Dr. Andrey Soldatenkov

Steklov Mathematical Institute

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Introduction to Kähler geometry

Exercise sheet 3

Exercise 3.1. Let M be a differentiable manifold and $\nabla\colon TM \to \Lambda^1 M \otimes TM$ a connection. Define its torsion $T_\nabla(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$ for any $X, Y \in TM$.

1. Prove that $T_\nabla \in \Lambda^2 M \otimes TM$.
2. Consider the induced connection on $\Lambda^1 M$, also denoted by ∇, and prove that $T_\nabla = 0$ if and only if $\mathrm{Alt}(\nabla\alpha) = d\alpha$ for any $\alpha \in \Lambda^1 M$. Here $\mathrm{Alt}\colon \Lambda^1 M \otimes \Lambda^1 M \to \Lambda^2 M$ is the exterior product map. A connection satisfying this condition is called torsion-free.

Exercise 3.2. Let g be a Riemannian metric on a manifold M.

1. Using a partition of unity, prove that there exists a connection $\nabla^\circ\colon TM \to \Lambda^1 M \otimes TM$ that preserves g, i.e. $\nabla^\circ g = 0$.
2. Recall that connections on TM form an affine space. Fixing ∇° as the origin, we may identify this space with the space of global sections of the vector bundle $\mathrm{Hom}(TM, \Lambda^1 M \otimes TM)$. Given a section $A \in \mathrm{Hom}(TM, \Lambda^1 M \otimes TM)$, the corresponding connention is $\nabla = \nabla^\circ + A$. Using the isomorphism $TM \simeq \Lambda^1 M$ induced by g, we may view A as a section of $\Lambda^1 M^{\otimes 3}$. Explicitly, it is given by $(X, Y, Z) \mapsto g(A_X Y, Z)$. Prove that ∇ preserves g if and only if $A \in \Lambda^1 M \otimes \Lambda^2 M$.
3. Consider the map $T\colon \nabla \mapsto T_\nabla$. Identifying the space of connections preserving g with the space of sections of $\Lambda^1 M \otimes \Lambda^2 M$ as above, prove that T is the antisymmetrization of the first two arguments. Prove that this antisymmetrization induces an isomorphism $\Lambda^1 M \otimes \Lambda^2 M \xrightarrow{\sim} \Lambda^2 M \otimes \Lambda^1 M$. Deduce that there exists a unique torsion-free connection preserving g, the Levi-Civita connection.

Exercise 3.3. Let M be a differentiable manifold and $\nabla\colon TM \to \Lambda^1 M \otimes TM$ a torsion-free connection.

1. For $\alpha \in \Lambda^1 M$ prove that $d\alpha(X,Y) = (\nabla_X \alpha)(Y) - (\nabla_Y \alpha)(X)$.
2. More generally, for $\alpha \in \Lambda^k M$ prove the following formula:

$$d\alpha(X_0, \ldots, X_k) = \sum_{i=0}^{k} (-1)^i (\nabla_{X_i}\alpha)(X_0, \ldots, \check{X}_i, \ldots, X_k).$$

Exercise 3.4. Let M be a differentiable manifold and $\nabla\colon TM \to \Lambda^1 M \otimes TM$ a torsion-free connection. Let $e_1, \ldots, e_n$ be a local frame in TM and $e_1^*, \ldots, e_n^*$ the dual frame in $\Lambda^1 M$. Prove the following formula for the de Rham differential:

$$d\alpha = \sum_{i=1}^{n} e_i^* \wedge \nabla_{e_i}\alpha$$

for any $\alpha \in \Lambda^k M$.

Exercise 3.5. Let I be an almost-complex structure on a manifold M and ∇ a torsion-free connection on TM. Assume that $\nabla I = 0$. Prove that I is integrable.