Prof. Dr. Daniel Huybrechts
Dr. Andrey Soldatenkov

Summer term 2016

Exercises, Algebraic Geometry II – Week 4

Exercise 15. (3 points) *Regular and smooth.*
Determine whether the following curves are smooth over k.

1. $V(x_0^n + x_1^n + x_2^n) \subset \mathbb{P}^2_k$.
2. $V(x_1^2 + f(x_2)) \subset \mathbb{A}^2_k$, assuming $\mathrm{char}(k) \neq 2$.
3. $V(ax_0^2 + bx_1^2 + cx_2^2) \subset \mathbb{P}^2_k$, assuming $\mathrm{char}(k) \neq 2$.

Exercise 16. (2 points) *Tangent spaces under base change.*
Let X be a scheme of finite type over a field k. Consider a closed point $x \in X$ and a $\bar{k}$-rational point $y \in Y = X_{\bar{k}}$ over x. Show that there exists a natural map between their maximal ideals $\mathfrak{m}_x \to \mathfrak{m}_y$ and an induced $\bar{k}$-linear map

$$T_{Y,y} \to T_{X,x} \otimes_{k(x)} \bar{k}.$$

Is this map always an isomorphism?

Exercise 17. (6 extra points) *Normal bundles of rational curves.*
Show, as a warm-up, that the restriction of $T_{\mathbb{P}^n_k}$ to any line $\mathbb{P}^1_k \subset \mathbb{P}^n_k$ is isomorphic to $\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus n-1}$.
Now, consider a closed immersion $\mathbb{P}^1_k \hookrightarrow \mathbb{P}^3_k$ defined by sections $s_0, \ldots, s_3 \in H^0(\mathbb{P}^1_k, \mathcal{O}(n))$.

1. We know that locally free sheaves on $\mathbb{P}^1_k$ are always isomorphic to sums of invertible sheaves. So, $\mathcal{N}_{\mathbb{P}^1_k/\mathbb{P}^3_k} \cong \mathcal{O}(a) \oplus \mathcal{O}(b)$. Show that $a + b = 4n - 2$.
2. Show that the Euler sequence on $\mathbb{P}^1_k$ and the restriction of the Euler sequence on $\mathbb{P}^3_k$ fit into a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \Omega_{\mathbb{P}^3_k}|_{\mathbb{P}^1_k} \arrow[r] \arrow[d] & \mathcal{O}(-1)^4|_{\mathbb{P}^1_k} \arrow[r] \arrow[d] & \mathcal{O} \arrow[r] \arrow[d, "\cong"] & 0 \\ 0 \arrow[r] & \Omega_{\mathbb{P}^1_k} \arrow[r] & \mathcal{O}(-1)^2 \arrow[r] & \mathcal{O} \arrow[r] & 0. \end{tikzcd}$$

   Here, the first vertical arrow is the canonical map and the second is given by the Jacobi matrix $\left(\frac{\partial s_i}{\partial x_j}\right)$, $i = 0, \ldots, 3$, $j = 0, 1$, (using $\mathcal{O}(1)|_{\mathbb{P}^1_k} \cong \mathcal{O}(n)$).
3. Let now $n = 3$. In this case the image of $\mathbb{P}^1_k$ in $\mathbb{P}^3_k$ is the *rational normal curve.* Compute a and b in this case. (With a little more work one can show that $0 \leq \rho \leq n - 1$ where $\rho = |a - b|/2$ for any n, see the article by Ghione, Sacchiero Manuscripta Math. 33 (1980/81).)

Please turn over

Due Monday 9 May, 2016. Before the lecture.

Exercise 18. (3 points) *Varieties without smooth points.*
Construct a variety X over k without any smooth closed point and with $H^0(X, \mathcal{O}_X) \cong k$.

Exercise 19. (6 points) *Complete intersection.*
Let X be a smooth variety over a field k. A closed subscheme $Y \subset X$ is a complete intersection if it is the (scheme theoretic) intersection $H_1 \cap \ldots \cap H_c$ of c hyperfsurfaces $H_i \subset X$ with $c = \dim(X) - \dim(Y)$.

1. Show that the normal bundle $\mathcal{N}_{Y/X}$ of a complete intersection Y (which by definition is $\mathcal{H}om(\mathcal{I}_Y/\mathcal{I}_Y^2, \mathcal{O}_Y)$) is isomorphic to $(\mathcal{O}(H_1) \oplus \ldots \oplus \mathcal{O}(H_c))|_Y$.
2. Let $X = \mathbb{P}^n_k$ and $Y = H_1 \cap \ldots \cap H_c$ be a smooth complete interesection with $H_i \subset \mathbb{P}^n_k$ hypersurfaces of degree d_i. Show that $\omega_{Y/k} \cong \mathcal{O}(\sum d_i - n - 1)$. (You may for simplicity assume that the H_i themselves are also smooth.)
3. Determine all $(d_1, \ldots, d_{n-2})$ for which there exists a smooth complete intersection $Y :=$ $H_1 \cap \ldots \cap H_{d_{n-2}} \subset \mathbb{P}^n_k$ with trivial canonical bundle. Can you compute $H^1(Y, \mathcal{O}_Y)$ in these cases?

Exercise 20. (4 points) *Isomorphism type of hypersurfaces.*
Let $Y_1, Y_2 \subset \mathbb{P}^n_k$ be smooth hypersurfaces of degrees $d_1 \neq d_2$.

1. Show that Y_1 and Y_2 are not isomorphic under additional assumption that $d_1 \geq n+1$ and $d_2 < n+1$;
2. Show that they are not isomorphic when $d_1 \neq d_2$ in general.