On the boundary of the ample cone of a hyperkähler manifold

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Hyperkähler manifolds

Definition

A hyperkähler structure on a C^∞-manifold X is a tuple (g, I, J, K), where:

- *g is a Riemannian metric;*
- *I, J and K are complex structures s.t. $IJ = -JI = K$;*
- *g is Kähler w.r.t. I, J and K.*

We have two-forms ω_I, ω_J and ω_K:

$$\omega_I(u, v) = g(Iu, v),$$
$$\omega_J(u, v) = g(Ju, v),$$
$$\omega_K(u, v) = g(Ku, v).$$

These forms are closed:

$$d\omega_I = d\omega_J = d\omega_K = 0.$$

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A Riemannian metric g as above is called hyperkähler.

Equivalently: g is hyperkähler if $\mathrm{Hol}(\nabla^g) \subset Sp(n)$,

∇^g is the Levi-Civita connection for g.

$Sp(n)$ = the group of quaternionic-linear transformations of $\mathbb{H}^n$ that preserve the quaternionic-Hermitian scalar product.

Consider the 2-form $\sigma_I = \omega_J + \sqrt{-1}\omega_K$.

σ_I is a non-degenerate closed (2,0)-form on X_I, i.e. a holomorphic symplectic form.

Today we assume: a hyperkähler manifold X is compact and of maximal holonomy, i.e. $\mathrm{Hol}(\nabla^g) = Sp(n)$.

This implies: $\pi_1(X) = 1$ and $H^0(X_I, \Omega^2_{X_I}) = \mathbb{C}\sigma_I$.

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Since σ_I is symplectic, we have:

- $\dim_{\mathbb{C}}(X_I) = 2n$,
- σ_I^n is a nowhere vanishing section of $K_{X_I} = \Omega^{2n}_{X_I}$.

Theorem (Beauville, Bogomolov, Fujiki)

There exists $c_X \in \mathbb{Q}$ such that for all $a \in H^2(X, \mathbb{Q})$

$$\int_X a^{2n} = c_X q(a)^n,$$

where q is a quadratic form on $H^2(X, \mathbb{Q})$, the Beauville–Bogomolov–Fujiki form, or the BBF form.

We may assume: q is primitive and integral on $H^2(X, \mathbb{Z})$.

The signature of q is $(3, b_2(X) - 3)$.

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Examples of hyperkähler manifolds

- Complex $K3$-surfaces: S a compact simply connected complex surface with $K_S \simeq \mathcal{O}_S$. For example:

$$S = \{(x_0 : x_1 : x_2 : x_3) \in \mathbb{C}P^3 \mid x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0\}.$$

- $K3^{[n]}$-type. Let S be a complex K3 surface. $S^{[n]}$ = the Hilbert scheme of length n subschemes of S.
- Kum^n-type. Let $T = \mathbb{C}^2/\mathbb{Z}^4$. The Albanese morphism:

$$a \colon T^{[n+1]} \to T, \quad (x_0, \ldots, x_n) \mapsto \sum x_i$$

  $K^nT = a^{-1}(0)$ = the generalized Kummer variety.
- $OG6$ and $OG10$-types. O'Grady's exceptional hyperkähler manifolds of dimensions 6 and 10.

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The ample cone

From now on: X is a projective hyperkähler manifold.

The Néron–Severi group:

$$\mathrm{NS}(X) = \mathrm{im}\left(\mathrm{Pic}(X) \xrightarrow{c_1} H^2(X,\mathbb{Z})\right) \subset H^{1,1}(X,\mathbb{Z})$$

The restriction of q to $\mathrm{NS}_{\mathbb{R}}(X) = \mathrm{NS}(X) \otimes \mathbb{R} \subset H^2(X,\mathbb{R})$ has signature

$$(1, \rho - 1)$$

where ρ is the Picard number of X.

The positive cone:

$$\mathcal{C}_X = \{x \in \mathrm{NS}_{\mathbb{R}}(X) \mid q(x) > 0\}^\circ$$

The ample cone: $\mathcal{A}_X \subset \mathcal{C}_X$ is the convex cone spanned by the classes $c_1(L)$ of ample line bundles L.

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The ample cone

We have: $\mathbb{P}(\mathcal{C}_X) = \mathbb{H}^{\rho-1}$ is a hyperbolic space. What is $\mathbb{P}(\mathcal{A}_X)$?

Amerik, Verbitsky: there is a collection of integral classes

$$\mathrm{MBM} \subset H^2(X, \mathbb{Z})$$

called monodromy birationally minimal or MBM classes with the following properties.

- MBM is $\mathscr{Diff}(X)$-invariant;
- There exists a constant $M > 0$ such that for all $x \in \mathrm{MBM}$ we have $-M \leqslant q(x) < 0$;
- Let $\mathrm{MBM}^{1,1} = \mathrm{MBM} \cap \mathrm{NS}(X)$. Then the hyperplanes $x^\perp$, where $x \in \mathrm{MBM}^{1,1}$, cut $\mathcal{C}_X$ into open chambers.
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- $\mathcal{A}_X$ is one of the chambers.

The ample cone

We have: $\mathbb{P}(\mathcal{C}_X) = \mathbb{H}^{\rho-1}$ is a hyperbolic space. What is $\mathbb{P}(\mathcal{A}_X)$?

Amerik, Verbitsky: there is a collection of integral classes

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The ample cone of a K3 surface

Let S be a projective K3 surface.
Then q is the intersection form and

$$H^2(S,\mathbb{Z}) \simeq (-E_8)^{\oplus 2} \oplus U^{\oplus 3},$$

where U is the rank two hyperbolic lattice $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
We have

$$\mathrm{MBM} = \{x \in H^2(S,\mathbb{Z}) \mid q(x) = -2\}.$$

Theorem (Nikulin)

Let Λ be an even lattice of signature $(1,\rho-1)$ where $\rho \leqslant 10$. Then there exists a projective K3 surface S with $\mathrm{NS}(S) \simeq \Lambda$.

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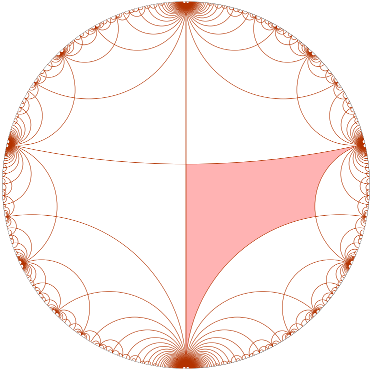
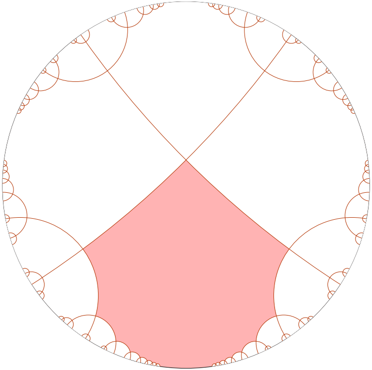
For example, when $\rho = 3$, we have $\mathbb{P}(\mathcal{C}_S) = \mathbb{H}^2$ and $\mathbb{P}(\mathcal{A}_S)$ may look like this:





The ideal boundary of the ample cone

Identify $\mathrm{NS}_{\mathbb{R}}(X)$ with $\mathbb{R}^{1,\rho-1}$, so that

$$q(x) = x_0^2 - x_1^2 - \ldots - x_{\rho-1}^2.$$

Then $\mathbb{P}(\mathcal{C}_X) = \mathbb{H}^{\rho-1}$ is identified with the unit ball $\mathbb{B}^{\rho-1}$ via the stereographic projection from the point $(-1, 0, \ldots, 0)$ and

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The following result was classically known for K3 surfaces due to S. Kovács, and was more recently generalized to an arbitrary hyperkähler manifold by F. Denisi.

Theorem (Kovách, Denisi)

Assume that X is a projective hyperkähler manifold with Picard number $\rho > 2$. Then we have the following dichotomy.

- *If $\mathcal{B}_X$ contains an open subset of $\mathbb{S}^{\rho-2}$, then $\mathcal{B}_X = \mathbb{S}^{\rho-2}$ and $\mathcal{A}_X = \mathcal{C}_X$. This happens if and only if $\mathrm{NS}(X)$ does not contain MBM classes.*
- *Otherwise $\mathcal{B}_X$ is nowhere dense in $\mathbb{S}^{\rho-2}$.*

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- *Otherwise* $\mathcal{B}_X$ *is nowhere dense in* $\mathbb{S}^{\rho-2}$*.*

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In general, $\mathcal{B}_X$ is a fractal in $\mathbb{S}^{\rho-2}$.

Let $v \in MBM \cap \mathrm{NS}(X) = MBM^{1,1}$. Then:

- $\mathbb{P}(v^\perp \cap \mathcal{C}_X)$ is a hyperbolic subspace $\mathbb{H}_v^{\rho-2} \subset \mathbb{H}^{\rho-1}$.
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Denote by $H_v \subset \mathbb{R}^{\rho-1}$ the hyperplane such that

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Under the stereographic projection:

$$\mathcal{B}_X = \mathbb{R}^{\rho-2} \setminus \bigcup_{v \in MBM^{1,1}} D_v.$$

Remark: If $\rho > 2$ and $MBM^{1,1} \neq \emptyset$, then $MBM^{1,1}$ is infinite, because it is stable under the action of a finite index subgroup in $\mathrm{O}(\mathrm{NS}(X), q)$.

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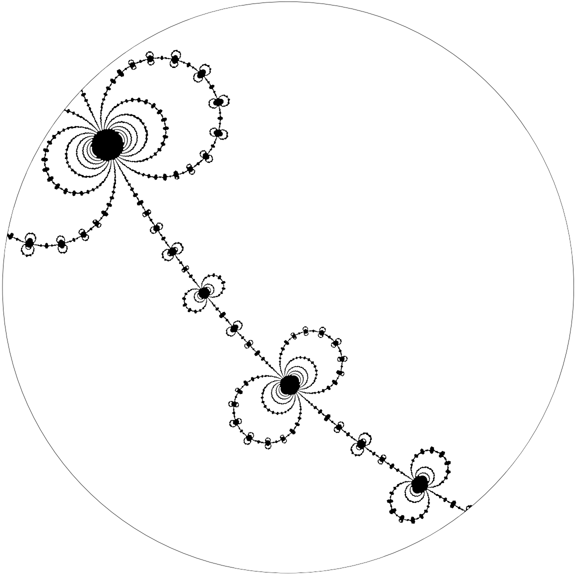
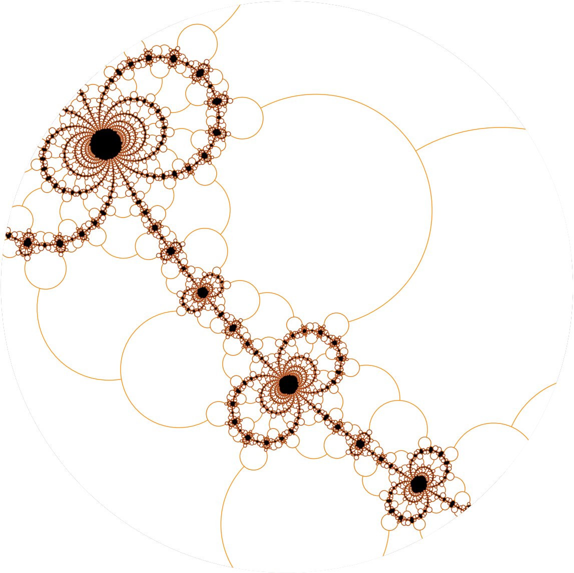
For example, if $\rho = 4$, then $\mathcal{B}_X$ may look like this:



The ideal boundary of the ample cone

The discs D_v in the above example look like this:



The ideal boundary of the ample cone

In the above example we used the lattice Λ with the intersection matrix

$$A = \begin{pmatrix} -2 & 4 & 0 & 0 \\ 4 & -2 & 2 & 0 \\ 0 & 2 & -4 & 2 \\ 0 & 0 & 2 & -4 \end{pmatrix}$$

If we start from the intersection matrix

$$A = \begin{pmatrix} -2 & 2 & 2 & 2 \\ 2 & -2 & 2 & 2 \\ 2 & 2 & -2 & 2 \\ 2 & 2 & 2 & -2 \end{pmatrix},$$

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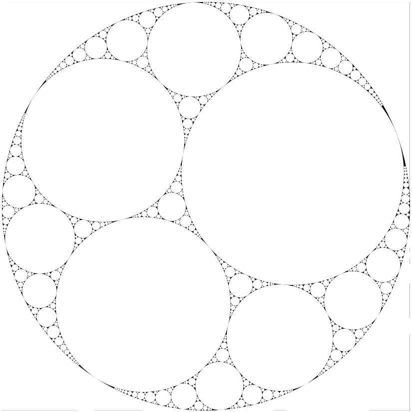
The Apollonian gasket as the ideal boundary



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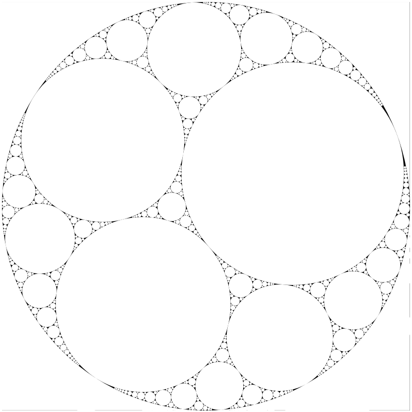

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The Apollonian carpet

Theorem (Amerik, S., Verbitsky)

Let X be a projective hyperkähler manifold with $\rho \geqslant 4$, and Z a germ of a positive-dimensional irreducible real-analytic subset of $\mathcal{B}_X$. Then:

- *There exists a sublattice $\Lambda \subset \mathrm{NS}(X)$ of signature $(1, d)$ for some $d \leqslant \rho$, such that*

$$Z \subset \mathbb{S}^d_\Lambda \subset \mathcal{B}_X,$$

  where $\mathbb{S}^d_\Lambda = \partial\mathbb{P}((\Lambda \otimes \mathbb{R}) \cap \mathcal{C}_X)$ is the ideal boundary of the corresponding hyperbolic subspace;
- *The Apollonian carpet of X is the union of the spheres $\mathbb{S}^d_\Lambda \subset \mathcal{B}_X$ as above;*
- *If the Apollonian carpet of X is non-empty, then its closure is equal to $\mathcal{B}_X$;*
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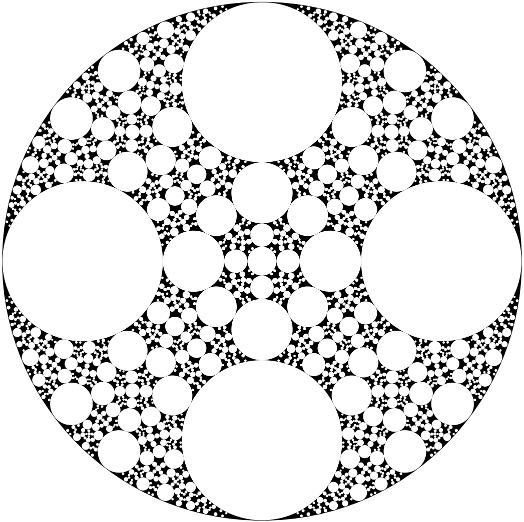
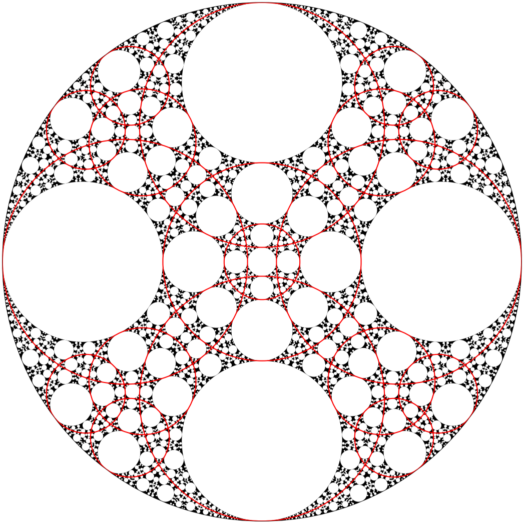
The boundary spheres

There are two types of boundary spheres $\mathbb{S}^d_\Lambda \subset \mathcal{B}_X$.
The first type corresponds to the case when there exists $v \in \mathrm{MBM}^{1,1} \cap \Lambda^\perp$.

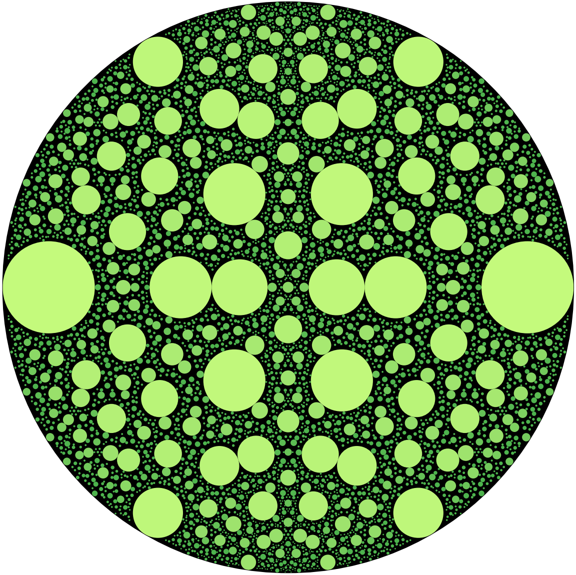


The boundary spheres

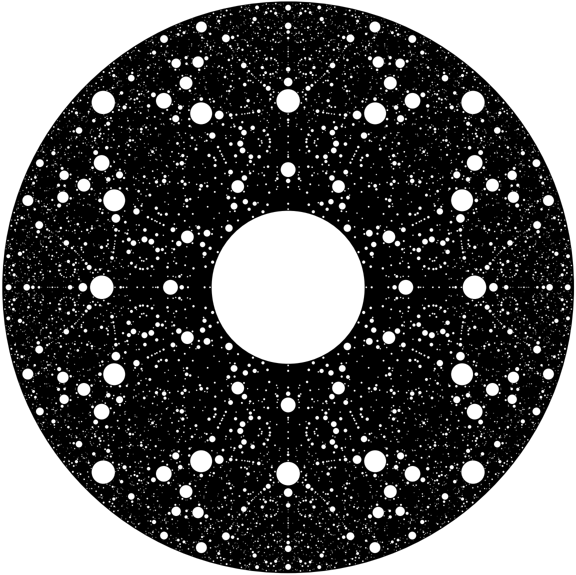
The second type: there is no $v \in \mathrm{MBM}^{1,1} \cap \Lambda^{\perp}$.



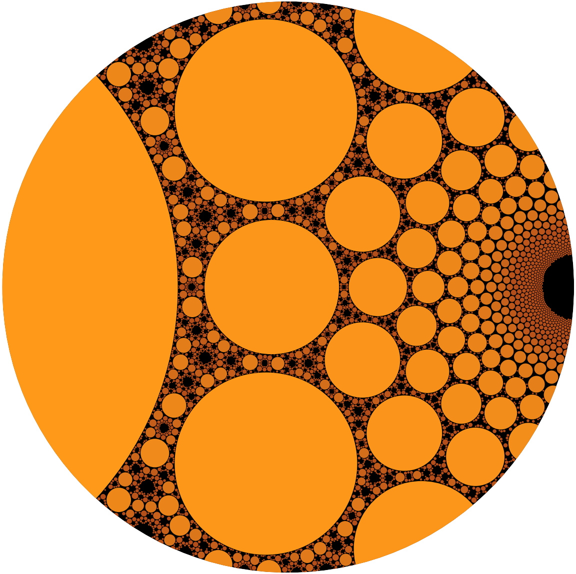
Examples



Examples



Examples



Thank you!