Solutions of the retry exam problems (V3A1, Algebra I)

Exercise A. i) We have $\operatorname{Spec}\mathbb{Z}_{(p)} = \{\mathfrak{q} \in \operatorname{Spec}\mathbb{Z} \colon \mathfrak{q} \subset (p)\} = \{(0), (p)\}$, so the dimension $\dim \mathbb{Z}_{(p)} = 1$, and the chain is $(0) \subset (p)$.

ii) $\operatorname{Spec}\mathbb{Z}_p = \{\mathfrak{q} \in \operatorname{Spec}\mathbb{Z} \colon p \notin \mathfrak{q}\}$. The only inclusions between prime ideals in $\mathbb{Z}$ are $(0) \subset (q)$, and for $q \neq p$ this gives a chain of maximal length in $\mathbb{Z}_p$, so $\dim \mathbb{Z} = 1$.

iii) Consider the morphism $\mathbb{Z}_p[T] \to \mathbb{Z}_p[X, 1/X]$, $T \mapsto X + 1/X$. This is an integral ring extension, since the generators X and $1/X$ are integral over $\mathbb{Z}_p[T]$: $X^2 - TX + 1 = 0$, $(1/X)^2 - T(1/X) + 1 = 0$. Hence the dimension is the same as for $\mathbb{Z}_p[T]$, which is $\dim \mathbb{Z}_p[T] = \dim \mathbb{Z}_p + 1 = 2$, because $\mathbb{Z}_p$ is Noetherian. The maximal chain: $(0) \subset (X-1) \subset (X-1, l)$ for a prime $l \neq p$.

iv) $\mathbb{Z}_{(p)}[1/p] = \mathbb{Q}$, so the dimension $\dim \mathbb{Q} = 0$, the only prime ideal is (0).

Exercise B. i) Note that $\mathfrak{p}_1 = \{F \in A \colon F(1,1) = 0\}$. Let $F \in \mathfrak{p}_2$, then $F = X \cdot G(X,Y)$, so $F(1,1) = F(0,0) = 0$ and $F \in \mathfrak{p}_1$.

ii) We have $X(X-1), X(Y-1) \in \mathfrak{p}_2$. If $(f) \cap A = \mathfrak{p}_2$, then $X(X-1) = f \cdot g_1$, $X(Y-1) = f \cdot g_2$ and since f is irreducible, $f = X$ (up to a constant).

iii) Let $\mathfrak{q}_1 = (X-1, Y-1)$. Suppose we have an ideal $\mathfrak{q}_2 \subset \mathfrak{q}_1$ with $\mathfrak{q}_2 \cap A = \mathfrak{p}_2$. Then $X(X-1) \in \mathfrak{q}_2$ and $X(Y-1) \in \mathfrak{q}_2$ and necessarily $X \in \mathfrak{q}_2$: otherwise we would have $X-1, Y-1 \in \mathfrak{q}_2$, hence $X - Y \in \mathfrak{q}_2 \cap A = \mathfrak{p}_2$ which is false. But $X \notin \mathfrak{q}_1$, so this is a contradiction.

iv) The going-down theorem states that if $A \subset B$ is an integral extension of domains, where A is a normal, then for any pair of prime ideals $\mathfrak{p}_2 \subset \mathfrak{p}_1 \subset A$ and a prime ideal $\mathfrak{q}_1 \subset B$ with $\mathfrak{q}_1 \cap A = \mathfrak{p}_1$ there exists a prime ideal $\mathfrak{q}_2 \subset \mathfrak{q}_1$ with $\mathfrak{q}_2 \cap A = \mathfrak{p}_2$.

In our case A is not normal: the element X lies in the quotient field of A, $X = \frac{X^2(X-1)}{X(X-1)}$, and satisfies the equation $t^2 - t - (X-1)X = 0$, but $X \notin A$.

Exercise C. i) By Nakayama's lemma there exists an element $e \in \mathfrak{a}$, such that $(1-e)\mathfrak{a} = 0$. In particular, $(1-e)e = 0$, so e is idempotent, and for any $a \in \mathfrak{a}$ we have $(1-e)a = 0$, so $a \in (e)$.

ii) For any prime ideal $\mathfrak{p} \in A$ the elements of the quotient $A/\mathfrak{p}$ satisfy the same identity: for $b \in A/\mathfrak{p}$ there exist $n > 0$ such that $b(b^{n-1} - 1) = 0$. $A/\mathfrak{p}$ has no zero divisors, so b is either zero or invertible, so $A/\mathfrak{p}$ is a field and $\mathfrak{p}$ is maximal. A ring for which any prime ideal is maximal has dimension zero.

Exercise D. A ring is called normal if it is an integral domain which is integrally closed in its field of fractions.

i) Not normal. The element X/Y satisfies the equation $t^2 - Y^5 = 0$, but $X/Y \notin A$, because otherwise $X = Y \cdot G(X,Y)$ which is impossible.

ii) Not normal. The element $i = 3i/3$ lies in the field of fractions and satisfies $t^2 + 1 = 0$, but $i \notin A$.

iii) Normal. The ring is isomorphic to $k[X]$.

Exercise E. i) Since $k[X,Y]$ is UFD, any element $h \in k(X,Y)$ can be uniquely represented in the form $h = (X-Y)^n f/g$ with f and g not divisible by $(X-Y)$. Define $\nu(h) = n$. Let's check that this is a discrete valuation. First, $\nu(h_1 h_2) = \nu(h_1) + \nu(h_2)$ is obvious. Second, for $h_1 = (X-Y)^{n_1} f_1/g_1$ and $h_2 = (X-Y)^{n_2} f_2/g_2$ we have $h_1 + h_2 = (X-Y)^{\min(n_1,n_2)} f_3/g_3$ with $f_3, g_3 \in k[X,Y]$, and g_3 not divisible by $(X-Y)$, so $\nu(h_1 + h_2) \geq \min(n_1, n_2) = \min(\nu(h_1), \nu(h_2))$. Finally, the map ν is surjective onto $\mathbb{Z}$, since $\nu((X-Y)^n) = n$. Clearly the ring A is the valuation ring for ν.

ii) Any element h with $\nu(h) = 1$ is uniformizing, for example $h = X - Y$.

Exercise F. i) To simplify things, define $y' = y - x^2$. Then $A = k[x, y', z]/(zy', z^2)$, and $\mathfrak{a}_1 = (y - x^2, z)^2 = (y', z)^2 = (y'^2, y'z, z^2)$. Then any $f \in A/\mathfrak{a}_1$ can be written in the form $f = p(x) + q(x)y' + r(x)z$. It is clear that if f is a zero-divisor, then $p(x) = 0$. In this case $f^2 = 0$, so any zero divisor in $A/\mathfrak{a}_1$ is nilpotent and $\mathfrak{a}_1$ is primary.
The ideal (z) is prime because $A/(z) = k[x, y]$ is a domain.

ii) Let us show that $(0) = (y', z)^2 \cap (z)$ in A. It is clearly enough to prove, that in $k[x, y', z]$ we have $(y', z)^2 \cap (z) = (zy', z^2)$. Any $f \in (y', z)^2$ is of the form $f = p(x, y', z)y'^2 + q(x, y', z)zy' + r(x, y', z)z^2$ and such f is also in (z) if and only if $p = 0$. This proves the claim.
The prime ideals corresponding to the primary ideals of the decomposition are: $\mathfrak{p}_1 = (y', z)$ and $\mathfrak{p}_2 = (z)$. We have $\mathfrak{p}_2 \subset \mathfrak{p}_1$, so $\mathfrak{p}_1$ is embedded and $\mathfrak{p}_2$ is isolated.

Exercise G. i) Since M is a submodule of A, and the ring A has no zero divisors, the elements of M have trivial annihilators, so localization $M_{\mathfrak{p}}$ at any prime ideal $\mathfrak{p} \subset M$ is non-zero. Hence $\mathrm{Ann}(M) = (0)$, $\mathrm{Supp}(M) = \mathrm{Spec}\, A$. The associated prime ideals are exactly the minimal elements of the support, so $\mathrm{Ass}(M) = \{(0)\}$.

ii) For an ideal $I \subset A$ the module A/I is supported at $\mathfrak{p} \in \mathrm{Spec}\, A$ if and only if $(A/I)_{\mathfrak{p}} = A_{\mathfrak{p}}/IA_{\mathfrak{p}} \neq 0$, that is when $IA_{\mathfrak{p}} \neq A_{\mathfrak{p}}$. This is the case exactly when $I \subset \mathfrak{p}$. In our case $I = (X - Y)$, so $\mathrm{Supp}(M) = V(I) = \mathrm{Spec}\, A/(X - Y)$. Trivially $\mathrm{Ann}(A/I) = I$, so $\mathrm{Ann}(M) = (X - Y)$. The ideal $(X - Y)$ is prime, so it is the only minimal prime of $\mathrm{Supp}(M)$, hence $\mathrm{Ass}(M) = \{(X - Y)\}$.

iii) As in the previous case, $M = A/I$ with $I = (X^2, Y^3)$, $\mathrm{Supp}(M) = V(I) = V(\sqrt{I})$. But $\sqrt{I} = (X, Y)$ is maximal, so $\mathrm{Supp}(M) = \{(X, Y)\}$. The annihilator $\mathrm{Ann}(M) = (X^2, Y^3)$ and $\mathrm{Ass}(X) = \{(X, Y)\}$.