-varying risk premia in the literature: shocks to risk-bearing capacity and redistribution between heterogeneous agents. I decompose risk premium dynamics into these two parts and show that risk limit shocks do not redistribute directly, only changing investors' leverage, which then determines how output shocks affect the wealth distribution.

Key Words: risk premium, value-at-risk, aggregation, robustness, model misspecification

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1 Introduction

One of the main objectives of the macro-finance literature is to construct and solve models with volatile risk premia. Existing approaches can be grouped into two categories. The first group of models relies on wealth dynamics in setups with heterogeneous risk-taking capacity. In the tradition of Holmstrom and Tirole (1997), Shleifer and Vishny (1997), and Xiong (2001), asset prices are determined by the wealth of financial intermediaries, who have superior capacity to handle risky assets. He and Krishnamurthy (2013) and Brunnermeier and Sannikov (2014) construct general equilibrium models where asset prices are determined by the intermediaries' wealth share. Real shocks affect risk premia and asset price volatility by redistributing wealth in the economy. Kekre and Lenel (2022) characterize and quantify this force.

The second group of models generates fluctuations in asset prices through risk preference or risk limit shocks. A long-standing tradition, tracing back at least to Abel (1990), Abel (1999), Jermann (1998), and Campbell and Cochrane (1999), uses consumption habits to induce time variation in risk aversion. More recently, segmented markets models of Itskhoki and Mukhin (2021) and Kekre and Lenel (2024) feature direct shocks to risk aversion parameters that move currency risk premia in models of exchange rate determination, an approach that has become highly influential.

I propose a way to unify these approaches in a tractable dynamic general equilibrium model, building on the growing literature using value-at-risk limits. The literature dates back at least to Adrian and Shin (2010) and has recently shown considerable progress in direct empirical evidence on these risk limits, a notable example being Barbiero, Bräuning, Joaquim, and Stein (2024).

Specifically, I suggest a form of risk-taking constraint on financial intermediaries that leads to tractable mean-variance portfolios with time-varying effective risk tolerance, which is exogenous and equals a primitive risk limit parameter. These portfolios work in a setup with long-lived agents, meaning that all asset prices and interest rates can be endogenous. This in contrast to the large body of work where effective risk-tolerance is endogenous and requires additional steps to characterize, sometimes depending on multiple differential equations that can only be solved numerically. The setup easily lends itself to aggregation, allowing me to separately characterize two sources of risk premium dynamics: redistributive forces and time variation in preferences.

I show three main results. First, I derive the value-at-risk constraint from concerns about model misspecification. I start with an investor with logarithmic utility who is not sure that her statistical model of asset returns is correct. She chooses from alternative, more pessimistic, models to make her choices robust to potential adverse scenarios. I show that this robust behavior is identical to self-imposing a value-at-risk constraint. The tightness of the resulting constraint maps into investor's willingness to consider alternative models, a potentially stochastic preference parameter that disciplines model choice in the face of statistical data arriving in real time.

Second, I show that economies with multiple risky assets and investors with heterogeneous and stochastic risk limits aggregate. Asset prices are identical to those in a world with a representative investor, whose risk limit is the wealth-weighted average of the economy's risk limits. The market's effective risk aversion is the harmonic average of individual ones. Alternatively, this investor's robustness preference parameter is the wealth-weighted harmonic average of all robustness parameters. The economy admits a stochastic discount factor that only depends on this average. Local dynamics of asset prices are fully determined by a scalar summary of the wealth distribution.

Third, I show that the evolution of the wealth distribution is determined by a single process. This process is the forecast error for aggregate market returns in the alternative model chosen by the representative robust investor. Because of the pessimism of the alternative model, this forecast error exhibits positive drift: the robust investor underestimates expected returns on the market portfolio. This generates systematic mispricing and delivers additional payoffs to risk-takers.

The fact that a single process drives the evolution of all wealth shares is important because the average risk limit is not Markovian. Dynamics of the entire wealth distribution matter for asset price dynamics. This opens the door for redistributive forces to affect risk premia. However, characterizing the evolution of wealth distributions is usually challenging, and the single-process feature offers a substantial reduction of computational complexity.

Using this result, I show that the drift in the forecast error process induces drift in wealth shares, making more risk-tolerant agents richer on average. This, in turn, induces a negative drift in risk premia. I explicitly characterize this drift and show that it is proportional to the wealth-weighted variance of the risk limits, or robustness parameters, in the economy. Realized positive shocks to aggregate market returns also decrease risk premia. This is because less constrained, or less pessimistic, agents take up more leverage and bet on aggregate growth. Positive growth surprises increase their wealth share, and their looser risk limits come to dominate the market.

Finally, I show that in my setup, shocks to risk limits or robustness parameters do not redistribute wealth. There are two reasons for this. First, with logarithmic utility, total wealth is proportional to total output, which shocks to risk limits do not affect. Second, all investors take on risky assets in the same proportions as the market portfolio. Their portfolios only differ in total risk exposure. Consequently, shocks that do not change aggregate wealth also do not redistribute. This leads to a clear demarcation: risk limit or robustness shocks change portfolios and agents' exposure to output shocks, and these output shocks, in turn, change their relative wealth.

I define the value-at-risk constraint in Section 2, show equivalence with robustness in Section 3, and aggregate in Section 4. Section 5 finishes with a simple two-agent example that illustrates the two drivers of risk premia, risk limit shocks and redistribution, analytically. This example relates to recent work on risk-centric macro models by Caballero and Simsek (2020), delineating risk premium dynamics in their model. I then review the literature in Section 6.

2 Value-at-risk constraint

Time is continuous and runs forever. The exogenous state of the economy is a d-dimensional vector x_t that evolves as

$$dx_t = \mu_X(x_t)dt + \sigma_X(x_t)dZ_t$$

Here $\{Z_t\}_{t\geq 0}$ is a standard b-dimensional Brownian process, so $\sigma_X(x_t)$ is a $(d \times b)$-dimensional matrix, and $\mu_X(x_t)$ is a vector of length d. The investor has access to a riskless asset that pays an instantaneous return $r(x_t)dt$ and a collection of k risky assets with a vector of instantaneous excess returns dR_t given by

$$dR_t = \mu_R(x_t)dt + \sigma_R(x_t)dZ_t$$

Here $\sigma_R(x_t)$ is a $(k \times b)$-dimensional matrix and $\mu_R(x_t)$ is a vector of length k. Returns and exogenous states do not necessarily load on all shocks, so some columns of $\sigma_R(x_t)$ and $\sigma_X(x_t)$ may be zero. I only require $\sigma_R(x_t)\sigma_R(x_t)'$ to have full rank for all x_t. Investor's wealth w_t evolves as

$$dw_t = (r(x_t)w_t - c_t)dt + w_t\theta_t' dR_t$$

Here c_t is consumption and θ_t is a k-dimensional vector of portfolio weights on risky assets, both chosen at time t. The pair (c_t, θ_t) are the only control variables.

The problem of the investor is to solve

$$\max_{\{c_t,\theta_t\}_{t\geq 0}} \mathbb{E}\left[\rho \int_0^\infty e^{-\rho t}\log(c_t)dt\right] \tag{1}$$

$$\text{s.t. } \mathbb{V}_t[\theta_t' dR_t] \leq \gamma_t \cdot \mathbb{E}_t[\theta_t' dR_t] \text{ for all } t \geq 0 \tag{2}$$

The value-at-risk constraint equation (2) is the key feature. It is imposed continuously on incremental returns and caps the variance of returns by a multiple of expected profits. Both variance and expectation are of order dt, so dt cancels out. The multiplier $\gamma_t < 1$ is one of the components of x_t. It is exogenous and stochastic, potentially driving returns and other macroeconomic outcomes. In Appendix A, I provide a heuristic explanation for why this constraint limits value-at-risk.

Investor's wealth w_t potentially impacts returns and prices in this economy too, although I do not allow her to internalize the price impact of her actions and wealth dynamics. To formally capture this, I let one of the components of x_t be a fictitious process $\hat{w}_t$ that coincides with w_t on all sample paths. The investor still treats it as exogenous, not realizing how her actions that change the evolution of w_t also change that of $\hat{w}_t$.

Proposition 1. *Investor's consumption and portfolio choice are*

$$c_t(w_t, x_t) = \rho w_t$$
$$\theta_t(w_t, x_t) = \gamma_t[\sigma_R(x_t)\sigma_R(x_t)']^{-1}\mu_R(x_t)$$

Investor's value function is separable over wealth and exogenous states: $V(w_t, x_t) = \log(w_t) + \eta(x_t)$, *where* $\eta(\cdot)$ *solves a second-order partial differential equation.*

I relegate the proof to Appendix B. The resulting portfolio is identical to one that a myopic mean-variance investor with risk aversion $1/\gamma_t$ would choose, so γ_t can be treated as time-varying effective risk tolerance. However, the investor is also forward-looking and solves a consumption-savings problem too, which is crucial for endogenizing asset prices and the interest rate $r(x_t)$. The fact that consumption is a constant fraction of wealth allows for simple linear aggregation, as is always the case with a unit elasticity of intertemporal substitution. The value-at-risk constraint adds to that a simple portfolio choice with a possibility to vary risk appetite over time.

Relation to other value-at-risk constraints. A literal interpretation of value-at-risk in the literature is a cap on the probability of losses exceeding certain proportion of wealth. Examples of this approach include Danielsson, Shin, and Zigrand (2011), Danielsson, Shin, and Zigrand (2012), and Adrian and Boyarchenko (2018). In practice, with normally distributed shocks, this leads to equation (2) with another upper bound for variance instead of expected returns. The upper bound is usually a stock variable, such as wealth, instead of a flow variable, such as expected returns, making it less interpretable with Brownian shocks, which make any realistic losses too small compared to total capital. Interestingly, without additional constraints, this formulation leads to the same mean-variance portfolios as is Proposition 1, except the effective risk tolerance γ_t is endogenous and includes Lagrange multipliers that require additional solution steps.

Relation to recursive preferences. Another way to allow for time-varying risk tolerance is using recursive preferences of Duffie and Epstein (1992). Given a process for consumption $\{c_t\}_{t\geq 0}$, they define the investor's value process $\{V_t\}_{t\geq 0}$ as follows:

$$V_t = \mathbb{E}\left[\int_t^\infty \varphi(c_s, V_s)ds\right]$$

The problem is to maximize V_t over $\{c_s, \theta_s\}_{s\geq t}$, while keeping $w_s \geq 0$ for all $s \geq t$. To be consistent with consumption choice of log investors, choose a form of $\varphi(\cdot)$ that keeps the elasticity of intertemporal substitution equal to one while allowing for non-unitary relative risk aversion:

$$\varphi(c, V) = \rho(1 - 1/\gamma)V\left[\log(c) - \frac{\log((1 - 1/\gamma)V)}{1 - 1/\gamma}\right]$$

The solution to this problem is usually guessed and verified. The value function is

$$V(w,x) = \frac{(w\eta(x))^{1-1/\gamma}}{1-1/\gamma}$$

Here $\eta(x)$ is marginal value of wealth that solves a second-order partial differential equation. The optimal consumption rule is $c(x_t) = \rho w_t$. The optimal portfolio is

$$\theta = \gamma[\sigma_R(x)\sigma_R(x)']^{-1}\mu_R(x) + \underbrace{\frac{\gamma-1}{\eta(x)}[\sigma_R(x)\sigma_R(x)']^{-1}\sigma_R(x)\sigma_X(x)'\eta_{x'}(x)}_{\text{hedging motive}} \tag{3}$$

The optimal portfolio consists of the mean-variance part $\gamma[\sigma_R(x)\sigma_R(x)']^{-1}\mu_R(x)$, sometimes called the myopic portfolio, and the hedging part. Computing the latter requires knowing the marginal value of wealth $\eta(x)$, which solves a second-order partial differential equation. The presence of this term increases computational complexity and reduces tractability. The only case without the hedging motive is $\gamma = 1$, but fixing γ makes it impossible to use fluctuations in preferences for risk as a driver of dynamics. Eliminating the hedging motive while preserving dynamics of γ is the main advantage of using the value-at-risk constraint.

Incorporating external income. Assuming that investors only have asset income and consumption expenditures is restrictive. The simple form of consumption and portfolio choice in Proposition 1 extends to some cases when a part of income is not chosen by the agent. Specifically,

$$dw_t = (r(x_t)w_t - c_t)dt + w_t\theta_t' dR_t - w_t\varsigma(x_t)dt - w_t\tau(x_t)'dZ_t$$

Here $w_t\varsigma(x_t)dt$ and $w_t\tau(x_t)'dZ_t$ are new terms that the agent does not choose. They can represent taxes, with $\varsigma(x_t)$ being a locally deterministic tax and $\tau(x_t)$ being a vector of taxes that load on the exogenous shocks. These taxes can be useful for inducing stationarity in the model through wealth redistribution: they can prevent the least constrained agent from taking over the entire economy. Taxes are lump-sum in the sense that the agent does not directly choose the tax base, which would be the case, for instance, with proportional taxes on portfolio returns. They still scale with wealth, however, and this feature is key to preserving consumption and portfolio choice.

The locally deterministic tax $\varsigma(x_t)$ clearly does not change anything in the agent's decision problem as it is simply isomorphic to a change in the interest rate. The stochastic tax in general affects portfolio choice. For a clearer characterization, I focus on a special case

$$\tau(x_t) = \zeta(x_t)\gamma_t \cdot \sigma_R(x_t)'[\sigma_R(x_t)\sigma_R(x_t)']^{-1}\mu_R(x_t) \tag{4}$$

If taxes are set up this way, they generate the same exposure to shocks as the optimal portfolio:

$$\tau(x_t)'dZ_t = \zeta(x_t)\theta(w_t, x_t)'\sigma_R(x_t)dZ_t \equiv \zeta(x_t)\theta(w_t, x_t)'(dR_t - \mu_R(x_t)dt)$$

Here $\theta(w_t, x_t)$ is the optimal vector of portfolio weights in the baseline model. This allows a potential government to effectively tax away a share $\zeta(x_t)$ of stochastic returns without imposing proportional taxes. I next show the effect of imposing this taxes on the agent.

Proposition 2. *Suppose the stochastic tax rate is given by equation (4) and there is a locally deterministic tax $\varsigma(x_t)$. Then, consumption choice is $c(w_t, x_t) = \rho w_t$. The optimal portfolio is*

$$\theta(w_t, x_t) = \min\{\gamma_t, 1 + \zeta(x_t)\gamma_t\} \cdot [\sigma_R(x_t)\sigma_R(x_t)']^{-1}\mu_R(x_t)$$

The value-at-risk constraint is slack if $\zeta(x_t) < 1 - 1/\gamma_t$ and binds otherwise.

Portfolio choice does not change unless the rate $\zeta(x_t)$ is very negative. The intuition is that taxing away a part of random returns decreases the agent's overall exposure to shocks, making her more willing to take risk. But the value-at-risk constraint is binding already without the taxes, so it continues to bind when $\zeta(x_t)$ is positive and even becomes tighter as measured by the size of the multiplier. A negative $\zeta(x_t)$, on the contrary, increases the agent's exposure to shocks and makes the constraint less tight. If $\zeta(x_t)$ falls below $1 - 1/\gamma_t$, the constraint stops binding, and the tax rate shows up in portfolio choice directly.

3 A foundation through robust choice

I now provide a microfoundation for the value-at-risk constraint in a setup with robustness preferences. I mostly follow Hansen, Khorrami, and Tourre (2024). Importantly, I slightly modify the traditional setup to simplify portfolio choice and eliminate hedging motives. The agent is allowed to consider misspecified processes for returns and choose potential worst-case scenarios. She is not allowed to misspecify the process for other states. This prevents her from imagining a non-zero correlation between her marginal value of wealth and returns in the future when she considers worst-case scenarios, and the hedging motive does not enter portfolio choice. As a result, a robust agent in my setup does not behave exactly like an agent with Kreps and Porteus (1978) or Duffie and Epstein (1992) preferences, unlike in most existing models.

Consider an investor who does not face a value-at-risk constraint but instead entertains alternative models of the underlying shock. Specifically, she thinks that the true b-dimensional Brownian process underlying the dynamics might be $\{B_t\}_{t\geq 0}$, and increments of $\{Z_t\}_{t\geq 0}$ differ from those of $\{B_t\}_{t\geq 0}$ by a time-varying drift: $dZ_t = dB_t - h_t dt$. The b-dimensional model correction process

$\{h_t\}_{t\geq 0}$ investor's choice. She assumes the following dynamics for returns:

$$dR_t = \mu_R(x_t)dt + \sigma_R(x_t)dZ_t \equiv (\mu_R(x_t) - \sigma_R(x_t)h_t)dt + \sigma_R(x_t)dB_t$$

This means she entertains alternative models for shocks driving returns and wishes to choose portfolios that are robust to potential model corrections h_t. These corrections make her assessment of excess returns pessimistic.

The investor assumes the following dynamics for exogenous states:

$$dx_t = \mu_X(x_t)dt + \sigma_X(x_t)dB_t$$

Crucial here is that the underlying shocks are dB_t instead of dZ_t as in Section 2. Hence, under alternative models that make $\{B_t\}_{t\geq 0}$ the true Brownian motion instead of $\{Z_t\}_{t\geq 0}$, there is no drift correction h_t to the dynamics of aggregate states. When the investor considers alternative models for shocks to returns, she automatically assumes that she was always right about the dynamics of aggregate states. This is the key difference compared to the standard setup in Hansen and Sargent (2001) and Hansen, Khorrami, and Tourre (2024).

Formally, let the original process $Z = \{Z_t\}_{t\geq 0}$ be a standard b-dimensional Brownian motion on the basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. Let $\{h_t\}_{t\geq 0}$ be a process adapted to $\{\mathcal{F}_t\}_{t\geq 0}$. By Girsanov's theorem, under a condition on $\{h_t\}_{t\geq 0}$, the process $B = \{B_t\}_{t\geq 0}$ given by $B_0 = Z_0$ and $dB_t = dZ_t - h_t dt$ is a standard b-dimensional Brownian motion on another statistical basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{Q})$. Here the measure $\mathbb{Q}$ satisfies $\mathbb{E}^{\mathbb{Q}}[\varphi_t] = \mathbb{E}^{\mathbb{P}}[M_t \varphi_t]$ for all $t \geq 0$ and for all bounded processes $\{\varphi_t\}_{t\geq 0}$ adapted to $\{\mathcal{F}\}_{t\geq 0}$. The process $\{M_t\}_{t\geq 0}$ is adapted to $\{\mathcal{F}_t\}_{t\geq 0}$ and given by $M_0 = 1$ and $dM_t = -h_t M_t dZ_t$. This re-weighting process is a likelihood ratio, and its logarithm $m_t = \log(M_t)$ evolves as

$$dm_t = -\frac{|h_t|^2}{2}dt - h_t' dZ_t = \frac{|h_t|^2}{2}dt - h_t' dB_t$$

I use this log-likelihood ratio to impose discipline on the investor's choices, limiting how far she can go in accounting for potential losses. One restriction that using log-likelihood deviations imposes on the environment is that the investor cannot entertain models with different quantities of risk. With Brownian shocks, volatility and correlation are instantly learnable, and hence create infinite log-likelihood ratios. This is what limits model adjustments to drift corrections h_t.

The problem of the investor is of the "multiplier" type:

$$\max_{\{c_t, \theta_t\}_{t\geq 0}} \inf_{\mathbb{Q}} \mathbb{E}^{\mathbb{Q}} \left[\rho \int_0^\infty e^{-\rho t} \log(c_t) dt + \int_0^\infty e^{-\rho t} \psi_t dm_t \right]$$

She chooses the alternative measure $\mathbb{Q}$ with potentially large losses and then maximizes utility over consumption and portfolio vectors that are robust to these losses. The discipline is provided by the cost of deviating from the original measure as measured by $\psi_t dm_t$ for each increment of time. Small values of the multiplier ψ_t make it easier to select pessimistic probability measures $\mathbb{Q}$, and the investor underestimates expected excess returns, which makes risky assets less attractive.

Proposition 3. *Consider a robust investor with a cost process $\{\psi_t\}_{t\geq 0}$. Her consumption and portfolio choice coincide with those of an investor with a value-at-risk constraint given by a multiplier process $\{\gamma_t\}_{t\geq 0}$, where $\gamma_t = \psi_t/(\psi_t+1)$:*

$$\begin{aligned} c(w_t, x_t) &= \rho w_t \\ \theta(w_t, z_t) &= \frac{\psi_t}{\psi_t + 1}[\sigma_R(x_t)\sigma_R(x_t)']^{-1}\mu_R(x_t) \end{aligned}$$

The drift corrections she chooses for instantaneous returns are

$$h(w_t, x_t) = \frac{1}{\psi_t + 1}\sigma_R(x_t)'[\sigma_R(x_t)\sigma_R(x_t)']^{-1}\mu_R(x_t)$$

Her value function is separable over wealth and exogenous states: $V(w_t, x_t) = \log(w_t) + \eta(x_t)$, where $\eta(\cdot)$ solves a second-order partial differential equation.

This proposition establishes the following mapping between the robustness parameter and the value-at-risk multiplier:

$$\psi_t = \frac{\gamma_t}{1 - \gamma_t}$$

The case $\psi_t = 0$ corresponds to infinite effective risk aversion, while $\psi_t \longrightarrow \infty$ corresponds to a standard log investor who does not make any model adjustments because it is prohibitively costly.

The resulting corrections account for the correlation structure of returns. Since making corrections to all fundamental factors is equally costly, marginal gains in terms of pessimism should be equalized across factors as well. These marginal gains depend on the correlation structure, hence the weighting matrix $\sigma_R(x_t)'[\sigma_R(x_t)\sigma_R(x_t)']^{-1}$ in the expression for $h(w_t, x_t)$. For example, if there is an element of dZ_t that no asset loads on, the corresponding column of $\sigma_R(x_t)$ is zero, and the corresponding element of $h(w_t, x_t)$ is zero too. The result of equalizing marginal gains in pessimism across factors is that portfolio shares on risky assets are simply scaled down compared to a regular log investor with no changes in relative weights.

Relation to standard robust preferences. Divorcing the shocks to returns dZ_t from aggregate shocks $d\tilde{Z}_t$ is key to obtaining tractable mean-variance portfolios. In the standard setup, where dZ_t and $d\tilde{Z}_t$ are not just perfectly correlated but coincide as processes, the investor considers alternative models for exogenous states as well as returns:

$$dR_t = \mu_R(x_t)dt + \sigma_R(x_t)dZ_t \equiv (\mu_R(x_t) - \sigma_R(x_t)h_t)dt + \sigma_R(x_t)dB_t$$
$$dx_t = \mu_X(x_t)dt + \sigma_X(x_t)dZ_t \equiv (\mu_X(x_t) - \sigma_X(x_t)h_t)dt + \sigma_X(x_t)dB_t$$

The optimal choice of h_t will now pick up the evolution of x_t and its impact on the investor's value through the gradient $V_{x'}(w,x)$. Since h_t affects the optimal choice of θ_t, the optimal portfolio will pick up $V_{x'}(w,x)$ too, and this will resurrect the hedging motive in portfolio choice. The solution will coincide with that under Kreps and Porteus (1978) and Duffie and Epstein (1992) preferences. I provide more detail in Appendix A.

Hedging motives disappear in the limit $\psi \longrightarrow \infty$. Another case where they disappear is one with $\sigma_R(x_t)\sigma_X(x_t)' = 0$, meaning that returns and aggregate states load on different shocks: some columns of $\sigma_R(x_t)$ and $\sigma_X(x_t)$ are zero, and the sets of their non-zero columns do not intersect. This happens when returns are purely idiosyncratic, and aggregates do not load on idiosyncratic shocks due to a large population. An example is the model of Di Tella, Malgieri, and Tonetti (2024). In their economy, entrepreneurs face idiosyncratic productivity risk realized after they commit to wages. The hiring decision is akin to an investment problem with a risky asset. There is a continuum of entrepreneurs, so idiosyncratic shocks do not drive aggregate dynamics. Potential drift corrections applied to idiosyncratic shocks would not appear in any process for aggregates, and standard robust preferences would also be equivalent to a value-at-risk constraint.

4 Aggregation with value-at-risk

I will now illustrate aggregative properties of the value-at-risk constraints in general equilibrium. Suppose there are n investors indexed by i. Their value-at-risk multipliers $\{\gamma_{it}\}$ are potentially different and stochastic. The vector of multipliers $\boldsymbol{\gamma}_t$ evolves as

$$d\,\boldsymbol{\gamma}_t = \mu_\gamma(\boldsymbol{\gamma}_t)dt + \sigma_\gamma(\boldsymbol{\gamma}_t)dZ_t^\gamma$$

Here dZ_t^γ is an increment of a $g-$dimensional standard Brownian motion, $\mu_\gamma(\boldsymbol{\gamma}_t)$ is an $n \times 1$ column vector, and $\sigma_\gamma(\boldsymbol{\gamma}_t)$ is an $n \times g$ matrix of loadings.

The rest of the exogenous states are the dividends of risky assets. There are k risky assets are indexed by j. The supply s_j of each asset j is fixed, with $s_j > 0$ for at least one j. The dividends are denoted by $\{y_{jt}\}$, and the vector of dividends $\boldsymbol{y}_t$ evolves as

$$d\,\boldsymbol{y}_t = \mu_y(\boldsymbol{y}_t)dt + \sigma_y(\boldsymbol{y}_t)dZ_t^y$$

Here dZ_t^y is an increment of a $d-$dimensional standard Brownian motion independent of dZ_t^γ, drift $\mu_y(\boldsymbol{y}_t)$ is a $k\times 1$ column vector, and $\sigma_y(\boldsymbol{y}_t)$ is a $k\times d$ matrix of loadings. The state space is completed by the vector of wealth. Investor i's wealth is w_{it}, and the state vector is $x_t' = (\boldsymbol{\gamma}_t' \;\; \boldsymbol{y}_t' \;\; \boldsymbol{w}_t')$.

Investors solve the problem in equation (1) subject to equation (2):

$$\max_{\{c_{it},\theta_{ijt}\}_{t\geq 0}} \mathbb{E}\left[\rho\int_0^\infty e^{-\rho t}\log(c_{it})dt\right]$$
$$\text{s.t. } \mathbb{V}_t[\boldsymbol{\theta}_{it}' d\boldsymbol{R}_t] \leq \gamma_{it}\mathbb{E}_t[\boldsymbol{\theta}_{it}' d\boldsymbol{R}_t] \text{ for all } t\geq 0$$

Their choice of portfolio shares $\boldsymbol{\theta}_{it} = \{\theta_{ijt}\}_{j=1}^k$ corresponds to a choice of share holdings in risky assets $\boldsymbol{h}_{it} = \{h_{ijt}\}_{j=1}^k$ with $p_{jt}h_{ijt} = \theta_{ijt}w_{it}$. The risk-free bond holdings $b_{it} = (1-\boldsymbol{\theta}_{it}' 1_k)w_{it}$ take up the complementary portfolio share.

Given initial holdings, an equilibrium is a collection of processes for prices $\{p_{jt}, r_t\}$, wealth $\{w_{it}\}$, and quantities $\{c_{it}, \boldsymbol{h}_{it}, b_{it}\}$ adapted to the filtration generated by $\{x_t\}_{t\geq 0}$ and satisfying the following conditions. First, quantities are chosen optimally by agents, who take prices as given. Second, the evolution of wealth is consistent with portfolio and consumption choices. Third, markets for all assets and consumption goods clear:

$$s_j = \sum\nolimits_{i=1}^n h_{ijt} \text{ for all } t\geq 0 \text{ and all } j\in\{1,...k\}$$
$$0 = \sum\nolimits_{i=1}^n b_{it} \text{ for all } t\geq 0$$
$$\sum\nolimits_{j=1}^k s_j y_{jt} = \sum\nolimits_{i=1}^n c_{it} \text{ for all } t\geq 0$$

I will characterize equilibrium prices $r(x_t)$ and $\boldsymbol{p}(x_t) = \{p_{jt}(x_t)\}$ as functions of aggregate states. These aggregate states generally include the wealth vector $\boldsymbol{w}_t = \{w_{it}\}$, and the epistemic assumption is that agents treat these wealth processes as exogenous, not realizing that their own wealth dynamics affect prices. This is what "taking prices as given" means.

Equilibrium characterization. The evolution of prices is

$$d\boldsymbol{p}(x_t) = \mu_p(x_t)dt + \sigma_{p,y}(x_t)dZ_t^y + \sigma_{p,\gamma}(x_t)dZ_t^\gamma$$

This defines the k-dimensional drift $\mu_p(x_t)$, the $(k\times d)$-dimensional matrix of loadings $\sigma_{p,y}(x_t)$, and the $(k\times g)$-dimensional matrix of loadings $\sigma_{p,\gamma}(x_t)$. I look for equilibria in which prices are diffusions. Given the drift and loadings of prices, the j-th component of excess returns is

$$dR_{jt} = \frac{[\mu_p(x_t)+\boldsymbol{y}(x_t)-r(x_t)\boldsymbol{p}(x_t)]_j}{[\boldsymbol{p}(x_t)]_j}dt + \frac{1}{[\boldsymbol{p}(x_t)]_j}[\sigma_{p,z}(x_t)dZ_t^y]_j + \frac{1}{[\boldsymbol{p}(x_t)]_j}[\sigma_{p,\gamma}(x_t)dZ_t^\gamma]_j$$

Let $\boldsymbol{s} = \{s_j\}$ be the vector of asset supply and denote the total wealth by $w_t = \boldsymbol{p}(x_t)'\boldsymbol{s}$. Since agents consume a constant multiple of their wealth, as shown in Section 2, total wealth is exogenous: $\rho w_t = \boldsymbol{y}(x_t)'\boldsymbol{s}$. Its evolution depends on $\boldsymbol{y}_t$ and can be written as

$$\frac{dw_t}{w_t} \equiv \mu_w(\boldsymbol{y}_t)dt + \sigma_w(\boldsymbol{y}_t)dZ_t^y$$

Two properties of wealth dynamics are important. First, the scalar drift $\mu_w(\boldsymbol{y}_t)$ and the $(1 \times d)$-dimensional row vector of loadings $\sigma_w(\boldsymbol{y}_t)$ directly map into the primitives of the environment. Endogenous dynamics are fully described by wealth shares of agents, making levels redundant. The dimensionality of the endogenous state space is $n-1$. Second, aggregate wealth does not depend on $\boldsymbol{\gamma}_t$. This is a consequence of assuming a unitary elasticity of intertemporal substitution.

The final step toward characterizing asset prices is defining wealth shares $\nu_{it} \equiv w_{it}/w_t$ and the wealth-weighted average of value-at-risk multipliers Γ_t:

$$\Gamma_t \equiv \sum\nolimits_{i=1}^{n} \gamma_{it}\nu_{it}$$

The next proposition characterizes the dynamics of asset prices and the interest rate.

Proposition 4. *Risky asset prices and the interest rate satisfy*

$$r(x_t)\boldsymbol{p}(x_t) = \mu_p(x_t) + \boldsymbol{y}_t - \frac{\sigma_{p,y}(x_t)\sigma_w(\boldsymbol{y}_t)'}{\Gamma_t} \tag{5}$$

$$r(x_t) = \rho + \mu_w(\boldsymbol{y}_t) - \frac{|\sigma_w(\boldsymbol{y}_t)|^2}{\Gamma_t} \tag{6}$$

I explain solving this system in Appendix A. Equation (5) is the main asset pricing equation. One part of the equation is present in risk-neutral pricing models, where $r(x_t)\boldsymbol{p}(x_t) = \mu_p(x_t) + \boldsymbol{y}_t$. The new content of equation (5) is the last term, which has two important properties.

First, risk adjustment in prices is given by the covariance of prices with the total wealth. Shocks only generate risk premia if they generate co-movements of prices with aggregate wealth. Importantly, since total wealth does not depend on $\boldsymbol{\gamma}_t$, price loadings on $\boldsymbol{\gamma}_t$ do not enter this expression. Uncertainty coming from shocks to value-at-risk multipliers is not priced in. This fact, however, does not imply that asset prices are independent of $\boldsymbol{\gamma}_t$. Shocks to any individual value-at-risk multiplier do induce individual price changes, but these price changes are all relative.

Second, the pricing of risk only depends on the wealth-weighted average Γ_t of all value-at-risk multipliers. The impact of the entire wealth distribution on prices is summarized by this scalar, which acts as the market's risk tolerance coefficient in both equation (5) and equation (6). In equation (6), Γ_t is the only endogenous object. The last term in equation (6) determines how costly uninsurable aggregate risk is for the economy: the interest rate is depressed when Γ_t is low and riskless debt commands a high safety premium. The market's risk aversion $1/\Gamma_t$ is the wealth-weighted harmonic average of individual effective risk aversion $1/\gamma_t$.

With hedging motives, aggregation would be considerably harder, since it would have to account for the heterogeneous marginal value of wealth, which usually requires one extra partial differential equation per agent. The fact that only Γ_t enters equation (5) and equation (6) means that a form of Gorman aggregation obtains in the model. Asset prices would be the same in an economy with a single investor with a value-at-risk multiplier equal to Γ_t on all possible histories. The economy also admits a stochastic discount factor that only dpeends on Γ_t:

Proposition 5. *Asset prices satisfy*

$$\Lambda_t \boldsymbol{p}(x_t) = \mathbb{E}_t \int_t^\infty \Lambda_s \boldsymbol{y}_s \, ds$$

Here $\Lambda_0 = 1$ *and*

$$d\Lambda_t = -r(x_t)\Lambda_t dt - \frac{1}{\Gamma_t}\Lambda_t \sigma_w(\boldsymbol{y}_t)' dZ_t^y$$

This discount factor is not necessarily unique, since markets do not have to be complete for any of the results in this model. It can be interpreted as belonging to the same fictitious representative agent with a value-at-risk multiplier $\gamma_t = \Gamma_t$ at all possible histories. Notably, the evolution of Λ_t does not load on the shocks to $\boldsymbol{\gamma}_t$ itself, reflecting the fact that $\boldsymbol{\gamma}_t$ does not affect aggregate wealth.

Evolution of the wealth distribution. Gorman aggregation does not imply that the economy collapses to a representative agent in the strong sense. Even though asset pricing is summarized by one scalar, the evolution of this scalar depends on the entire wealth distribution. Put another way, Γ_t is not a Markov process, and the minimal set of variables that have the Markov property is $x_t = (\boldsymbol{y}_t, \boldsymbol{\gamma}_t, \boldsymbol{\nu}_t)$, where $\boldsymbol{\nu}_t$ is the vector of wealth shares. This allows the model to capture movements in risk premia resulting from wealth redistribution. It turns out though that the evolution of the wealth distribution is driven by a single stochastic process, which I describe next.

As a first step, define aggregate market excess returns dR_t as

$$dR_t = \frac{1}{\sum_j s_j p_j(x_t)} \sum_j s_j p_j(x_t) dR_{jt}$$

Using Proposition 4,

$$dR_t = \frac{1}{\Gamma_t} |\sigma_w(\boldsymbol{y}_t)|^2 dt + \sigma_w(\boldsymbol{y}_t) dZ_t^y$$

Aggregate expected excess returns are larger if the hypothetical investor's constraint is tighter.

Consider now a hypothetical investor who has model misspecification concerns instead of a value-at-risk constraint. Suppose her robustness parameter Ψ_t maps into the aggregate value-at-risk multiplier Γ_t according to Proposition 3:

$$\Psi_t = \frac{\Gamma_t}{1 - \Gamma_t}$$

The amplification term $1/\Gamma_t = 1 + 1/\Psi_t$ in the expected excess returns can be interpreted as the robustness parameter of this hypothetical investor. The inverse $1/\Psi_t$ is hence the wealth-weighted harmonic average of individual robustness parameters.

Suppose this hypothetical robust investor has access to this one asset, the total market index. Let $dW_t \equiv dR_t - \mathbb{E}^{\mathbb{Q}_t}[dR_t]$ be the forecast error process the hypothetical robust investor chooses for this asset's return, where $\mathbb{Q}_t$ is the alternative probability measure corresponding to her optimally chosen drift correction. The next proposition characterizes it.

PROPOSITION 6. *The forecast error process dW_t chosen by the robust investor are*

$$dW_t = \frac{1 - \Gamma_t}{\Gamma_t} |\sigma_w(\boldsymbol{y}_t)|^2 dt + \sigma_w(\boldsymbol{y}_t) dZ_t^y$$

The error term dW_t has a predictable part, a drift correction, reflecting the robust investor's pessimistic choice of the model. This drift correction disappears when the robustness cost Ψ_t goes to infinity, which sets $\Gamma_t = 1$ and recovers log utility.

It turns out that in this economy, the forecast error process dW_t drives wealth dynamics. Define the leverage of investor i as $\lambda_{it} = \boldsymbol{\theta}_{it}' 1_k$. The next proposition characterizes leverage and individual asset holdings and relates the dynamics of wealth to dW_t.

PROPOSITION 7. *In equilibrium, total leverage of each agent i is $\lambda_{it} = \gamma_{it}/\Gamma_t$. Individual holdings of risky assets are $h_{ijt} = s_j \nu_{it} \lambda_{it}$. Wealth shares $\{\nu_{it}\}$ follow*

$$\frac{d\nu_{it}}{\nu_{it}} = (\lambda_{it} - 1) dW_t \tag{7}$$

The first property of the value-at-risk constraint is that the equilibrium leverage is given by a simple expression γ_{it}/Γ_t. The wealth-weighted leverage is always equal to one: the economy overall cannot hold a non-zero position in risk-free bonds. Underlying this is considerable heterogeneity between investors. Since Γ_t is a convex combination of γ_{it}, the investor with the highest γ_{it} always borrows in the risk-free asset, and the one with lowest γ_{it} always lends. More generally, excess leverage can be rewritten as

$$\lambda_{it} - 1 = (1 - \nu_{it})(\gamma_{it} - \gamma_{-it})$$

Here γ_{-it} is the wealth-weighted average of all value-at-risk multiplier except for i's. Excess leverage is positive if $\gamma_{it} > \gamma_{-it}$, which is equivalent to $\gamma_{it} > \Gamma_t$. Excess leverage falls if ν_{it} approaches one because all other agents become small and there is nobody to borrow from. If all investors have the same γ_{it}, everyone's leverage is one, and none has a gross position in the risk-free.

The share of asset j held by investor i is $h_{ijt}/s_j = \nu_{it}\lambda_{it}$. This does not depend on j: investors have the same portfolios of risky assets up to scale, only differing in the risky-riskless split. This implies that the risky part of everyone's portfolio is a scaled copy of the aggregate market portfolio: each investor assigns the same relative weights to all risky assets as the entire market. If everyone's γ_{it} is the same, then everyone holds exactly zero in the risk-free bond, and everyone's portfolio weights are the same as the market's.

The vector of wealth shares $\boldsymbol{\nu}_t = \{\nu_{it}\}$ is the endogenous state. The most interesting result in Proposition 7 is that the dynamics of $d\nu_{it}$ do not load on dZ_t^{γ}, since the forecast error dW_t does not. Shocks to value-at-risk multipliers do not induce wealth redistribution. The reason is that all investors hold risky portfolios with the same weights, which are also the aggregate market weights. Hence, any investor's wealth only changes when aggregate wealth does, and this does not happen since shocks to value-at-risk multipliers only reshuffle asset prices without affecting the total.

Unsurprisingly, equation (7) shows that, if $\lambda_{it} > 1$, ν_{it} is positively exposed to $\sigma_w(\boldsymbol{y}_t)dZ_t^y$ through the forecast error dW_t. Wealth shares of levered investors are positively correlated with aggregate wealth. The new part in equation (7) compared to the log utility case is that there is a drift in the wealth shares. This drift is positive for levered investors, reflecting the deterministic risk compensation they receive from the rest of the market. Levered investors exploit the market's robustness concerns, or the value-at-risk limits, and get ahead of others by pocketing the risk premium. With $\Gamma_t = 1$, or $\Psi_t = \infty$, this drift in wealth shares disappears.

Evolution of risk premia. Armed with Proposition 7, I can characterize the dynamics of risk premia in this economy and the two forces driving it. I do it by deriving the law of motion for Γ_t. This wealth-weighted average changes due to wealth redistribution by aggregate shocks and due to changes in individual value-at-risk multipliers, encapsulating both channels present in the literature. The following proposition characterizes its dynamics with both forces.

PROPOSITION 8. *The evolution of Γ_t is*

$$d\Gamma_t = \frac{\Delta_t}{\Gamma_t} dW_t + \boldsymbol{\nu}_t' \, d\,\boldsymbol{\gamma}_t$$

Here Δ_t is the wealth-weighted dispersion of the multipliers:

$$\Delta_t = \sum\nolimits_{i=1}^{n} \nu_{it}\gamma_{it}^2 - \left(\sum\nolimits_{i=1}^{n} \nu_{it}\gamma_{it}\right)^2 \geq 0$$

The first part of equation (8) is the impact of redistribution. Aggregate shocks induce wealth redistribution if investors choose heterogeneous leverage, taking different bets on the market. For this, they must have heterogeneous γ_{it}, implying $\Delta_t > 0$. In this case, aggregate risk tolerance is positively exposed to shocks to aggregate wealth $\sigma_w(\boldsymbol{y}_t)dZ_t^y$ through the expectation error term dW_t. This is because investors with higher γ_{it} become relatively richer after positive shocks, driving up the weighted average. There is a positive drift in the redistributive part of $d\Gamma_t$ as well, reflecting the upward drift in the wealth shares of less constrained investors due to the risk premia they collect. This drift disappears in the log utility case, when $\Gamma_t = 1$ or $\Psi_t = \infty$.

The second part of equation (8) is the change in the value-at-risk limits, or in the robustness preferences. This part could be non-zero even if everyone had the same γ_{it}, and wealth shares were be fixed forever. In this special case, risk premia would still be volatile, but the weighted average Γ_t would only move exogenously. This could be desirable in applications because it would reduce the dimensionality of the problem. Section 5 presents an example with an exogenously driven Γ_t.

5 Financial cycle

I now use a simple special case to separately illustrate how shocks to risk preferences and wealth redistribution impact risk premium. The claim to total output is the only risky asset. There are two investors with different value-at-risk parameters. In equilibrium, the agent with a less tight value-at-risk constraint borrows from the other one to bet on output growth. One of the value-at-risk parameters changes exogenously, following a mean-reverting process. This mean-reverting process generates a cycle in the market's effective risk-tolerance through two channels. First, it directly changes effective risk-tolerance of one of the agents. Second, without instantly affecting wealth shares, it makes agents rebalance their portfolios and take different exposure to aggregate shocks. Output shocks then redistribute wealth and change the risk premium by changing the wealth weights in the wealth-weighted average of value-at-risk multipliers. I characterize the law of motion of wealth shares, the risk premium, and the interest rate analytically.

The first agent's value-at-risk coefficient, denoted by γ_t, varies over time stochastically:

$$d\gamma_t = \mu_\gamma(\gamma_t)dt + \sigma_\gamma(\gamma_t)dZ_t^\gamma$$

Here γ_t varies within $[\underline{\gamma}, \overline{\gamma}] \subset (0, 1]$. The second agent's coefficient is fixed at $\hat{\gamma} \in [\underline{\gamma}, \overline{\gamma}]$. Output is produced by a Lucas tree in unit supply, $\boldsymbol{s} = 1$. The flow output of the tree is $y_t dt$, where the rate of production evolves as

$$\frac{dy_t}{y_t} = \mu dt + \sigma dZ_t$$

The tree price is p_t, and excess returns on the tree are $dR_t = (dp_t + y_t dt)/p_t - r_t dt$. Since the supply is normalized to one, total wealth is equal to the tree's price: $\rho p_t = y_t$. The price-dividend ratio is constant, and the capital gains process coincides with that of output growth. Excess returns transform into $dR_t = (\rho + \mu - r_t)dt + \sigma dZ_t$.

To induce stationarity, I make agents pay small wealth taxes. Agent i's budget constraint is

$$dw_{it} = (r_t w_{it} - c_{it})dt + \theta_{it} w_{it} dR_{it} - T_{it} w_{it} dt$$

Here, as before, θ_{it} is the share of i's portfolio allocated to the risky asset. The tax rate T_{it} is the following function of i's wealth share ν_{it}: $T_{it} = T(\nu_{it})$, where

$$T(\nu_{it}) = \tau \left(\nu_{it} - \frac{1}{2}\right) \sqrt{\frac{1-\nu_{it}}{\nu_{it}}}$$

Here τ is a small positive parameter. This tax policy balances the budget:

$$\sum\nolimits_{i\in\{1,2\}} T_{it} w_{it} = w_t \cdot \sum\nolimits_{i\in\{1,2\}} T_{it} \nu_{it} = \tau w_t \cdot \sum\nolimits_{i\in\{1,2\}} \left(\nu_{it} - \frac{1}{2}\right) \sqrt{\nu_{it}(1-\nu_{it})} = 0$$

Importantly, agent i takes T_{it} as given and does not realize how her wealth accumulation affects the tax rate. The fact that agents still perceive dw_{it} as linear in w_{it} preserves the functional forms of consumption and portfolio choice, as shown by Proposition 2.

There are two state variables in this economy: the endogenous wealth share of the first agent, below denoted by ν_t, and her exogenous value-at-risk multiplier γ_t. Denote the expected excess returns by $\pi(\nu_t, \gamma_t) \equiv \rho + \mu - r(\nu_t, \gamma_t)$. This is the equilibrium risk premium. With a constant price-dividend ratio, the dynamics of the interest rate exactly mirror those of $\pi(\nu_t, \gamma_t)$.

Equation (6) from Proposition 4 leads to

$$\pi(\nu_t, \gamma_t) = \frac{\sigma^2}{\Gamma(\nu_t, \gamma_t)} \tag{8}$$

The risk premium is inversely proportional to the wealth-weighted effective risk tolerance. When the agent with a less tight value-at-risk constraint accumulates more wealth, the market's aggregate risk tolerance rises, and the risk premium falls. Since the price-dividend ratio is fixed, this is achieved through a higher interest rate.

With just one risky asset, every investor's leverage is her risky portfolio share: $\lambda_{it} = \theta_{it}$, and hence $\theta_{it} = \gamma_{it}\pi(\nu_t, \gamma_t)/\sigma^2$ for both $\gamma_{it} \in \{\gamma_t, \hat{\gamma}\}$. Wealth dynamics can be represented as

$$\frac{dw_{it}}{w_{it}} = \left(\mu - \pi(\nu_t, \gamma_t) + \gamma_{it} \frac{\pi(\nu_t, \gamma_t)^2}{\sigma^2}\right) dt + \gamma_{it} \frac{\pi(\nu_t, \gamma_t)}{\sigma} dZ_t - T_{it} dt \tag{9}$$

The term $\mu - \pi(\nu_t, \gamma_t)$ reflects the consumption-savings trade-off. A high risk premium $\pi(\nu_t, \gamma_t)$ corresponds to a low interest rate, reflecting precautionary motives. The term $\gamma_{it}\pi(\nu_t, \gamma_t)^2/\sigma^2$ is the compensation for risk. Higher γ_{it} leads the investor to take a longer position in the tree and raises the compensation together with exposure of her wealth to dZ_t^y.

I now apply a version of Proposition 7 to find the evolution of the first agent's wealth share.

COROLLARY 1. *The evolution of the first agent's wealth share ν_t is*

$$d\nu_t = \underbrace{\frac{\nu_t(1-\nu_t)(\gamma_t - \hat{\gamma})}{\Gamma(\nu_t, \gamma_t)}}_{\text{excess leverage}} \cdot \left[\frac{1-\Gamma(\nu_t, \gamma_t)}{\Gamma(\nu_t, \gamma_t)}\sigma^2 dt + \sigma dZ_t^y\right] - \tau\nu_t(1-\nu_t)\left(\nu_t - \frac{1}{2}\right)dt \qquad (10)$$

The first agent's wealth share is positively exposed to output growth dZ_t^y. This is because, when $\gamma_t > \hat{\gamma}$, she borrows from the second agent to take on more exposure to aggregate risk. Besides the impact of shocks dZ_t^y, abstracting from taxes, her wealth share drifts up if and only if she currently has higher risk-taking capacity, $\gamma_t > \hat{\gamma}$. This is because she receives deterministic compensation for taking on aggregate risk, which makes her relatively richer over time.

Zooming in on specific trajectories without realized (by chance) output shocks, I illustrate the financial cycle in this economy. Consider a realized path of $\{\gamma_t\}_{t\in[t_1,t_2]}$ with fluctuations on $[\gamma', \gamma'']$ for some $\gamma' > \hat{\gamma} > \gamma''$: at first, γ_t linearly increases from γ' to γ'', then linearly goes back to γ', and repeats this a few times. Consider also a particular realization of $\{dZ_t^y\}_{t\in[t_1,t_2]}$ with all dZ_t^y realized to be zero. The left panel of Figure 1 illustrates this cycle. The first agent's wealth share drift up when she is more risk-tolerant, inducing rightward movement on the phase diagram. When she becomes less risk-tolerant, she rebalances out of the risky asset and starts to rely more on interest payments, gradually losing her wealth. This moves the economy leftwards on the diagram.

The right panel additionally illustrates the impact of a negative shock $dZ_\tau^y < 0$ for some τ in the middle of the timeline $[t_1, t_2]$. The shock happens when the first agent is more risk-tolerant, $\gamma_\tau > \hat{\gamma}$, and is levered up. Her relatively high exposure to aggregate output makes her wealth share decrease on impact. The cycle then continues, with the wealth share slowly drifting up on average across the half-periods because of wealth taxes. The shock instantly raises the risk premium by decreasing the wealth-weighted average $\Gamma(\nu_t, \gamma_t)$ since the first agent has a higher γ_t at the time.

I next characterize the dynamics of the risk premium $\pi(\nu_t, \gamma_t)$. Denote

$$\frac{d\pi(\nu_t, \gamma_t)}{\pi(\nu_t, \gamma_t)} = \mu_\pi(\nu_t, \gamma_t)dt + \sigma_{\pi,y}(\nu_t, \gamma_t)dZ_t^y + \sigma_{\pi,\gamma}(\nu_t, \gamma_t)dZ_t^\gamma$$

Applying Itô's lemma to the result of Proposition 8 and supposing $\tau = 0$ for illustration, I get the following evolution of $\pi(\nu_t, \gamma_t)$.

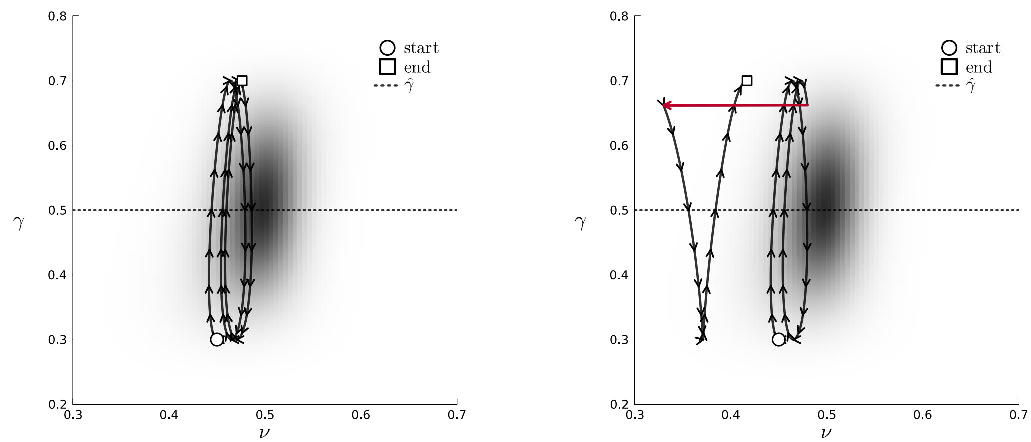

Figure 1: Left panel: path of (ν_t, γ_t) with no realized output shocks dZ_t^y. Right panel: path with a realized output contraction $dZ_\tau^y < 0$ at some $\tau \in (t_1, t_2)$. I use the following parameters to solve for the stationary distributions: $\sigma = 0.2$, $\overline{\gamma} = 0.8$, $\underline{\gamma} = 0.2$, $\tau = 0.1$. The drift an volatility of γ are $\mu_\gamma(\gamma_t) = 2\varsigma(\beta - \gamma_t)$ and $\sigma_\gamma(\gamma_t) = \varsigma\sqrt{(\overline{\gamma} - \gamma_t)(\gamma_t - \underline{\gamma})}$, where $\varsigma = 0.2$ and $\beta = 5$.

Corollary 2. *The shock loadings of $\pi(\nu_t, \gamma_t)$ are*

$$\sigma_{\pi,y}(\nu_t, \gamma_t) = -\sigma \cdot \frac{\Delta(\nu_t, \gamma_t)}{\Gamma(\nu_t, \gamma_t)^2}$$

$$\sigma_{\pi,\gamma}(\nu_t, \gamma_t) = -\sigma_\gamma(\gamma_t) \cdot \frac{\nu_t}{\Gamma(\nu_t, \gamma_t)}$$

$$\mu_\pi(\nu_t, \gamma_t) = \Gamma(\nu_t, \gamma_t) \cdot \Bigg(\underbrace{\sigma_{\pi,\gamma}(\nu_t, \gamma_t)^2 \left[1 - \frac{\mu_\gamma(\gamma_t)}{\sigma_\gamma(\gamma_t)^2 \nu_t}\right]}_{\textit{impact of preferences}} + \underbrace{\sigma_{\pi,y}(\nu_t, \gamma_t)^2 \left[1 - \frac{(1 - \Gamma(\nu_t, \gamma_t))\Gamma(\nu_t, \gamma_t)}{\Delta(\nu_t, \gamma_t)}\right]}_{\textit{impact of redistribution}} \Bigg)$$

The second term, impact of redistribution, is always negative.

The risk premium decreases after positive output shocks because they redistribute to the more risk-tolerant agent, who levers up to bet on output growth. Of course, it also decreases following positive shocks to the value-at-risk multiplier. The drift in the risk premium has two parts. The part coming from the evolution of the value-at-risk multiplier can take either sign, since the multiplier is mean-reverting. The part coming from wealth redistribution is always positive, since deterministic risk compensation makes the more risk-tolerant agent relatively richer over time.

I make a final note on the leverage cyclicality in this economy. Aggregate leverage is always fixed, since the quantity of risky assets is constant. However, one can interpret the more risk-tolerant agent as a financial intermediary. Proposition 7 implies that her leverage increases in her value-at-risk multiplier and decreases in her wealth share. The former means that it is procyclical with respect to the risk premium: as γ_t increases, leverage rises, while risk premium falls. Leverage is countercyclical with respect to output because it decreases in the first agent's wealth share

ν_t: the other agent becomes too small following positive output shocks and cannot lend enough for the first agent to lever up as much. Procyclical leverage is the empirically relevant case, as shown by Geanakoplos (2010), Adrian and Shin (2014), and Kalemli-Ozcan, Sorensen, and Yesiltas (2012). Countercyclical leverage typical of models in Brunnermeier and Sannikov (2014) and He and Krishnamurthy (2013) is also nested in my model when the cycle is measured by output.

Vasicek (1977) model with a single agent. Consider a special case of this model with $\nu_t = 1$: the investor with a time-varying multiplier dominating the market. The state space reduces to one dimension, with γ_t being the only state variable. In this model, the risk premium fluctuates for exogenous reasons, and so does the interest rate. It is possible to obtain any process for the interest rate by choosing an appropriate process for γ_t, subject to the natural restrictions $\gamma_t \in (0, 1]$, which implies $r_t \in (-\infty, \rho + \mu - \sigma^2]$. One example is the interest rate process from the Vasicek (1977) model. This Ornstein-Uhlenbeck process has been widely used in models of term structure, including a recent preferred-habitat models in the style of Vayanos and Vila (2021).

Corollary 3. *Let $\kappa_r > 0$ and σ_r be parameters and suppose the process for γ_t is*

$$\frac{d\gamma_t}{\gamma_t} = \left(\kappa_r + \frac{\sigma_r^2}{\sigma^4}\gamma_t^2\right) dt + \frac{\sigma_r}{\sigma^2}\gamma_t dZ_t^\gamma$$

with a reflecting boundary at $\gamma_t = 1$. Then, the process for the interest rate is

$$dr_t = \kappa_r(\rho + \mu - r_t)dt + \sigma_r dZ_t^\gamma$$

with a reflecting boundary at $r_t = \rho + \mu - \sigma^2$.

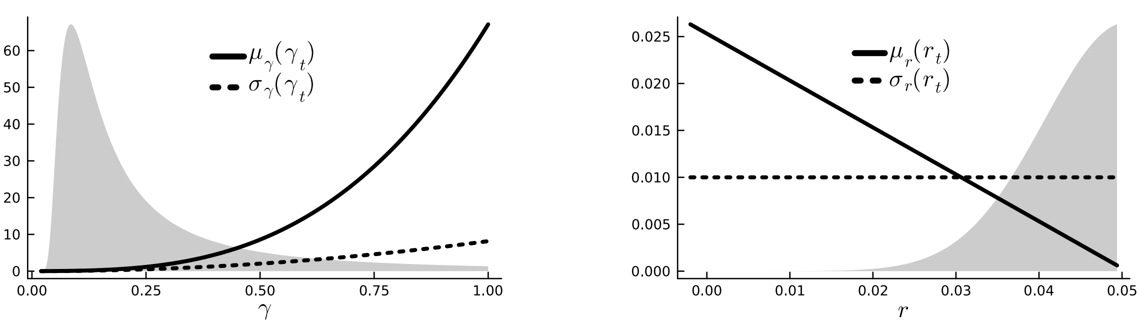

Figure 2: drift and volatility forms in Corollary 3. Stationary distributions in the background. I use the following parameters: $\sigma = 0.035$, $\kappa_r = 0.5$, $\sigma_r = 0.01$, and $\rho + \mu = 0.05 + \sigma^2$.

Caballero and Simsek (2020) model with fixed multipliers. Consider the converse case, where ν_t fluctuates within $(0, 1)$ but γ_t is fixed at $\overline{\gamma}$, the upper bound of the support. For simplicity, let the second investor's multiplier be at the lower bound: $\hat{\gamma} = \underline{\gamma}$. The state space again reduces to one dimension, but the state variable is now the wealth share ν_t, which is endogenous.

This reduction of the state space to one dimension means that the wealth share ν_t and the risk premium π_t are both Markov processes that be characterized in the following form:

$$
\begin{aligned}
d\nu_t &= \mu_\nu(\nu_t)dt + \sigma_\nu(\nu_t)dZ_t \\
\frac{d\pi_t}{\pi_t} &= \mu_\pi(\pi_t)dt + \sigma_\pi(\pi_t)dZ_t
\end{aligned}
$$

The risk premium fluctuates on $[\underline{\pi}, \overline{\pi}]$, where $\underline{\pi} = \sigma^2/\overline{\gamma}$ and $\overline{\pi} = \sigma^2/\underline{\gamma}$. Corollary 2 implies

Corollary 4. *Supposing $\tau = 0$ for simplicity, the drift and volatility of ν_t are*

$$
\begin{aligned}
\mu_\nu(\nu_t) &= \nu_t\,(1-\nu_t) \cdot \frac{\sigma^2(\gamma - \underline{\gamma})\,(1 - \nu_t\overline{\gamma} - (1-\nu_t)\underline{\gamma})}{\left(\nu_t\overline{\gamma} + (1-\nu_t)\underline{\gamma}\right)^2} \\
\sigma_\nu(\nu_t) &= \nu_t\,(1-\nu_t) \cdot \frac{\sigma(\overline{\gamma} - \underline{\gamma})}{\nu_t\overline{\gamma} + (1-\nu_t)\underline{\gamma}}
\end{aligned}
$$

with $\mu_\nu(\nu_t) > 0$ and $\sigma_\nu(\nu_t) > 0$ for all ν_t. The drift and volatility of the risk premium are

$$
\begin{aligned}
\mu_\pi(\pi_t) &= \left(\frac{\sigma}{\overline{\pi}\underline{\pi}}\right)^2 \cdot \left(\sigma^2(\overline{\pi} + \underline{\pi} - \pi_t) - \overline{\pi}\underline{\pi}\right) \cdot (\overline{\pi} - \pi_t)(\pi_t - \underline{\pi}) \\
\sigma_\pi(\pi_t) &= -\frac{\sigma}{\overline{\pi}\underline{\pi}} \cdot (\overline{\pi} - \pi_t)(\pi_t - \underline{\pi})
\end{aligned}
$$

with $\mu_\pi(\pi_t) < 0$ and $\sigma_\pi(\pi_t) < 0$ for all π_t.

Both drift and volatility of the wealth share ν_t increase in the difference $\overline{\gamma} - \underline{\gamma}$, which measures polarization in attitudes to risk. The gap in risk appetite creates different portfolio allocations and ultimately allows output shocks to induce redistribution towards more risk-tolerant agents. This is a simplified version of the mechanism in Caballero and Simsek (2020), who induce two agents take differential bets on output growth because of disagreement on the underlying process. When the realized output growth is above average, the more optimistic agent becomes relatively richer, which suppresses the risk premium and raises the interest rate. Caballero and Simsek (2020) then mount a New-Keynesian block on this structure, generating additional output expansion if the monetary authority does not correspondingly raise the policy rate.

In my setup, differences in risk tolerance come from value-at-risk multipliers. Heterogeneous multipliers reflect the fact that agents choose different statistical models for asset returns. There is a disagreement over expected output growth just like in Caballero and Simsek (2020). The difference of my setup compared to Caballero and Simsek (2020) is that here agents have the same reference model, and the fundamental heterogeneity is in robustness preference parameters.

Discussion. This one-tree two-agent example highlights the benefit of the value-at-risk: endogenous processes only affect asset prices and the risk-free rate through a weighted average Γ_t. This

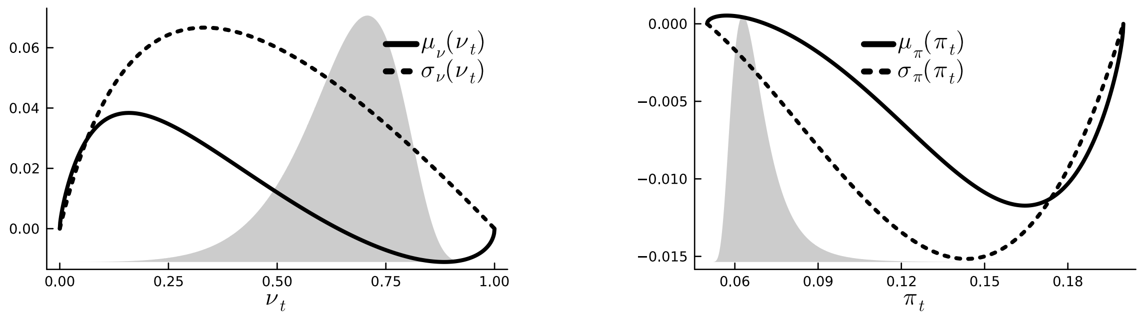

Figure 3: drift and volatility of the first agent's wealth share ν_t and the risk premium π_t. Stationary distributions in the background. Parameters are the same as those used for Figure 1.

leads to simple expressions for the interest rate in one-asset economies with geometric Brownian shocks paralleling those in Cochrane, Longstaff, and Santa-Clara (2008), to which my example is a converse: two agents and one tree instead of one agent and two trees. In Cochrane, Longstaff, and Santa-Clara (2008) dividend shares of the two trees move risk premia and the interest rate by changing equilibrium risk exposures of the agent: a high concentration of dividends increases the quantity of risk. In my example and in Caballero and Simsek (2020), the quantity of risk is fixed, and prices of risk change endogenously due to redistribution between heterogeneous agents.

6 Related literature

Value-at-risk measures are widely used in banking. Stulz (2016) explains the use of value-at-risk in banks' framework in detail. Sizova (2023) collects information on banks updating their models from financial reports and shows that banks often mention "value-at-risk models" in their reports. Their model revisions respond to changes in regulation. Barbiero, Bräuning, Joaquim, and Stein (2024) collect data on risk limits faced by the largest US banks. They show that 165 of 167 trading desks whose activities are related to currency trading face value-at-risk constraints. Adrian and Shin (2010) show that the ratio of value-at-risk to capital at the largest US banks was fairly constant in 2000-2007. Adrian and Shin (2014) extend measurements to the after-crisis times, when regulation changed significantly, and find that the value-at-risk to equity ratio seems relatively stable across regimes. They then provide a foundation for this constraint using the contractual framework from Holmstrom and Tirole (1997). Value-at-risk restrictions emerge endogenously when banks use debt financing, enjoy limited liability, and choose between extreme value-distributed investment opportunities with different riskiness.

The literature using value-at-risk constraints for portfolio choice in macroeconomics dates back to at least Danielsson, Shin, and Zigrand (2012) and Adrian and Boyarchenko (2018). These papers use continuous-time models, where intermediaries' risky positions are subject to the following constraint: the standard deviation of the excess returns cannot exceed a multiple of net worth. Intermediaries form mean-variance portfolios with the Lagrange multiplier on the constraint acting as the time-varying risk aversion coefficient. The leverage of the intermediaries is inversely proportional to the variance of returns. Danielsson, Shin, and Zigrand (2012) and Adrian and Boyarchenko (2018) make use of these properties to arrive at countercyclical leverage and obtain the "volatility paradox": low exogenous volatility coincides with high leverage, which leads to high systemic risk. Hofmann, Shim, and Shin (2022) use the same constraint to show the impact of dollar appreciation on portfolio inflows in emerging markets. My formulation of the value-at-risk constraint is different in that I cap the total variance of returns, rather than the standard deviation, and I use a flow (expected profits) rather than the stock of wealth as the upper bound on risk exposure. The result is mean-variance portfolios with time-varying risk aversion explicitly given as a primitive of the model instead of an endogenous Lagrange multiplier. This allows for more transparent aggregation and simplifies the use of attitudes to risk as a driver of shocks.

A desirable feature of the value-at-risk constraints is the procyclical leverage they generate. Kalemli-Ozcan, Sorensen, and Yesiltas (2012) show this empirically. Shin (2012) uses this property to argue that the leverage cycle of global banks drives credit supply and loan risk premia in the US. Coimbra (2020) models intermediaries with an occasionally binding value-at-risk constraint. In busts, when intermediaries are up against the constraint, risk-averse households absorb the residual supply of risky assets, which leads to a rise in risk premia. Coimbra and Rey (2024) model a cross-section of intermediaries with different value-at-risk constraint parameters. Some intermediaries are inactive, and active ones choose different risk exposures. Changes in expected productivity and interest rates induce changes in both extensive and intensive margins of their risk-taking, moving the aggregate leverage of the financial sector. Intermediaries are short-lived, so interest rate and productivity news lead to a reallocation of activity rather than wealth redistribution between them. My environment allows for wealth redistribution and exogenous changes in risk preferences but does not incorporate the extensive margin of activity. Another difference is that Coimbra (2020) and Coimbra and Rey (2024) use discrete time, which allows them to interpret value-at-risk literally: intermediaries face a limit on the probability of negative equity returns or failure. My continuous-time framework interprets the constraint as a limit on the instantaneous variance of returns, for which I provide a heuristic derivation.

Empirically, Coimbra, Kim, and Rey (2022) estimate value-at-risk parameters from bank balance sheet data and find substantial heterogeneity in cross-section. Barbiero, Bräuning, Joaquim, and Stein (2024) find that changes in value-at-risk limits of dealers impact currency returns. FX market is highly intermediated and depends on a concentrated industry of large dealers. Using regulatory data, Barbiero, Bräuning, Joaquim, and Stein (2024) find rather long-lived effects of value-at-risk limit tightenings of individual dealers. Bräuning and Stein (2024) find that limit changes also affect the functioning of the treasury markets.

A large set of models rely on constraints related to value-at-risk. Examples include Gromb and Vayanos (2002) and Gromb and Vayanos (2018), where arbitrageurs face financial constraints that disallow negative equity. This can be interpreted as setting a zero limit on value at risk. Vayanos (2004) incorporates performance-based liquidation into a model of fund management. End investors liquidate their fund holdings with a probability tied to returns. Fund managers are concerned about liquidation, and these concerns affect their risk-taking in a way similar to a value-at-risk penalty. Vayanos and Vila (2021), Gourinchas, Ray, and Vayanos (2022), Ray (2019), Kamdar and Ray (2024), and Greenwood, Hanson, Stein, and Sunderam (2023) endow arbitrageurs with mean-variance preferences, mentioning that this can capture value-at-risk constraints in reduced form. I operationalize this conjecture in a fully dynamic setup.

Kekre, Lenel, and Mainardi (2024) extend Vayanos and Vila (2021) by making arbitrageurs infinitely-lived agents with power utility. Abritrageurs' wealth has a positive duration in equilibrium, so positive interest rate shocks lead to an increase in term premia. This resolves a conflict between the data and the baseline model of Vayanos and Vila (2021), where term premia fall following contractionary monetary shocks. Portfolio choice in Kekre, Lenel, and Mainardi (2024) includes a hedging term, which they show disappears in the limit of infinite impatience. Value-at-risk constraints provide an alternative way to eliminate hedging motives.

A recent literature in international macro uses risk aversion shocks of myopic intermediaries to achieve time-varying risk premia. Itskhoki and Mukhin (2021) and Kekre and Lenel (2024) augment the framework developed by Gabaix and Maggiori (2015) with time variation in the risk aversion parameters to achieve desirable properties of exchange rates. Oskolkov (2024) develops a model of the global economy with long-lived global intermediaries and studies the financial cycle generated by shocks to the same value-at-risk parameter. Kekre and Lenel (2025) replace risk aversion shocks with monetary shocks that destroy intermediairy wealth with similar results.

The other force driving time variation in risk premia, wealth redistribution, is characterized in He and Krishnamurthy (2013) and Brunnermeier and Sannikov (2014) and the long literature that follows their general equilibrium frameworks. Kekre and Lenel (2022) provide a clear exposition of this force in a simple model and then study it in a quantitative model of the US economy.

My foundation for the value-at-risk constraint is related to a large literature on robust control and model misspecification. The classical reference for this framework is Hansen and Sargent (2001). They clarify the link between the maxmin decision rules of Gilboa and Schmeidler (1989) and economic applications of robust control, which typically adopt a "penalty" approach to model misspecification. Anderson, Hansen, and Sargent (2003) and Hansen, Sargent, Turmuhambetova, and Williams (2006) relate and compare different ways to apply robust control. A recent application of robust portfolio choice in macro-financial models is Hansen, Khorrami, and Tourre (2024). My setup falls into this long tradition. The main difference is that I separate model misspecification for returns and aggregate states. The agent explores alternative probability measures for shocks driving excess returns but ignores the implications of these alternatives for her views on aggregate states. This allows me to eliminate hedging motives and reduce portfolio choice to mean-variance, breaking away from Kreps and Porteus (1978) and Duffie and Epstein (1992).

Finally, there is a literature in finance studying value-at-risk in portfolio choice. Basak and Shapiro (2001) and Berkelaar, Cumperayot, and Kouwenberg (2002) analyze equilibrium consequences of the presence of value-at-risk-constrained agents for asset prices. Sentana (2001) conceptualizes iso-value-at-risk curves by analogy with iso-Sharpe curves in asset space. Yiu (2004) derives a dynamically imposed constraint, which resembles the constraint in Danielsson, Shin, and Zigrand (2012) with a standard deviation of returns instead of variance. Alexander and Baptista (2003) and Alexander and Baptista (2004) establish that imposing value-at-risk constraints can make mean-variance agents act as more or less risk averse depending on their initial preferences. Methodological literature includes Noyan and Rudolf (2013), Bernard, Rüschendorf, and Vanduffel (2017), Pirvu (2007), Krokhmal, Palmquist, and Uryasev (2002), Alexander and Baptista (2003), Alexander and Baptista (2008), Cuoco, He, and Isaenko (2008), and many others.

7 Conclusion

I describe a value-at-risk constraint that generates mean-variance portfolios with time-varying risk tolerance. These portfolios work in infinite horizon economies without generating hedging demand terms that are hard to aggregate. I provide a foundation for this constraint through a version of robustness concerns, suggesting an interpretation of shocks to risk limits that does not depend on changing regulation and instead captures changing model misspecification concerns.

I then show that prices of risky assets and the risk-free rate depend on the wealth distribution through a single scalar: the wealth-weighted average of the value-at-risk multipliers, which can be interpreted as the market's effective risk tolerance or, alternatively, the strength of robustness concerns of a hypothetical representative investor with model misspecification concerns instead of constraints. Despite the existence of an "as-if representative" agent, the evolution of risk premia depends on the entire wealth distribution. However, the distribution itself is driven by a single process: the hypothetical robust investor's forecast errors. This process has a drift, which induces drift in wealth shares and makes less constraint, or less robust, agents accumulate wealth over time. I decompose the evolution of risk premia into exogenous shocks to risk limits or robustness parameters and redistribution between heterogeneously constrained, or heterogeneously robust, agents. Shocks to risk limits or robustness parameters do not redistribute wealth directly, instead changing how much risk agents take, which then opens the door for redistribution by output shocks.

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A Additional details

Heuristic explanation for value-at-risk. A generic value-at-risk constraint caps the probability of a given level of losses at a certain level. For example, given constants L and α, the constraint is

$$\mathbb{P}\left\{\theta_t' dR_t \leq -\sqrt{Ldt}\right\} \leq \alpha \tag{A.1}$$

Choosing Ldt instead of $\sqrt{Ldt}$ in this expression would not work. Heuristically, the variance of Brownian shocks is proportional to dt, so the standard deviation of $\theta_t' dR_t$ is of order $\sqrt{dt}$. Hence, as $t \longrightarrow 0$, the probability on the left of equation (A.1) would just converge to 0 or 1 depending on how $\mathbb{E}[\theta_t' dR_t] = \theta_t' \mu_R(x_t) dt$ compares to Ldt. With $\sqrt{Ldt}$ instead, equation (A.1) becomes

$$\Phi\left(-\frac{\sqrt{Ldt} + \theta_t' \mu_R(x_t) dt}{\sqrt{\theta_t' \sigma_R(x_t) \sigma_R(x_t)' \theta_t dt}}\right) \leq \alpha$$

Here $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution. As $dt \longrightarrow 0$, the $O(dt)$ term in the numerator vanishes. For all $\alpha < 1/2$, the limit of this inequality becomes

$$\theta_t' \sigma_R(x_t) \sigma_R(x_t)' \theta_t \leq \frac{L}{(\Phi^{-1}(\alpha))^2}$$

One choice of L is a proportion of net worth w_t, capping the probability of losing a fraction of capital. In continuous time, flow losses are not commensurate with net worth, which is a stock. No amount of instantaneous Brownian losses will make a dent in the intermediary's capital. Instead, one could choose another flow, such as a multiple of expected profits, $L = \hat{\gamma}_t \theta_t' \mu_R(x_t)$, to make a flow-to-flow comparison. With $\gamma_t = \hat{\gamma}_t / (\Phi^{-1}(\alpha))^2$, the constraint takes its final form:

$$\theta_t' \sigma_R(x_t) \sigma_R(x_t)' \theta_t \leq \gamma_t \theta_t' \mu_R(x_t)$$

The last step is realizing that $\mathbb{V}[\theta_t' dR_t] = \theta_t' \sigma_R(x_t) \sigma_R(x_t)' \theta_t dt$ and $\mathbb{E}[\theta_t' dR_t] = \theta_t' \mu_R(x_t) dt$.

Standard robustness concerns and recursive preferences. The recursive representation of the robust problem when both return and aggregate state processes are misspecified is

$$\begin{aligned}
\rho V(w,x) = \max_{c,\theta} \min_{h} \; & \rho \log(c) + \frac{\psi |h|^2}{2} + (r(x) w - c + w\theta' \mu_R(x) - w\theta' \sigma_R(x) h) V_w(w,x) \\
& + \frac{\theta' \sigma_R(x) \sigma_R(x)' \theta}{2} w^2 V_{ww}(w,x) + \frac{1}{2} \mathrm{tr}[\sigma_X(x)' V_{xx'}(w,x) \sigma_X(x)] + w\theta' \sigma_R(x) \sigma_X(x)' V_{wx'}(w,x) \\
& + (\mu_X(x)' \underbrace{- h' \sigma_X(x)'}_{\text{new}}) V_{x'}(w,x)
\end{aligned}$$

The solution $V(w,x) = \log(w) + \log(\eta(x))$ can be guessed and verified. Given this, $c = \rho w$ and

$$h = \frac{1}{\psi\eta(x)}\sigma_X(x)'\eta_{x'}(x) + \frac{1}{\psi}\sigma_R(x)'\theta$$
$$\theta = [\sigma_R(x)\sigma_R(x)']^{-1}(\mu_R(x) - \sigma_R(x)h)$$

The fact that consumption is linear in w and (θ, h) only depend on x verifies the conjecture that $V(w,x)$ is log-separable over w and x. The optimal portfolio θ is given by

$$\theta = \frac{\psi}{1+\psi}[\sigma_R(x)\sigma_R(x)']^{-1}\mu_R(x) - \underbrace{\frac{1}{(1+\psi)\eta(x)}[\sigma_R(x)\sigma_R(x)']^{-1}\sigma_R(x)\sigma_X(x)'\eta_{x'}(x)}_{\text{hedging motive}}$$

This expression exactly coincides with equation (3) if $\gamma = \psi/(1+\psi)$.

Computation in general equilibrium. I make a brief note on solving for asset prices. Apply Itô's lemma to $\mu_p(\cdot)$ and $\sigma_{p,y}(\cdot)$. Omitting the time subscripts, define the drift and shock loadings of $x = (\boldsymbol{y}, \boldsymbol{\gamma}, \boldsymbol{\nu})$ as $\mu_x(x)$ and $\sigma_x(x)' \equiv (\sigma_{x,y}(x)'\ \sigma_{x,\gamma}(x)')$, where

$$dx = \mu_x(x)dt + \sigma_{x,y}(x)dZ^y + \sigma_{x,\gamma}(x)dZ^\gamma$$

With this definition, equation (5) can be transformed to

$$\left(\rho + \mu_w(\boldsymbol{y}) \underbrace{-\frac{|\sigma_w(\boldsymbol{y})|^2}{\Gamma(x)}}_{\text{prudence}}\right)p_j(x) = y_j + \mathcal{D}p_j(x)\left(\mu_x(x) \underbrace{-\frac{\sigma_{x,z}(x)\sigma_w(\boldsymbol{y})'}{\Gamma(x)}}_{\text{risk adjustment}}\right) + \frac{1}{2}\text{tr}[\mathcal{H}p_j(x)\sigma_x(x)\sigma_x(x)']$$

This formulation is reminiscent of asset pricing by risk-neutral agents and of Hamilton-Jacobi-Bellman equations used to solve for value functions. In my setup, the relevant parts of value functions can be characterized in closed form, and the key forward-looking objects are asset prices.

Two elements are new compared to risk-neutral pricing or HJB equations: overall discounts rates are suppressed due to demand for precautionary savings (the prudence term), and expected capital gains are attenuated by risk compensation (the risk adjustment term). The same result obtains in a model with homogeneous investors with log utility, which is nested as $\Gamma = 1$. The tractability gain of the value-at-risk model is that it generalizes the log case to time-varying effective risk tolerance without losing the convenience of the pricing equations and only expanding the state space just enough to account for agent heterogeneity. The linear partial differential equation for prices can be solved on a grid for $x = (\boldsymbol{y}, \boldsymbol{\gamma}, \boldsymbol{\nu})$ in conjunction with the exogenous laws of motion for $\boldsymbol{y}$ and $\boldsymbol{\gamma}$ and that for $\boldsymbol{\nu}$ given in Proposition 7. Conveniently, local dynamics of each coordinate of $\boldsymbol{\nu}$ are given only by the coordinate itself and $\Gamma = \boldsymbol{\nu}'\boldsymbol{\gamma}$, which simplifies computations.

B Proofs

Proof of Proposition 1. Take the recursive form of equation (1). Let $V(w,x)$ be the value of an agent with wealth w given the aggregate state x. The HJB equation is

$$\begin{aligned}\rho V(w,x) = \max_{c,\theta}\ &\rho\log(c) + (r(x)w - c + w\theta'\mu_R(x))V_w(w,x) + \frac{\theta'\sigma_R(x)\sigma_R(x)'\theta}{2}w^2 V_{ww}(w,x)\\ &+ \mu_X(x)'V_{x'}(w,x) + \frac{1}{2}\mathrm{tr}[\sigma_X(x)'V_{xx'}(w,x)\sigma_X(x)] + w\theta'\sigma_R(x)\sigma_X(x)'V_{wx'}(w,x) \qquad \text{(A.2)}\\ \text{s.t. }\ &\theta'\sigma_R(x)\sigma_R(x)'\theta \leq \gamma\theta'\mu_R(x) \qquad \text{(A.3)}\end{aligned}$$

Guess and verify the solution to equation (A.2) to be $V(w,x) = \log(w) + \eta(x)$. First, this implies $V_{wx'}(w,x) = 0$. Second, $V_x(w,x)$ and $V_{xx'}(w,x)$ are functions of x only: $V_x(w,x) = \eta_x(x)$ and $V_{xx'}(w,x) = \eta_{xx'}(x)$. Finally, consumption is a constant fraction of wealth: $c = \rho w$.

Next, consider portfolio choice. Let $\xi(x,w)$ be the multiplier on equation (A.3). Taking the first-order condition with respect to θ,

$$\theta = \frac{1+\gamma\xi(w,x)}{1+2\xi(w,x)}[\sigma_R(x)\sigma_R(x)']^{-1}\mu_R(x)$$

If the constraint were slack, then $\xi(w,x) = 0$ and θ would be the regular mean-variance portfolio typical for log investors, $\theta = [\sigma_R(x)\sigma_R(x)']^{-1}\mu_R(x)$. But by equation (A.3), this contradicts $\gamma < 1$. Hence, $\xi(w,x) > 0$. Plugging θ into equation (A.3) yields

$$\begin{aligned}\gamma &= \frac{1+\gamma\xi(w,x)}{1+2\xi(w,x)}\\ \theta &= \gamma[\sigma_R(x)\sigma_R(x)']^{-1}\mu_R(x) \qquad \text{(A.4)}\end{aligned}$$

The rest of investor's value function $\eta(x)$, which only matters for welfare accounting, solves

$$\begin{aligned}\rho\eta(x) = \rho\log(\rho) + r(x) - \rho + \frac{2\gamma-\gamma^2}{2}\mu_R(x)'[\sigma_R(x)\sigma_R(x)']^{-1}\mu_R(x)\\ + \mu_X(x)'\eta_{x'}(x) + \frac{1}{2}\mathrm{tr}[\sigma_X(x)'\eta_{xx'}(x)\sigma_X(x)]\end{aligned}$$

with appropriate boundary conditions. □

Proof of Proposition 2. Take the recursive form of equation (1) with the new budget constraint including $\tau(x)$. Let $V(w,x)$ be the value. The HJB equation is

$$\begin{aligned}\rho V(w,x) = \max_{c,\theta}\ & \rho\log(c) + (r(x)w - c + w\theta'\mu_R(x))V_w(w,x) \\ & + \frac{(\theta'\sigma_R(x) - \tau(x)')(\sigma_R(x)'\theta - \tau(x))}{2} w^2 V_{ww}(w,x) + w\theta'\sigma_R(x)\sigma_X(x)'V_{wx'}(w,x) \\ & + \mu_X(x)'V_{x'}(w,x) + \frac{1}{2}\mathrm{tr}[\sigma_X(x)'V_{xx'}(w,x)\sigma_X(x)] \qquad \text{(A.5)} \\ \text{s.t. } & \theta'\sigma_R(x)\sigma_R(x)'\theta \le \gamma\theta'\mu_R(x) \qquad \text{(A.6)}\end{aligned}$$

Like in the proof of Proposition 1, guess and verify the solution to equation (A.5) to be $V(w,x) = \log(w) + \eta(x)$. Again, the consequences are that $V_{wx'}(w,x) = 0$ and that $V_x(w,x)$ and $V_{xx'}(w,x)$ are functions of x only: $V_x(w,x) = \eta_x(x)$ and $V_{xx'}(w,x) = \eta_{xx'}(x)$. Consumption is still a constant fraction of wealth: $c = \rho w$. Moving to portfolio choice, let $\xi(x,w)$ be the multiplier on equation (A.6). The first-order condition with respect to θ is now

$$\theta = \frac{1+\gamma\xi(w,x)}{1+2\xi(w,x)}[\sigma_R(x)\sigma_R(x)']^{-1}\mu_R(x) + \frac{1}{1+2\xi(w,x)}[\sigma_R(x)\sigma_R(x)']^{-1}\sigma_R(x)\tau(x)$$

Now use the fact that $\tau(x) = \gamma\zeta(x)\sigma_R(x)'[\sigma_R(x)\sigma_R(x)']^{-1}\mu_R(x)$:

$$\theta = \frac{1+\gamma\xi(w,x)+\gamma\zeta(x)}{1+2\xi(w,x)}[\sigma_R(x)\sigma_R(x)']^{-1}\mu_R(x)$$

Suppose the constraint is slack and $\xi(w,x) = 0$. Then,

$$\theta = (1+\zeta(x)\gamma)[\sigma_R(x)\sigma_R(x)']^{-1}\mu_R(x)$$

Plug this into equation (A.6) to see that slackness is equivalent to

$$1 + \zeta(x)\gamma < \gamma$$

Alternatively, $\zeta(x) < 1 - 1/\gamma$. If $\zeta(x) > 1 - 1/\gamma$, the constraint binds, and

$$\frac{1+\gamma\xi(w,x)+\gamma\zeta(x)}{1+2\xi(w,x)} = \gamma$$

Portfolio choice in this case is hence the same as without taxes:

$$\theta = \gamma[\sigma_R(x)\sigma_R(x)']^{-1}\mu_R(x)$$

The tax vector happens to generate the same exposure to shocks as a share $\zeta(x)$ of the optimal portofolio, as designed: $\tau(x) = \zeta(x)\sigma_R(x)'\theta(x)$. The equation that the rest of investor's value function $\eta(x)$ solves changes to

$$\begin{aligned}\rho\eta(x) &= \rho\log(\rho) + r(x) - \rho + \frac{2\gamma - (1-\zeta(x))^2\gamma^2}{2}\mu_R(x)'[\sigma_R(x)\sigma_R(x)']^{-1}\mu_R(x) \\ &\quad + \mu_X(x)'\eta_{x'}(x) + \frac{1}{2}\text{tr}[\sigma_X(x)'\eta_{xx'}(x)\sigma_X(x)]\end{aligned}$$

with appropriate boundary conditions. □

Proof of Proposition 3. Alternative measures $\mathbb{Q}$ are indexed by drift correction processes $\{h_t\}_{t\geq 0}$, so the problem transforms into maximizing value over $\{c_t, \theta_t\}_{t\geq 0}$ after minimizing it over $\{h_t\}_{t\geq 0}$. Moreover, the cost function penalizing the log-likelihood ratio turns into $\psi|h|^2/2$ because $\mathbb{E}^{\mathbb{Q}}[\psi_t dm_t|\mathcal{F}_t] = \psi_t|h_t|^2/2$. The recursive representation of this problem is

$$\begin{aligned}\rho V(w,x) &= \max_{c,\theta}\ \min_{h}\ \rho\log(c) + \frac{\psi|h|^2}{2} + (r(x)w - c + w\theta'\mu_R(x) - w\theta'\sigma_R(x)h)V_w(w,x) \\ &\quad + \frac{\theta'\sigma_R(x)\sigma_R(x)'\theta}{2}w^2V_{ww}(w,x) + \mu_X(x)'V_{x'}(w,x) \\ &\quad + \frac{1}{2}\text{tr}[\sigma_X(x)'V_{xx'}(w,x)\sigma_X(x)] + w\theta'\sigma_R(x)\sigma_X(x)'V_{wx'}(w,x) \qquad \text{(A.7)}\end{aligned}$$

The solution to equation (A.7) can be guessed and verified. Conjecture that $V(w,x) = \log(w) + \eta(x)$. Then, $c = \rho w$ and $V_{wx'}(w,x) = 0$, so the guess that $V(\cdot)$ is separable over w and other states is correct. The model corrections and portfolio weights satisfy

$$\begin{aligned}h &= \frac{1}{\psi}\sigma_R(x)'\theta \\ \theta &= [\sigma_R(x)\sigma_R(x)']^{-1}(\mu_R(x) - \sigma_R(x)h)\end{aligned}$$

This linear system can be solved as

$$\begin{aligned}h &= \frac{1}{1+\psi}\sigma_R(x)'[\sigma_R(x)\sigma_R(x)']^{-1}\mu_R(x) \\ \theta &= \frac{\psi}{1+\psi}[\sigma_R(x)\sigma_R(x)']^{-1}\mu_R(x)\end{aligned}$$

Portfolio choice here coincides with that in equation (A.4), with $\gamma = \psi/(1+\psi) < 1$.

The exogenous part of the value function $\eta(\cdot)$ satisfies the following partial differential equation:

$$\rho\eta(x) = \rho\log(\rho) + r(x) - \rho + \frac{\psi}{2(1+\psi)}\mu_R(x)'[\sigma_R(x)\sigma_R(x)']^{-1}\mu_R(x)$$
$$+ \mu_X(x)'\eta_{x'}(x) + \frac{1}{2}\mathrm{tr}[\sigma_X(x)'\eta_{xx'}(x)\sigma_X(x)]$$

The partial differential equation is similar to one in the value-at-risk setup. The appropriate boundary conditions depend on the process for x. □

Proof of Proposition 4. Start with writing down excess returns in matrix form using prices and dividends only. The vector of excess returns $d\boldsymbol{R}_t = \overline{\mu}_R(x_t)dt + \overline{\sigma}_R(x_t)dZ_t$, where $(dZ_t)' = [(dZ_t^y)' \ (dZ_t^\gamma)']$, is

$$d\boldsymbol{R}_t \equiv D(\boldsymbol{p}(x_t))^{-1}[\mu_p(x_t) + \boldsymbol{y}(x_t) - r(x_t)\boldsymbol{p}(x_t)]dt + D(\boldsymbol{p}(x_t))^{-1}\sigma_p(x_t)dZ_t \tag{A.8}$$

Here $D(\boldsymbol{p}(x_t))$ is a diagonal matrix with $\boldsymbol{p}(x_t)$ on the main diagonal and zeros everywhere else, ans $\sigma_p(x_t) = [\sigma_{p,y}(x_t) \ \sigma_{p,\gamma}(x_t)]$. Expected excess returns on each asset j consist of capital gains $[\mu_p(x_t)]_j/[\boldsymbol{p}(x_t)]_j$ and dividend yield $[\boldsymbol{y}(x_t)]_j/[\boldsymbol{p}(x_t)]_j$ over and above the risk-free rate $r(x_t)$. The loadings of excess returns on the shocks are the loadings of capital gains $[\sigma_p(x_t)]_{.j}/[\boldsymbol{p}(x_t)]_j$ only.

Portfolio choice of every agent i is

$$\boldsymbol{\theta}_{it} = \gamma_{it}[\overline{\sigma}_R(x_t)\overline{\sigma}_R(x_t)']^{-1}\overline{\mu}_R(x_t) \tag{A.9}$$

Multiplying this by w_{it}, summing across i, and using the market clearing condition for each asset,

$$D(\boldsymbol{p}(x_t))\boldsymbol{s} = \sum\nolimits_{i=1}^{n} \gamma_{it}w_{it} \cdot [\overline{\sigma}_R(x_t)\overline{\sigma}_R(x_t)']^{-1}\overline{\mu}_R(x_t)$$

Multiply both sides by $\overline{\sigma}_R(x_t)\overline{\sigma}_R(x_t)'$ on the left:

$$\overline{\sigma}_R(x_t)\overline{\sigma}_R(x_t)'D(\boldsymbol{p}(x_t))\boldsymbol{s} = \sum\nolimits_{i=1}^{n} \gamma_{it}w_{it} \cdot \overline{\mu}_R(x_t)$$

Using the definition of Γ_t and the fact that total wealth is $\boldsymbol{p}(x_t)'\boldsymbol{s}$,

$$\overline{\sigma}_R(x_t)\overline{\sigma}_R(x_t)'D(\boldsymbol{p}(x_t))\boldsymbol{s} = \Gamma_t \cdot \boldsymbol{p}(x_t)'\boldsymbol{s} \cdot \overline{\mu}_R(x_t) \tag{A.10}$$

Multiply both sides by $D(\boldsymbol{p}(x_t))$ and replace $D(\boldsymbol{p}(x_t))\mu_R(x_t)$ using equation (A.8):

$$\mu_p(x_t) + \boldsymbol{y}(x_t) - r(x_t)\boldsymbol{p}(x_t) = \frac{1}{\Gamma_t \cdot \boldsymbol{p}(x_t)'\boldsymbol{s}} \cdot \sigma_p(x_t)\sigma_p(x_t)'\boldsymbol{s}$$

This is equation (5) because $\sigma_p(x_t)'\boldsymbol{s}/(\boldsymbol{p}(x_t)'\boldsymbol{s}) = \sigma_w(x_t)'$. Multiplying this by $\boldsymbol{s}'$ on the left,

$$\boldsymbol{s}'\mu_p(x_t) + \boldsymbol{s}'\boldsymbol{y}(x_t) - r(x_t)\boldsymbol{s}'\boldsymbol{p}(x_t) = \frac{1}{\Gamma_t} \cdot \frac{\boldsymbol{s}'\sigma_p(x_t)\sigma_p(x_t)'\boldsymbol{s}}{\boldsymbol{p}(x_t)'\boldsymbol{s}}$$

Using the fact that the total consumption $\boldsymbol{s}'\boldsymbol{y}(x_t)$ is a fraction ρ of total wealth $\boldsymbol{s}'\boldsymbol{p}(x_t)$, divide both sides by $\boldsymbol{s}'\boldsymbol{p}(x_t)$ to get

$$\frac{\boldsymbol{s}'\mu_p(x_t)}{\boldsymbol{s}'\boldsymbol{p}(x_t)} + \rho - r(x_t) = \frac{1}{\Gamma_t} \cdot \frac{\boldsymbol{s}'\sigma_p(x_t)\sigma_p(x_t)'\boldsymbol{s}}{(\boldsymbol{p}(x_t)'\boldsymbol{s})^2}$$

Reorganizing,

$$r(x_t) = \rho + \frac{\boldsymbol{s}'\mu_p(x_t)}{\boldsymbol{s}'\boldsymbol{p}(x_t)} - \frac{1}{\Gamma_t} \cdot \frac{|\sigma_p(x_t)'\boldsymbol{s}|^2}{(\boldsymbol{p}(x_t)'\boldsymbol{s})^2} = \rho + \mu_w(\boldsymbol{y}_t) - \frac{|\sigma_w(\boldsymbol{y}_t)|^2}{\Gamma_t}$$

This is equation (6). $\square$

Proof of Proposition 5. The proof is constructive. Take a function $\Lambda(\cdot)$ that satisfies

$$\Lambda(x_t)\,\boldsymbol{p}(x_t) = \mathbb{E}_t \int_t^\infty \Lambda(x_s)\,\boldsymbol{y}_s\,ds$$

This implies $\Lambda(x_t)\,\boldsymbol{y}_t\,dt = -\mathbb{E}_t[d(\Lambda(x_t)\,\boldsymbol{p}(x_t))]$. Use Itô's lemma:

$$\mathbb{E}_t[d(\Lambda(x_t)\,\boldsymbol{p}(x_t))] = \mu_\lambda(x_t)\Lambda(x_t)\,\boldsymbol{p}(x_t)dt + \mu_p(x_t)\Lambda(x_t)dt + \sigma_{p,y}(x_t)\sigma_\lambda(x_t)'\Lambda(x_t)dt$$

Here the drift and loadings $\mu_\lambda(x_t)$ and $\sigma_\lambda(x_t)$ are defined as

$$d\Lambda(x_t) = \mu_\lambda(x_t)\Lambda(x_t)dt + \Lambda(x_t)\sigma_{\lambda,y}(x_t)'dZ_t^y$$

Using $\Lambda(x_t)\,\boldsymbol{y}_t\,dt = -\mathbb{E}_t[d(\Lambda(x_t)\,\boldsymbol{p}(x_t))]$,

$$\mu_\lambda(x_t)\,\boldsymbol{p}(x_t) + \mu_p(x_t) + \boldsymbol{y}_t + \sigma_{p,y}(x_t)\sigma_\lambda(x_t)' = 0$$

One choice of $\mu_\lambda(x_t)$ and $\sigma_\lambda(x_t)$ that satisfies both this and equation (5) is $\mu_\lambda(x_t) = -r(x_t)$ and $\sigma_\lambda(x_t) = -\sigma_w(\boldsymbol{y}_t)/\Gamma(x_t)$. Using equation (6), we have

$$\begin{aligned}\frac{d\Lambda(x_t)}{\Lambda(x_t)} &= -r(x_t)dt - \frac{1}{\Gamma(x_t)}\sigma_w(\boldsymbol{y}_t)'dZ_t^y = -(\rho + \mu_w(\boldsymbol{y}_t))dt + \frac{1}{\Gamma(x_t)}(|\sigma_w(\boldsymbol{y}_t)|^2 dt - \sigma_w(\boldsymbol{y}_t)dZ_t^y) \\ &= -(\rho + \mu_w(\boldsymbol{y}_t) - |\sigma_w(x_t)|^2)dt + \frac{1}{\Gamma(x_t)}dW_t\end{aligned}$$

Proof of Proposition 6. Using the result of Proposition 3, we can characterize the drift correction H_t the hypothetical investor with a robustness parameter Ψ_t chooses for the aggregate market return $dR_t = \hat{\mu}_R(x_t)dt + \hat{\sigma}_R(x_t)dZ_t^y$:

$$H_t = \frac{1}{\Psi_t + 1}\hat{\sigma}_R(x_t)'[\hat{\sigma}_R(x_t)\hat{\sigma}_R(x_t)']^{-1}\hat{\mu}_R(x_t) \tag{A.11}$$

From Proposition 4, we know that $\hat{\sigma}_R(x_t) = \sigma_w(\boldsymbol{y}_t)$ and $\hat{\mu}_R(x_t) = |\sigma_w(\boldsymbol{y}_t)|^2/\Gamma_t$. Hence,

$$H_t = \frac{1}{\Psi_t + 1} \cdot \frac{1}{\Gamma_t}\sigma_w(\boldsymbol{y}_t)' = \frac{1 - \Gamma_t}{\Gamma_t}\sigma_w(\boldsymbol{y}_t)' \tag{A.12}$$

The last equality uses $\Psi_t = \Gamma_t/(1 - \Gamma_t)$. The final step is recognizing that the drift correction maps to the forecast error in the following way: since $\mathbb{E}^{\mathbb{Q}_t}[dZ_t^y] = -H_t dt$,

$$\begin{aligned} dR_t - \mathbb{E}^{\mathbb{Q}_t}[dR_t] &= dR_t - \hat{\mu}_R(x_t)dt + \hat{\sigma}_R(x_t)H_t dt = \hat{\mu}_R(x_t)dt + \hat{\sigma}_R(x_t)dZ_t^y - \hat{\mu}_R(x_t)dt + \hat{\sigma}_R(x_t)H_t dt \\ &= \frac{1 - \Gamma_t}{\Gamma_t}|\sigma_w(\boldsymbol{y}_t)|^2 dt + \sigma_w(\boldsymbol{y}_t)' dZ_t^y \end{aligned}$$

This completes the proof. □

Proof of Proposition 7. This proof follows the notation introduced in the proof of Proposition 4. To obtain total leverage λ_{it} of investor i, use equation (A.9) for portfolio choice:

$$\lambda_{it} = \gamma_{it} \cdot [\overline{\sigma}_R(x_t)\overline{\sigma}_R(x_t)']^{-1}\overline{\mu}_R(x_t)1_k$$

The market clearing condition for risk-free bonds implies

$$0 = \sum\nolimits_{i=1}^n w_{it}(1 - \lambda_{it}) = \sum\nolimits_{i=1}^n w_{it} - \sum\nolimits_{i=1}^n \gamma_{it} w_{it} \cdot [\overline{\sigma}_R(x_t)\overline{\sigma}_R(x_t)']^{-1}\overline{\mu}_R(x_t)1_k$$

Dividing by total wealth,

$$1 = \Gamma_t \cdot [\overline{\sigma}_R(x_t)\overline{\sigma}_R(x_t)']^{-1}\overline{\mu}_R(x_t)1_k$$

Plugging this back into the expression for leverage,

$$\lambda_{it} = \frac{\gamma_{it}}{\Gamma_t}$$

To get the expression for holdings $\{h_{ijt}\}$, notice that $h_{ijt} = \theta_{ijt} w_{it}/p_{jt}$, which means

$$h_{ijt} = \gamma_{it} w_{it} \cdot [D(\boldsymbol{p}(x_t))[\overline{\sigma}_R(x_t)\overline{\sigma}_R(x_t)']^{-1}\overline{\mu}_R(x_t)]_j$$

For every j, holdings h_{ijt} are proportional to $\gamma_{it} w_{it}$ with a common j-specific coefficient of proportionality. Since holdings sum to s_j,

$$h_{ijt} = s_j \cdot \frac{\gamma_{it} w_{it}}{\sum_{i=1}^n \gamma_{it} w_{it}} = s_j \cdot \frac{\gamma_{it} \nu_{it}}{\Gamma_t} = s_j \nu_{it} \lambda_{it}$$

The last remaining result to establish is equation (7) for the dynamics of wealth shares. Start with using equation (A.10) to replace $\mu_R(x_t)$ in equation (A.9):

$$\boldsymbol{\theta}_{it} = \frac{\gamma_{it}}{\Gamma_t \cdot \boldsymbol{p}(x_t)'\boldsymbol{s}} \cdot D(\boldsymbol{p}(x_t))\boldsymbol{s} = \frac{\lambda_{it}}{\boldsymbol{p}(x_t)'\boldsymbol{s}} \cdot D(\boldsymbol{p}(x_t))\boldsymbol{s}$$

This implies

$$\begin{aligned}
\boldsymbol{\theta}_{it}'\mu_R(x_t) &= \frac{\lambda_{it}\boldsymbol{s}' D(\boldsymbol{p}(x_t))\overline{\sigma}_R(x_t)\overline{\sigma}_R(x_t)' D(\boldsymbol{p}(x_t))\boldsymbol{s}}{\Gamma_t(\boldsymbol{p}(x_t)'\boldsymbol{s})^2} = \frac{\lambda_{it}\boldsymbol{s}'\sigma_p(x_t)\sigma_p(x_t)'\boldsymbol{s}}{\Gamma_t(\boldsymbol{p}(x_t)'\boldsymbol{s})^2} = \frac{\lambda_{it}}{\Gamma_t} \cdot \frac{|\sigma_p(x_t)'\boldsymbol{s}|^2}{(\boldsymbol{p}(x_t)'\boldsymbol{s})^2} \\
\boldsymbol{\theta}_{it}'\sigma_R(x_t) &= \lambda_{it} \cdot \frac{\boldsymbol{s}'\sigma_p(x_t)}{\boldsymbol{p}(x_t)'\boldsymbol{s}}
\end{aligned}$$

Here I used the fact that $D(\boldsymbol{p}(x_t))\overline{\sigma}_R(x_t) = \sigma_p(x_t)$. The dynamics of individual wealth are

$$\begin{aligned}
dw_{it} &= (r(x_t) - \rho)w_{it}dt + w_{it}\boldsymbol{\theta}_{it}'\overline{\mu}_R(x_t)dt + w_{it}\boldsymbol{\theta}_{it}'\overline{\sigma}_R(x_t)dZ_t \\
&= (r(x_t) - \rho)w_{it}dt + \frac{\lambda_{it}w_{it}}{\Gamma_t} \cdot \frac{|\sigma_p(x_t)'\boldsymbol{s}|^2}{(\boldsymbol{p}(x_t)'\boldsymbol{s})^2}dt + \lambda_{it}w_{it} \cdot \frac{\boldsymbol{s}'\sigma_p(x_t)}{\boldsymbol{p}(x_t)'\boldsymbol{s}}dZ_t
\end{aligned}$$

Summing this across i and denoting the total wealth by w_t,

$$dw_t = (r(x_t) - \rho)w_t dt + \frac{w_t}{\Gamma_t} \cdot \frac{|\sigma_p(x_t)'\boldsymbol{s}|^2}{(\boldsymbol{p}(x_t)'\boldsymbol{s})^2}dt + w_t \cdot \frac{\boldsymbol{s}'\sigma_p(x_t)}{\boldsymbol{p}(x_t)'\boldsymbol{s}}dZ_t$$

Here I use the fact that the average leverage weighted with wealth shares is one:

$$\sum_{i=1}^n \lambda_{it} w_{it} = \sum_{i=1}^n \lambda_{it}\nu_{it} w_t = \sum_{i=1}^n \frac{\gamma_{it}\nu_{it}}{\Gamma_t} w_t = w_t$$

Now consider the dynamics of $\nu_{it} = w_{it}/w_t$:

$$\begin{aligned}
d\nu_{it} &= \nu_{it}\left(\frac{dw_{it}}{w_{it}} - \frac{dw_t}{w_t} + \frac{(dw_t)^2}{w_t^2} - \frac{dw_t}{w_t}\frac{dw_{it}}{w_{it}}\right) \\
&= \nu_{it}\left(\frac{\lambda_{it}-1}{\Gamma_t} \cdot \frac{|\sigma_p(x_t)'\boldsymbol{s}|^2}{(\boldsymbol{p}(x_t)'\boldsymbol{s})^2}dt + (\lambda_{it}-1) \cdot \frac{\boldsymbol{s}'\sigma_p(x_t)}{\boldsymbol{p}(x_t)'\boldsymbol{s}}dZ_t + (1-\lambda_{it}) \cdot \frac{|\sigma_p(x_t)'\boldsymbol{s}|^2}{(\boldsymbol{p}(x_t)'\boldsymbol{s})^2}dt\right) \\
&= \nu_{it}(\lambda_{it}-1) \cdot \left[\frac{1-\Gamma_t}{\Gamma_t} \cdot \frac{|\sigma_p(x_t)'\boldsymbol{s}|^2}{(\boldsymbol{p}(x_t)'\boldsymbol{s})^2}dt + \frac{\boldsymbol{s}'\sigma_p(x_t)}{\boldsymbol{p}(x_t)'\boldsymbol{s}}dZ_t\right] \\
&= \nu_{it}(\lambda_{it}-1) \cdot \left[\frac{1-\Gamma_t}{\Gamma_t}|\sigma_w(\boldsymbol{y}_t)|^2dt + \sigma_w(\boldsymbol{y}_t)dZ_t^{\boldsymbol{y}}\right] = \nu_{it}(\lambda_{it}-1)dW_t
\end{aligned}$$

This completes the proof. □

Proof of Proposition 8. Apply Itô's lemma to Γ_t:

$$d\Gamma_t = \sum_{i=1}^n d\nu_{it}\gamma_{it} + \sum_{i=1}^n \nu_{it}d\gamma_{it} + \sum_{i=1}^n d\nu_{it}d\gamma_{it} \tag{A.13}$$

Since wealth shares do not load on dZ_t^γ, which follows from Proposition 7, the last term is zero. The second term is $\boldsymbol{\nu}_t'\,d\,\boldsymbol{\gamma}_t$. The first term is

$$\sum_{i=1}^n d\nu_{it}\gamma_{it} = \frac{1}{\Gamma_t}\sum_{i=1}^n \nu_{it}\gamma_{it}(\gamma_{it}-\Gamma_t)dW_t = \frac{1}{\Gamma_t}\sum_{i=1}^n(\nu_{it}\gamma_{it}^2 - \Gamma_t^2)dW_t = \frac{\Delta_t}{\Gamma_t}dW_t \tag{A.14}$$

Here Δ_t is the wealth-weighted variance of γ_t. □

Proof of Corollary 1. The statement directly follows Proposition 7 after plugging in the tax policy. □

Proof of Corollary 2. The statement follows from Proposition 8 and applying Itô's lemma to $\pi(\nu_t, \gamma_t) = \sigma^2/\Gamma(\nu_t, \gamma_t)$. □

Proof of Corollary 3. The statement follows from Proposition 4 and applying Itô's lemma to $\pi(\nu_t, \gamma_t) = \sigma^2/\Gamma(\nu_t, \gamma_t)$. □

Proof of Corollary 4. The statement follows from Proposition 7 and Proposition 8 after a change of variables. □