Prof. Dr. Daniel Huybrechts SS 2015

Dr. Andrey Soldatenkov

Retry Exam: Commutative Algebra (V3A1, Algebra I)

Solutions can be written in English or German

Exercise A. $(1 + 1 + 2 + 1$ points$)$
Compute the dimensions of the following rings and provide a chain of prime ideals of maximal length in each case (p is a prime number and $\mathbb{Z}_p$ is the localization with respect to $S = \{p^n\}$):
i) $\mathbb{Z}_{(p)}$; ii) $\mathbb{Z}_p$; iii) $\mathbb{Z}_p[X, 1/X]$; iv) $\mathbb{Z}_{(p)}[1/p]$.

Exercise B. $(1 + 2 + 2 + 3$ points$)$
Let $B = k[X, Y]$, k a field, and $A = \{F(X, Y) \in B \mid F(0, 0) = F(1, 1)\}$. Show that $A \subset B$ does not have the going-down property by considering the prime ideals $\mathfrak{p}_2 := (X) \cap A$ and $\mathfrak{p}_1 := (X - 1, Y - 1) \cap A$. Proceed in three steps:
i) Show $\mathfrak{p}_2 \subset \mathfrak{p}_1$;
ii) Show that if $f(X, Y) \in k[X, Y]$ is irreducible with $(f) \cap A = \mathfrak{p}_2$, then $(f) = (X)$;
iii) Use i) and ii) to prove that the going-down property does not hold.
iv) Which of the assumptions of the going-down theorem is not satisfied?

Exercise C. $(2 + 2$ points$)$
i) Let A be a ring and $\mathfrak{a} \subset A$ a finitely generated ideal with $\mathfrak{a}^2 = \mathfrak{a}$. Show that there exists an element $e \in A$ with $\mathfrak{a} = (e)$ and $e^2 = e$ (i.e. $\mathfrak{a}$ is generated by an idempotent).
ii) Let A be a ring such that for all $a \in A$ there exists an $n > 0$ with $a^n = a$. Show that then $\dim(A) = 0$.

Exercise D. $(2 + 1 + 1$ points$)$
Define when a ring is called *normal* and decide which of the following rings are normal:
i) $k[X, Y]/(X^2 + Y^7)$; ii) $\mathbb{Z}[3i]$; iii) $k[X, Y]/(X^3 - Y)$

Exercise E. $(2 + 2$ points$)$
Consider the subring $A \subset k(X, Y)$ of all $h = f/g$ with $h(Y, Y) \in k(Y)$, that is all those h for which $g(Y, Y) \neq 0$ (assuming that f and g have no common factors).
i) Describe a discrete valuation $\nu\colon k(X, Y)^* \to \mathbb{Z}$ such that A is its valuation ring.
ii) Determine a uniformizing parameter.

Exercise F. $(2 + 2$ points$)$
Consider the ring $A := k[x, y, z]/(z(y - x^2), z^2)$, where k is a field.
i) Show that $(y - x^2, z)^2$ is a primary ideal and that (z) is a prime ideal.
ii) Determine a minimal primary decomposition of (0) and decide which of the associated prime ideals are isolated and which ones are embedded.

Exercise G. $(2 + 2 + 2$ points$)$
Let $A := k[X, Y]$. Determine $\mathrm{Supp}(M)$, $\mathrm{Ann}(M)$, and $\mathrm{Ass}(M)$ for each of the following three A-modules:
i) $M = (X - Y)$; ii) $M = A/(X - Y)$; iii) $M = A/(X^2, Y^3)$.

Klausureinsicht (review of corrected exam): Monday September 28, 11.15 – 12.45, seminar room 0.011.