Second Exam, Algebraic Geometry I — Solutions

Problem A. (4 points) Let $X \subset \mathbb{P}^3_k$ be a hypersurface defined by a homogeneous cubic polynomial $F \in k[x_0, x_1, x_2, x_3]$ with k a field. Compute the *Hilbert polynomial* $P(n) := \chi(X, \mathcal{O}_X(n))$.

Solution. Using the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3_k}(n-3) \to \mathcal{O}_{\mathbb{P}^3_k}(n) \to \mathcal{O}_X(n) \to 0,$$

we find that $\chi(X, \mathcal{O}_X(n)) = \chi(\mathbb{P}^3_k, \mathcal{O}_X(n)) = \chi(\mathbb{P}^3_k, \mathcal{O}_{\mathbb{P}^3}(n)) - \chi(\mathbb{P}^3_k, \mathcal{O}_{\mathbb{P}^3_k}(n-3))$.

Let us compute $\chi(\mathbb{P}^3_k, \mathcal{O}_{\mathbb{P}^3}(n))$. For $n \geq 0$ we have $\chi(\mathbb{P}^3_k, \mathcal{O}_{\mathbb{P}^3}(n)) = \dim H^0(X, \mathcal{O}_{\mathbb{P}^3}(n)) = \binom{n+3}{3} = (n+3)(n+2)(n+1)/3$. For $n = -1$, -2 and -3 all cohomology groups of $\mathcal{O}_{\mathbb{P}^3_k}(n)$ vanish, so $\chi(\mathbb{P}^3_k, \mathcal{O}_{\mathbb{P}^3}(n)) = 0$. For $n \leq -4$ we have $\chi(\mathbb{P}^3_k, \mathcal{O}_{\mathbb{P}^3}(n)) = -\dim H^3(X, \mathcal{O}_{\mathbb{P}^3}(n)) = -\binom{-n-4}{3} = -(-n-1)(-n-2)(-n-3)/3 = (n+3)(n+2)(n+1)/3$. We conclude that $\chi(\mathbb{P}^3_k, \mathcal{O}_{\mathbb{P}^3}(n)) = \binom{n+3}{3}$ for all n.

Analogously, $\chi(\mathbb{P}^3_k, \mathcal{O}_{\mathbb{P}^3}(n-3)) = \binom{n}{3}$ for all n. We find that $\chi(X, \mathcal{O}_X(n)) = \binom{n+3}{3} - \binom{n}{3} = \frac{3}{2}n^2 + \frac{3}{2}n + 1$.

Problem B. (3 points) Consider the standard open set $\mathrm{Spec}(k[x]) = D_+(x_0) \subset \mathbb{P}^1_k = \mathrm{Proj}(k[x_0, x_1])$, where $x = x_1/x_0$. Let $\mathcal{F} = \tilde{M}$ be the coherent sheaf on $D_+(x_0)$ corresponding to the $k[x]$-module $M = k[x]/(x^2 - 1)$. Describe a coherent extension of $\mathcal{F}$ to $\mathbb{P}^1_k$, i.e. a coherent sheaf $\mathcal{G}$ on $\mathbb{P}^1_k$ with $\mathcal{G}|_{D_+(x_0)} \cong \mathcal{F}$, in terms of a $k[x_0, x_1]$-module. Is this extension unique?

Solution. Consider the graded $k[x_0, x_1]$-module $N = k[x_0, x_1]/(x_1^2 - x_0^2)$. By definition of $\tilde{N}$, its restriction to $D_+(x_0)$ is the zero degree component of the localizatoin N_{x_0}. It is isomorphic to M, hence $\tilde{N}$ is an extension of $\tilde{M}$. The extension is not unique: one can consider for example $N' = N \oplus (k[x_0, x_1]/(x_0))$. Then $N'_{x_0} = N_{x_0}$, so $\tilde{N}'$ also is an extension. The module $k[x_0, x_1]/(x_0)$ defines a skyscraper sheaf supported at the point $x_0 = 0$.

Problem C. (3 points) Let Z be a scheme over an algebraically closed field k. Let $Y = \mathbb{P}^1_k \times_k Z$ and $X \subset Y$ be a closed subscheme. Assume that X does not contain any closed fibre of the projection $Y \to Z$. Denote by $\pi : X \to Z$ the restriction to X of this projection. Prove that π is an affine morphism. (Extra 2 points for proving that π is a finite morphism.)

Solution. i) Let $z \in Z$ be a closed point. We have to find an open affine subscheme $U \subset Z$, such that $\pi^{-1}(U)$ is affine. Denote by $Y_z \simeq \mathbb{P}^1_k$ the fibre of Y over z. Since X does not contain Y_z, we can find a closed point $p \in \mathbb{P}^1_k$, such that $(\{p\} \times_k Z) \cap X$ is disjoint from Y_z. Then $W = \pi((\{p\} \times_k Z) \cap X)$ is a closed subset of Z that does not contain z. Let U be an affine open neighbourhood of z in $Z \backslash W$. By construction, $\pi^{-1}(U)$ is contained in $(\mathbb{P}^1_k \times_k U) \backslash (\{p\} \times_k U) \simeq \mathbb{A}^1_k \times_k U$. So $\pi^{-1}(U)$ is affine. Since z was arbitrary closed point of Z, any point of Z is contained in an open subset as above, so π is an affine morphism.

ii) We consider a closed point $z \in Z$ and its affine open neighborhood $U \subset Z$ as in i). Assume $U = \mathrm{Spec}(A)$. Then $X_U = \pi^{-1}(U)$ is contained in $\mathbb{A}^1_k \times_k U = \mathrm{Spec}(A[t])$. Let $I \subset A[t]$ be the ideal defining X_U and $\mathfrak{m} \subset A$ be the ideal corresponding to z. The fibre $X_z = \pi^{-1}(z)$ is defined by the image of I in $(A/\mathfrak{m})[t] \simeq k[t]$. We can assume that the fibre X_z does not contain the point $t = 0$ (this can be achieved by a linear change of coordinate $t' = t + a$ if necessary). This means that I contains a polynomial $f = a_0 + a_1 t + \ldots + a_n t^n$ with $a_0 \notin \mathfrak{m}$. Pass to the opposite affine chart on $\mathbb{P}^1_k$ by setting $t = 1/s$. In that affine chart the ideal defining X contains some power of f which is of the form $a_0^m s^{mn} + \ldots + a_n^m$. In the localization A_{a_0} the leading coefficient of this polynomial is invertible, hence the algebra $A_{a_0}[s]/I$ is finite over A_{a_0}. But $\operatorname{Spec}(A_{a_0})$ is an open neighborhood of z because $a_0 \notin \mathfrak{m}$. So we have found an open neighborhood of z over which X is finite. This proves the claim.

Problem D. (4 points) Let X be a projective regular curve of genus one over an algebraically closed field k and let $x_1, x_2 \in X$ be two closed points. Show that $H^1(X, \mathcal{O}(x_1 - x_2)) \neq 0$ if and only if $x_1 = x_2$.

Solution. By Riemann-Roch

$$\dim H^0(X, \mathcal{O}(x_1 - x_2)) - \dim H^1(X, \mathcal{O}(x_1 - x_2)) = \deg(\mathcal{O}(x_1 - x_2)) = 0.$$

We see that $H^1(X, \mathcal{O}(x_1 - x_2)) \neq 0$ if and only if $H^0(X, \mathcal{O}(x_1 - x_2)) \neq 0$. If $x_1 = x_2$, then $\mathcal{O}(x_1 - x_2) = \mathcal{O}$, so $H^1(X, \mathcal{O}(x_1 - x_2)) = H^0(X, \mathcal{O}(x_1 - x_2)) = k$. Conversely, suppose these exists a section s of $\mathcal{O}(x_1 - x_2)$. Then the vanishing locus of s is empty, since it is an effective divisor of degree zero. Hence $\mathcal{O}(x_1 - x_2) \simeq \mathcal{O}$ and $x_1 = x_2$.

Problem E. (4 points) Let k be an algebraically closed field. Describe (as the zero locus of a section of a line bundle) the scheme $X \subset \mathbb{P}^1_k \times \mathbb{P}^1_k \times \mathbb{P}^1_k$ for which the fibre of the first projection $X \to \mathbb{P}^1_k$ over a closed point $[t_0 : t_1]$ is the curve $X_{[t_0:t_1]} \subset \mathbb{P}^1_k \times \mathbb{P}^1_k$ described by $x_0^2 y_1 t_1 = (x_0^2 + x_1^2) y_0 t_0$ (where x_0, x_1 and y_0, y_1 are the coordinates on the two factors). Find closed points for which the fibre is irreducible, non-reduced, and reducible, respectively.

Solution. The polynomial defining X has degree two in x_i, one in y_i and one in t_i. So the line bundle is $p_1^*\mathcal{O}_{\mathbb{P}^1_k}(2) \otimes p_2^*\mathcal{O}_{\mathbb{P}^1_k}(1) \otimes p_3^*\mathcal{O}_{\mathbb{P}^1_k}(1)$, where $p_i : \mathbb{P}^1_k \times \mathbb{P}^1_k \times \mathbb{P}^1_k \to \mathbb{P}^1_k$ denotes the projection to the i-th factor.

Consider the fibre over the point $[0 : 1]$. It is given by the equation $x_0^2 y_1 = 0$. This defines a union of two lines, one of them with non-reduced structure. Hence the fibre is non-reduced and reducible.

Consider the fibre over the point $[1 : 1]$. It has an equation $x_0^2 y_1 = (x_0^2 + x_1^2) y_0$. Passing to the affine chart $x_0 \neq 0$, $y_0 \neq 0$, we get a curve $y = 1 + x^2$, where $x = x_1/x_0$, $y = y_1/y_0$. This curve is irreducible, since it is isomorphic to $\mathbb{A}^1_k$. The complement to the affine chart consists of two lines $x_0 = 0$ and $y_0 = 0$. They are not contained in the fibre, so the fibre is irreducible.

Problem F. (4 points) Let X be an arbitrary integral scheme. Decide which of the following sheaves are quasi-coherent: i) $\mathcal{O}_X^*$; ii) $\mathcal{K}_X$; iii) The sheaf $i_*(\mathcal{O}_{X,x})$ for $x \in X$ a point and $i : \{x\} \to X$ the inclusion.

Solution. i) This is not a sheaf of $\mathcal{O}_X$-modules, hence not quasi-coherent.

ii) This sheaf is the push-forward of the structure sheaf of the generic point, hence quasi-coherent.

iii) In case when x is the generic point this is the same as ii). In all other cases the sheaf is not quasi-coherent. To see this, consider an affine chart $U = \operatorname{Spec}(A)$ containing the point x. Let $\mathfrak{p}$ be the corresponding ideal. Sections of $i_*(\mathcal{O}_{X,x})$ over U form an A-module $M = A_{\mathfrak{p}}$. If the sheaf was quasi coherent, its restriction to U would be $\tilde{M}$. But the module M is supported at the generic point: $M_{(0)} \simeq A_{(0)} \neq 0$. This is a contradiction, because the support of $i_*(\mathcal{O}_{X,x})$ does not contain the generic point (restriction of this sheaf to the complement of the closure of x is trivial).

Problem G. (3 points) Determine the base locus of the linear system $\{t_0(x_0^4 + x_1^4 + x_2^4 + x_3^4) + t_1 x_0 x_1 x_2 x_3\} \subset |\mathcal{O}(4)|$ on $\mathbb{P}^3_k$ where $\mathrm{char}(k) \neq 2$. Prove that the irreducible components of the base locus are curves, find their genera.

Solution. The base locus is the intersection of two surfaces $x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0$ and $x_0x_1x_2x_3 = 0$. The last one is the union of four planes $x_i = 0$. The intersection of the plane $x_i = 0$ with the surface $x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0$ is a curve of degree 4 in $\mathbb{P}^2_k$. Let us prove that these curves are irreducible. Consider the curve in the plane $x_0 = 0$, it is given by the polynomial $p = x_1^4 + x_2^4 + x_3^4$. If p was reducible, its derivatives $\partial_{x_i} p = 4x_i^3$, $i = 1, 2, 3$ would all vanish at some point on the curve. Since this is not the case, the curve is irreducible. Analogously for the other three curves.

We conclude that the irreducible components of the base locus are 4 plane curves of degree 4, they are of genus 3.