Dr. Andrey Soldatenkov

Steklov Mathematical Institute

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Introduction to Kähler geometry

Exercise sheet 6

In all exercises below we work in the category of $\mathbb{Q}$-Hodge structures, i.e. all Hodge structures are assumed rational.

Exercise 6.1. Prove that there exists a unique Hodge structure of weight 2 whose underlying vector space is one-dimensional. Denote this Hodge structure by $\mathbb{Q}(-1)$ and let $\mathbb{Q}(-k) = \mathbb{Q}(-1)^{\otimes k}$, $\mathbb{Q}(k) = (\mathbb{Q}(-1)^*)^{\otimes k}$ for $k \geqslant 0$. Find Hodge numbers of $\mathbb{Q}(k)$ for all k.

Exercise 6.2. Let H be a Hodge structure of weight k. Let $Q\colon H \otimes H \to \mathbb{Q}(-k)$ be a morphism of Hodge structures. It is called a polarization if:

1. For k even Q is symmetric and for k odd Q is skew-symmetric;
2. For any non-zero $v \in H^{p,q}$ we have $\sqrt{-1}^{p-q} Q(v, \bar{v}) > 0$.

A Hodge stucture is called polarizable if it admits a polarization. A Hodge structure is called simple if it contains no non-trivial sub-Hodge structures. Prove that a polarizable Hodge structure is a direct sum of simple Hodge structures.

Exercise 6.3. Construct an example of a Hodge structure that is neither simple nor a direct sum of two proper sub-Hodge structures.

Exercise 6.4. Prove that Hodge structures on the cohomology groups of complex projective manifolds are polarizable.

Exercise 6.5. Let $\mathcal{K} \subset H^{1,1}(M)$ be the Kähler cone of a compact Kähler manifold M. Prove that a line bundle $L \in \mathrm{Pic}(M)$ is positive (i.e. admits a Hermitian metric whose curvature form is positive) if and only if $c_1(L) \in \mathcal{K}$.

Exercise 6.6. Find an example of a compact Kähler manifold and a complex line bundle that does not admit a holomorphic structure.

Exercise 6.7. Let E be a holomorphic vector bundle over a compact Kähler manifold M. Consider an arbitrary Hermitian metric on E and construct an analogue of the Hodge $*$-operator for E-valued differential forms

$$*\colon \Lambda^{p,q} M \otimes E \to \Lambda^{n-q,n-p} M \otimes \overline{E}^*.$$

Construct a natural pairing between $\Lambda^{p,q} M \otimes E$ and $\Lambda^{n-p,n-q} M \otimes E^*$ and use the $*$-operator to prove that the pairing is non-degenerate. Deduce the Serre duality isomorphism: $H^q(M, \Omega^p_M \otimes E) \simeq H^{n-q}(M, \Omega^{n-p}_M \otimes E^*)^*$.