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Winter term 2015/16

Second Exam, Algebraic Geometry I

Problem A. (4 points)
Let $X \subset \mathbb{P}^3_k$ be a hypersurface defined by a homogeneous cubic polynomial $F \in k[x_0, x_1, x_2, x_3]$ with k a field. Compute the *Hilbert polynomial* $P(n) := \chi(X, \mathcal{O}_X(n))$.

Problem B. (3 points)
Consider the standard open set $\mathrm{Spec}(k[x]) = D_+(x_0) \subset \mathbb{P}^1_k = \mathrm{Proj}(k[x_0, x_1])$, where $x = x_1/x_0$. Let $\mathcal{F} = \tilde{M}$ be the coherent sheaf on $D_+(x_0)$ corresponding to the $k[x]$-module $M = k[x]/(x^2 - 1)$. Describe a coherent extension of $\mathcal{F}$ to $\mathbb{P}^1_k$, i.e. a coherent sheaf $\mathcal{G}$ on $\mathbb{P}^1_k$ with $\mathcal{G}|_{D_+(x_0)} \cong \mathcal{F}$, in terms of a $k[x_0, x_1]$-module. Is this extension unique?

Problem C. (3 points)
Let Z be a scheme over an algebraically closed field k. Let $Y = \mathbb{P}^1_k \times_k Z$ and $X \subset Y$ be a closed subscheme. Assume that X does not contain any closed fibre of the projection $Y \to Z$. Denote by $\pi : X \to Z$ the restriction to X of this projection. Prove that π is an affine morphism. (Extra 2 points for proving that π is a finite morphism.)

Problem D. (4 points)
Let X be a projective regular curve of genus one over an algebraically closed field k and let $x_1, x_2 \in X$ be two closed points. Show that $H^1(X, \mathcal{O}(x_1 - x_2)) \neq 0$ if and only if $x_1 = x_2$.

Problem E. (4 points)
Let k be an algebraically closed field. Describe (as the zero locus of a section of a line bundle) the scheme $X \subset \mathbb{P}^1_k \times \mathbb{P}^1_k \times \mathbb{P}^1_k$ for which the fibre of the first projection $X \to \mathbb{P}^1_k$ over a closed point $[t_0 : t_1]$ is the curve $X_{[t_0:t_1]} \subset \mathbb{P}^1_k \times \mathbb{P}^1_k$ described by $x_0^2 y_1 t_1 = (x_0^2 + x_1^2) y_0 t_0$ (where x_0, x_1 and y_0, y_1 are the coordinates on the two factors). Find closed points for which the fibre is irreducible, non-reduced, and reducible, respectively.

Problem F. (4 points)
Let X be an arbitrary integral scheme. Decide which of the following sheaves are quasi-coherent: i) $\mathcal{O}_X^*$; ii) $\mathcal{K}_X$; iii) The sheaf $i_*(\mathcal{O}_{X,x})$ for $x \in X$ a point and $i : \{x\} \to X$ the inclusion.

Problem G. (3 points)
Determine the base locus of the linear system $\{t_0(x_0^4 + x_1^4 + x_2^4 + x_3^4) + t_1 x_0 x_1 x_2 x_3\} \subset |\mathcal{O}(4)|$ on $\mathbb{P}^3_k$ where $\mathrm{char}(k) \neq 2$. Prove that the irreducible components of the base locus are curves, find their genera.

Klausureinsicht (review of corrected exam): Tuesday April 5, 11:15-12:00, seminar room 0.011