The twistor space of a compact hypercomplex manifold is never Moishezon

IMPA

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Hypercomplex manifolds

Let $\mathbb{H}$ be the algebra of quaternions generated by I, J and K, which satisfy

$$I^2 = J^2 = K^2 = -\mathrm{Id}, \qquad IJ = -JI = K.$$

A manifold M is called **almost hypercomplex** if the algebra $\mathbb{H}$ acts on TX. It is called **hypercomplex** if every complex structure on X induced from $\mathbb{H}$ is integrable.

A Hopf manifold X is a compact complex manifold obtained as a quotient of $\mathbb{C}^n \backslash \{0\}$ by a cyclic group $\langle\gamma\rangle$ generated by an invertible holomorphic contraction γ

$$X \cong \mathbb{C}^n \backslash \{0\} / \langle\gamma\rangle$$

When the dimension n is even and $\gamma \in \mathrm{GL}_n(\mathbb{H})$, it is, in fact, a compact hypercomplex manifold:

$$\mathbb{H}^n \backslash \{0\} / \langle\gamma\rangle.$$

Twistor space of hypercomplex manifold

Let M be a hypercomplex manifold.

For any $(a, b, c) \in S^2$, the linear combination $L := aI + bJ + cK$ defines *a* $\mathbb{C}P^1$*-family* of complex structures, which is called **the twistor deformation**.

The twistor space of the hypercomplex manifold M is a new complex manifold $\mathrm{Tw}(M)$ diffeomorphic to $M \times \mathbb{C}\mathrm{P}^1$ with an almost complex structure defined as follows.

For any point $(x, L) \in M \times \mathbb{C}\mathrm{P}^1$ the complex structure on $T_{(x,L)}\,\mathrm{Tw}(M)$ is given by L on $T_x M$ and the standard complex structure $I_{\mathbb{C}P^1}$ on $T_L \mathbb{C}\mathrm{P}^1$.

Moishezon manifolds I

Naive observation The twistor space is rich in rational curves.

A compact complex manifold X is called **Moishezon** if it is bimeromorphically equivalent to a projective manifold.

The first example *a non-projective* Moishezon twistor spaces Z was first produced by Y.S. Poon:

$$M = \mathbb{C}P^2 \# \mathbb{C}P^2, \qquad Z = \text{bimeromorphic to } \mathbb{C}P^3.$$

Theorem (F. Campana) A twistor space is Moishezon only when the 4-manifold is S^4 or $\#_n \mathbb{C}P^2$.

Hyperkähler case and the theorem we prove

Let X be a compact complex manifold.

An algebraic dimension $a(X)$ of X is a transcendence degree of the field of global meromorphic functions $k(X)$ on X.

A compact complex manifold X is called **Moishezon** if the algebraic dimension $a(X)$ is equal to the complex dimension $\dim_{\mathbb{C}}(X)$.

Theorem (M. Verbitsky) The twistor space $\mathrm{Tw}(M)$ of a compact hyperkähler manifold M has an algebraic dimension $a(\mathrm{Tw}(M)) = 1$.

Theorem ['24] Let (X, I, J, K) be a compact hypercomplex manifold. Then $\mathrm{Tw}(X)$ cannot be Moishezon.

Algebraic dimension of the twistor space of a Hopf manifold

Let $X = \mathbb{H}^n \backslash \{0\} / \langle \mu \rangle$ be a hypercomplex elliptic Hopf manifold

$$X \longrightarrow \mathbb{C}P^{2n-1}, \quad (z_1, \cdots, z_{2n}) \mapsto [z_1 : \cdots z_{2n}]$$

with the fibers $\mathbb{C}^* / \langle \mu \rangle$ elliptic curves.

An algebraic reduction of X is a projective variety X^{red} together with a meromorphic dominant map $\varphi : X \longrightarrow X^{red}$ such that the associated map $\varphi^* : k(X^{red}) \longrightarrow k(X)$ of the fields of meromorphic functions is an isomorphism.

From the main result it follows that $a(\mathrm{Tw}(X)) \leqslant 2n$. However, there is an algebraic reduction of $\mathrm{Tw}(X)$ to $\mathbb{C}P^1 \times \mathbb{C}P^{2n-1}$, hence $a(\mathrm{Tw}(X)) = 2n$.

Hodge structures and polarization – I

Let $V_{\mathbb{Z}}$ be free $\mathbb{Z}$-module of finite rank, $V_{\mathbb{C}} = V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$ its complexification. **A Hodge structure of weight** $k \in \mathbb{Z}$ is a finitely generated $\mathbb{Z}$-module V together with a direct sum decomposition on $V_{\mathbb{C}}$

$$V_{\mathbb{C}} = \bigoplus_{p+q=k} V^{p,q}, \quad \text{with} \quad V^{p,q} = \overline{V^{q,p}} \quad \text{the Hodge decomposition.}$$

A **Hodge filtration** on $V_{\mathbb{Z}}$ is a decreasing filtration

$$F^{\bullet}: \qquad \cdots \supseteq F^{p-1}V \supseteq F^{p}V \supseteq F^{p+1}V \supseteq \cdots$$

on $V_{\mathbb{C}}$, such that

$$V_{\mathbb{C}} = F^{p}V_{\mathbb{C}} \oplus \overline{F^{k-p+1}V_{\mathbb{C}}}$$

holds, where $F^{P}V_{\mathbb{C}} := \bigoplus_{i \geqslant p} V^{i,n-i}$ and $V^{p,q} = F^{P}V_{\mathbb{C}} \cap \overline{F^{n-p}V_{C}}$.

Hodge structures and polarization – II

A Hodge structure is equipped with **$U(1)$-action**, with $z \in \mathrm{U}(1)$ acting as z^{p-q} on $V^{p,q}$.

A polarization of a Hodge structure V of weight k is a $\mathrm{U}(1)$-invariant bilinear form $Q : V \otimes V \longrightarrow \mathbb{Z}$ which is $(-1)^k$-symmetric such that for its $\mathbb{C}$-bilinear extension to $V_{\mathbb{C}}$

1. $Q(u, v) = 0$ for $u \in V^{p,q}, v \in V^{a,b}$, where $p \neq b$ and $q \neq a$;
2. the form $u \mapsto (\sqrt{-1})^{p-q} Q(u, \overline{u})$ is positive definite on $V^{p,q}$.

Primitive cohomology

Let L be an ample line bundle on X. Denote

$$\omega := c_1(L) \in H^2(X, \mathbb{Z}) \cap H^{1,1}.$$

Then we could define **the primitive cohomology**:

$$H^k_{prim}(X, \mathbb{Q}) := \ker(H^k(X) \longrightarrow H^{2n-k+2}(X)), \quad x \mapsto x \wedge \omega^{n-k+1}$$

and decompose $H^k(X, \mathbb{Q}) = \bigoplus_{i \leqslant 0} H^{k-2i}_{prim}(X, \mathbb{Q})$.

On each component there is a bilinear pairing on a $\mathbb{Q}$-Hodge structure $H^k_{prim}(X, \mathbb{Q})$:

$$Q(x, y) := (-1)^{\frac{k(k-1)}{2}} \int_X x \wedge y \wedge \omega^{n-k}.$$

Families

Let X be a compact complex manifold, and B a compact complex manifold.

A holomorphic submersive map

$$f : \mathcal{X} \longrightarrow B, \quad \mathcal{X}_0 \cong X$$

is called **a holomorphic family**. We assume that it is proper and all fibers satisfy dd^c-lemma[1].

Cohomology $H^k(X_t, \mathbb{Z})$ of these fibers form *a local system* on B.

[1] $\operatorname{im} dd^c = \ker d \cap \ker d^c \cap \operatorname{im} d$; then $H^k(X, \mathbb{Z}) \otimes \mathbb{C} = \bigoplus_{p+q=k} H^{p,q}(X)$

Cohomology of the fiber

Let $\mathbb{Z}$ be a constant sheaf. Take its k-th higher direct image

$$\mathbb{V}_{\mathbb{Z}}^k := R^k f_* \mathbb{Z}, \quad \text{a local system over } B.$$

Its stalk at a point $t \in B$ is isomorphic to the integer cohomology of the fiber:

$$(\mathbb{V}_{\mathbb{Z}}^k)_t := H^k(X_t, \mathbb{Z}).$$

Denote a free $\mathbb{Z}$-module obtained from $\mathbb{V}_{\mathbb{Z}}^k$ by

$$\mathbb{V}^k := \frac{R^k f_* \underline{\mathbb{Z}}}{\text{torsion}} \cong \mathbb{Z}^r.$$

Let $\mathbb{V}_{\mathbb{C}}^k := \mathbb{V}^k \otimes_{\mathbb{Z}} \mathbb{C}$ be the complexification.

Corresponding stalks admit a Hodge decomposition:

$$(\mathbb{V}_{\mathbb{C}}^k)_t = (R^k f_* \mathbb{C})_t \cong H^k(X_t, \mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}(X_t).$$

How do the cohomology of the fibers depend on $t \in B$?

If the family is not locally trivial, the Hodge decomposition has to vary in a nontrivial way.

We go from a locally constant sheaf $R^k f_* \mathbb{C}$ to a holomorphic vector bundle $\mathcal{V}^k := \mathcal{O}_B \otimes_{\mathbb{C}} R^k f_* \mathbb{C}$; it comes together with the flat connection $\nabla : \mathcal{V}^k \longrightarrow \Omega^1_B \otimes_{\mathcal{O}_B} \mathcal{V}^k$.

Griffits transversality condition:

$$\nabla(F^p \mathcal{V}^k) \subset \Omega^1_B \otimes_{\mathcal{O}_B} F^{p-1} \mathcal{V}^k.$$

Theorem Let M be a Moishezon manifold, $\pi : M \longrightarrow B$ a proper holomorphic submersion[2]. Consider a local system $\mathbb{V}_{\mathbb{C}} = R^k \pi_*(\mathbb{C})$. This construction gives a VHS.

Proof: Check the Griffits transversality condition.

[2] the fibers M_b of π are also Moishezon manifolds

Monodromy and the theorem of fixed part

Let $\mathbb{V}^k$ be a Hodge structure of weight k, $t \in B$ a base point. The morphism

$$\rho : \pi_1(B, t) \longrightarrow \mathsf{GL}(\mathbb{V}_t^k)$$

is called **the monodromy representation**.

The image $\Gamma := \rho(\pi_1(B, t)) \subseteq \mathsf{GL}(\mathbb{V}_t^k, \mathbb{Z})$ is called **the monodromy group**.

Theorem (P. Deligne) Let B be a smooth quasiprojective variety, $\mathbb{V}$ a polarized VHS on B with the trivial monodromy. Then $\mathbb{V}$ is trivial.

Sketch of the proof of the theorem: Step 1

Theorem Let (X, I, J, K) be a compact hypercomplex manifold. Then $\mathrm{Tw}(X)$ cannot be Moishezon.

Proof:

Ad absurdum. Assume that $\mathrm{Tw}(X)$ is Moishezon. The Hodge-to-de Rham spectral sequence of Moishezon manifolds degenerates in E_1. This defines a VHS over $\mathbb{C}P^1$.

Sketch of the proof: Step 2

This VHS is in fact polarized. There exists a bimeromorphic map (blow ups with smooth centers)

$$\mu : \widetilde{\mathrm{Tw}(X)} \longrightarrow \mathrm{Tw}(X), \quad \widetilde{\mathrm{Tw}(X)} \text{ projective.}$$

The map of the cohomology μ^* is an embedding:

$$\mu^* : H^k(\mathrm{Tw}(X), \mathbb{Z}) \longrightarrow H^k(\widetilde{\mathrm{Tw}(X)}, \mathbb{Z}).$$

Projective manifolds admit the natural polarization. The map

$$\widetilde{\pi} := \pi \circ \mu : \widetilde{\mathrm{Tw}(X)} \longrightarrow \mathbb{C}P^1$$

is a holomorphic submersion outside the algebraic subset (Sard's theorem).

Hence, we obtain a polarized VHS associated with π as a substructure of polarized VHS, associated with $\widetilde{\pi}$.

Sketch of the proof: Step 3

Deligne's theorem implies that any polarized rational VHS with trivial monodromy over a quasiprojective manifold is **trivial**.

It remains to prove that the VHS is **non-trivial**.

Sketch of the proof: Step 4

If $H^1(X_I) \neq 0$, we have a non-trivial VHS on the first cohomology.

Indeed, consider two points $\pm I \in \mathbb{C}P^1$ and the corresponding fibers (X, I) and $(X, -I)$.

Let $[\alpha] \in H_I^{1,0}$. Then $I\alpha = \sqrt{-1}\alpha = -(-I)\alpha$. Hence, $-\sqrt{-\alpha} = -I\alpha$ and $\alpha \in H_{-I}^{0,1}$.

Therefore, **the VHS is non-trivial**.

Assume that $H^1(X_I) = 0$,

$$H^{0,1}(X_I) = H^1(X_I, \mathcal{O}_{X_I}) = 0.$$

Note that the canonical bundle of X_I is topologically trivial. From the exponential exact sequence of sheaves we get $\mathrm{Pic}^0(X_I) = 0$. Hence, there is a non-zero holomorphic section $\Phi \in \Omega^n(X_I)$, i.e. it is holomorphically trivial.

Sketch of the proof: Step 5

Consider the VHS over $\mathbb{C}P^1$ associated with the middle cohomology of the fiber X_I.

By **Step 3**, it has to be trivial; by **Step 4**, $H^{n,0}(X_I) = \langle \Phi \rangle$, and Φ is a nowhere degenerate holomorphic section of the canonical bundle of X_I. In particular,

$$\int_{X_I} \Phi \wedge \overline{\Phi} = \int_{X_{-I}} \Phi \wedge \overline{\Phi} > 0,$$

i.e. there exists is a non-trivial VHS such that $[\Phi] \in H^{0,n}(X_{-I})$. This implies that the VHS on the middle cohomology of the fibers of π is non-trivial, contradicting **Step 3**.

Fujiki class $\mathcal{C}$ and Corollary

A complex manifold X is called **Fujiki class** $\mathcal{C}$ if it is birationally equivalent to a Kähler manifold.

An ample rational curve on a complex manifold is a smooth rational curve C, such that its normal bundle is positive, i.e. $NC \cong \bigoplus \mathcal{O}(i_k)$, where $i_k > 0$.

Theorem(Campana) A Fujiki class $\mathcal{C}$ manifold is Moishezon if and only if it is *algebraically connected*.

Theorem (Verbitsky) Twistor space of a compact hypercomplex manifold contains an ample curve.

Corollary The twistor space of a compact hypercomplex manifold is never of Fujiki class $\mathcal{C}$. In particular, it is never Kähler.

$\mathscr{F}!\mathscr{N}$