Value-at-Risk Constraints, Robustness, and Aggregation*

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Abstract

I describe a value-at-risk constraint that induces long-lived investors to choose static mean-variance portfolios with time-varying risk tolerance. This allows incorporating risk aversion shocks into dynamic models while keeping portfolios easy to aggregate across heterogeneous investors. In equilibrium, asset prices follow standard risk-neutral pricing equations with one additional term that depends on the wealth distribution through a single scalar. I provide a foundation for the value-at-risk constraint through a version of robustness concerns, where investors fear model misspecification and try to account for adverse alternative scenarios. I then illustrate the practicality of value-at-risk in a sovereign debt model with a cross-section of countries. Aggregate shocks to lenders' constraints create endogenous pricing risk, global interest rate risk, and a volatile common component in spreads that is orthogonal to fundamentals. I show that, despite this rich upside, solving for global equilibria with value-at-risk constraints requires minimal departures from models with risk-neutral lenders.

Key Words: value-at-risk, aggregation, robustness

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1 Introduction

I describe a tractable version of the value-at-risk portfolio constraint for macroeconomic models. Investors with log utility trade a risk-free bond with instant maturity and a portfolio of risky assets with Brownian returns. The value-at-risk constraint caps the variance of returns by a multiple of expected excess returns. The multiple is exogenous and potentially stochastic. I show that investors consume a constant fraction of their wealth and choose mean-variance portfolios of risky assets with the multiplier acting as the effective risk-tolerance coefficient.

This formulation of investors' problem combines myopic portfolios and time-varying effective risk tolerance with meaningful consumption choice, which makes intertemporal substitution a force in the model and allows for long-lived agents. Log utility alone allows for meaningful consumption choice and myopic portfolios but fixes effective risk tolerance at one. Departing from unit risk tolerance requires power utility or recursive preferences, which creates additional hedging terms in portfolios. These terms depend on investors' marginal value of wealth, an endogenous state variable that solves a potentially complicated differential equation. Habits, another way to generate time-varying risk aversion, similarly generate additional state variables. Value-at-risk constraints help avoid these complications.

I provide a foundation for the value-at-risk constraint through a version of robustness concerns. Investors are concerned about model misspecification and entertain alternative probability measures for returns. They first pick an adverse alternative model subject to a penalty for deviating too much from a reference measure and then choose consumption and portfolio weights. This is a version of maxmin decision rules of Gilboa and Schmeidler (1989) operationalized by Hansen and Sargent (2001), Anderson, Hansen, and Sargent (2003), and Hansen and Sargent (2008).

My main result is that an investor with log utility and robustness concerns behaves exactly like a value-at-risk-constrained investor. I relate the value-at-risk multiplier to the penalty parameter in the robust problem. Importantly, I deviate from the traditional way of modeling robustness concerns by allowing agents to misspecify their models of returns but not aggregate quantities. They ignore the implications of model misspecification for the dynamics of state variables in their problems, which is what usually creates hedging terms under robustness concerns. This allows me to break away from the equivalence between robust choice and Kreps and Porteus (1978) and Duffie and Epstein (1992) preferences and obtain myopic portfolios.

I then show how consumption and portfolio choice under value-at-risk aggregate in equilibrium. Risky asset prices and the risk-free rate depend on the wealth distribution through one scalar: the average value-at-risk multiplier weighted with wealth shares. Each investor's leverage is the ratio of her value-at-risk multiplier to the same weighted average, which can be interpreted as aggregate effective risk tolerance. Myopic portfolios ensure that it only depends on the allocation of wealth between agents with different attitudes to risk. In turn, wealth shares of all investors load on the shocks in the same proportions. The sign and the total scale of the loadings depend exclusively on their leverage minus one. Wealth shares of levered investors, who borrow risk-free to invest in risky assets, increase in total wealth and drift up due to the risk premia they collect. It is exactly the opposite for investors with tighter value-at-risk constraints, who lend risk-free in equilibrium.

In a special case when heterogeneous value-at-risk multipliers are fixed over time, I show that aggregate risk tolerance also loads positively on total wealth and drifts up with the speed proportional to the dispersion of the value-at-risk multipliers. I illustrate this in a special example inspired by Caballero and Simsek (2020): a Lucas tree, a risk-free bond, and two investors with different value-at-risk multipliers. The less constrained investor's wealth share is the only state variable. There are closed-form solutions for the risk premium, the interest rate, and the evolution of the wealth share itself. The less constrained investor borrows from the other one to lever up and invest in the Lucas tree. Negative output shocks decrease her wealth share, which raises the equilibrium risk premium and lowers the interest rate. Using value-at-risk constraints makes this mechanism extremely simple to expose.

I then shift the focus from heterogeneity in the value-at-risk multipliers to time variation. For illustration, I apply value-at-risk constraints to pricing sovereign bonds. The representative investor has a stochastic value-at-risk multiplier. She lends to a cross-section of sovereigns with idiosyncratic shocks to productivity, interpreted as fiscal surpluses. With major simplifications on the borrowers' side, I arrive at bond prices as a function of two exogenous states: the country's productivity and the lender's value-at-risk multiplier. I show that prices follow the usual risk-neutral pricing equation in the tradition of Leland (1994) with one additional term representing the risk correction. The stochastic global risk-free rate is a function of the value-at-risk multiplier only. The equilibrium spreads as well as the risk-free rate are easily computed given prices.

The practicality of value-at-risk is that it combines simple portfolio choice, which leads to tractable pricing equations in a small state space, with long-lived investors and a meaningful consumption-savings trade-off, which makes it easy to endogenize the risk-free rate. The time-varying value-at-risk multiplier adds interest rate and pricing risk to default risk, so it dynamically changes both the quantity and price components of the risk premium with no technical complications. This offers a direct, if somewhat mechanical, way to approach the conflict between high and volatile spreads and low historical incidence of default in the data, as stated by Aguiar, Chatterjee, Cole, and Stangebye (2016).

There is a long literature in mathematical finance studying value-at-risk constraints and a relatively new literature in financial economics applying them to equilibrium analysis. Examples include Danielsson, Shin, and Zigrand (2012), Adrian and Shin (2014), Adrian and Boyarchenko (2018), Hofmann, Shim, and Shin (2022), Coimbra (2020), and Coimbra and Rey (2024). Models in these papers use two different types of constraints, which I review in Section 6. A much larger set of papers uses mean-variance preferences and justifies them by alluding to value-at-risk. I contribute to this literature by suggesting another particularly convenient formulation of the constraint, formalizing the mean-variance portfolios, analyzing their aggregation properties, and providing a foundation through robust choice.

The connection between value-at-risk and robust choice can be helpful because it offers an interpretation of shocks to prices of risk that does not depend on regulation. Barbiero, Bräuning, Joaquim, and Stein (2024) and Bräuning and Stein (2024) show that regulatory limits of this type do indeed affect asset prices. Coimbra, Kim, and Rey (2022) document substantial cross-sectional heterogeneity in risk limits, which means that reallocation of activity between financial intermediaries should affect aggregate risk premia because of uneven exposure to regulation. However, these effects are not necessarily strong enough to explain the large global variation in prices of risk, and changes in parameters regulating attitudes to risk offer a complementary mechanism.

I formulate the value-at-risk constraint in Section 2, present the foundation in Section 3, describe aggregation in Section 4, and provide a sovereign debt example in Section 5. Section 6 reviews the literature using the value-at-risk constraints.

2 Value-at-risk constraint

Time is continuous and runs forever. The exogenous state of the economy is a d-dimensional vector x_t that evolves as

$$dx_t = \mu_X(x_t)dt + \sigma_X(x_t)dZ_t$$

Here $\{Z_t\}_{t\geq 0}$ is a standard b-dimensional Brownian process, so $\sigma_X(x_t)$ is a $(d \times b)$-dimensional matrix, and $\mu_X(x_t)$ is a vector of length d. The investor has access to a riskless asset that pays an instantaneous return $r(x_t)dt$ and a collection of k risky assets with a vector of instantaneous excess returns dR_t given by

$$dR_t = \mu_R(x_t)dt + \sigma_R(x_t)dZ_t$$

Here $\sigma_R(x_t)$ is a $(k \times b)$-dimensional matrix and $\mu_R(x_t)$ is a vector of length k. Returns and exogenous states do not necessarily load on all shocks, so some columns of $\sigma_R(x_t)$ and $\sigma_X(x_t)$ may be zero. In what follows, I only require $\sigma_R(x_t)\sigma_R(x_t)'$ to have full rank for all x_t.

Investor's wealth w_t evolves as

$$dw_t = (r(x_t)w_t - c_t)dt + w_t\theta_t' dR_t$$

Here c_t is consumption and θ_t is a k-dimensional vector of portfolio weights on risky assets, both chosen at time t. The pair (c_t, θ_t) are the only control variables.

The problem of the investor is to keep $w_t \geq 0$ and to solve

$$\max_{\{c_t,\theta_t\}_{t\geq 0}} \mathbb{E}\left[\rho \int_0^\infty e^{-\rho t} \log(c_t) dt\right] \tag{1}$$

$$\text{s.t. } \mathbb{V}_t[\theta_t' dR_t] \leq \gamma_t \cdot \mathbb{E}_t[\theta_t' dR_t] \text{ for all } t \geq 0 \tag{2}$$

The value-at-risk constraint equation (2) is the key feature. It is imposed continuously on incremental returns and caps the variance of returns by a multiple of expected profits. Both variance and expectation are of order dt, so dt cancels out. The multiplier $\gamma_t < 1$ is one of the components of x_t. It is exogenous and stochastic, potentially driving returns and other macroeconomic outcomes.

Investor's wealth w_t potentially impacts returns and prices in this economy too, although I do not allow her to internalize the price impact of her actions and wealth dynamics. To formally capture this, I let one of the components of x_t be a fictitious process $\hat{w}_t$ that coincides with w_t on all sample paths. The investor still treats it as exogenous, not realizing how her actions that change the evolution of w_t also change that of $\hat{w}_t$.

Proposition 1. *Investor's consumption and portfolio choice are*

$$c_t(w_t, x_t) = \rho w_t$$
$$\theta_t(w_t, x_t) = \gamma_t[\sigma_R(x_t)\sigma_R(x_t)']^{-1}\mu_R(x_t)$$

Investor's value function is separable over wealth and exogenous states: $V(w_t, x_t) = \log(w_t) + \eta(x_t)$, *where* $\eta(\cdot)$ *solves a second-order partial differential equation.*

Proof. Take the recursive form of equation (1). Let $V(w, x)$ be the value of an agent with wealth w given the aggregate state x. The HJB equation is

$$\begin{aligned}\rho V(w,x) = \max_{c,\theta}\ & \rho\log(c) + (r(x)w - c + w\theta'\mu_R(x))V_w(w,x) + \frac{\theta'\sigma_R(x)\sigma_R(x)'\theta}{2}w^2 V_{ww}(w,x) \\ & + \mu_X(x)'V_{x'}(w,x) + \frac{1}{2}\text{tr}[\sigma_X(x)'V_{xx'}(w,x)\sigma_X(x)] + w\theta'\sigma_R(x)\sigma_X(x)'V_{wx'}(w,x) \end{aligned} \tag{3}$$

$$\text{s.t. } \theta'\sigma_R(x)\sigma_R(x)'\theta \leq \gamma\theta'\mu_R(x) \tag{4}$$

Guess and verify the solution to equation (3) to be $V(w,x) = \log(w) + \eta(x)$. The first implication of this conjecture is that $V_{wx'}(w,x) = 0$. Second, $V_x(w,x)$ and $V_{xx'}(w,x)$ are functions of x only: $V_x(w,x) = \eta_x(x)$ and $V_{xx'}(w,x) = \eta_{xx'}(x)$. Finally, consumption is a constant fraction of wealth: $c = \rho w$.

Next, consider portfolio choice. Let $\xi(x,w)$ be the multiplier on equation (4). Taking the first-order condition with respect to θ,

$$\theta = \frac{1+\gamma\xi(w,x)}{1+2\xi(w,x)}[\sigma_R(x)\sigma_R(x)']^{-1}\mu_R(x)$$

If the constraint were slack, then $\xi(w,x) = 0$ and θ would be the regular mean-variance portfolio typical for log investors, $\theta = [\sigma_R(x)\sigma_R(x)']^{-1}\mu_R(x)$. But by equation (4), this contradicts $\gamma < 1$. Hence, $\xi(w,x) > 0$. Plugging θ into equation (4) yields

$$\begin{aligned}\gamma &= \frac{1+\gamma\xi(w,x)}{1+2\xi(w,x)} \\ \theta &= \gamma[\sigma_R(x)\sigma_R(x)']^{-1}\mu_R(x)\end{aligned} \tag{5}$$

The rest of investor's value function $\eta(x)$, which only matters for welfare accounting, solves

$$\begin{aligned}\rho\eta(x) = \rho\log(\rho) + r(x) - \rho + \frac{2\gamma-\gamma^2}{2}\mu_R(x)'[\sigma_R(x)\sigma_R(x)']^{-1}\mu_R(x) \\ + \mu_X(x)'\eta_{x'}(x) + \frac{1}{2}\mathrm{tr}[\sigma_X(x)'\eta_{xx'}(x)\sigma_X(x)]\end{aligned}$$

with appropriate boundary conditions. □

The resulting portfolio is identical to one that a myopic mean-variance investor with risk aversion $1/\gamma_t$ would choose, so γ_t can be treated as time-varying effective risk tolerance. However, the investor is also forward-looking and solves a consumption-savings problem too, which is crucial for the determination of asset prices and the interest rate $r(x_t)$ in general equilibrium. The fact that consumption is a constant fraction of wealth allows for simple linear aggregation, as is always the case with a unit elasticity of intertemporal substitution. The value-at-risk constraint adds to that a simple portfolio choice with a possibility to vary attitudes to risk over time.

The key assumption is that the terms in dw_t that the agent cannot choose are linear in w_t. In the basic formulation of the problem above, this applies to interest income $r(x_t)w_t$. In a richer setup with, for example, taxes or labor income, these additional terms would have to be linear in w_t for the value function to be separable over w_t and other states. This property is standard, and the value-at-risk constraint in the form of equation (2) does not add new restrictions.

Heuristic explanation for value-at-risk. Why is equation (2) a value-at-risk constraint if it looks like a constraint on variance? A generic value-at-risk constraint caps the probability of a given level of losses at a certain level. For example, given constants L and α, the constraint is

$$\mathbb{P}\left\{\theta_t' dR_t \leq -\sqrt{Ldt}\right\} \leq \alpha \tag{6}$$

Choosing Ldt instead of $\sqrt{Ldt}$ in this expression would not work. Heuristically, the variance of Brownian shocks is proportional to dt, so the standard deviation of $\theta_t' dR_t$ is of order $\sqrt{dt}$. Hence, as $t \longrightarrow 0$, the probability on the left of equation (6) would just converge to 0 or 1 depending on how $\mathbb{E}[\theta_t' dR_t] = \theta_t' \mu_R(x_t)dt$ compares to Ldt. With $\sqrt{Ldt}$ instead, equation (6) becomes

$$\Phi\left(-\frac{\sqrt{Ldt} + \theta_t'\mu_R(x_t)dt}{\sqrt{\theta_t'\sigma_R(x_t)\sigma_R(x_t)'\theta_t dt}}\right) \leq \alpha$$

Here $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution. As $dt \longrightarrow 0$, the $O(dt)$ term in the numerator vanishes. For all $\alpha < 1/2$, the limit of this inequality becomes

$$\theta_t'\sigma_R(x_t)\sigma_R(x_t)'\theta_t \leq \frac{L}{(\Phi^{-1}(\alpha))^2}$$

One candidate for the benchmark level of losses L is some multiple of expected profits per unit of time, $L = \hat{\gamma}_t\theta_t'\mu_R(x_t)$. With $\gamma_t = \hat{\gamma}_t/(\Phi^{-1}(\alpha))^2$, the constraint takes its final form:

$$\theta_t'\sigma_R(x_t)\sigma_R(x_t)'\theta_t \leq \gamma_t\theta_t'\mu_R(x_t)$$

The last step is realizing that $\mathbb{V}[\theta_t' dR_t] = \theta_t'\sigma_R(x_t)\sigma_R(x_t)'\theta_t dt$ and $\mathbb{E}[\theta_t' dR_t] = \theta_t'\mu_R(x_t)dt$.

Relation to recursive preferences. Another way to achieve lower risk tolerance than with log utility is using recursive preferences of Duffie and Epstein (1992). I will mostly follow Hansen, Khorrami, and Tourre (2024). Given a process for consumption $\{c_t\}_{t\geq 0}$, define the investor's value process $\{V_t\}_{t\geq 0}$ as

$$V_t = \mathbb{E}\left[\int_t^\infty \varphi(c_s, V_s)ds\right]$$

The problem is to maximize V_t over $\{c_s, \theta_s\}_{s\geq t}$, while keeping $w_s \geq 0$ for all $s \geq t$. The recursive representation of this problem is

$$\begin{aligned} 0 = \max_{c,\theta}\ & \varphi(c, V(w,x)) + (r(x)w - c + w\theta'\mu_R(x))V_w(w,x) + \frac{\theta'\sigma_R(x)\sigma_R(x)'\theta}{2}w^2 V_{ww}(w,x) \\ & + \mu_X(x)'V_{x'}(w,x) + \frac{1}{2}\mathrm{tr}[\sigma_X(x)'V_{xx'}(w,x)\sigma_X(x)] + w\theta'\sigma_R(x)\sigma_X(x)'V_{wx'}(w,x) \end{aligned} \tag{7}$$

I choose a form of $\varphi(\cdot)$ that keeps the elasticity of intertemporal substitution equal to one while allowing for a lower risk tolerance coefficient:

$$\varphi(c, V) = \rho(1 - 1/\gamma)V\left[\log(c) - \frac{\log((1-1/\gamma)V)}{1 - 1/\gamma}\right]$$

The solution to equation (7) can be guessed and verified too. Conjecture that

$$V(w,x) = \frac{(w\eta(x))^{1-1/\gamma}}{1-1/\gamma}$$

This implies $c = \rho w$ and $\varphi(c,V) = \rho(w\eta(x))^{1-1/\gamma}[\log(\rho) - \log(\eta(x))]$. Hence, $w^{1-1/\gamma}$ cancels out from equation (7), and the conjecture that $V(\cdot)$ is multiplicatively separable over w and other states proves correct. The optimal portfolio is

$$\theta = \gamma[\sigma_R(x)\sigma_R(x)']^{-1}\mu_R(x) + \underbrace{\frac{\gamma-1}{\eta(x)}[\sigma_R(x)\sigma_R(x)']^{-1}\sigma_R(x)\sigma_X(x)'\eta_{x'}(x)}_{\text{hedging motive}} \tag{8}$$

The optimal portfolio consists of the mean-variance part $\gamma[\sigma_R(x)\sigma_R(x)']^{-1}\mu_R(x)$, sometimes called the myopic portfolio, and the hedging part. Computing the latter requires knowing the marginal value of wealth $\eta(x)$, which solves a second-order partial differential equation. The presence of this term increases computational complexity and reduces tractability. The only case without the hedging motive is $\gamma = 1$, but fixing γ makes it impossible to use fluctuations in preferences for risk as a driver of dynamics. Eliminating the hedging motive while preserving dynamics of γ is the main advantage of using the value-at-risk constraint.

Incorporating external income. The simple form of consumption and portfolio choice in Proposition 1 extends to certain cases when a part of income is not chosen by the agent. Specifically,

$$dw_t = (r(x_t)w_t - c_t)dt + w_t\theta_t' dR_t - w_t\varsigma(x_t)dt - w_t\tau(x_t)'dZ_t$$

Here $w_t\varsigma(x_t)dt$ and $w_t\tau(x_t)'dZ_t$ are new terms that the agent does not choose. They can represent taxes and subsidies, with $\varsigma(x_t)$ being a locally deterministic tax and $\tau(x_t)$ being a vector of taxes that load on the exogenous shocks. These taxes are lump-sum in the sense that the agent does not directly choose the tax base, which would be the case, for instance, with proportional taxes on portfolio returns. They still scale with wealth, however, and this feature is key to preserving consumption and portfolio choice.

The locally deterministic tax $\varsigma(x_t)$ clearly does not change anything in the agent's decision problem as it is simply isomorphic to a change in the interest rate. The stochastic tax in general affects portfolio choice. For a clearer characterization, I focus on a special case

$$\tau(x_t) = \zeta(x_t)\gamma_t \cdot \sigma_R(x_t)'[\sigma_R(x_t)\sigma_R(x_t)']^{-1}\mu_R(x_t) \tag{9}$$

If taxes are set up this way, they generate the same exposure to shocks as the optimal portfolio:

$$\tau(x_t)' dZ_t = \zeta(x_t)\theta(w_t, x_t)'\sigma_R(x_t)dZ_t \equiv \zeta(x_t)\theta(w_t, x_t)'(dR_t - \mu_R(x_t)dt)$$

Here $\theta(w_t, x_t)$ is the optimal vector of portfolio weights in the baseline model. This allows a potential government to effectively tax away a share $\zeta(x_t)$ of stochastic returns without imposing proportional taxes. I next show the effect of imposing this taxes on the agent.

Proposition 2. *Suppose the stochastic tax rate is given by equation (9) and there is a locally deterministic tax $\varsigma(x_t)$. Then, consumption choice is $c(w_t, x_t) = \rho w_t$. The optimal portfolio is*

$$\theta(w_t, x_t) = \min\{\gamma_t, 1 + \zeta(x_t)\gamma_t\} \cdot [\sigma_R(x_t)\sigma_R(x_t)']^{-1}\mu_R(x_t)$$

The value-at-risk constraint is slack if $\zeta(x_t) < 1 - 1/\gamma_t$ and binds otherwise.

I relegate the proof to Appendix A. Portfolio choice does not change unless the rate $\zeta(x_t)$ is very negative. The intuition is that taxing away a part of random returns decreases the agent's overall exposure to shocks, making her more willing to take risk. But the value-at-risk constraint is binding already without the taxes, so it continues to bind when $\zeta(x_t)$ is positive and even becomes tighter as measured by the size of the multiplier. A negative $\zeta(x_t)$, on the contrary, increases the agent's exposure to shocks and makes the constraint less tight. If $\zeta(x_t)$ falls below $1 - 1/\gamma_t$, the constraint stops binding, and the tax rate shows up in portfolio choice directly.

3 A foundation through robust choice

I now provide a microfoundation for the value-at-risk constraint in a setup with robustness preferences. I mostly follow Hansen, Khorrami, and Tourre (2024). Importantly, I slightly modify the traditional setup to simplify portfolio choice and eliminate hedging motives. The agent is allowed to consider misspecified processes for returns and choose potential worst-case scenarios. She is not allowed to misspecify the process for other states. This prevents her from imagining a non-zero correlation between the marginal value of wealth and returns in the future when she considers worst-case scenarios, and the hedging motive does not enter portfolio choice. As a result, a robust agent in my setup does not behave exactly like an agent with Kreps and Porteus (1978) or Duffie and Epstein (1992) preferences, unlike in most existing models.

Consider an investor who does not face a value-at-risk constraint but instead entertains alternative models of the underlying shock. Specifically, she thinks that the true b-dimensional Brownian process underlying the dynamics might be $\{B_t\}_{t\geq 0}$, and increments of $\{Z_t\}_{t\geq 0}$ differ from those of $\{B_t\}_{t\geq 0}$ by a time-varying drift: $dZ_t = dB_t - h_t dt$. The b-dimensional model correction process

$\{h_t\}_{t\geq 0}$ investor's choice. She assumes the following dynamics for returns:

$$dR_t = \mu_R(x_t)dt + \sigma_R(x_t)dZ_t \equiv (\mu_R(x_t) - \sigma_R(x_t)h_t)dt + \sigma_R(x_t)dB_t$$

This means she entertains alternative models for shocks driving returns and wishes to choose portfolios that are robust to potential model corrections h_t. These corrections make her assessment of excess returns pessimistic.

The investor assumes the following dynamics for exogenous states:

$$dx_t = \mu_X(x_t)dt + \sigma_X(x_t)dB_t$$

Crucial here is that the underlying shocks are dB_t instead of dZ_t as in Section 2. Hence, under alternative models that make $\{B_t\}_{t\geq 0}$ the true Brownian motion instead of $\{Z_t\}_{t\geq 0}$, there is no drift correction h_t to the dynamics of aggregate states. When the investor considers alternative models for shocks to returns, she automatically assumes that she was always right about the dynamics of aggregate states. This is the key difference compared to the standard setup in Hansen and Sargent (2001) and Hansen, Khorrami, and Tourre (2024).

Formally, let the original process $Z = \{Z_t\}_{t\geq 0}$ be a standard b-dimensional Brownian motion on the basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. Let $\{h_t\}_{t\geq 0}$ be a process adapted to $\{\mathcal{F}_t\}_{t\geq 0}$. By Girsanov's theorem, under a condition on $\{h_t\}_{t\geq 0}$, the process $B = \{B_t\}_{t\geq 0}$ given by $B_0 = Z_0$ and $dB_t = dZ_t - h_t dt$ is a standard b-dimensional Brownian motion on another statistical basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{Q})$. Here the measure $\mathbb{Q}$ satisfies $\mathbb{E}^{\mathbb{Q}}[\varphi_t] = \mathbb{E}^{\mathbb{P}}[M_t\varphi_t]$ for all $t \geq 0$ and for all bounded processes $\{\varphi_t\}_{t\geq 0}$ adapted to $\{\mathcal{F}\}_{t\geq 0}$. The process $\{M_t\}_{t\geq 0}$ is adapted to $\{\mathcal{F}_t\}_{t\geq 0}$ and given by $M_0 = 1$ and $dM_t = -h_t M_t dZ_t$. This re-weighting process is a likelihood ratio, and its logarithm $m_t = \log(M_t)$ evolves as

$$dm_t = -\frac{|h_t|^2}{2}dt - h_t' dZ_t = \frac{|h_t|^2}{2}dt - h_t' dB_t$$

I use this log-likelihood ratio to impose discipline on the investor's choices, limiting how far she can go in accounting for potential losses. One restriction that using log-likelihood deviations imposes on the environment is that the investor cannot entertain models with different quantities of risk. With Brownian shocks, volatility and correlation are instantly learnable, and hence create infinite log-likelihood ratios. This is what limits model adjustments to drift corrections h_t.

The problem of the investor is of the "multiplier" type:

$$\max_{\{c_t,\theta_t\}_{t\geq 0}} \inf_{\mathbb{Q}} \mathbb{E}^{\mathbb{Q}}\left[\rho \int_0^\infty e^{-\rho t}\log(c_t)dt + \int_0^\infty e^{-\rho t}\psi_t dm_t\right]$$

She chooses the alternative measure $\mathbb{Q}$ with potentially large losses and then maximizes utility over consumption and portfolio vectors that are robust to these losses. The discipline is provided by the cost of deviating from the original measure as measured by $\psi_t dm_t$ for each increment of time. Small values of the multiplier ψ_t make it easier to select pessimistic probability measures $\mathbb{Q}$, and the investor underestimates expected excess returns, which makes risky assets less attractive.

PROPOSITION 3. *Consider a robust investor with a cost process $\{\psi_t\}_{t\geq 0}$. Her consumption and portfolio choice coincide with those of an investor with a value-at-risk constraint given by a multiplier process $\{\gamma_t\}_{t\geq 0}$, where $\gamma_t = \psi_t/(\psi_t+1)$:*

$$\begin{aligned} c(w_t, x_t) &= \rho w_t \\ \theta(w_t, z_t) &= \frac{\psi_t}{\psi_t+1}[\sigma_R(x_t)\sigma_R(x_t)']^{-1}\mu_R(x_t) \end{aligned}$$

The drift corrections she chooses for instantaneous returns are

$$h(w_t, x_t) = \frac{1}{\psi_t+1}\sigma_R(x_t)'[\sigma_R(x_t)\sigma_R(x_t)']^{-1}\mu_R(x_t)$$

Her value function is separable over wealth and exogenous states: $V(w_t, x_t) = \log(w_t) + \eta(x_t)$, where $\eta(\cdot)$ solves a second-order partial differential equation.

Proof. Alternative measures $\mathbb{Q}$ are indexed by drift correction processes $\{h_t\}_{t\geq 0}$, so the problem transforms into maximizing value over $\{c_t, \theta_t\}_{t\geq 0}$ after minimizing it over $\{h_t\}_{t\geq 0}$. Moreover, the cost function that penalizes the log-likelihood ratio turns into the quadratic expression $\psi|h|^2/2$ because $\mathbb{E}^{\mathbb{Q}}[\psi_t dm_t|\mathcal{F}_t] = \psi_t|h_t|^2/2$. The recursive representation of this problem is

$$\begin{aligned} \rho V(w,x) = \max_{c,\theta}\ \min_{h}\ & \rho\log(c) + \frac{\psi|h|^2}{2} + (r(x)w - c + w\theta'\mu_R(x) - w\theta'\sigma_R(x)h)V_w(w,x) \\ & + \frac{\theta'\sigma_R(x)\sigma_R(x)'\theta}{2}w^2 V_{ww}(w,x) + \mu_X(x)'V_{x'}(w,x) \\ & + \frac{1}{2}\mathrm{tr}[\sigma_X(x)'V_{xx'}(w,x)\sigma_X(x)] + w\theta'\sigma_R(x)\sigma_X(x)'V_{wx'}(w,x) \end{aligned} \tag{10}$$

The solution to equation (10)) can be guessed and verified. Conjecture that $V(w,x) = \log(w) + \eta(x)$. Then, $c = \rho w$ and $V_{wx'}(w,x) = 0$, so the guess that $V(\cdot)$ is separable over w and other states is correct. The model corrections and portfolio weights satisfy

$$\begin{aligned} h &= \frac{1}{\psi}\sigma_R(x)'\theta \\ \theta &= [\sigma_R(x)\sigma_R(x)']^{-1}(\mu_R(x) - \sigma_R(x)h) \end{aligned}$$

This linear system can be solved as

$$h = \frac{1}{1+\psi}\sigma_R(x)'[\sigma_R(x)\sigma_R(x)']^{-1}\mu_R(x)$$
$$\theta = \frac{\psi}{1+\psi}[\sigma_R(x)\sigma_R(x)']^{-1}\mu_R(x)$$

Portfolio choice here coincides with that in equation (5), with $\gamma = \psi/(1+\psi) < 1$.

The exogenous part of the value function $\eta(\cdot)$ satisfies the following partial differential equation:

$$\rho\eta(x) = \rho\log(\rho) + r(x) - \rho + \frac{\psi}{2(1+\psi)}\mu_R(x)'[\sigma_R(x)\sigma_R(x)']^{-1}\mu_R(x)$$
$$+ \mu_X(x)'\eta_{x'}(x) + \frac{1}{2}\mathrm{tr}[\sigma_X(x)'\eta_{xx'}(x)\sigma_X(x)]$$

The partial differential equation is similar to one in the value-at-risk setup. The appropriate boundary conditions depend on the process for x. $\square$

The case $\psi_t = 0$ corresponds to infinite effective risk aversion, and the limit $\psi_t \longrightarrow \infty$ corresponds to a standard log investor who does not make any model adjustments because it is prohibitively costly. For all positive ψ_t, portfolio weights are below those of a regular log investor, meaning that robustness adjustments always decrease effective risk tolerance.

The resulting corrections account for the correlation structure of returns. Since making corrections to all fundamental factors is equally costly, marginal gains in terms of pessimism should be equalized across factors as well. These marginal gains depend on the correlation structure, hence the weighting matrix $\sigma_R(x_t)'[\sigma_R(x_t)\sigma_R(x_t)']^{-1}$ in the expression for $h(w_t, x_t)$. For example, if there is an element of dZ_t that no asset loads on, the corresponding column of $\sigma_R(x_t)$ is zero, and the corresponding element of $h(w_t, x_t)$ is zero too. The result of equalizing marginal gains in pessimism across factors is that portfolio shares on risky assets are simply scaled down compared to a regular log investor with no changes in relative weights.

Relation to standard robust preferences. Divorcing the shocks to returns dZ_t from aggregate shocks $d\tilde{Z}_t$ is key to obtaining tractable mean-variance portfolios. In the standard setup, where dZ_t and $d\tilde{Z}_t$ are not just perfectly correlated but coincide as processes, the investor considers alternative models for exogenous states as well as returns:

$$dR_t = \mu_R(x_t)dt + \sigma_R(x_t)dZ_t \equiv (\mu_R(x_t) - \sigma_R(x_t)h_t)dt + \sigma_R(x_t)dB_t$$
$$dx_t = \mu_X(x_t)dt + \sigma_X(x_t)dZ_t \equiv (\mu_X(x_t) - \sigma_X(x_t)h_t)dt + \sigma_X(x_t)dB_t$$

The optimal choice of h_t will now pick up the evolution of x_t and its impact on the investor's value through the gradient $V_{x'}(w, x)$. Since h_t affects the optimal choice of θ_t, the optimal portfolio will pick up $V_{x'}(w,x)$ too, and this will resurrect the hedging motive in portfolio choice. The solution will coincide with that under Kreps and Porteus (1978) and Duffie and Epstein (1992) preferences.

The recursive representation of the problem is

$$\begin{aligned}\rho V(w,x) = \max_{c,\theta}\ \min_{h}\ & \rho \log(c) + \frac{\psi |h|^2}{2} + (r(x)w - c + w\theta'\mu_R(x) - w\theta'\sigma_R(x)h)V_w(w,x) \\ & + \frac{\theta'\sigma_R(x)\sigma_R(x)'\theta}{2}w^2 V_{ww}(w,x) + \frac{1}{2}\mathrm{tr}[\sigma_X(x)'V_{xx'}(w,x)\sigma_X(x)] + w\theta'\sigma_R(x)\sigma_X(x)'V_{wx'}(w,x) \\ & + (\mu_X(x)' \underbrace{-h'\sigma_X(x)'}_{\text{new}})V_{x'}(w,x)\end{aligned}$$

The solution $V(w,x) = \log(w) + \log(\eta(x))$ can be guessed and verified. Given this, $c = \rho w$ and

$$\begin{aligned}h &= \frac{1}{\psi\eta(x)}\sigma_X(x)'\eta_{x'}(x) + \frac{1}{\psi}\sigma_R(x)'\theta \\ \theta &= [\sigma_R(x)\sigma_R(x)']^{-1}(\mu_R(x) - \sigma_R(x)h)\end{aligned}$$

The fact that consumption is linear in w and (θ, h) only depend on x verifies the conjecture that $V(w,x)$ is log-separable over w and x. The optimal portfolio θ is given by

$$\theta = \frac{\psi}{1+\psi}[\sigma_R(x)\sigma_R(x)']^{-1}\mu_R(x) - \underbrace{\frac{1}{(1+\psi)\eta(x)}[\sigma_R(x)\sigma_R(x)']^{-1}\sigma_R(x)\sigma_X(x)'\eta_{x'}(x)}_{\text{hedging motive}}$$

This expression exactly coincides with equation (8) if $\gamma = \psi/(1+\psi)$.

Hedging motives disappear in the limit $\psi \longrightarrow \infty$. Another case where they disappear is one with $\sigma_R(x_t)\sigma_X(x_t)' = 0$, meaning that returns and aggregate states load on different shocks: some columns of $\sigma_R(x_t)$ and $\sigma_X(x_t)$ are zero, and the sets of their non-zero columns do not intersect. This happens when returns are purely idiosyncratic, and aggregates do not load on idiosyncratic shocks due to a large population. The application I provide in Section 5 is one such example. Another example is that of Di Tella, Malgieri, and Tonetti (2024). In their economy, entrepreneurs face idiosyncratic productivity risk realized after they decide how much labor to hire and commit to pay given wages. The hiring decision is isomorphic to an investment problem with a risky asset. There is a continuum of entrepreneurs, so idiosyncratic shocks do not drive aggregate dynamics, and only individual returns load on them. For this reason, potential drift corrections applied to idiosyncratic shocks would not appear in any process for aggregates, and there would be no need to duplicate the shock processes to make robust preferences equivalent to a value-at-risk constraint.

4 Aggregation with value-at-risk

I will now illustrate aggregative properties of the value-at-risk constraints in a simple environment. Suppose there are n investors indexed by i. Investor i's wealth is w_{it}. Their value-at-risk multipliers $\{\gamma_{it}\}$ are potentially different and stochastic.

Maintain the assumption that there are b standard Brownian motions driving the d-dimensional exogenous state $x_t \in X \subseteq \mathbb{R}^d$. The k risky assets are indexed by j. The supply s_j of each asset j is fixed, with $s_j > 0$ for at least one j. The dividend y_{jt} of asset j is a twice differentiable function of aggregate states $y_j(\cdot) : X \mapsto \mathbb{R}^+$. Investors solve the problem in equation (1) subject to equation (2):

$$\max_{\{c_{it},\theta_{ijt}\}_{t\geq 0}} \mathbb{E}\left[\rho \int_0^\infty e^{-\rho t}\log(c_{it})dt\right]$$
$$\text{s.t. } \mathbb{V}_t[\boldsymbol{\theta}'_{it} d\boldsymbol{R}_t] \leq \gamma_{it}\mathbb{E}_t[\boldsymbol{\theta}'_{it} d\boldsymbol{R}_t] \text{ for all } t \geq 0$$

Their choice of portfolio shares $\boldsymbol{\theta}_{it} = \{\theta_{ijt}\}_{j=1}^k$ corresponds to a choice of share holdings in risky assets $\boldsymbol{h}_{it} = \{h_{ijt}\}_{j=1}^k$ with $p_{jt}h_{ijt} = \theta_{ijt}w_{it}$. The risk-free bond holdings $b_{it} = (1 - \boldsymbol{\theta}'_{it}1_k)w_{it}$ take up the complementary portfolio share.

Given initial holdings, an equilibrium is a collection of processes for prices $\{p_{jt}, r_t\}$, wealth $\{w_{it}\}$, and quantities $\{c_{it}, \boldsymbol{h}_{it}, b_{it}\}$ adapted to the filtration generated by $\{x_t\}_{t\geq 0}$ and satisfying the following conditions. First, quantities are chosen optimally by agents, who take prices as given. Second, the evolution of wealth is consistent with portfolio and consumption choices. Third, markets for all assets and consumption goods clear:

$$s_j = \sum\nolimits_{i=1}^n h_{ijt} \text{ for all } t \geq 0 \text{ and all } j \in \{1, ...k\}$$
$$0 = \sum\nolimits_{i=1}^n b_{it} \text{ for all } t \geq 0$$
$$\sum\nolimits_{j=1}^k s_j y_{jt} = \sum\nolimits_{i=1}^n c_{jt} \text{ for all } t \geq 0$$

I will characterize equilibrium prices $r(x_t)$ and $\boldsymbol{p}(x_t) = \{p_{jt}(x_t)\}$ as functions of aggregate states. These aggregate states generally include the wealth vector $\boldsymbol{w}_t = \{w_{it}\}$, and the epistemic assumption is that agents treat these wealth processes as exogenous, not realizing that their own wealth dynamics affect prices. This is what "taking prices as given" means. The evolution of prices is

$$d\boldsymbol{p}(x_t) = \mu_p(x_t)dt + \sigma_p(x_t)dZ_t$$

This defines the k-dimensional drift $\mu_p(x_t)$ and the $(k \times b)$-dimensional matrix of loadings $\sigma_p(x_t)$. I look for equilibria in which prices are diffusions, which is why I require the dividend processes

$\boldsymbol{y}(x_t)$ to be twice differentiable functions of x_t. Given the drift and loadings of prices, the j-th component of excess returns is $\mathbb{E}_t[dR_{jt}] = [\mu_p(x_t) + \boldsymbol{y}(x_t) - r(x_t)\boldsymbol{p}(x_t)]_j/[\boldsymbol{p}(x_t)]_j$, and its loading on dZ_{lt} is $[\sigma_p(x_t)]_{lj}/[\boldsymbol{p}(x_t)]_j$ for all components $l \in \{1, ...b\}$ of the underlying diffusion.

Let $\boldsymbol{s} = \{s_j\}$ be the vector of asset supply. Let $w_t = \boldsymbol{p}(x_t)'\boldsymbol{s}$ denote the total wealth with

$$\frac{dw_t}{w_t} = \mu_w(x_t)dt + \sigma_w(x_t)dZ_t \equiv \frac{\mu_p(x_t)'\boldsymbol{s}}{\boldsymbol{p}(x_t)'\boldsymbol{s}}dt + \frac{\boldsymbol{s}'\sigma_p(x_t)}{\boldsymbol{s}'\boldsymbol{p}(x_t)}dZ_t$$

Define Γ_t as an average of value-at-risk parameters $\{\gamma_{it}\}$ weighted with wealth shares $\nu_{it} = w_{it}/w_t$:

$$\Gamma_t = \sum\nolimits_{i=1}^{n} \gamma_{it}\nu_{it}$$

Since agents optimally consume a fraction ρ of their wealth, $\rho w_t = \boldsymbol{y}(x_t)'\boldsymbol{s}$, and hence total wealth is exogenous. It follows that the endogenous dynamics are fully described by wealth shares.

To complete the description of equilibrium, define the total leverage of investor i as $\lambda_{it} = \boldsymbol{\theta}_{it}' 1_k$. Straightforward manipulation of the market clearing conditions leads to the following characterization of leverage λ_{it}, individual holdings h_{ijt}, wealth shares, and equilibrium prices.

Proposition 4. *In equilibrium, total leverage of each agent i is $\lambda_{it} = \gamma_{it}/\Gamma_t$. Individual holdings of risky assets are $h_{ijt} = s_j \cdot \nu_{it}\lambda_{it}$. Risky asset prices and the interest rate satisfy*

$$r(x_t)\boldsymbol{p}(x_t) = \mu_p(x_t) + \boldsymbol{y}(x_t) - \frac{\sigma_p(x_t)\sigma_w(x_t)'}{\Gamma_t w_t} \tag{11}$$

$$r(x_t) = \rho + \mu_w(x_t) - \frac{|\sigma_w(x_t)|^2}{\Gamma_t} \tag{12}$$

Wealth shares $\{\nu_{it}\}$ follow

$$d\nu_{it} = \nu_{it}(\lambda_{it} - 1) \cdot \left[\frac{1-\Gamma_t}{\Gamma_t}|\sigma_w(x_t)|^2 dt + \sigma_w(x_t)dZ_t\right] \tag{13}$$

The first property of the value-at-risk constraint is that the equilibrium leverage is given by a simple expression γ_{it}/Γ_t. The wealth-weighted leverage is always equal to one: the economy overall cannot hold a non-zero position in risk-free bonds. Underlying this is considerable heterogeneity between investors. Since Γ_t is a convex combination of γ_{it}, the investor with the highest γ_{it} always borrows in the risk-free asset, and the one with lowest γ_{it} always lends. If all investors have the same γ_{it}, everyone's leverage is one, and none has a gross position in the risk-free.

The share of asset j held by investor i is $h_{ijt}/s_j = \nu_{it}\lambda_{it}$. It does not depend on j. Investors have the same portfolios of risky assets up to scale. If everyone's γ_{it} is the same then everyone holds zero in the risk-free bond, implying everyone holds the same market portfolio, and differences in positions only reflect wealth differences.

Equation (11) is the main asset pricing equation of the model. One part of the equation is well known from risk-neutral pricing models, where $r(x_t)\boldsymbol{p}(x_t) = \mu_p(x_t) + \boldsymbol{y}(x_t)$. The content of equation (11) is the last term: the risk adjustment to prices $\boldsymbol{p}(x_t)$ is given by the variance-covariance matrix of prices, with Γ_t acting as the aggregate risk tolerance of the market. This is the main tractability gain of the value-at-risk constraint: the impact of endogenous variables on prices is summarized by a single weighted average Γ_t. With hedging motives, aggregation would be considerably harder, since it would have to account for the heterogeneous marginal value of wealth, which usually requires one extra partial differential equation per agent.

Equation (12) shows that the interest rate has three components: time discounting, growth, and precautionary motives. Growth is given by the drift $\mu_p(x_t)'\boldsymbol{s}$ in total wealth $\boldsymbol{p}(x_t)'\boldsymbol{s}$, while precautionary motives depend on the variance of total wealth $|\boldsymbol{s}'\sigma_p(x_t)|^2$, with $\Gamma_t < 1$ amplifying the depression in $r(x_t)$. The source of this depression, as usual, is uninsured risk.

The vector of wealth shares $\{\nu_{it}\}$ is the full description of endogenous states. Unsurprisingly, equation (13) shows that ν_{it} is positively exposed to $\sigma_w(x_t)dZ_t$, which represents shocks to total wealth $w_t = \boldsymbol{s}'\boldsymbol{p}(x_t)$, if $\lambda_{it} > 1$. Wealth shares of levered investors are positively correlated with aggregate wealth. What is less obvious is that ν_{it} also drifts up if and only if $\lambda_{it} > 1$. This happens because i in this case goes short in the risk-free bond and earns a net risk premium. With $\Gamma_t < 1$ amplifying the depression in the risk-free rate, going short in the risk-free is a strategy that puts a levered investor ahead of the average in expectation. Hence the term $1/\Gamma_t - 1$ in the drift of ν_{it}.

Heterogeneity in γ_{it} is necessary for wealth redistribution after shocks dZ_t. If everyone had the same γ_{it}, even varying over time, then everyone's leverage would be one, and wealth shares would be fixed forever. The weighted average Γ_t would then move exogenously. This could be desirable in applications because it would reduce the dimensionality of the problem. Section 5 presents an example that uses an exogenously driven Γ_t.

A converse case to homogeneous γ_{it} varying over time is the one with heterogeneity in γ_{it} but no time variation: $d\gamma_{it} = 0$. I present one such example below. In this case, it is possible to sign the drift of Γ_t.

Corollary 1. *If $d\gamma_{it} = 0$ for all i and t, then*

$$d\Gamma_t = \frac{\Theta_t}{\Gamma_t} \cdot \left[\frac{1-\Gamma_t}{\Gamma_t} |\sigma_w(x_t)|^2 dt + \sigma_w(x_t) dZ_t \right]$$

Here Θ_t is the wealth-weighted dispersion of the multipliers:

$$\Theta_t = \sum\nolimits_{i=1}^{n} \nu_{it}\gamma_i^2 - \left(\sum\nolimits_{i=1}^{n} \nu_{it}\gamma_i \right)^2 \geq 0$$

Aggregate risk tolerance is positively exposed to shocks to aggregate wealth $\sigma_w(x_t)dZ_t$. This is because investors with higher γ_{it} become relatively richer after positive shocks, driving up the weighted average. The drift is positive too, reflecting the upward drift in the wealth shares of less constrained investors due to the risk premia they collect.

A combination of the two cases is when time-variation further shuts down and all γ_{it} are constant. Then Γ_t is fixed. This nests, for example, Cochrane, Longstaff, and Santa-Clara (2008), where a single investor with log utility trades two Lucas trees.

A final observation is that a general economy with heterogeneous and stochastic $\{\gamma_{it}\}$ is comparable to one with a representative agent in the following, very particular, sense. For every economy with a collection of processes $\{\gamma_{it}\}$, there exists a counterfactual single-investor economy with the same processes for all prices. The value-at-risk multiplier of the investor in the counterfactual economy equals Γ_t from the original one on all sample paths. Given the exogenous processes for $\{\gamma_{it}\}$ and those for wealth shares given by equation (13), one can construct the process for Γ_t using Itô's lemma. Equation (11) and equation (12) are then trivially satisfied in the counterfactual economy too. Of course, in a truly representative-agent economy, the law of motion of Γ_t would only depend on Γ_t itself and exogenous parameters. This is not generally true: the evolution of Γ_t depends on the dispersion Θ_t, which in turn depends on higher moments and so on.

Example. I now illustrate a tractable special case with one part of the setup from Caballero and Simsek (2020). The claim to total output is the only risky asset. There are two investors with different value-at-risk parameters. In equilibrium, the one with a less tight value-at-risk constraint borrows from the other one to bet on output growth. Output shocks redistribute wealth and change the equilibrium risk premium, which leads to changes in the interest rate. I characterize the law of motion of wealth shares, the risk premium, and the interest rate in closed form.

The value-at-risk coefficients $\gamma_i \in \{1, 2\}$ are constant. Output is produced by a Lucas tree in unit supply, $\boldsymbol{s} = 1$. The flow output of the tree is $y_t dt$, where the rate of production evolves as

$$\frac{dy_t}{y_t} = \mu dt + \sigma dZ_t$$

The tree price is p_t, and excess returns on the tree are $dR_t = (dp_t + y_t dt)/p_t - r_t dt$. Since the supply is normalized to one, total wealth is equal to the tree's price: $\rho p_t = y_t$. The price-dividend ratio is constant, and the capital gains process coincides with that of output growth. Excess returns transform into $dR_t = (\rho + \mu - r_t)dt + \sigma dZ_t$. Denote the expected excess returns by $x_t \equiv \rho + \mu - r_t$. This is the equilibrium risk premium. With a constant price-dividend ratio, the dynamics of the interest rate exactly mirror those of x_t.

Taking equation (12) from Proposition 4 and using the fact that $p_t = y_t/\rho$, $\mu_p(y_t) = \mu y_t/\rho$, and $\sigma_p(y_t) = \sigma y_t/\rho$ leads to $x_t = \sigma^2/\Gamma_t$. The risk premium is inversely proportional to the wealth-weighted effective risk tolerance. When the agent with a less tight value-at-risk constraint accumulates more wealth, the market's aggregate risk tolerance rises, and the risk premium falls. Since the price-dividend ratio is fixed, this is achieved through a higher interest rate.

With just one risky asset, every investor's leverage is equal to her risky portfolio share: $\lambda_{it} = \theta_{it}$, and hence $\theta_{it} = \gamma_i x_t/\sigma^2$. As a result, wealth dynamics can be represented as

$$\frac{dw_{it}}{w_{it}} = (r_t - \rho + \theta_{it}x_t)dt + \theta_{it}\sigma dZ_t = \left(\mu - x_t + \gamma_i \frac{x_t^2}{\sigma^2}\right)dt + \gamma_i \frac{x_t}{\sigma} dZ_t \tag{14}$$

The term $\mu - x_t$ reflects the consumption-savings trade-off. A high risk premium x_t lowers the interest rate due to precautionary motives, inducing individual investors to save less. The term $\gamma_i x_t^2/\sigma^2$ is the compensation for risk. Higher γ_i leads the investor to take a longer position in the tree and raises the compensation. The exposure of i's wealth to dZ_t also rises with γ_i.

The growth of wealth only depends on the risk premium x_t and fixed parameters. This implies that the evolution of wealth shares only depends on x_t, and so does the evolution of x_t itself. Denote the two values of γ_i by $\{\overline{\gamma}, \underline{\gamma}\}$ and assume $\overline{\gamma} > \underline{\gamma}$. Denote the corresponding wealth shares by $\overline{\nu}_t$ and $\underline{\nu}_t \equiv 1 - \overline{\nu}_t$. Define the drift and volatility of wealth shares and risk premium:

$$\begin{aligned} d\overline{\nu}_t &= \mu_\nu(\overline{\nu}_t)dt + \sigma_\nu(\overline{\nu}_t)dZ_t \\ \frac{dx_t}{x_t} &= \mu_x(x_t)dt + \sigma_x(x_t)dZ_t \end{aligned}$$

A straightforward application of Itô's lemma leads to the following result.

Proposition 5. *The drift and volatility of $\overline{\nu}(x_t)$ are*

$$\begin{aligned} \mu_\nu(\overline{\nu}_t) &= \overline{\nu}_t\,(1 - \overline{\nu}_t) \cdot \frac{\sigma^2(\overline{\gamma} - \underline{\gamma})\,(1 - \overline{\nu}_t\overline{\gamma} - (1 - \overline{\nu}_t)\underline{\gamma})}{(\overline{\nu}_t\overline{\gamma} + (1 - \overline{\nu}_t)\underline{\gamma})^2} \\ \sigma_\nu(\overline{\nu}_t) &= \overline{\nu}_t\,(1 - \overline{\nu}_t) \cdot \frac{\sigma(\overline{\gamma} - \underline{\gamma})}{\overline{\nu}_t\overline{\gamma} + (1 - \overline{\nu}_t)\underline{\gamma}} \end{aligned}$$

with $\mu_\nu(\overline{\nu}_t) > 0$ and $\sigma_\nu(\overline{\nu}_t) > 0$ for all x_t. The drift and volatility of the risk premium are

$$\begin{aligned} \mu_x(x_t) &= \frac{(\overline{\gamma}x_t - \sigma^2)\,(\sigma^2 - \underline{\gamma}x_t)}{\sigma^6} \cdot x_t\left(\sigma^2(\overline{\gamma} + \underline{\gamma} - 1) - \overline{\gamma}\underline{\gamma}x_t\right) < 0 \\ \sigma_x(x_t) &= \frac{(\sigma^2 - \overline{\gamma}x_t)\,(\sigma^2 - \underline{\gamma}x_t)}{\sigma^3} \end{aligned}$$

with $\mu_x(x_t) < 0$ and $\sigma_x(x_t) < 0$ for all x_t.

The less constrained investor's wealth share is positively exposed to output growth. The drift of her wealth share is also positive since she earns a higher compensation for risk, as evident from equation (14). This makes the drift in x_t negative: the less constrained investor is gradually becoming richer, raising the market's risk tolerance.

Both drift and volatility increase in the difference $\overline{\gamma}-\underline{\gamma}$, which measures polarization in attitudes to risk. The difference in effective risk tolerance is what creates different portfolio allocations and ultimately allows output shocks to induce redistribution towards more risk-tolerant agents. This is a simplified version of the mechanism that Caballero and Simsek (2020) use to make shocks raise the interest rate. Good shocks redistribute wealth towards the less constrained agent, who is always betting on growth. This lowers the risk premium in equilibrium, but with a constant price-dividend ratio, changes in risk premia fully pass through to the interest rate.

Inducing stationarity. The setup above is not stationary in general. As the wealth share of the less constrained agent approaches one, its volatility decreases toward zero too fast, and it cannot bounce back with high enough probability. The risk premium approaches its lower bound $\sigma^2/\overline{\gamma}$. To induce stationarity in this setup, one can use simple redistributive wealth taxes. By Proposition 2, they do not change portfolio choice. Change agent i's budget constraint to

$$dw_{it} = (r_t w_{it} - c_{it})dt + \theta_{it} w_{it} dR_{it} - T_{it} w_{it} dt$$

Specify the tax rate T_{it} as a following function of i's wealth share ν_{it}: $T_{it} = T(\nu_{it})$, where

$$T(\nu_{it}) = \tau \left(\nu_{it} - \frac{1}{2}\right) \sqrt{\frac{1-\nu_{it}}{\nu_{it}}}$$

This tax policy balances the budget:

$$\sum\nolimits_{i\in\{1,2\}} T_{it} w_{it} = w_t \cdot \sum\nolimits_{i\in\{1,2\}} T_{it} \nu_{it} = \tau w_t \cdot \sum\nolimits_{i\in\{1,2\}} \left(\nu_{it} - \frac{1}{2}\right) \sqrt{\nu_{it}(1-\nu_{it})} = 0$$

Importantly, agents i takes T_{it} as given and does not realize how her wealth accumulation affects the tax rate. The fact that agents still perceive dw_{it} as linear in w_{it} preserves the functional forms of consumption and portfolio choice. The law of motion of $\nu(x_t)$ and x_t are only slightly altered:

Proposition 6. *The dynamics of $\overline{\nu}_t$ and x_t are*

$$d\overline{\nu}_t = \mu_\nu(\overline{\nu}_t)dt + \sigma_v(\overline{\nu}_t)dZ_t - \tau\sqrt{\overline{\nu}_t(1-\overline{\nu}_t)} \cdot \frac{2\overline{\nu}_t - 1}{2} dt$$
$$dx_t = \mu_x(x_t)dt + \sigma_x(x_t)dZ_t - \tau\sqrt{\left(\overline{\gamma}x_t - \sigma^2\right)\left(\sigma^2 - \underline{\gamma}x_t\right)} \cdot \frac{2\sigma^2 - (\overline{\gamma}+\underline{\gamma})x_t}{2\sigma^2(\overline{\gamma}-\underline{\gamma})} dt$$

where $\mu_\nu(\cdot)$, $\sigma_\nu(\cdot)$, $\mu_x(\cdot)$, and $\sigma_x(\cdot)$ are given in Proposition 5.

Taxes push the risk premium x_t away from its boundaries $\sigma^2/\overline{\gamma}$ and $\sigma^2/\underline{\gamma}$ by pushing the wealth shares away from their boundaries 0 and 1. Taxes converge to zero as $\overline{\nu}_t$ approaches 0 or 1, but they do it at a slower rate than volatility $\sigma_\nu(\cdot)$, and this induces stationarity.

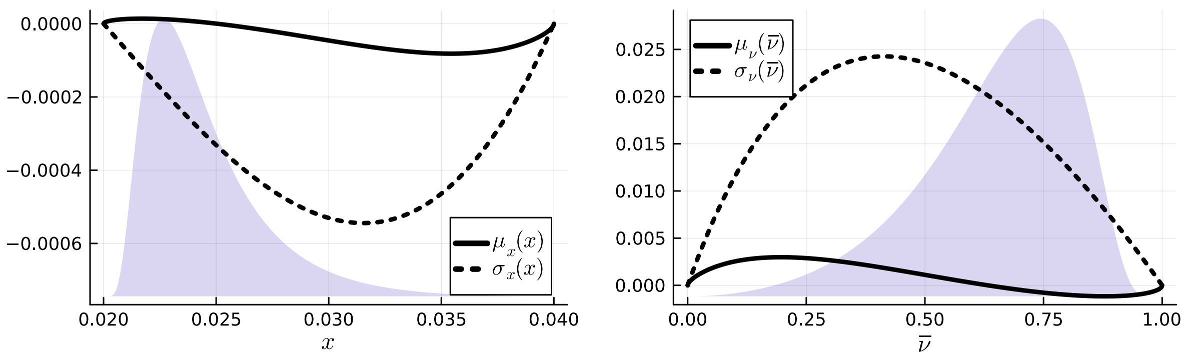


Figure 1: Left panel: drift and volatility of x_t. Right panel: drift and volatility of $\overline{\nu}_t$.

Figure (1) shows $\mu_x(\cdot)$, $\sigma_x(\cdot)$, and the ergodic density of x_t on the left, and the same for the wealth share $\overline{\nu}_t$ on the right. I take $\overline{\gamma} = 1$, $\underline{\gamma} = 0.2$, $\sigma^2 = 0.02$, and $\tau = 0.01$ for this example. The values of μ and ρ do not affect the dynamics and levels of the risk premium and only determine the interest rate through $r_t = \rho + \mu - x_t$.

Discussion. This example highlights one cost and one benefit of the value-at-risk. An empirically undesirable feature of the model is the constant price-dividend ratio and the exact mirror relationship between the risk-free rate and the risk premium. This feature is typical of one-asset economies with log utility, and value-at-risk cannot break it. A clear benefit of the value-at-risk is that endogenous processes only affect asset prices and the risk-free rate through a weighted average Γ_t. This leads to simple expressions for the interest rate in one-asset economies with geometric Brownian shocks paralleling those in Cochrane, Longstaff, and Santa-Clara (2008), to which my example is a natural converse: two agents and one tree instead of one agent and two trees. In Cochrane, Longstaff, and Santa-Clara (2008) dividend shares of the two trees move risk premia and the interest rate by changing equilibrium risk exposures of the agent: a high concentration of dividends increases the quantity of risk. In my example and in Caballero and Simsek (2020), the quantity of risk is fixed, and prices of risk change endogenously due to redistribution between heterogeneous agents.

There are two reasons for the acyclicality of leverage in my setup: markets are fully integrated and risk-free assets are in zero net supply. The fact that wealth-weighted λ_{it} is equal to one is an accounting identity. To make the discussion of leverage cyclicality meaningful, one would need to segment the markets or designate a subset of the agents as intermediaries, or natural buyers in the terminology of Geanakoplos (2010a), focusing on their leverage only. For instance, designating the more risk-tolerant agent in the two-agent example as an intermediary creates countercyclical leverage typical of models in Brunnermeier and Sannikov (2014) and He and Krishnamurthy (2013). Good output shocks redistribute wealth to the risk-tolerant agents, depressing the market risk premium and lowering her risky portfolio share. The empirically relevant case is procyclical leverage, as shown by Geanakoplos (2010b), Adrian and Shin (2014), and Kalemli-Ozcan, Sorensen, and Yesiltas (2012). Obtaining this would require more agents, more assets, or market segmentation.

5 Application: cross-section of defaultable bonds

In this section, I apply the value-at-risk constraint to a setting with sovereign default. Multiple countries issue defaultable bonds to intermediaries with time-varying value-at-risk constraints. Simplifying the model in all dimensions other than the lenders' preferences, I compute the cross-section of bond prices by solving the usual pricing equation found in models of the Leland (1994) type with one additional term. This term reflects the effective risk aversion of the market and introduces a common time-varying component to the price of risk.

I then solve for the global interest rate as a function of the value-at-risk multiplier. Importantly, the equilibrium interest rate is stochastic, adding aggregate interest rate risk to idiosyncratic default risk. Time-varying effective risk tolerance of the lenders additionally generates endogenous pricing risk typical for Brunnermeier and Sannikov (2014) models. One implication is that the value-at-risk constraint changes both the price and the quantity of risk in equilibrium. Another implication is that aggregate shocks introduce precautionary motives, depressing interest rates and inflating asset prices. As a result, away from the default threshold, defaultable bond prices are higher than they would be in a counterfactual economy with risk-neutral lenders.

The global economy has a single consumption good. It is populated by two types of agents: sovereigns and financial intermediaries. There is a unit measure of sovereigns, indexed by i. Each of them produces x_{it} units of consumption goods per unit of time when business goes as usual. Sovereigns are in debt: every one of them has one bond outstanding and can be either current or in default. When current, sovereigns pay coupons κ per unit of time, and their economies operate as usual, producing fiscal surpluses at full potential. Their productivity x_{it} evolves as

$$\frac{dx_{it}}{x_{it}} = \mu dt + \sigma dZ_{it}$$

Shocks dZ_{it} are independent across i. When in default, sovereigns do not make coupon payments and cannot produce and consume surpluses. Productivity in the default state evolves as

$$\frac{dx_{it}}{x_{it}} = \overline{\mu} dt + \sigma dZ_{it}$$

Here $\overline{\mu} > \mu$, capturing reform or reorganization that the country undergoes when in default. Countries can go into default and back at any time. Their bonds are not written down, and there is no cost of default other than the inability to produce. The problem of a sovereign is

$$V_{i,0}^{sov} = \max_{\{\Delta_{it}\}_{t\geq 0}} \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t}(x_{it} - \kappa)(1 - \Delta_{it})dt \right]$$
$$\text{s.t. } \frac{dx_{it}}{x_{it}} = (\mu + (\overline{\mu} - \mu)\Delta_{it})dt + \sigma dZ_{it}$$

Sovereigns are risk-neutral. They only choose when to go in and out of the default state ($\Delta_{it} = 1$ or $\Delta_{it} = 0$), do not participate in other trades, and consume output minus coupons:

$$c_{it}^{sov} = (x_{it} - \kappa)(1 - \Delta_{it})$$

The second type of agent are intermediaries, who act as lenders. Each country i is matched to a unit measure of intermediaries. This assignment means that only intermediaries of type i have access to country i's bond, which is trading at a price p_{it}. They also trade instant-maturity risk-free debt, which pays an interest rate r_t.

All intermediaries are part of the same large financial firm. They can be interpreted as trading desks, and the market for short-term bonds can be interpreted as an intra-firm funding market. The headquarters exogenously set up two other types of flows between desks. First, there is default coverage: each intermediary of type i pays a premium $\pi_t dt$ and receives payments κdt when bond i is in default. Second, there is a profit tax: each intermediary of type i sends payments $d\tau_{it}$ to the headquarters. These redistributive payments re-capitalize losing trading desks at the expense of winning ones. Wealth evolves as

$$dw_{it} = (r_t w_{it} - c_{it})dt + w_{it}\theta_{it}dR_{it} - \frac{w_{it}}{\hat{w}_{it}}(\pi_t - \kappa\Delta_{it})dt - \frac{w_{it}}{\hat{w}_{it}}d\tau_{it}$$

The problem of an intermediary of type i is

$$V_{i,0}^{int} = \max_{\{c_{it},\theta_{it}\}_{t\geq 0}} \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \log(c_{it})dt \right] \text{ s.t.}$$
$$\mathbb{V}_t \left[\theta_{it}dR_{it}\right] \leq \gamma_t \cdot \mathbb{E}_t \left[\theta_{it}dR_{it}\right]$$

Each intermediary of type i chooses her consumption and portfolio share θ_{it} of the defaultable bond. Here $\hat{w}_{it}$ is the aggregate wealth of the intermediaries that each one of them takes as given. Taxes and default coverage payments are distributed in proportion to wealth within each type i. Looking ahead, in a symmetric equilibrium $w_{it} = \hat{w}_{it}$, but budget constraints have to look this way to preserve consumption and portfolio choice in keeping with Proposition 2.

The internal default swap premium π is chosen to make default coverage self-financing on aggregate:

$$\pi_t = \kappa \cdot \int \Delta_{it} di$$

The profit taxes have the following form:

$$d\tau_{it} = dp_{it} - r_t p_{it} dt - \left(\int dp_{jt} dj - \int p_{jt} dj \cdot r_t dt \right)$$

Trading desks send their excess capital gains to the headquarters. These redistributive payments are also self-financing on aggregate.

Excess returns dR_{it} on sovereign bonds are given by

$$dR_{it} = \frac{\kappa(1 - \Delta_{it})dt + dp_{it}}{p_{it}} - r_t dt$$

Portfolio share θ_{it} implies risky holdings $h_{it} = \theta_{it} w_{it}/p_{it}$ and risk-free debt holdings $b_{it} = (1-\theta_{it})w_{it}$. The tightness γ_t of the value-at-risk constraint is common to all i and stochastic. It evolves as

$$d\gamma_t = \mu_\gamma(\gamma_t)dt + \sigma_\gamma(\gamma_t)dW_t$$

Given initial conditions $\{h_{i,0}, b_{i,0}, x_{i,0}\}$ and the filtration generated by $\{x_{it}\}_{t\geq 0}$, an equilibrium is a collection of adapted price processes $\{\{p_{it}\}, r_t\}_{t\geq 0}$, wealth processes $\{\{w_{it}, \hat{w}_{it}\}\}_{t\geq 0}$, quantity processes $\{\{c_{it}, h_{it}, b_{it}\}\}_{t\geq 0}$, and default processes $\{\{\Delta_{it}\}\}_{t\geq 0}$ satisfying the following conditions. First, sovereigns optimally choose their default state. Second, intermediaries optimally choose their holdings, taking the default states, prices, and aggregate wealth of their type as given. Third, the evolution of individual and aggregate wealth of each type is consistent with the quantity choices. Fourth, markets for every bond i, risk-free debt, and consumption goods clear:

$$1 = h_{it} \text{ for all } i, t \geq 0$$
$$0 = \int b_{it} di \text{ for all } t \geq 0$$
$$\int x_{it}(1 - \Delta_{it}) di = \int (c^{sov}_{it} + c_{it}) di \text{ for all } t \geq 0$$

Equilibrium characterization. The sovereigns' block of the model is a simple decision problem that does not depend on equilibrium objects. They follow a standard threshold default policy $\Delta_{it} = \mathbb{1}\{x_{it} < \hat{x}\}$. Individual productivity follows $dx_{it} = (\mu + (\overline{\mu} - \mu)\mathbb{1}\{x_{it} < \hat{x}\})x_{it}dt + \sigma x_{it} dZ_{it}$. I summarize this block of the model in the following proposition.

PROPOSITION 7. *Suppose $\rho > \overline{\mu} \geq \mu$. The optimal default threshold $\hat{x}$ is*

$$\hat{x} = \kappa \cdot \frac{\rho - \mu}{\rho} \cdot \frac{\overline{\zeta}\zeta}{\big(\overline{\zeta} - 1\big)\big(\zeta - 1\big)} > \kappa$$

Here $\overline{\zeta} > 1$ and $\zeta < 0$ only depend on $(\mu, \overline{\mu}, \sigma, \rho)$. The invariant distribution of x has density

$$g(x) = \frac{(1+\xi)(1+\overline{\xi})}{(\xi - \overline{\xi})\hat{x}} \cdot \Big(\frac{x}{\hat{x}}\Big)^{\overline{\xi}\mathbb{1}\{x<\hat{x}\}+\xi\mathbb{1}\{x\geq\hat{x}\}}$$

Here $\overline{\xi} > 0$ and $\xi < -1$ only depend on $(\mu, \overline{\mu}, \sigma)$. The default $\hat{\Delta}$ share at this distribution is

$$\hat{\Delta} = \frac{\xi + 1}{\xi - \overline{\xi}} = \frac{\sqrt{(\mu - \sigma^2)^2 + \sigma^4} - \mu + \sigma^2}{\sqrt{(\mu - \sigma^2)^2 + \sigma^4} - \mu + \sqrt{(\overline{\mu} - \sigma^2)^2 + \sigma^4} + \overline{\mu}}$$

I will focus on the productivity steady state, assuming that the economy starts at the invariant distribution. This will shut down aggregate shocks to the quantity of default risk.

Unlike in the model of Leland (1994), the default threshold for cash flows is above the coupon. Sovereigns do not have to abandon the country when they go into default, so betting on resurrection does not have option value. Instead, countries go into default when cash flows are still above the break-even level to take advantage of faster productivity growth $\overline{\mu} > \mu$. The technical motivation for using this law of motion for x_{it} is that it induces a stationary distribution. Another option would be to reset countries in default to some initial productivity level x_0, but this would create discontinuous wealth dynamics for intermediaries attached to them. Discontinuities in the sample paths of returns destroy the tractable portfolios afforded by value-at-risk.

Consumption and portfolio choice of intermediaries is standard. Moreover, if they start with the same initial wealth $w_{i,0}$, the way taxes $d\tau_{it}$ and insurance coverage are set up induces wealth to be constant over time and across i. To see this, observe first that the aggregate wealth is constant: consumption choice $c_{it} = \rho w_{it}$ and consumption market clearing imply

$$\int w_{it} di = \frac{\kappa - \pi}{\rho}$$

Since risk-free bonds are in zero net supply, total wealth is the average bond price. Hence,

$$\int dp_{it} di = \int dw_{it} di = 0$$

$$d\tau_{it} = dp_{it} - r_t p_{it} dt + \frac{r_t}{\rho}(\kappa - \pi) dt$$

Next, market clearing for defaultable bonds implies $h_{it} = 1$, which means $\theta_{it} w_{it} = p_{it}$. Plugging this and consumption choice $c_{it} = \rho w_{it}$ into the budget constraint of the intermediaries,

$$\begin{aligned} dw_{it} &= (r_t - \rho) w_{it} dt + p_{it} dR_{it} + (\kappa \Delta_{it} - \pi) dt - d\tau_{it} \\ &= (r_t - \rho) w_{it} dt + dp_{it} - r_t p_{it} dt + (\kappa - \pi) dt - d\tau_{it} \\ &= (\rho - r_t) \left(w_{it} - \frac{\kappa - \pi}{\rho} \right) \end{aligned}$$

If all intermediaries start with the same wealth, it must be equal to $(\kappa - \pi)/\rho$. But then $dw_{it} = 0$ for all $t \geq 0$, implying $w_{it} = (\kappa - \pi)/\rho$ for all i and $t > 0$. Profit taxes shut down wealth heterogeneity across trading desks. The technical importance of this is that solving the model does not require tracking the wealth distribution, which would complicate solving for dynamics of p_{it} and r_t.

The two remaining state variables in the model are x_{it} and γ_t. I can now characterize defaultable bond prices as functions of (x, γ) and the interest rate as a function of γ only, with $dp(x, \gamma)$ loading on the corresponding driving shocks:

$$dp(x, \gamma) = \mu_p(x, \gamma) dt + \sigma_{px}(x, \gamma) dZ_t + \sigma_{p\gamma}(x, \gamma) dW_t \equiv \mu_p(x, \gamma) dt + \sigma_p(x, \gamma) \cdot dX_t$$

Here $dX_t = (dZ_t \; dW_t)'$ combines both shocks that $p(x, \gamma)$ is exposed to in equilibrium. Abusing notation, I drop the i subscript on $dZ_{it} \equiv dZ_t$. Expected excess returns and their variance are

$$\begin{aligned} \mathbb{E}[dR(x, \gamma) | x, \gamma] &= \frac{\mathbb{E}[dp(x, \gamma) + (1 - \Delta(x)) \kappa dt - r(\gamma) p(x, \gamma) dt | x, \gamma]}{p(x, \gamma)} \\ \mathbb{V}[dR(x, \gamma) | x, \gamma] &= \frac{\mathbb{V}[dp(x, \gamma) | x, \gamma]}{p(x, \gamma)^2} \end{aligned}$$

Market clearing and optimal portfolio choice imply $\gamma w \cdot \mathbb{E}[dR(x, \gamma) | x, \gamma] = p(x, \gamma) \cdot \mathbb{V}[dR(x, \gamma) | x, \gamma]$. Applying Itô's lemma to get $\mu_p(\cdot)$ and $\sigma_{(\cdot)}$,

$$\begin{aligned} r(\gamma) p(x, \gamma) = \kappa (1 - \Delta(x)) &+ p_x(x, \gamma) \mu(x) + p_\gamma(x, \gamma) \mu_\gamma(\gamma) + p_{xx}(x, \gamma) \frac{\sigma^2 x^2}{2} + p_{\gamma\gamma}(x, \gamma) \frac{\sigma_\gamma(\gamma)^2}{2} \\ & \underbrace{- \frac{(p_x(x, \gamma) \sigma x)^2 + (p_\gamma(x, \gamma) \sigma_\gamma(\gamma))^2}{\gamma w}}_{\text{risk correction}} \end{aligned} \tag{15}$$

The first line in equation (15) is standard and appears in all models with risk-neutral investors pricing the bonds. Expected gains on holding the bond are equal to the coupon plus the capital gains. The drift in the underlying states leads to expected capital gains, hence the terms including $p_x(\cdot)$ and $p_\gamma(\cdot)$. Since $p(\cdot)$ is non-linear, volatility in the underlying states creates more drift in capital gains, which is reflected in the terms including $p_{xx}(\cdot)$ and $p_\gamma(\cdot)$.

The last term, the risk correction, comes from investors' risk aversion. This risk correction is a product of the price of risk $(\gamma w)^{-1}$, where $w = (\kappa - \pi)/\rho$ is investors' wealth, and the quantity of risk captured by the loadings of the price on x and γ. All local risk away from the default threshold $\hat{x}$ here is endogenous in the terminology of Brunnermeier and Sannikov (2014): it reflects the volatility of prices. Using a simple log investor would also lead to risk correction with $\gamma = 1$, but risk would only be coming from cash flows x. Using the value-at-risk constraint allows the model to incorporate aggregate shocks to risk-taking capacity that lead to time-varying risk premia and interest rates, adding interest rate risk to default risk.

The interest rate $r(\gamma)$ is the last remaining piece of equilibrium characterization.

Proposition 8. *The interest rate is*

$$r(\gamma) = \rho - \frac{\rho^2}{\gamma(\kappa - \pi)^2} \int |\sigma_p(x, \gamma)|^2 g(x) dx \tag{16}$$

where $|\sigma_p(x, \gamma)|^2 = (p_x(x, \gamma)\sigma x)^2 + (p_\gamma(x, \gamma)\sigma_\gamma(\gamma))^2$.

As usual, the interest rate is depressed relative to the subjective discount rate ρ because of aggregate risk that is uninsurable and hence creates precautionary saving motives. The gap between ρ and $r(\gamma)$ is proportional to the total amount of risk given by the variance of price growth and inversely proportional to global risk-taking capacity γ.

Solving equation (15) numerically requires a two-dimensional grid, and the algorithm is minimally demanding. The key simplification is that wealth heterogeneity between intermediaries is shut down, and both remaining state variables (x, γ) are essentially exogenous. This means that there is no need to iterate over the equilibrium distribution of states. The only loop required to solve for equilibrium alternates solving equation (15) given a conjecture for $r(\cdot)$ and updating this conjecture using the resulting $p(\cdot)$ in equation (16).

I choose the following dynamics of γ_t for a numerical example:

$$\mu_\gamma(\gamma) = \frac{\alpha\varsigma^2}{2}(\underline{\gamma} + \overline{\gamma} - 2\gamma)$$
$$\sigma_\gamma(\gamma) = \varsigma\sqrt{(\overline{\gamma} - \gamma)(\gamma - \underline{\gamma})}$$

With the drift and volatility specified this way, the invariant distribution of γ is a Beta-distribution on $[\underline{\gamma}, \overline{\gamma}]$ with parameters (α, α). The model is parameterized by $(\rho, \kappa, \mu, \overline{\mu}, \sigma, \alpha, \varsigma, \underline{\gamma}, \overline{\gamma})$. I choose the following values for a numerical example: $\rho = 0.1$, $\kappa = 0.08$, $\sigma = 0.1$, $\mu = 0$, and $\overline{\mu} = 0.06$: productivity is growing at the rate of 6% when restructuring and stagnating on average when current. These numbers lead to approximately 80% of countries paying coupons at any given time. For the dynamics of the value-at-risk parameter, I choose $\underline{\gamma} = 0.2$, $\overline{\gamma} = 1.0$, $\alpha = 5$, and $\varsigma = 0.1$, which induces a symmetric Beta distribution.

Figure (2) describes equilibrium prices. Panel (a) shows the risk-free rate with the invariant distribution of γ in the background. The interest rate is lower than the subjective discount rate ρ, and the difference increases as the effective risk tolerance γ falls. Second, panel (b) shows the risk correction term in equation (15) with the invariant distribution of x in the background. The magnitude of this term should be compared to coupons and $r(\gamma)p(x,\gamma)$. It is predictably larger for lower γ and spikes around the default threshold, where coupon risk is tangible. Away from the threshold, default risk becomes a distant possibility.. It spikes around the default threshold due to a discontinuity in the drift: $\overline{\mu} > \mu$, reflecting faster growth during restructuring.

Panels (c) and (d) slice the common component $\sigma_{p\gamma}/p$ of return volatility along x and γ. First, panel (c) shows $\sigma_{p\gamma}(x,\cdot)/p(x,\cdot)$ as a function of γ for three different levels of x. The loading $\sigma_{p\gamma}/p$ becomes very small at the extremes since the volatility of γ itself goes to zero there. Between the extremes, the sign of the loading depends on x, and this dependence is non-monotonic. For x far enough from the default threshold, returns load on risk-taking capacity negatively since rising γ raises the interest rate. A change in the default status becomes more probable as x approaches the default threshold from either side, and in that region, changes in γ also tangibly affect the risk premium, so the loading $\sigma_{p\gamma}/p$ is positive. Panel (d) makes this clear by plotting $\sigma_{p\gamma}(\cdot,\gamma)/p(\cdot,\gamma)$ as a function of x for two different levels of γ.

Spreads. I next describe spreads and illustrate their common component. Define the spread $y(x,\gamma)$ as the price difference between the bonds in the model and counterfactual securities without default risk:

$$y(x,\gamma) = \log(\hat{p}(\gamma)) - \log(p(x,\gamma))$$

Here the counterfactual price function $\hat{p}(\gamma)$ solves

$$r(\gamma)\hat{p}(\gamma) = \kappa + \hat{p}'(\gamma)\mu_\gamma(\gamma) + \hat{p}''(\gamma)\frac{\sigma_\gamma(\gamma)^2}{2} - \frac{(\hat{p}'(\gamma)\sigma_\gamma(\gamma))^2}{\gamma w}$$

The risk correction is still present, reflecting lenders' risk aversion and the interest rate risk. Comparing $p(x,\gamma)$ to $\hat{p}(\gamma)$ only strips out default risk.

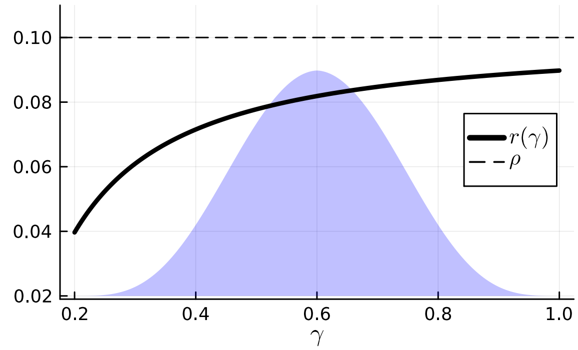
Define the loadings on the local and global shocks $\sigma_{yx}(\cdot)$ and $\sigma_{y\gamma}(\cdot)$:

$$dy(x,\gamma) = \mu_y(x,\gamma)dt + \sigma_{yx}(x,\gamma)dZ_t + \sigma_{y\gamma}(x,\gamma)dW_t$$

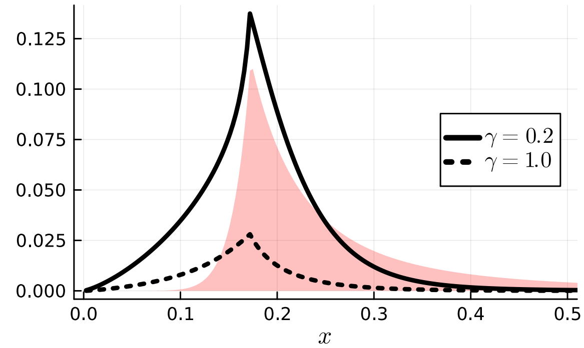
The common component is $\sigma_y(\gamma)dW_t$ averaging the loadings on the global shock:

$$\sigma_\gamma(\gamma) = \int \sigma_{y\gamma}(x,\gamma)g(x)dx = -\sigma_\gamma(\gamma)\int\left(\frac{\kappa p_\gamma(x,\gamma)}{p(x,\gamma)^2} + r'(\gamma)\right)g(x)dx$$

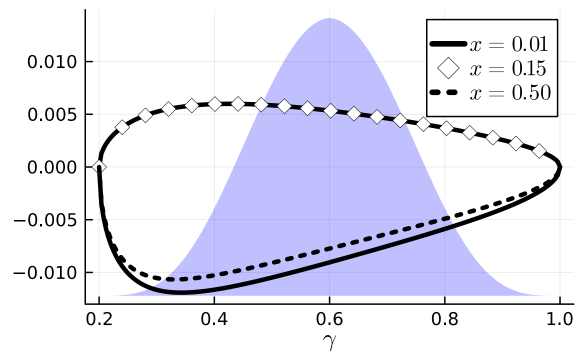
Figure 2: solution of the model.



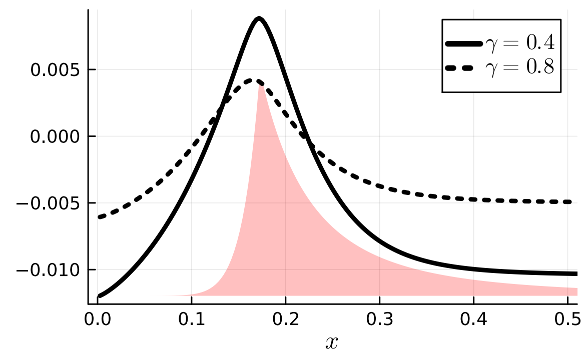
(a) Risk-free rate as a function of γ



(b) Risk correction in equation (15)



(c) Common component $\sigma_{p\gamma}(x, \cdot)/p(x, \cdot)$ as a function of γ for different levels of x



(d) Common component $\sigma_{p\gamma}(\cdot, \gamma)/p(\cdot, \gamma)$ as a function of x for different levels of γ

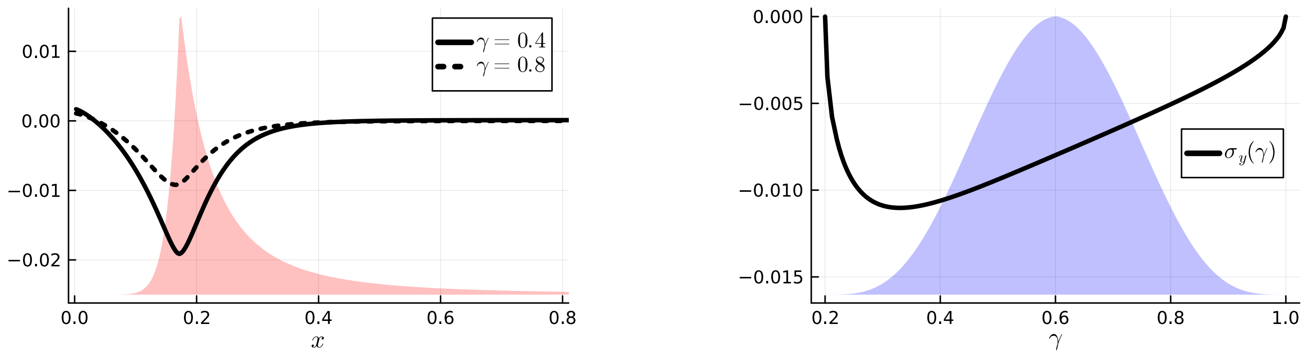
These loadings are easy to compute once one has $p(\cdot)$ and $r(\cdot)$. Figure (3) plots $\sigma_{y\gamma}(\cdot, \gamma)$ as a function of x for two different levels of γ and the common component $\sigma_y(\cdot)$.

Risk-neutral benchmarks. How different is $p(\cdot)$ with risk-neutral lenders? I illustrate the differences in two ways with Figure (4). First, panel (a) reports the log difference between $p(x, \gamma)$ and a counterfactual risk-neutral valuation of the bond that solves the standard pricing equation:

$$r(\gamma)\overline{p}(x, \gamma) = \kappa(1 - \Delta(x)) + \overline{p}_x(x, \gamma)\mu(x) + \overline{p}_\gamma(x, \gamma)\mu_\gamma(\gamma) + \overline{p}_{xx}(x, \gamma)\frac{\sigma^2 x^2}{2} + \overline{p}_{\gamma\gamma}(x, \gamma)\frac{\sigma_\gamma(\gamma)^2}{2}$$

To compute this counterfactual valuation, I keep the interest rate the same function of γ as in the baseline. This exercise shows how departing from risk neutrality leads to $p(x, \gamma)$ pricing in both default and interest rate risk. The latter necessarily arises in general equilibrium with risk aversion and tangibly changes prices. Panel (b) makes this clear by eliminating the changes in the interest rate. The counterfactual valuation $\tilde{p}(\cdot)$ on this figure solves

Figure 3: comparison to counterfactual prices.



(a) loadings $\sigma_{y\gamma}(\cdot, \gamma)$ as a function of x

(b) common component in spreads $\sigma_{\gamma}(\cdot)$

$$\rho\tilde{p}(x) = \kappa(1 - \Delta(x)) + \tilde{p}'(x)\mu(x) + \tilde{p}''(x)\frac{\sigma^2 x^2}{2}$$

The interest rate is set at the level ρ, which would happen in equilibrium if intermediaries were risk-neutral. Since γ does not affect the interest rate now, it drops from the arguments of $\tilde{p}(\cdot)$. In fact, there is an explicit solution for the bond price in this well-studied case:

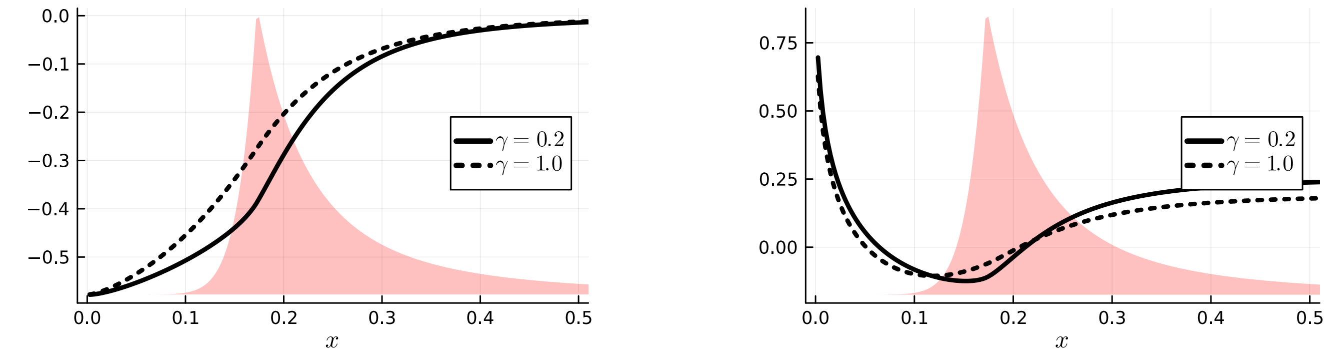
$$\tilde{p}(x) = \begin{cases} \frac{\kappa}{\rho} \cdot \frac{\zeta}{\zeta - \overline{\zeta}} \cdot \left(\frac{x}{\hat{x}}\right)^{\overline{\zeta}} & \text{for } x \leq \hat{x} \\ \frac{\kappa}{\rho} \cdot \left(1 + \frac{\overline{\zeta}}{\zeta - \overline{\zeta}} \cdot \left(\frac{x}{\hat{x}}\right)^{\zeta}\right) & \text{for } x \geq \hat{x} \end{cases}$$

Here $\zeta < 0$ and $\overline{\zeta} > 0$ are the coefficients from Proposition 7 that only depend on $(\mu, \overline{\mu}, \sigma, \rho)$. I provide their functional forms in the proof.

The takeaway from panel (d) is that the equilibrium price of defaultable bonds is only lower than its risk-neutral counterpart around the default threshold. Departing from the threshold, default risk subsides, and bond prices in the risk-averse economy exceed the risk-neutral benchmark due to lower interest rates. Precautionary motives inflate all asset prices, and this effect dominates when the default threshold is far away.

Discussion. Value-at-risk allows one to add aggregate shocks to risk preferences to the model while keeping it tractable. Using intermediaries that maximize log utility would also add the risk correction term to the pricing equation (15), but risk aversion would be fixed at one. Using recursive preferences or power utility to vary the risk correction would cause equation (15) to lose

Figure 4: comparison to counterfactual prices.



(a) Log bond price relative to partly risk-neutral valuation $\log(p(x, \gamma)) - \log(\overline{p}(x, \gamma))$

(b) Log bond price relative to fully risk-neutral valuation $\log(p(x, \gamma)) - \log(\tilde{p}(x))$

its tractable form. Value-at-risk allows one to keep the functional form of the risk correction term, simply adding the time index to γ and adding γ to state variables. General equilibrium is then easy to characterize and compute numerically.

A crucial assumption that makes general equilibrium easily computable with aggregate shocks and a whole cross-section of countries is the absence of endogenous distributions. The distribution of cash flows is exogenous and does not have to be explicitly tracked. In a related model, Oskolkov (2024) uses Lucas trees located in different countries instead of defaultable bonds. Countries also have local investors who hold the trees together with intermediaries. There is a non-degenerate wealth distribution across countries, which creates a full cross-section of risk premia in equilibrium but makes aggregate shocks to γ unpalatable computationally.

6 Related literature

Value-at-risk measures are widely used in banking. Stulz (2016) explains the use of value-at-risk in banks' framework in detail. Sizova (2023) collects information on banks updating their models from financial reports. She shows that banks often mention "value-at-risk models" in their reports. Their model revisions respond to changes in regulation and to events affecting the banks' own performance. Barbiero, Bräuning, Joaquim, and Stein (2024) collect data on risk limits faced by the largest US banks. They show that 165 of 167 trading desks whose activities are related to currency trading face value-at-risk constraints. Adrian and Shin (2010) show that the ratio of value-at-risk to capital at the largest US banks was fairly constant in 2000-2007. Adrian and Shin (2014) extend measurements to the after-crisis times, when regulation changed significantly, and find that the value-at-risk to equity ratio seems relatively stable across regimes. They then provide a foundation for this constraint using the contractual framework from Holmstrom and

Tirole (1997). Value-at-risk restrictions emerge endogenously when banks use debt financing, enjoy limited liability, and choose between extreme value-distributed investment opportunities with different riskiness.

The literature using value-at-risk constraints for modeling portfolio choice dates back to at least Danielsson, Shin, and Zigrand (2012) and Adrian and Boyarchenko (2018). These papers use continuous-time models, where intermediaries' risky positions are subject to the following constraint: the standard deviation of the excess returns cannot exceed a multiple of net worth. Intermediaries form mean-variance portfolios with the Lagrange multiplier on the constraint acting as the time-varying risk aversion coefficient. The leverage of the intermediaries is inversely proportional to the variance of returns. Danielsson, Shin, and Zigrand (2012) and Adrian and Boyarchenko (2018) make use of these properties to arrive at countercyclical leverage and obtain the "volatility paradox": low exogenous volatility coincides with high leverage, which leads to high systemic risk. Hofmann, Shim, and Shin (2022) use the same constraint to show the impact of dollar appreciation on portfolio inflows in emerging markets. My formulation of the value-at-risk constraint is different in that I cap the total variance of returns, rather than the standard deviation, and I use a flow (expected profits) rather than the stock of wealth as the upper bound on risk exposure. The result is mean-variance portfolios with time-varying risk aversion explicitly given as a primitive of the model instead of an endogenous Lagrange multiplier. This allows for more transparent aggregation and simplifies the use of attitudes to risk as a driver of shocks.

A desirable feature of the value-at-risk constraints is the procyclical leverage they generate. Kalemli-Ozcan, Sorensen, and Yesiltas (2012) show this empirically. Shin (2012) uses this property to argue that the leverage cycle of global banks drives credit supply and loan risk premia in the US. Coimbra (2020) models intermediaries with an occasionally binding value-at-risk constraint. In busts, when intermediaries are up against the constraint, risk-averse households absorb the residual supply of risky assets, which leads to a rise in risk premia. Coimbra and Rey (2024) model a cross-section of intermediaries with different value-at-risk constraint parameters. Some intermediaries are inactive, and active ones choose different risk exposures. Changes in expected productivity and interest rates induce changes in both extensive and intensive margins of their risk-taking, moving the aggregate leverage of the financial sector. Intermediaries are short-lived, so interest rate and productivity news lead to a reallocation of activity rather than wealth redistribution between them. My environment allows for wealth redistribution and exogenous changes in risk preferences but does not incorporate the extensive margin of activity. Another difference is that Coimbra (2020) and Coimbra and Rey (2024) use discrete time, which allows them to interpret value-at-risk literally: intermediaries face a limit on the probability of negative equity returns or failure. My continuous-time framework interprets the constraint as a limit on the instantaneous variance of returns, for which I provide a heuristic derivation.

Empirically, Coimbra, Kim, and Rey (2022) estimate value-at-risk parameters from bank balance sheet data and find substantial heterogeneity in cross-section. Barbiero, Bräuning, Joaquim, and Stein (2024) find that changes in value-at-risk limits of dealers impact currency returns. FX market is highly intermediated and depends on a concentrated industry of large dealers. Using regulatory data, Barbiero, Bräuning, Joaquim, and Stein (2024) find rather long-lived effects of value-at-risk limit tightenings of individual dealers. Bräuning and Stein (2024) find that limit changes also affect the functioning of the treasury markets.

A large set of models rely on constraints related to value-at-risk. Examples include Gromb and Vayanos (2002) and Gromb and Vayanos (2018), where arbitrageurs face financial constraints that disallow negative equity. This can be interpreted as setting a zero limit on value at risk. Vayanos (2004) incorporates performance-based liquidation into a model of fund management. End investors liquidate their fund holdings with a probability tied to returns. Fund managers are concerned about liquidation, and these concerns affect their risk-taking in a way similar to a value-at-risk penalty. Vayanos and Vila (2021), Gourinchas, Ray, and Vayanos (2022), Ray (2019), Kamdar and Ray (2024), and Greenwood, Hanson, Stein, and Sunderam (2023) endow arbitrageurs with mean-variance preferences, mentioning that this can capture value-at-risk constraints in reduced form. I operationalize this conjecture in a fully dynamic setup.

Kekre, Lenel, and Mainardi (2024) extend Vayanos and Vila (2021) by making arbitrageurs infinitely-lived agents with power utility. Abritrageurs' wealth has a positive duration in equilibrium, so positive interest rate shocks lead to an increase in term premia. This resolves a conflict between the data and the baseline model of Vayanos and Vila (2021), where term premia fall following contractionary monetary shocks. Portfolio choice in Kekre, Lenel, and Mainardi (2024) includes a hedging term that complicates computations. They show that it disappears in the limit of infinite impatience and focus on this case for simplicity. Value-at-risk constraints provide an alternative way to eliminate hedging motives.

My foundation for the value-at-risk constraint is related to a large literature on robust control and model misspecification. The classical reference for this framework is Hansen and Sargent (2001). They clarify the link between the maxmin decision rules of Gilboa and Schmeidler (1989) and economic applications of robust control, which typically adopt a "penalty" approach to model misspecification. Anderson, Hansen, and Sargent (2003) and Hansen, Sargent, Turmuhambetova, and Williams (2006) relate and compare different ways to apply robust control. A recent application of robust portfolio choice in macro-financial models is Hansen, Khorrami, and Tourre (2024). My setup falls into this long tradition. The main difference is that I separate model misspecification for returns and aggregate states. The agent explores alternative probability measures for shocks driving excess returns but ignores the implications of these alternatives for her views on aggregate states. This allows me to eliminate hedging motives and reduce portfolio choice to mean-variance, breaking away from Kreps and Porteus (1978) and Duffie and Epstein (1992).

My sovereign default application is related to Grosse-Steffen and Podstawski (2016), Pouzo and Presno (2016), Coimbra (2020), and Roch and Roldán (2023). Pouzo and Presno (2016) and Roch and Roldán (2023) use lenders with robust choice in their models. These models successfully account for the fact that spreads on sovereign bonds are too high and volatile to be explained by historical default probabilities, as documented by Aguiar, Chatterjee, Cole, and Stangebye (2016). Grosse-Steffen and Podstawski (2016) introduce ambiguity about borrower's productivity for a similar purpose. Coimbra (2020) directly models lenders as risk-neutral banks with a value-at-risk constraint. When default risk is low and the constraint is slack, banks hold the total stock of debt. Spreads only incorporate expected haircuts in the case of default. When the default risk is high, the constraint binds, and households pick up the residual supply. This increases spreads because households are risk-averse. A similar mechanism is at work in Schneider (2023), where risk-averse households and risk-neutral intermediaries with a financial constraint from Gertler and Kiyotaki (2015) trade bonds of multiple maturities.

Another closely related paper is Tourre (2017). He endows lenders with an exogenous stochastic discount factor to price a cross-section of sovereign bonds. His environment produces a time-varying global risk-free rate and common variation in sovereign risk premia, while recursive preferences deliver high and volatile spreads. My setup uses exogenous shocks to risk tolerance as a driver of interest rates and spreads, which is an alternative to an exogenously specified SDF. My environment shares its use of continuous time with Aguiar, Amador, Farhi, and Gopinath (2015), Nuño and Thomas (2016), Aguiar and Amador (2020), Lorenzoni and Werning (2019), and Bornstein (2020). Compared to these models, my application is massively simplified. I add global risk tolerance as a state variable but strip out all interesting dynamics of default decisions to maximize tractability. My borrowers are risk-neutral, there is no punishment for default, and debt maturity is infinite. Default decisions are essentially exogenous. This allows me to derive a simple two-dimensional PDE for bond prices in a fully closed model. I can then easily compute the interest rate as a function of the global risk tolerance and extract common components from sovereign spreads, accounting for both default and interest rate risk.

Bai, Kehoe, and Perri (2019) and Morelli, Ottonello, and Perez (2022) study the transmission of global shocks in models with a cross-section of sovereign bonds. Bai, Kehoe, and Perri (2019) focus on long-run risk with two components: one in emerging markets that issue defaultable bonds and one in advanced economies that buy them. The long-run risk component in emerging markets induces common variation in spreads through anticipation of future default, affecting the quantity of risk. The component coming from advanced economies affects the price of risk through investors' desire to save in anticipation of slow growth. My application completely shuts down aggregate shocks originating in the issuing countries and attacks the price of risk directly through investros' preferences. The mechanism in Morelli, Ottonello, and Perez (2022) operates through the net worth of global intermediaries, who are the sole investors in sovereign bonds. In my application, the shock hits effective risk tolerance instead, which has similar effects because risk tolerance always enters pricing equations in tandem with net worth. Morelli, Ottonello, and Perez (2022) also show empirical evidence on the impact of investor net worth of sovereign spreads, finding substantial common variation that is driven by the balance sheets of global banks.

Finally, there is a long literature in finance analyzing value-at-risk constraints in portfolio choice. Basak and Shapiro (2001) and Berkelaar, Cumperayot, and Kouwenberg (2002) analyze equilibrium consequences of the presence of value-at-risk-constrained agents for asset prices. Sentana (2001) conceptualizes iso-value-at-risk curves by analogy with iso-Sharpe curves in assets space. Yiu (2004) derives a dynamically imposed constraint, which ends up resembling the constraint in Danielsson, Shin, and Zigrand (2012) with a standard deviation of returns instead of variance. Methodological literature includes Noyan and Rudolf (2013), Bernard, Rüschendorf, and Vanduffel (2017), Pirvu (2007), Krokhmal, Palmquist, and Uryasev (2002), Alexander and Baptista (2003), Alexander and Baptista (2008), Cuoco, He, and Isaenko (2008), and many others.

7 Conclusion

I describe a version of the value-at-risk constraint that generates mean-variance portfolios when investors maximize log utility and excess returns are driven by diffusions. I provide a foundation for this constraint through a version of robustness concerns, suggesting an interpretation of shocks to risk limits that does not depend on changing regulation. I then show aggregation properties of the value-at-risk constraint. Prices of risky assets and the risk-free rate depend on the wealth distribution through a single scalar: the average value-at-risk multiplier weighted with wealth shares, which can be interpreted as aggregate effective risk tolerance. Pricing equations for risky assets look like standard risk-neutral HJB equations with a single risk correction term. Wealth shares of investors have the same loadings on the shocks, scaled by their leverage minus one, where each investor's leverage is her value-at-risk multiplier divided by the wealth-weighted average.

The practicality of the value-at-risk constraints stems from their ability to shut down hedging motives. While convenient, this understandably cripples the model's empirical performance. Without hedging motives, the model will have a limited handle on the co-movement between the marginal value of wealth and other economic variables. A more alarming implication of optimal choice under value-at-risk, which really originates in log utility, is that price-dividend ratios tend to be stable. With a single risky asset, they are simply constant. This suggests that applications of the setup presented above should be chosen carefully because value-at-risk clearly gravitates to the simplicity side of the simplicity-precision trade-off.

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A Proofs

Proof of Proposition 2.

Take the recursive form of equation (1) with the new budget constraint including $\tau(x)$. Let $V(w,x)$ be the value. The HJB equation is

$$\begin{aligned}\rho V(w,x) = \max_{c,\theta}\ & \rho\log(c) + (r(x)w - c + w\theta'\mu_R(x))V_w(w,x) \\ &+ \frac{(\theta'\sigma_R(x) - \tau(x)')(\sigma_R(x)'\theta - \tau(x))}{2} w^2 V_{ww}(w,x) + w\theta'\sigma_R(x)\sigma_X(x)'V_{wx'}(w,x) \\ &+ \mu_X(x)'V_{x'}(w,x) + \frac{1}{2}\mathrm{tr}[\sigma_X(x)'V_{xx'}(w,x)\sigma_X(x)] \qquad \text{(A.1)}\\ \text{s.t. }\ & \theta'\sigma_R(x)\sigma_R(x)'\theta \leq \gamma\theta'\mu_R(x) \qquad \text{(A.2)}\end{aligned}$$

Like in the proof of Proposition 1, guess and verify the solution to equation (A.1) to be $V(w,x) = \log(w) + \eta(x)$. Again, the consequences are that $V_{wx'}(w,x) = 0$ and that $V_x(w,x)$ and $V_{xx'}(w,x)$ are functions of x only: $V_x(w,x) = \eta_x(x)$ and $V_{xx'}(w,x) = \eta_{xx'}(x)$. Consumption is still a constant fraction of wealth: $c = \rho w$.

Moving to portfolio choice, let $\xi(x,w)$ be the multiplier on equation (A.2). The first-order condition with respect to θ is now

$$\theta = \frac{1+\gamma\xi(w,x)}{1+2\xi(w,x)}[\sigma_R(x)\sigma_R(x)']^{-1}\mu_R(x) + \frac{1}{1+2\xi(w,x)}[\sigma_R(x)\sigma_R(x)']^{-1}\sigma_R(x)\tau(x)$$

Now use the fact that $\tau(x) = \gamma\zeta(x)\sigma_R(x)'[\sigma_R(x)\sigma_R(x)']^{-1}\mu_R(x)$:

$$\theta = \frac{1+\gamma\xi(w,x)+\gamma\zeta(x)}{1+2\xi(w,x)}[\sigma_R(x)\sigma_R(x)']^{-1}\mu_R(x)$$

Suppose the constraint is slack and $\xi(w,x) = 0$. Then,

$$\theta = (1+\zeta(x)\gamma)[\sigma_R(x)\sigma_R(x)']^{-1}\mu_R(x)$$

Plug this into equation (A.2) to see that slackness is equivalent to

$$1+\zeta(x)\gamma < \gamma$$

Alternatively, $\zeta(x) < 1 - 1/\gamma$. If $\zeta(x) > 1 - 1/\gamma$, the constraint binds, and

$$\frac{1+\gamma\xi(w,x)+\gamma\zeta(x)}{1+2\xi(w,x)} = \gamma$$

Portfolio choice in this case is hence the same as without taxes:

$$\theta = \gamma[\sigma_R(x)\sigma_R(x)']^{-1}\mu_R(x)$$

The tax vector happens to generate the same exposure to shocks as a share $\zeta(x)$ of the optimal portofolio, as designed: $\tau(x) = \zeta(x)\sigma_R(x)'\theta(x)$. The equation that the rest of investor's value function $\eta(x)$ solves changes to

$$\begin{aligned}\rho\eta(x) = \rho\log(\rho) + r(x) - \rho + \frac{2\gamma - (1-\zeta(x))^2\gamma^2}{2}\mu_R(x)'[\sigma_R(x)\sigma_R(x)']^{-1}\mu_R(x) \\ + \mu_X(x)'\eta_{x'}(x) + \frac{1}{2}\mathrm{tr}[\sigma_X(x)'\eta_{xx'}(x)\sigma_X(x)]\end{aligned}$$

with appropriate boundary conditions. □

Proof of Proposition 4.

Start with writing down excess returns in matrix form using prices and dividends only. The vector of excess returns $d\boldsymbol{R}_t = \mu_R(x_t)dt + \sigma_R(x_t)dZ_t$ is

$$d\boldsymbol{R}_t \equiv D(\boldsymbol{p}(x_t))^{-1}[\mu_p(x_t) + \boldsymbol{y}(x_t) - r(x_t)\boldsymbol{p}(x_t)]dt + D(\boldsymbol{p}(x_t))^{-1}\sigma_p(x_t)dZ_t \tag{A.3}$$

Here $D(\boldsymbol{p}(x_t))$ is a diagonal matrix with $\boldsymbol{p}(x_t)$ on the main diagonal and zeros everywhere else. Expected excess returns on each asset j consist of capital gains $[\mu_p(x_t)]_j/[\boldsymbol{p}(x_t)]_j$ and dividend yield $[\boldsymbol{y}(x_t)]_j/[\boldsymbol{p}(x_t)]_j$ over and above the risk-free rate $r(x_t)$. The loadings of excess returns on the shocks are the loadings of capital gains $[\sigma_p(x_t)]_{\cdot j}/[\boldsymbol{p}(x_t)]_j$ only.

Portfolio choice of every agent i is

$$\boldsymbol{\theta}_{it} = \gamma_{it}[\sigma_R(x_t)\sigma_R(x_t)']^{-1}\mu_R(x_t) \tag{A.4}$$

Multiplying this by w_{it}, summing across i, and using the market clearing condition for each asset,

$$D(\boldsymbol{p}(x_t))\boldsymbol{s} = \sum\nolimits_{i=1}^{n} \gamma_{it}w_{it} \cdot [\sigma_R(x_t)\sigma_R(x_t)']^{-1}\mu_R(x_t)$$

Multiply both sides by $\sigma_R(x_t)\sigma_R(x_t)'$ on the left:

$$\sigma_R(x_t)\sigma_R(x_t)'D(\boldsymbol{p}(x_t))\boldsymbol{s} = \sum\nolimits_{i=1}^{n} \gamma_{it}w_{it} \cdot \mu_R(x_t)$$

Using the definition of Γ_t and the fact that total wealth is $\boldsymbol{p}(x_t)'\boldsymbol{s}$,

$$\sigma_R(x_t)\sigma_R(x_t)'D(\boldsymbol{p}(x_t))\boldsymbol{s} = \Gamma_t \cdot \boldsymbol{p}(x_t)'\boldsymbol{s} \cdot \mu_R(x_t) \tag{A.5}$$

This can be reorganized by usingequation (A.3):

$$\mu_p(x_t) + \boldsymbol{y}(x_t) - r(x_t)\boldsymbol{p}(x_t) = \frac{1}{\Gamma_t \cdot \boldsymbol{p}(x_t)'\boldsymbol{s}} \cdot \sigma_p(x_t)\sigma_p(x_t)'\boldsymbol{s}$$

This is equation (11). Multiplying this by $\boldsymbol{s}'$ on the left,

$$\boldsymbol{s}'\mu_p(x_t) + \boldsymbol{s}'\boldsymbol{y}(x_t) - r(x_t)\boldsymbol{s}'\boldsymbol{p}(x_t) = \frac{1}{\Gamma_t} \cdot \frac{\boldsymbol{s}'\sigma_p(x_t)\sigma_p(x_t)'\boldsymbol{s}}{\boldsymbol{p}(x_t)'\boldsymbol{s}}$$

Using the fact that the total consumption $\boldsymbol{s}'\boldsymbol{y}(x_t)$ is a fraction ρ of total wealth $\boldsymbol{s}'\boldsymbol{p}(x_t)$, divide both sides by $\boldsymbol{s}'\boldsymbol{p}(x_t)$ to get

$$\frac{\boldsymbol{s}'\mu_p(x_t)}{\boldsymbol{s}'\boldsymbol{p}(x_t)} + \rho - r(x_t) = \frac{1}{\Gamma_t} \cdot \frac{\boldsymbol{s}'\sigma_p(x_t)\sigma_p(x_t)'\boldsymbol{s}}{(\boldsymbol{p}(x_t)'\boldsymbol{s})^2}$$

Reorganizing,

$$r(x_t) = \rho + \frac{\boldsymbol{s}'\mu_p(x_t)}{\boldsymbol{s}'\boldsymbol{p}(x_t)} - \frac{1}{\Gamma_t} \cdot \frac{|\sigma_p(x_t)'\boldsymbol{s}|^2}{(\boldsymbol{p}(x_t)'\boldsymbol{s})^2}$$

This is equation (12).

To obtain total leverage λ_{it} of investor i, use equation (A.4) for portfolio choice:

$$\lambda_{it} = \gamma_{it} \cdot [\sigma_R(x_t)\sigma_R(x_t)']^{-1}\mu_R(x_t)1_k$$

The market clearing condition for risk-free bonds implies

$$0 = \sum_{i=1}^{n} w_{it}(1 - \lambda_{it}) = \sum_{i=1}^{n} w_{it} - \sum_{i=1}^{n} \gamma_{it} w_{it} \cdot [\sigma_R(x_t)\sigma_R(x_t)']^{-1}\mu_R(x_t)1_k$$

Dividing by total wealth,

$$1 = \Gamma_t \cdot [\sigma_R(x_t)\sigma_R(x_t)']^{-1}\mu_R(x_t)1_k$$

Plugging this back into the expression for leverage,

$$\lambda_{it} = \frac{\gamma_{it}}{\Gamma_t}$$

To get the expression for holdings $\{h_{ijt}\}$, notice that $h_{ijt} = \theta_{ijt} w_{it}/p_{jt}$, which means

$$h_{ijt} = \gamma_{it} w_{it} \cdot [D(\boldsymbol{p}(x_t))[\sigma_R(x_t)\sigma_R(x_t)']^{-1}\mu_R(x_t)]_j$$

For every j, holdings h_{ijt} are proportional to $\gamma_{it}w_{it}$ with a common j-specific coefficient of proportionality. Since holdings sum to s_j,

$$h_{ijt} = s_j \cdot \frac{\gamma_{it}w_{it}}{\sum_{i=1}^{n}\gamma_{it}w_{it}} = s_j \cdot \frac{\gamma_{it}\nu_{it}}{\Gamma_t} = s_j\nu_{it}\lambda_{it}$$

The last remaining result to establish is equation (13) for the dynamics of wealth shares. Start with using equation (A.5) to replace $\mu_R(x_t)$ in equation (A.4):

$$\boldsymbol{\theta}_{it} = \frac{\gamma_{it}}{\Gamma_t \cdot \boldsymbol{p}(x_t)'\boldsymbol{s}} \cdot D(\boldsymbol{p}(x_t))\boldsymbol{s} = \frac{\lambda_{it}}{\boldsymbol{p}(x_t)'\boldsymbol{s}} \cdot D(\boldsymbol{p}(x_t))\boldsymbol{s}$$

This implies

$$\begin{aligned}
\boldsymbol{\theta}_{it}'\mu_R(x_t) &= \frac{\lambda_{it}\boldsymbol{s}'D(\boldsymbol{p}(x_t))\sigma_R(x_t)\sigma_R(x_t)'D(\boldsymbol{p}(x_t))\boldsymbol{s}}{\Gamma_t(\boldsymbol{p}(x_t)'\boldsymbol{s})^2} = \frac{\lambda_{it}\boldsymbol{s}'\sigma_p(x_t)\sigma_p(x_t)'\boldsymbol{s}}{\Gamma_t(\boldsymbol{p}(x_t)'\boldsymbol{s})^2} = \frac{\lambda_{it}}{\Gamma_t} \cdot \frac{|\sigma_p(x_t)'\boldsymbol{s}|^2}{(\boldsymbol{p}(x_t)'\boldsymbol{s})^2} \\
\boldsymbol{\theta}_{it}'\sigma_R(x_t) &= \lambda_{it} \cdot \frac{\boldsymbol{s}'\sigma_p(x_t)}{\boldsymbol{p}(x_t)'\boldsymbol{s}}
\end{aligned}$$

Here I used the fact that $D(\boldsymbol{p}(x_t))\sigma_R(x_t) = \sigma_p(x_t)$. The dynamics of individual wealth are

$$\begin{aligned}
\frac{dw_{it}}{w_{it}} &= (r(x_t) - \rho)w_{it}dt + w_{it}\boldsymbol{\theta}_{it}'\mu_R(x_t)dt + w_{it}\boldsymbol{\theta}_{it}'\sigma_R(x_t)dZ_t \\
&= (r(x_t) - \rho)w_{it}dt + \frac{\lambda_{it}w_{it}}{\Gamma_t} \cdot \frac{|\sigma_p(x_t)'\boldsymbol{s}|^2}{(\boldsymbol{p}(x_t)'\boldsymbol{s})^2}dt + \lambda_{it}w_{it} \cdot \frac{\boldsymbol{s}'\sigma_p(x_t)}{\boldsymbol{p}(x_t)'\boldsymbol{s}}dZ_t
\end{aligned}$$

Summing this across i and denoting the total wealth by w_t,

$$dw_t = (r(x_t) - \rho)w_t dt + \frac{w_t}{\Gamma_t} \cdot \frac{|\sigma_p(x_t)'\boldsymbol{s}|^2}{(\boldsymbol{p}(x_t)'\boldsymbol{s})^2}dt + w_t \cdot \frac{\boldsymbol{s}'\sigma_p(x_t)}{\boldsymbol{p}(x_t)'\boldsymbol{s}}dZ_t$$

Here I use the fact that the average leverage weighted with wealth shares is one:

$$\sum\nolimits_{i=1}^{n}\lambda_{it}w_{it} = \sum\nolimits_{i=1}^{n}\lambda_{it}\nu_{it}w_t = \sum\nolimits_{i=1}^{n}\frac{\gamma_{it}\nu_{it}}{\Gamma_t}w_t = w_t$$

Now consider the dynamics of $\nu_{it} = w_{it}/w_t$:

$$\begin{aligned}
d\nu_{it} &= \nu_{it}\left(\frac{dw_{it}}{w_{it}} - \frac{dw_t}{w_t} + \frac{(dw_t)^2}{w_t^2} - \frac{dw_t}{w_t}\frac{dw_{it}}{w_{it}}\right) \\
&= \nu_{it}\left(\frac{\lambda_{it}-1}{\Gamma_t} \cdot \frac{|\sigma_p(x_t)'\boldsymbol{s}|^2}{(\boldsymbol{p}(x_t)'\boldsymbol{s})^2}dt + (\lambda_{it}-1) \cdot \frac{\boldsymbol{s}'\sigma_p(x_t)}{\boldsymbol{p}(x_t)'\boldsymbol{s}}dZ_t + (1-\lambda_{it}) \cdot \frac{|\sigma_p(x_t)'\boldsymbol{s}|^2}{(\boldsymbol{p}(x_t)'\boldsymbol{s})^2}dt\right) \\
&= \nu_{it}(\lambda_{it}-1) \cdot \left[\frac{1-\Gamma_t}{\Gamma_t} \cdot \frac{|\sigma_p(x_t)'\boldsymbol{s}|^2}{(\boldsymbol{p}(x_t)'\boldsymbol{s})^2}dt + \frac{\boldsymbol{s}'\sigma_p(x_t)}{\boldsymbol{p}(x_t)'\boldsymbol{s}}dZ_t\right]
\end{aligned}$$

Realizing that $\boldsymbol{s}'\sigma_p(x_t) = \sigma_w(x_t)$ and $\boldsymbol{p}(x_t)'\boldsymbol{s} = w_t$ completes the proof. $\square$

Proof of Proposition 5.

Denote the total wealth in the economy by w_t and the wealth of the less constrained investor by $\overline{w}_t$. The evolution of her wealth share $\overline{\nu}_t = \overline{w}_t/w_t$ is

$$d\left(\frac{\overline{w}_t}{w_t}\right) = \frac{d\overline{w}_t}{w_t} - \frac{\overline{w}_t dw_t}{w_t^2} + \frac{\overline{w}_t dw_t^2}{w_t^3} - \frac{d\overline{w}_t dw_t}{w_t^2} = \overline{\nu}_t\left(\frac{d\overline{w}_t}{\overline{w}_t} - \frac{dw_t}{w_t} + \frac{dw_t^2}{w_t^2} - \frac{d\overline{w}_t}{\overline{w}_t}\frac{dw_t}{w_t}\right)$$

The evolution of $\overline{w}_t$ and w_t is

$$\begin{aligned} \frac{d\overline{w}_t}{\overline{w}_t} &= \left(\mu - x_t + \overline{\gamma}\frac{x_t^2}{\sigma^2}\right)dt + \overline{\gamma}\frac{x_t}{\sigma}dZ_t \\ \frac{dw_t}{w_t} &= \mu dt + \sigma dZ_t \end{aligned}$$

The second equation follows from $\rho w_t = y_t$, which is the consumption good market clearing condition. Plugging these into the evolution of $\overline{\nu}_t$,

$$\begin{aligned} d\overline{\nu}_t &= \overline{\nu}_t\left(\sigma^2 - x_t + \overline{\gamma}\frac{x_t^2}{\sigma^2} - \overline{\gamma}x_t\right)dt + \overline{\nu}_t\left(\overline{\gamma}\frac{x_t}{\sigma} - \sigma\right)dZ_t \\ &= \overline{\nu}_t\frac{(\overline{\gamma}x_t - \sigma^2)(x_t - \sigma^2)}{\sigma^2}dt + \overline{\nu}_t\frac{\overline{\gamma}x_t - \sigma^2}{\sigma}dZ_t \end{aligned}$$

Invert $x_t = \sigma^2/\Gamma_t = \sigma^2/(\overline{\nu}_t\overline{\gamma} + (1-\overline{\nu}_t)\underline{\gamma})$ to express wealth shares as functions of x_t:

$$\begin{aligned} \overline{\nu}_t &= \frac{\sigma^2 - \underline{\gamma}x_t}{(\overline{\gamma} - \underline{\gamma})x_t} \\ \underline{\nu}_t &= \frac{\overline{\gamma}x_t - \sigma^2}{(\overline{\gamma} - \underline{\gamma})x_t} \end{aligned}$$

Plugging this,

$$d\overline{\nu}_t = \overline{\nu}_t(1-\overline{\nu}_t)\frac{(\overline{\gamma} - \underline{\gamma})(x_t - \sigma^2)x_t}{\sigma^2}dt + \overline{\nu}_t(1-\overline{\nu}_t)\frac{(\overline{\gamma} - \underline{\gamma})x_t}{\sigma}dZ_t$$

Using $x_t = \sigma^2/(\overline{\nu}_t\overline{\gamma} + (1-\overline{\nu}_t)\underline{\gamma})$ again,

$$d\overline{\nu}_t = \frac{1 - \overline{\nu}_t\overline{\gamma} - (1-\overline{\nu}_t)\underline{\gamma}}{\left(\overline{\nu}_t\overline{\gamma} + (1-\overline{\nu}_t)\underline{\gamma}\right)^2}\cdot\overline{\nu}_t(1-\overline{\nu}_t)\sigma^2 dt + \frac{\overline{\gamma} - \underline{\gamma}}{\overline{\nu}_t\overline{\gamma} + (1-\overline{\nu}_t)\underline{\gamma}}\cdot\overline{\nu}_t(1-\overline{\nu}_t)\sigma dZ_t$$

This proves the first part of the proposition.

To see the second part, use Itô's lemma on $x_t = \sigma^2/\Gamma_t = \sigma^2/(\overline{\nu}_t\overline{\gamma} + (1-\overline{\nu}_t)\underline{\gamma})$:

$$dx_t = -\frac{\sigma^2(\overline{\gamma}-\underline{\gamma})}{(\overline{\nu}_t\overline{\gamma} + (1-\overline{\nu}_t)\underline{\gamma})^2}d\overline{\nu}_t + \frac{\sigma^2(\overline{\gamma}-\underline{\gamma})^2}{(\overline{\nu}_t\overline{\gamma} + (1-\overline{\nu}_t)\underline{\gamma})^3}d\overline{\nu}_t^2 = -\frac{(\overline{\gamma}-\underline{\gamma})x_t^2}{\sigma^2}d\overline{\nu}_t + \frac{(\overline{\gamma}-\underline{\gamma})^2x_t^3}{\sigma^4}d\overline{\nu}_t^2$$

The drift and volatility of $\overline{\nu}_t$ expressed as functions of x_t are

$$\begin{aligned}
\mu_\nu(\overline{\nu}(x_t)) &= \frac{(\sigma^2-\underline{\gamma}x_t)(\overline{\gamma}x_t-\sigma^2)(x_t-\sigma^2)}{(\overline{\gamma}-\underline{\gamma})\sigma^2x_t} \\
\sigma_\mu(\overline{\nu}(x_t)) &= \frac{(\sigma^2-\underline{\gamma}x_t)(\overline{\gamma}x_t-\sigma^2)}{(\overline{\gamma}-\underline{\gamma})\sigma x_t}
\end{aligned}$$

Using this,

$$\begin{aligned}
\mu_x(x_t) &= \frac{(\sigma^2-\underline{\gamma}x_t)^2(\overline{\gamma}x_t-\sigma^2)^2x_t}{\sigma^6} - \frac{(\sigma^2-\underline{\gamma}x_t)(\overline{\gamma}x_t-\sigma^2)(x_t-\sigma^2)x_t}{\sigma^4} \\
&= \frac{(\sigma^2-\underline{\gamma}x_t)(\overline{\gamma}x_t-\sigma^2)x_t}{\sigma^6}((\sigma^2-\underline{\gamma}x_t)(\overline{\gamma}x_t-\sigma^2)-(x_t-\sigma^2)\sigma^2) \\
&= \frac{(\sigma^2-\underline{\gamma}x_t)(\overline{\gamma}x_t-\sigma^2)x_t^2}{\sigma^6}(\sigma^2(\overline{\gamma}+\underline{\gamma}-1)-\overline{\gamma}\underline{\gamma}x_t) \\
\sigma_x(x_t) &= -\frac{(\sigma^2-\underline{\gamma}x_t)(\overline{\gamma}x_t-\sigma^2)x_t}{\sigma^3}
\end{aligned}$$

This proves that dx_t/x_t evolves as in the statement of the proposition. $\square$

Proof of Proposition 6.

The tax changes the process for individual wealth to the following:

$$\frac{d\overline{w}_t}{\overline{w}_t} = \left(\mu - x_t + \overline{\gamma}\frac{x_t^2}{\sigma^2}\right)dt + \left(\frac{1}{2}-\overline{\nu}_t\right)\tau\sqrt{\overline{\nu}_t(1-\overline{\nu}_t)}dt + \overline{\gamma}\frac{x_t}{\sigma}dZ_t$$

There is one additional term in the drift of $\overline{w}_t$, which translates into one additional term in the drift of $\overline{\nu}_t$. Since this term already only depends on $\overline{\nu}_t$, it does not require additional simplification:

$$\begin{aligned}
d\overline{\nu}_t = &\frac{1-\overline{\nu}_t\overline{\gamma}-(1-\overline{\nu}_t)\underline{\gamma}}{(\overline{\nu}_t\overline{\gamma}+(1-\overline{\nu}_t)\underline{\gamma})^2}\cdot\overline{\nu}_t(1-\overline{\nu}_t)\sigma^2dt + \frac{\tau(1-2\overline{\nu}_t)}{2}\sqrt{\overline{\nu}_t(1-\overline{\nu}_t)}dt \\
&+ \frac{\overline{\gamma}-\underline{\gamma}}{\overline{\nu}_t\overline{\gamma}+(1-\overline{\nu}_t)\underline{\gamma}}\cdot\overline{\nu}_t(1-\overline{\nu}_t)\sigma dZ_t
\end{aligned}$$

To see the second part of the result, notice that the relationship between dx_t and $d\overline{\nu}_t$ has not changed with the addition of the tax:

$$dx_t = -\frac{(\overline{\gamma}-\underline{\gamma})x_t^2}{\sigma^2}d\overline{\nu}_t + \frac{(\overline{\gamma}-\underline{\gamma})^2x_t^3}{\sigma^4}d\overline{\nu}_t^2$$

The only thing that changes is $\mu_\nu(\overline{\nu}(x_t))$, which is reflected in

$$\begin{aligned}\mu_\nu(\overline{\nu}(x_t)) &= \frac{(\sigma^2-\underline{\gamma}x_t)(\overline{\gamma}x_t-\sigma^2)(x_t-\sigma^2)}{(\overline{\gamma}-\underline{\gamma})\sigma^2 x_t} + \frac{\tau(1-2\overline{\nu}(x_t))}{2}\sqrt{\overline{\nu}(x_t)(1-\overline{\nu}(x_t))} \\ &= \frac{(\sigma^2-\underline{\gamma}x_t)(\overline{\gamma}x_t-\sigma^2)(x_t-\sigma^2)}{(\overline{\gamma}-\underline{\gamma})\sigma^2 x_t} - \tau(2\sigma^2-(\overline{\gamma}+\underline{\gamma})x_t)\frac{\sqrt{(\overline{\gamma}x_t-\sigma^2)(\sigma^2-\underline{\gamma}x_t)}}{2x_t^2(\overline{\gamma}-\underline{\gamma})^2}\end{aligned}$$

Hence, the drift of x_t changes to

$$\begin{aligned}\mu_x(x_t) = \frac{(\sigma^2-\underline{\gamma}x_t)(\overline{\gamma}x_t-\sigma^2)x_t^2}{\sigma^6}&(\sigma^2(\overline{\gamma}+\underline{\gamma}-1)-\overline{\gamma}\underline{\gamma}x_t) \\ +\tau\sqrt{(\overline{\gamma}x_t-\sigma^2)(\sigma^2-\underline{\gamma}x_t)}&\frac{2\sigma^2-(\overline{\gamma}+\underline{\gamma})x_t}{2\sigma^2}\end{aligned}$$

This completes the proof. $\square$

Proof of Proposition 7.

The recursive representation of a sovereign's value is

$$\rho v(x) = \begin{cases} x-\kappa+\mu x v'(x)+\frac{\sigma^2}{2}x^2 v''(x) & \text{for } x \geq \hat{x} \\ \overline{\mu} x v'(x)+\frac{\sigma^2}{2}x^2 v''(x) & \text{for } x \leq \hat{x} \end{cases} \tag{A.6}$$

Optimality requires that $v(\cdot)$ be continuous and twice differentiable at $\hat{x}$.

The solution on $[\hat{x},\infty)$ has a homogeneous and a non-homogeneous parts:

$$v(x) = \frac{x}{\rho-\mu} - \frac{\kappa}{\rho} + \beta x^\zeta$$

Here ζ is the negative root of the characteristic polynomial for equation (A.6):

$$\zeta = \frac{\sigma^2-2\mu-\sqrt{(2\mu-\sigma^2)^2+8\rho\sigma^2}}{2\sigma^2}$$

On $[0,\hat{x}]$, there is only the homogeneous part:

$$v(x) = \overline{\beta}x^{\overline{\zeta}}$$

The surviving root of the characteristic polynomial on this segment is positive:

$$\overline{\zeta} = \frac{\sigma^2-2\overline{\mu}+\sqrt{(2\overline{\mu}-\sigma^2)^2+8\rho\sigma^2}}{2\sigma^2}$$

There are three smoothness conditions and three unknonws: $\hat{x}$, ζ, and $\overline{\zeta}$.

The smoothness conditions are value matching, smooth pasting, and super contact:

$$\frac{\hat{x}}{\rho-\mu} - \frac{\kappa}{\rho} + \beta\hat{x}^{\zeta} = \overline{\beta}\hat{x}^{\overline{\zeta}} \tag{A.7}$$

$$\frac{1}{\rho-\mu} + \zeta\beta\hat{x}^{\zeta-1} = \overline{\zeta\beta}\hat{x}^{\overline{\zeta}-1} \tag{A.8}$$

$$(\zeta-1)\zeta\beta\hat{x}^{\zeta-2} = (\overline{\zeta}-1)\overline{\zeta\beta}\hat{x}^{\overline{\zeta}-2} \tag{A.9}$$

Multiplying the equation (A.8) by $\hat{x}$, subtracting it from equation (A.7), and using equation (A.9) for elimination,

$$\begin{aligned}\beta\hat{x}^{\zeta} &= \frac{\kappa}{\rho}\cdot\frac{\overline{\zeta}}{(1-\zeta)(\overline{\zeta}-\zeta)}\\ \overline{\beta}\hat{x}^{\overline{\zeta}} &= \frac{\kappa}{\rho}\cdot\frac{\zeta}{(1-\overline{\zeta})(\overline{\zeta}-\zeta)}\end{aligned}$$

Using equation (A.7) again leads to

$$\hat{x} = \kappa\cdot\frac{\rho-\mu}{\rho}\cdot\frac{\overline{\zeta}\zeta}{(\overline{\zeta}-1)(\zeta-1)}$$

For $\hat{x}>0$, it is necessary and sufficient that $\overline{\zeta}>1$, which is achieved when $\rho>\overline{\mu}$.

Since $\hat{x}$ decreases in $\overline{\zeta}$, which itself decreases in $\overline{\mu}$, $\hat{x}$ increases in $\overline{\mu}$. Since $\overline{\mu}>\mu$,

$$\begin{aligned}\frac{\hat{x}}{\kappa} &> \frac{\rho-\mu}{\rho}\cdot\frac{(\sqrt{(2\mu-\sigma^2)^2+8\rho\sigma^2}-2\mu+\sigma^2)(\sqrt{(2\mu-\sigma^2)^2+8\rho\sigma^2}+2\mu-\sigma^2)}{(\sqrt{(2\mu-\sigma^2)^2+8\rho\sigma^2}-2\mu-\sigma^2)(\sqrt{(2\mu-\sigma^2)^2+8\rho\sigma^2}+2\mu+\sigma^2)}\\ &= \frac{\rho-\mu}{\rho}\cdot\frac{8\rho\sigma^2}{(2\mu-\sigma^2)^2+8\rho\sigma^2-(2\mu+\sigma^2)^2} = 1\end{aligned}$$

This proves that $\hat{x}>\kappa$.

Next, the density $g(\cdot)$ corresponding to the invariant distribution of x solves

$$(\mu(x)g(x))' = \frac{\sigma^2}{2}(x^2g(x))''$$

Here $\mu(x) = (\mu+(\overline{\mu}-\mu)\mathbb{1}\{x<\hat{x}\})$. Taking the derivatives,

$$\frac{\sigma^2}{2}g''(x)x^2 + (2\sigma^2-\mu)g'(x)x + (\sigma^2-\mu)g(x) = 0$$

for $x \geq \hat{x}$. On $[0, \hat{x}]$, $g(\cdot)$ solves the same equation with $\overline{\mu}$ instead of μ. The solution is

$$g(x) = \beta x^{\xi} \mathbb{1}\{x \geq \hat{x}\} + \overline{\beta} x^{\overline{\xi}} \mathbb{1}\{x < \hat{x}\}$$

Here ξ and $\overline{\xi}$ are the roots of the characteristic polynomials with the right sign: $\xi < 0$ and $\overline{\xi} > 0$:

$$\begin{aligned} \xi &= \frac{\mu - 2\sigma^2 - \sqrt{(\mu - \sigma^2)^2 + \sigma^4}}{\sigma^2} \\ \overline{\xi} &= \frac{\overline{\mu} - 2\sigma^2 + \sqrt{(\overline{\mu} - \sigma^2)^2 + \sigma^4}}{\sigma^2} \end{aligned}$$

It holds that $\xi < -1$ whenver $\sigma > 0$.

The density must be continuous at $x = \hat{x}$ and it must integrate to one. These conditions read

$$\begin{aligned} \beta \hat{x}^{\xi} &= \overline{\beta} \hat{x}^{\overline{\xi}} \\ \frac{\overline{\beta} \hat{x}^{\overline{\xi}+1}}{\overline{\xi}+1} - \frac{\beta \hat{x}^{\xi+1}}{\xi+1} &= 1 \end{aligned}$$

Combining them,

$$\begin{aligned} \beta &= \frac{(1+\xi)(1+\overline{\xi})}{\xi - \overline{\xi}} \cdot \hat{x}^{-1-\xi} \\ \overline{\beta} &= \frac{(1+\xi)(1+\overline{\xi})}{\xi - \overline{\xi}} \cdot \hat{x}^{-1-\overline{\xi}} \end{aligned}$$

The share $\hat{\Delta}$ of firms in default is the integral of the density up to $\hat{x}$:

$$\hat{\Delta} = \frac{\overline{\beta} \hat{x}^{\overline{\xi}+1}}{\overline{\xi}+1} = \frac{\xi+1}{\xi - \overline{\xi}} = \frac{\sqrt{(\mu - \sigma^2)^2 + \sigma^4} - \mu + \sigma^2}{\sqrt{(\mu - \sigma^2)^2 + \sigma^4} - \mu + \sqrt{(\overline{\mu} - \sigma^2)^2 + \sigma^4} + \overline{\mu}}$$

This completes the proof. $\square$

Proof of Proposition 8.
Integrating equation (15) pre-multiplied by $g(x)$, the invariant density of x,

$$\begin{aligned} r(\gamma) \cdot \int p(x,\gamma) g(x) dx = \kappa \cdot (1 - \hat{\Delta}) &+ \int \left[p_x(x,\gamma)\mu(x) + p_{xx}(x,\gamma)\frac{\sigma^2 x^2}{2} \right] g(x) dx \\ &+ \mu_\gamma(\gamma) \partial_\gamma \left(\int p(x,\gamma) g(x) dx \right) + \frac{\sigma_\gamma(\gamma)^2}{2} \partial_{\gamma\gamma} \left(\int p(x,\gamma) g(x) dx \right) \\ &- \frac{\rho}{\gamma(\kappa - \pi)} \cdot \int \left[(p_x(x,\gamma)\sigma x)^2 + (p_\gamma(x,\gamma)\sigma_\gamma(\gamma))^2 \right] g(x) dx \end{aligned}$$

The fact that the average price equals total wealth, which is to $(\kappa - \pi)/\rho$, means that the derivatives with respect to γ in the second line are zero. Integrating the first line by parts,

$$r(\gamma) \cdot \frac{\kappa - \pi}{\rho} = = \kappa \cdot (1 - \hat{\Delta}) + \int \Big[\underbrace{\frac{1}{2}(\sigma^2 x^2 g(x))'' - (\mu(x) g(x))'}_{=0} \Big] p(x) dx$$

$$- \frac{\rho}{\gamma(\kappa - \pi)} \cdot \int \left[(p_x(x,\gamma)\sigma x)^2 + (p_\gamma(x,\gamma)\sigma_\gamma(\gamma))^2 \right] g(x) dx$$

The fact that the expression under the integral sign is zero follows from the fact that $g(\cdot)$ is an invariant distribution. Using $\pi = \kappa \cdot \hat{\Delta}$ and rearranging again leads to equation (16):

$$r(\gamma) = \rho - \frac{\rho^2}{\gamma(\kappa - \pi)^2} \int \left[(p_x(x,\gamma)\sigma x)^2 + (p_\gamma(x,\gamma)\sigma_\gamma(\gamma))^2 \right] g(x) dx$$

This completes the proof. $\square$