Dr. Andrey Soldatenkov

Steklov Mathematical Institute

Spring 2022

# Introduction to Kähler geometry

## Exercise sheet 7

**Exercise 7.1.** Assume that $D \subset M$ is a smooth hypersurface in a compact Kähler manifold $M$ of complex dimension $n$. Assume that $L = \mathcal{O}_M(D)$ is ample.

1. Construct the following exact triples of coherent sheaves on $M$:

$$0 \to \Omega^p_M \otimes \mathcal{O}_M(-D) \to \Omega^p_M \to \Omega^p_M|_D \to 0,$$

$$0 \to \Omega^{p-1}_D \otimes \mathcal{O}_M(-D)|_D \to \Omega^p_M|_D \to \Omega^p_D \to 0.$$

2. Using the above exact triples, Kodaira vanishing and Serre duality, prove that the natural maps $H^q(M, \Omega^p_M) \to H^q(D, \Omega^p_D)$ are bijective for $p + q + 1 < n$ and injective for $p + q < n$.

3. Deduce the "weak Leschetz theorem": $H^k(M, \mathbb{C}) \to H^k(D, \mathbb{C})$ is bijective for $k \leqslant n - 2$ and injective for $k \leqslant n - 1$.

**Exercise 7.2.** A Hermitian metric $h$ on a holomorphic vector bundle $E$ over a compact Kähler manifold $M$ is called Hermite-Einstein if

$$\sqrt{-1}\Lambda_\omega R_\nabla = \lambda \operatorname{Id}_E$$

for some $\lambda \in \mathbb{R}$. Prove that any holomorphic line bundle over a compact Kähler manifold admits a Hermite-Einstein metric.

In the following exercises $\omega$, $\omega'$ denote Kähler metrics on a compact Kähler manifold $M$ of dimension $n$, and $\rho$, $\rho'$ denote the corresponding Ricci forms.

**Exercise 7.3.** Assume that $\lambda[\omega] = 2\pi c_1(M)$, where $\lambda = \pm 1$. Show that there exists a real-valued function $f$ that satisfies $\rho = \lambda\omega + \sqrt{-1}\partial\overline{\partial} f$. Show that the metric $\omega' = \omega + \sqrt{-1}\partial\overline{\partial} u$ satisfies $\rho' = \lambda\omega'$ if and only if $u$ is a solution of the equation

$$(\omega + \sqrt{-1}\partial\overline{\partial} u)^n = e^{f - \lambda u + C}\omega^n$$

for some constant $C$.

**Exercise 7.4.** Prove that for $\lambda = -1$ the equation from the previous exercise has at most one solution.

**Exercise 7.5.** Consider the operator $\mathcal{MA}_\lambda \colon u \mapsto e^{\lambda u}(\omega + \sqrt{-1}\partial\overline{\partial} u)^n$. Compute its differential at the point $u$ and prove that the differential is surjective if $\omega + \sqrt{-1}\partial\overline{\partial} u$ is positive.