Prof. Dr. Daniel Huybrechts
Dr. Andrey Soldatenkov Summer term 2016

Exercises, Algebraic Geometry II – Week 6

Exercise 26. (4 points) *Flatness of projections.*
Consider the projection $\pi\colon \mathbb{A}^2_k \to \mathbb{A}^1_k$, $(x_1, x_2) \mapsto x_1 + x_2$. Decide whether the restriction of π to $X \subset \mathbb{A}^2_k$ is flat or smooth, where X is: i) $X = V(x_1^2 - x_2^2)$; ii) $X = V(x_1^2 + x_2^2 + 2x_1x_2 - x_2 + x_1)$; iii) $X = \mathbb{A}^2_k \setminus V(x_1 - x_2)$; iv) $X = V((x_1 - x_2)(x_1 - 1), (x_1 - x_2)(x_1 + x_2))$.

Exercise 27. (3 points) *Uniqueness of flat extensions.*
Suppose $f\colon X \to \operatorname{Spec}(A)$ is a morphism and $Z_1, Z_2 \subset X$ are two closed subschemes satisfying the following conditions: (i) $f_i := f|_{Z_i}\colon Z_i \to \operatorname{Spec}(A)$, $i = 1, 2$ are flat; (ii) There exists a non-zero divisor $t \in A$ for which $f_1^{-1}(\operatorname{Spec}(A_t)) = f_2^{-1}(\operatorname{Spec}(A_t))$. Show that then $Z_1 = Z_2$. (*Hint*: Reduce to $X = \operatorname{Spec}(B)$ and show that the ideals $\mathfrak{a}_i$, $i = 1, 2$, defining Z_i satisfy $\mathfrak{a}_1 B_t = \mathfrak{a}_2 B_t$.)

Exercise 28. (4 points) *Irreducibility for flat morphisms.*
Describe an example of a morphism $f\colon X \to Y$ of finite type k-schemes such that f is surjective, Y is irreducible, all fibres X_y are irreducible (even geometrically), but X is not irreducible.

Show that if f is in addition flat, then X has to be irreducible as well.

Exercise 29. (2 points) *Flatness of finite morphisms.*
Let $f\colon X \to Y$ be a finite morphism with Y Noetherian. Show that f is flat if and only if $f_*\mathcal{O}_X$ is locally free. If Y is integral this is equivalent to $\dim_{k(y)}(f_*\mathcal{O}_X \otimes k(y)) \equiv \text{const}$.

Exercise 30. (3 points) *Conic bundle.*
Let E be a vector bundle of rank 3 on $\mathbb{P}^n_k$ where k is an algebraically closed field of characteristic zero. Let $\det(E) = \mathcal{O}_{\mathbb{P}^n_k}(d)$, $d \neq 0$. Consider the projectivisation $\pi : \mathbb{P}(E) \to \mathbb{P}^n_k$ and a section $s \in H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)/\mathbb{P}^n_k}(2))$, $s \neq 0$. Let $X \subset \mathbb{P}(E)$ be the zero locus of s (if the restriction of π to X is a flat morphism, then X is called a *conic bundle* over $\mathbb{P}^n_k$). Consider the set $U \subset \mathbb{P}^n_k$ of points $z \in \mathbb{P}^n_k$ such that the fibre of X over z is a smooth curve. Assuming that U is non-empty, prove that the complement of U is a divisor. Determine the degree of this divisor.

Due Monday 30 May, 2016.

Easy test questions. (no points)

1. Let X be a scheme. For which points $x \in X$ is $\operatorname{Spec}(k(x)) \to X$ a flat morphism.
2. Give an example of a quasi-projective variety that is neither projective nor quasi-affine.
3. Describe an example of a birational morphism $f\colon X \to Y$ whose image is neither open nor closed.
4. Write down an example of a field extension $K_1 \subset K_2$ with K_2/K_1 algebraic but $\Omega_{K_2/K_1} \neq 0$.
5. Let A be a k-algebra. Compare $\Omega_{k[x_1,\ldots,x_n]/k}$ with $\Omega_{A[x_1,\ldots,x_n]/A}$.
6. Let $f_1, \ldots, f_r \in k[x_1, \ldots, x_n]$ and $x \in \operatorname{Spec}(k[x_1, \ldots, x_n])$. Where does the Jacobian J_x live?
7. Let X be a scheme and $x \in X$. Compare $\dim_{k(x)} T_{X,x}$ and $\dim \mathcal{O}_{X,x}$.
8. Let X be a scheme over a field k. What is the relation between smoothness of X over k and regularity of $X_{\bar{k}}$?
9. Consider morphisms of schemes $f\colon X \to Y$ and $g\colon Y \to Z$. Is the natural morphism $f^*\Omega_{Y/Z} \to \Omega_{X/Y}$ always injective?
10. Let X be an irreducible scheme of finite type over a field k. Is X smooth over k if $\Omega_{X/k}$ is locally free?
11. Let X be an integral scheme of finite type over a field k and $x \in X$. Compare $\dim_{K(X)} \Omega_{K(X)/k}$ and $\dim_{k(x)}(\Omega_{X/k} \otimes k(x))$.
12. Find an example of a DVR $(A, \mathfrak{m})$ and an A-module M such that $\dim_{Q(A)}(M \otimes_A Q(A)) \neq \dim_{A/\mathfrak{m}}(M \otimes_A (A/\mathfrak{m}))$.
13. Find an example of a non-empty, integral, finite type k-scheme X for which there exists no non-empty open subset $U \subset X$ which is smooth over k.
14. What is the canonical bundle $\omega_{X/k}$ of $X = \mathbb{P}^n \times_k \mathbb{P}^m$?
15. Let $C \subset \mathbb{P}^3_k$ be a smooth intersection of two quadric hypersurfaces. What is the genus of C?
16. Is it true that the blow-up of a smooth variety in arbitrary ideal sheaf is also a smooth variety?
17. Give an example of a morphism of two schemes $f : X \to Y$, such that $\Omega_{X/Y}$ is a non-zero torsion sheaf.
18. Consider the exact sequence $0 \to \mathcal{O}_{\mathbb{P}^1_k}(1) \to E \to \mathcal{O}_{\mathbb{P}^1_k} \to 0$ of vector bundles on $\mathbb{P}^1_k$. Is it true that $E \simeq \mathcal{O}_{\mathbb{P}^1_k} \oplus \mathcal{O}_{\mathbb{P}^1_k}(1)$?
19. Is the normalization $f : \tilde{X} \to X$ of a variety X a flat morhism?
20. Give an example of a vector bundle on $\mathbb{P}^2_k$ that is not a direct sum of line bundles.