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Summer term 2016

Exercises, Algebraic Geometry II – Week 7

Exercise 31. (3 points) *Unramified morphisms.*
In class we proved that a morphism locally of finite type $f\colon X \to Y$ between locally Noetherian schemes is unramified if and only if $\Omega_{X/Y} = 0$. Prove that this is also equivalent to the diagonal morphism $\Delta\colon X \to X \times_Y X$ being an open immersion.

Exercise 32. (5 points) *Étale morphisms.*
Decide which of the following morphisms are étale or at least unramified.

1. $\mathbb{A}^1_k \setminus \{0\} \to \mathbb{A}^1_k \setminus \{0\}$, $t \mapsto t^2$.
2. $\mathbb{A}^2_k \to \mathbb{A}^2_k$, $(x, y) \mapsto (x, xy)$
3. $\mathbb{P}^n_k \to \mathbb{P}^n_k$, $[x_0 : \cdots : x_n] \mapsto [x_0^\ell : \cdots : x_n^\ell]$.
4. $\mathrm{Spec}(\mathcal{O}_{\mathbb{Q}(\sqrt{5})}) \to \mathrm{Spec}(\mathcal{O}_{\mathbb{Q}})$.
5. $\mathrm{Spec}(k[t]) \to \mathrm{Spec}(k[x, y]/(x^3 - y^2))$ given by $x \mapsto t^2$, $y \mapsto t^3$.

Exercise 33. (2 points) *Hurwitz formula.*
Let $f\colon X \to Y$ be a finite separable morphism of smooth complete curves of degree d. Show that $g(X) \geq g(Y)$ and that equality is only possible if $g(Y) = 0, 1$ or $d = 1$.

Study the morphism $f\colon \mathbb{P}^1_k \to \mathbb{P}^1_k$ which on $\mathbb{A}^1_k$ is given by $x \mapsto x^{2p} - x$, where $p = \mathrm{char}(k)$. Determine the ramification divisor and, in particular, decide whether the morphism is tamely ramified.

Exercise 34. (3 points) *Canonical bundle of projective bundles.*
Use the relative Euler sequence (see Exercise 11) to compute the canonical bundle $\omega_{\mathbb{P}(\mathcal{E})}$ of a projective bundle $X = \mathbb{P}(\mathcal{E}) \to Y$, where $\mathcal{E}$ is a locally free sheaf on Y. Can $\omega_{\mathbb{P}(\mathcal{E})}$ ever be ample? What about its dual? Compute $\omega_{\mathbb{P}(\mathcal{T}_{\mathbb{P}^n})}$.

Exercise 35. (3 points) *Ramification divisor of the blow-up.*
Let $f\colon X \to \mathbb{A}^n_k$ be the blow-up of the origin with exceptional divisor $E = f^{-1}(0)$. Show that the ramification divisor R_f is $(n - 1)E$.

Let now Y be smooth and of finite type over a field k and let $f\colon X \to Y$ be the blow-up at a (k-rational) closed point $y \in Y$. Express ω_X in terms of ω_Y and the exceptional divisor $E = f^{-1}(y)$.

Due Monday 6 June, 2016.