An elementary proof that $\mathbb{P}^1$ is proper

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It is clear that $\mathbb{P}^1$ is separated and of finite type over the base ring. To prove that the structural morphism is universally closed, it is necessary to check that for any ring A the morphism $\mathbb{P}^1_A \to \mathrm{Spec}(A)$ is closed.

Cover $\mathbb{P}^1_A$ by two open subschemes $\mathrm{Spec}(A[t])$ and $\mathrm{Spec}(A[s])$. The open subsets $\mathrm{Spec}(A[t,t^{-1}])$ and $\mathrm{Spec}(A[s,s^{-1}])$ are identified by the map which sends t to s^{-1}. A closed subset $Z \subset \mathbb{P}^1_A$ is given by a pair of ideals $I_t \subset A[t]$ and $I_s \subset A[s]$ which define the same subvariety in $\mathrm{Spec}(A[t,t^{-1}]) \simeq \mathrm{Spec}(A[s,s^{-1}])$.

Pick a prime ideal $\mathfrak{p} \subset A$ which is not contained in the image of Z. We need to show that some neighborhood of $\mathfrak{p}$ is also not in the image of Z. Localizing at $\mathfrak{p}$ we can reduce to the case when $(A, \mathfrak{m})$ is a local ring and $\mathfrak{p} = \mathfrak{m}$. The condition that $\mathfrak{m}$ is not in the image of Z can be rewritten as $\mathfrak{m} \cdot A[t] + I_t = A[t]$ and $\mathfrak{m} \cdot A[s] + I_s = A[s]$. Using the first equality we find an element $m \in \mathfrak{m}$ and two polynomials $p \in A[t]$ and $q \in I_t$, such that $mp + q = 1$. Suppose theat $q = q_0 t^n + \ldots + q_{n-1}t + q_n$. Then the equality $mp + q = 1$ implies that q_n is invertible. Consider the image of q in $A[s,s^{-1}]$ that is $q_0 s^{-n} + \ldots + q_{n-1}s^{-1} + q_n$. By our assumption this polynomial vanishes on the subscheme defined by I_s, so some power of it is contained in I_s. Multiplying by a sufficiently large power of s, we see that I_s contains a polynomial of the form $r = a_0 s^N + \ldots + a_N$ where a_0 is invertible. Then $V(I_s)$ is contained in $V(r)$, but $A \to A[s]/(r)$ is an integral extension, so the morphism $V(r) \to \mathrm{Spec}(A)$ is closed (this follows from the going-up theorem for integral extensions and the lemma below). This implies that the image of $V(I_s)$ is closed in $\mathrm{Spec}(A)$ and does not contain $\mathfrak{m}$ (this actually means that the image is empty and that the ideal I_s became trivial when we localized at $\mathfrak{p}$). Since everything is symmetric in s and t, the same is true about the image of $V(I_t)$. This completes the proof of closedness.

Lemma 0.1. *Let $f\colon \mathrm{Spec}(A) \to \mathrm{Spec}(B)$ be a morphism of affine schemes. Assume that the image of f is closed under specialization. Then the image of f is closed.*

Proof. We may assume that f is dominant, in other words that B is a subring of A. In this case we need to prove that f is surjective. Pick a prime ideal $\mathfrak{p} \subset B$. Localize both rings with respect to the multiplicative system $S = B \backslash \mathfrak{p}$. Then we get an embedding $B_{\mathfrak{p}} \to S^{-1}A$. The ring $S^{-1}A$ is non-trivial, since S does not contain zero, so $S^{-1}A$ contains a prime ideal $\mathfrak{q}$. The preimage of $\mathfrak{q}$ in B will be contained in $\mathfrak{p}$. This preimage also lies in the image of f, and since the image of f is closed under specialization, the ideal $\mathfrak{p}$ is also in the image of f. $\square$

1 Questions for the students

1. Let $\mathcal{F}$ be a sheaf of sets on a topological space. What is $\mathcal{F}(\emptyset)$?
2. Let $X = \mathrm{Spec}(k[x])$ and fix a closed point $p \in X$. Consider the sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules defined by setting $\mathcal{F}(U) = 0$ if $p \in U$ and $\mathcal{F}(U) = \mathcal{O}_X(U)$ otherwise. Is $\mathcal{F}$ quasicoherent? Same question if p is the generic point.
3. Let $X = \mathrm{Spec}(\mathbb{Z})$ and $p \in \mathbb{Z}$ be a prime. What is the fiber $\mathcal{O}_{X,(p)}$ of $\mathcal{O}_X$ at $(p) \in X$? What is the fibre at the generic point $\mathcal{O}_{X,(0)}$?
4. Let $A = k[[t]]$ be the ring of formal power series over a field. Describe the topological space $\mathrm{Spec}(A)$.
5. Let $X = \mathrm{Spec}(\mathbb{Z}[x])$. Describe the fibre product $X \times_{\mathrm{Spec}\mathbb{Z}} \mathbb{F}_p$.
6. Let $X = \mathrm{Spec}(k[x])$, $Y = \mathrm{Spec}(k[y])$ and $f : X \to Y$ be induced by the map $y \mapsto x^2$. Describe the fibre of f over the closed point $y = 0$.
7. Let $X = \mathrm{Spec}(k[x])$. Consider the two projections $p_1, p_2 : X \times X \to X$. Let $a \in X$ be a point, such that $p_1(a) = p_2(a)$. Is it true that a lies in the image of the diagonal morphism $\Delta : X \to X \times X$?
8. Let $X = \mathrm{Spec}(k[x, y]/(xy))$ and $o \in X$ be the point corresponding to the maximal ideal $\mathfrak{m} = (x, y)$. What is the dimension of the tangent space of X at o?
9. Let k be a field, $X = \mathrm{Spec}(k[x])$ and $f : X \to \mathrm{Spec}(k)$ be the natural morphism. Describe $f_*\mathrm{O}_X$? Same question for $X = \mathrm{Proj}(k[x, y])$.
10. Let $X = \mathrm{Spec}(k[x, y]/(xy^2))$. Describe the reduction X_{red}. Describe the normalization of the reduction $\widetilde{X_{red}}$.
11. Let the morphism $f : \mathrm{Spec}(k[x, y, t]/(y^2 - tx(x^2 - 1))) \to \mathrm{Spec}(k[t])$ be induced by the inclusion $k[t] \to k[x, y, t]$. Is the fibre of f over the generic point $(0) \subset k[t]$ reduced? Is it normal?
12. Let the morphism $f : \mathrm{Spec}(\mathbb{C}[x, y]) \to \mathrm{Spec}(\mathbb{C}[z, w])$ be given by $z \mapsto x$, $w \mapsto xy$. Is f an open morphism?
13. Let $\mathcal{I}$ be a non-zero ideal sheaf on an integral scheme X. What is the support of $\mathcal{I}$?
14. Let $I = (x, y) \subset \mathbb{C}[x, y]$ be the ideal and $\widetilde{I}$ the corresponding ideal sheaf on $X = \mathrm{Spec}(\mathbb{C}[x, y])$. Compute the dimensions $\dim(\widetilde{I}_p \otimes \mathbb{C})$ for all closed points $p \in X$.
15. Consider the ideal $I = (x) \subset k[x]$ and the corresponding ideal sheaf $\widetilde{I}$ on $\mathrm{Spec}(k[x])$. Is $\widetilde{I}$ locally free? Same question for $I = (x, y) \subset k[x, y]$.
16. Let $A = \oplus_{i \geqslant 0} A_i$ be a graded algebra over a field $k = A_0$. If A is finite-dimensional as a k-vector space, what is $\mathrm{Proj}(A)$?.
17. If for a graded algebra $A = \oplus_{i \geqslant 0} A_i$ over a field $k = A_0$ we have $\mathrm{Proj}(A) = \emptyset$, is it true that A is a finite-dimensional vector space?
18. Let X and Z be schemes and $i : Z \to X$ a closed immersion. Consider the push-forward $i_*\mathcal{O}_Z$ of the structure sheaf of Z. What is $i_*\mathcal{O}_Z \otimes_{\mathcal{O}_X} i_*\mathcal{O}_Z$?
19. For a graded ring A, is the nilradical a homogeneous ideal?
20. Is the scheme $\mathrm{Spec}(\bar{\mathbb{Q}} \otimes_{\mathbb{Q}} \bar{\mathbb{Q}})$ connected?