Questions for Christmas

1. Let $\mathscr{F}$ be a sheaf of sets on a topological space. What is $\mathscr{F}(\varnothing)$?

2. Let $X = \mathrm{Spec}(k[x])$ and fix a closed point $p \in X$. Consider the sheaf $\mathscr{F}$ of $\mathscr{O}_X$-modules defined by setting $\mathscr{F}(U) = 0$ if $p \in U$ and $\mathscr{F}(U) = \mathscr{O}_X(U)$ otherwise. Is $\mathscr{F}$ quasicoherent? Same question if p is the generic point.

3. Let X be the complement to a closed point in $\mathrm{Spec}(\mathbb{C}[x, y])$. Is X affine?

4. Let $X = \mathrm{Spec}(\mathbb{Z})$ and $p \in \mathbb{Z}$ be a prime. What is the fibre $\mathscr{O}_{X,(p)}$ of $\mathscr{O}_X$ at $(p) \in X$? What is the fibre $\mathscr{O}_{X,(0)}$ at the generic point?

5. Let $A = k[[t]]$ be the ring of formal power series over a field. Describe the topological space $\mathrm{Spec}(A)$.

6. Let $X = \mathrm{Spec}(\mathbb{Z}[x])$. Describe the fibre product $X \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(\mathbb{F}_p)$.

7. Let $X = \mathrm{Spec}(k[x])$, $Y = \mathrm{Spec}(k[y])$ and $f : X \to Y$ be induced by the map $y \mapsto x^2$. Describe the fibre of f over the closed point $y = 0$.

8. Let $X = \mathrm{Spec}(k[x])$. Consider the two projections $p_1, p_2 : X \times X \to X$ (the product is over $\mathrm{Spec}(k)$). Let $a \in X \times X$ be a point, such that $p_1(a) = p_2(a)$. Is it true that a lies in the image of the diagonal morphism $\Delta : X \to X \times X$?

9. Let $X = \mathrm{Spec}(k[x, y]/(xy))$ and $o \in X$ be the point corresponding to the maximal ideal $\mathfrak{m} = (x, y)$. What is the dimension of the tangent space of X at o?

10. Let k be a field, $X = \mathrm{Spec}(k[x])$ and $f : X \to \mathrm{Spec}(k)$ be the natural morphism. Describe $f_*\mathscr{O}_X$. Same question for $X = \mathrm{Proj}(k[x, y])$.

11. Let $X = \mathrm{Spec}(k[x, y]/(xy^2))$. Describe the reduction X_{red}. Describe the normalization of the reduction $\widetilde{X_{red}}$.

12. Let the morphism $f : \mathrm{Spec}(k[x, y, t]/(y^2 - tx(x^2 - 1))) \to \mathrm{Spec}(k[t])$ be induced by the inclusion $k[t] \to k[x, y, t]$. Is the fibre of f over the generic point $(0) \subset k[t]$ reduced? Is it normal?

13. Let the morphism $f : \mathrm{Spec}(\mathbb{C}[x, y]) \to \mathrm{Spec}(\mathbb{C}[z, w])$ be given by $z \mapsto x$, $w \mapsto xy$. Is f an open morphism?

14. Let $\mathscr{I}$ be a non-zero ideal sheaf on an integral scheme X. What is the support of $\mathscr{I}$?

15. Let $I = (x, y) \subset \mathbb{C}[x, y]$ be the ideal and $\widetilde{I}$ the corresponding ideal sheaf on $X = \mathrm{Spec}(\mathbb{C}[x, y])$. Compute the dimensions $\dim_{k(p)}(\widetilde{I}_p \otimes_{\mathscr{O}_{X,p}} k(p))$ for all points $p \in X$.

16. Let X be a Noetherian scheme and $\mathscr{F}$ a coherent sheaf on it. Is it true that the subset
$$X_n = \{p \in X \mid \dim_{k(p)}(\mathscr{F}_p \otimes_{\mathscr{O}_{X,p}} k(p)) = n\} \subset X$$
for $n \in \mathbb{Z}$ is locally closed (that is, an intersection of a closed and an open subsets)?

17. Consider the ideal $I = (x) \subset k[x]$ and the corresponding ideal sheaf $\widetilde{I}$ on $\mathrm{Spec}(k[x])$. Is $\widetilde{I}$ locally free? Same question for $I = (x, y) \subset k[x, y]$.

18. Let $A = \oplus_{i \geq 0} A_i$ be a graded algebra over a field $k = A_0$. If A is finite-dimensional as a k-vector space, what is $\mathrm{Proj}(A)$?

19. If for a graded algebra $A = \oplus_{i \geq 0} A_i$ over a field $k = A_0$ we have $\mathrm{Proj}(A) = \varnothing$, is it true that A is finite-dimensional as a k-vector space?

20. Let X and Z be schemes and $i\colon Z \to X$ a closed immersion. Consider the push-forward $i_*\mathscr{O}_Z$ of the structure sheaf of Z. What is $i_*\mathscr{O}_Z \otimes_{\mathscr{O}_X} i_*\mathscr{O}_Z$?

21. For a graded ring A, is the nilradical a homogeneous ideal?

22. Give an example of two local rings and a ring homomorphism which is not local.

23. Give examples: of a reduced scheme which is not geometrically reduced; of an irreducible scheme which is not geometrically irreducible.

24. If k is a field of characteristic zero, does there exist a k-scheme possessing a non-trivial automorphism which acts identically on the underlying topological space?

25. Is the scheme $\mathrm{Spec}(\bar{\mathbb{Q}} \otimes_{\mathbb{Q}} \bar{\mathbb{Q}})$ connected?