Exercise 36. Consider the ring homomorphism $k[x_1, x_2] \to k[x]$, $x_1 \mapsto x^2$, $x_2 \mapsto x^3$. It is clear that this homomorphism factors through $\phi : A \to k[x]$. Moreover, ϕ is injective — just note that any element of A can be uniquely written as $p(x_1) + x_2 q(x_1)$ where p and q are polynommials and this maps to $p(x^2) + x^3 q(x^2)$, so all the monomials in the first summand are of even degree, in the second summand of odd degree, and we get zero iff $p = q = 0$.

We see that A is isomorphic to the subring $k[x^2, x^3]$ of $k[x]$. These rings clearly have the same field of fractions $k(x)$, and the element $x \in k(x)$ is integral over A, so the integral closure coincides with $k[x]$.

Let us consider the open subsets $U = D_{x^2} \subset \operatorname{Spec}(k[x^2, x^3])$ and $U' = D_x \subset \operatorname{Spec}(k[x])$. The localization of $k[x^2, x^3]$ at powers of x^2 clearly coincides with the localization of $k[x]$ at powers of x, so we see that the mentioned open subsets are isomorphic. The complement of U' is the point $x = 0$, and the complement of U is $V(x^2)$ which is also a point — $k[x^2, x^3]/(x^2) = k \oplus k \cdot x^3 \simeq k[\varepsilon]/(\varepsilon^2)$. These two points are identified by ϕ. So we see that ϕ induces a bijection of spectra (actually, a homeomorphism).

Exercise 37. Let's take an element $x = a + b\sqrt{n}$ with $a, b \in \mathbb{Q}$. The minimal polynomial of x over $\mathbb{Q}$ is $p(X) = X^2 - 2aX + a^2 - nb^2$. If x is integral over $\mathbb{Z}$, the polynomial which gives the integral dependence $q(X)$ has to be divisible by $p(X)$ (in the polynomial ring $\mathbb{Q}[X]$), so $q(X) = p(X)r(X)$. Since both p and q are monic and $q \in \mathbb{Z}[X]$, by Gauss lemma we must have $p \in \mathbb{Z}[X]$. Conclusion: x is integral over $\mathbb{Z}$ iff $2a \in \mathbb{Z}$ and $a^2 - nb^2 \in \mathbb{Z}$.

First consider the case when $a \in \mathbb{Z}$. Then we must have $nb^2 \in \mathbb{Z}$ and since n was square-free we see that $b \in \mathbb{Z}$. We conclude that the subring $\mathbb{Z}[\sqrt{n}]$ is always contained in the integral closure.

Next let $a = \frac{a'}{2}$ with a' odd. It is enough to consider $a = \frac{1}{2}$ because we can add to x elements from $\mathbb{Z}[\sqrt{n}]$. Let $b = \frac{p}{q}$ with p and q coprime, then $1 - 4n\frac{p^2}{q^2} \in 4\mathbb{Z}$ and we see that q must be equal to 2. Then $np^2 = 1 (\text{mod } 4)$ and this is impossible if $n = 2, 3 (\text{mod } 4)$, and will always be the case for odd p when $n = 1 (\text{mod } 4)$. This proves what was claimed.

Exercise 38. Note that we just consider the normalization of a nodal cubic curve $\operatorname{Spec}(A_0)$ where $A_0 := k[x_1, x_2]/(x_2^2 - x_1^2(x_1 + 1))$, and the normalization is given by $A_0 \to k[x]$, $x_1 \mapsto x^2 - 1$, $x_2 \mapsto x(x^2 - 1)$. Then we take the product of everything with the affine line $\operatorname{Spec}(k[y])$, so that $A = A_0 \otimes k[y]$, etc. The resulting map is the normalization of a singular surface.

Consider the prime ideals $I_1 = (x - 1, y)$ and $I_2 = (y - (x + 1))$. The first one is the ideal of a point and this point maps to the point P given by $x_1 = x_2 = y = 0$. Note that there is another point which maps to P — it is given by the ideal $I_1' = (x + 1, y)$. The ideal I_2 defines a line and is contained in I_1' but not in I_1. The image L of the line given by I_2 passes through P. The preimage of L is unique — this is because the normalization map identifies only one pair of lines: $x = 1$ and $x = -1$, and the line defined by I_2 is not one of those.

Now we take $\mathfrak{p}$ to be the ideal of L and $\mathfrak{p}'$ to be the ideal of P, so that $\mathfrak{p} \subset \mathfrak{p}'$. We have lifted $\mathfrak{p}'$ to I_1, but we can not lift $\mathfrak{p}$ so that the lift is contained in I_1, because $\mathfrak{p}$ has unique lift I_2, which is not contained in I_1.

Addendum: why I_2 is the unique ideal which lifts the preimage of I_2 in A. Let's consider the localized rings A_{x_1} and $k[x, y]_{x^2-1}$. Both localizations are non-zero and by the universal property of the localization we get a map $A_{x_1} \to k[x, y]_{x^2-1}$ (note that x_1 is sent exactly into $x^2 - 1$). I claim that these two rings are isomorphic. To see this consider a map in the opposite direction: $k[x, y]_{x^2-1} \to A_{x_1}$ given by $x \mapsto \frac{x_2}{x_1}$. Note that this is well-defined, because $x^2 - 1 \mapsto \frac{x_2^2}{x_1^2} - 1 = x_1$ (recall the definition of A), and x_1 is invertible in A_{x_1}. It is clear that the two maps are inverse to each other.

Next note, that the ideal I_2 extends to a non-trivial ideal in the localization $k[x,y]_{x^2-1}$. This is clear, because $(x^2-1)^n$ does not belong to I_2 for any n, so the multiplicative system does not intersect I_2. If we had a second ideal I_2' which has the same preimage as I_2 in A, then I_2 and I_2' would have the same extension to $k[x,y]_{x^2-1}$, since this ring is isomorhic to A_{x_1}. But this is possible only if $I_2 = I_2'$: recall, that for any ring R, a multiplicative set S in it and a prime ideal $\mathfrak{p}$ which does not meet S, we have $S^{-1}\mathfrak{p} \cap R = \mathfrak{p}$. So $I_2' = (I_2' \cdot k[x,y]_{x^2-1}) \cap k[x,y] = (I_2 \cdot k[x,y]_{x^2-1}) \cap k[x,y] = I_2$.

Exercise 39. Straightforward from the definition.

Exercise 40. i) All the ideals are of the form (m) and the radical of such an ideal is $(p_1 \cdot \ldots \cdot p_k)$ where p_i are distinct prime factors of m. We know from the previous exercise that the radical of a primary ideal must be prime, so there is only one prime factor and $m = p^n$.

ii) It is easy to see that the radical of $\mathfrak{q}$ is (x,y) which is maximal, so $A/\mathfrak{q}$ has unique prime ideal, hence all zero-divisors in $A/\mathfrak{q}$ are nilpotent. If $\mathfrak{q} = \mathfrak{p}^n$ then $\mathfrak{p}$ is the radical of $\mathfrak{q}$, which is (x,y), but $\mathfrak{q}$ is not $(x,y)^n$.

iii) Consider the elements $\bar{x}$ and $\bar{y}$. Their product is in $\mathfrak{p}^2$, $\bar{x}$ is not in $\mathfrak{p}^2$, and no power of $\bar{y}$ is in $\mathfrak{p}^2$.

Exercise 41. Straightforward.