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Introduction to Kähler geometry

Exercise sheet 5

Exercise 5.1. Prove that $b_{2k+1}(M)$ is even for any compact Kähler manifold M and any k.

Exercise 5.2. Denote by $\mathbb{S}$ the real Lie group of invertible complex numbers $\mathbb{C}^*$ (it is sometimes called *Deligne torus*). Prove that a $\mathbb{Q}$-Hodge structure of weight k on a vector space V over $\mathbb{Q}$ is the same thing as a representation of real Lie groups $\rho\colon \mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}})$ such that $\rho(\lambda)v = \lambda^k v$ for any $\lambda \in \mathbb{R}^* \subset \mathbb{S}$ and $v \in V_{\mathbb{R}}$, where $V_{\mathbb{R}} = V \otimes_{\mathbb{Q}} \mathbb{R}$.

Exercise 5.3. Compute the Hodge numbers of the following manifolds:

1. $\mathbb{C}P^n$;
2. An elliptic curve $E = \mathbb{C}/\Lambda$, where $\Lambda \subset \mathbb{C}$ is a lattice;
3. An abelian surface $A = \mathbb{C}^2/\Lambda$, where $\Lambda \subset \mathbb{C}^2$ is a lattice;
4. $\mathbb{C}P^n \times E$ for an elliptic curve E;
5. The blow up of $\mathbb{C}P^2$ in a point.

Exercise 5.4. Prove the following version of the $\partial\overline{\partial}$-lemma called dd^c-lemma. Let $\alpha \in \Lambda^k M$ be a k-form on a compact Kähler manifold M such that $d\alpha = 0$ and $\alpha = d^c\beta$ for some $\beta \in \Lambda^{k-1}M$. Then $\alpha = dd^c\gamma$ for some $\gamma \in \Lambda^{k-2}M$. Prove that the same conclusion holds if we assume that $d^c\alpha = 0$ and $\alpha = d\beta$.

Exercise 5.5. Let M be a compact Kähler manifold. For a Kähler form ω denote by $[\omega] \in H^{1,1}(M)$ its cohomology class. Let $\mathcal{K} \subset H^{1,1}(M)$ be the set of all classes $[\omega]$, where ω is a Kähler form of some Kähler metric on M (we fix the complex structure on M and vary only the metric). Prove that $\mathcal{K}$ is an open convex cone in $H^{1,1}(M) \cap H^2(M, \mathbb{R})$.