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Introduction to the Moduli Spaces of Sheaves on K3 Surfaces

Exercise sheet 1

Exercise 1.1. Let C be a curve and $F \in Coh(C)$. Prove that $F \simeq T \oplus E$ where T is a torsion sheaf and E is a vector bundle.

Exercise 1.2. Prove that any vector bundle E over $\mathbb{P}^1$ is isomorphic to $\oplus_{i=1}^{\mathrm{rk}(E)} \mathcal{O}_{\mathbb{P}^1}(a_i)$ for some $a_i \in \mathbb{Z}$ as follows:

1. Prove that there exists $k_0 \in \mathbb{Z}$ such that $H^0(\mathbb{P}^1, E(k_0)) \neq 0$ but $H^0(\mathbb{P}^1, E(k)) = 0$ for $k < k_0$;
2. Construct a short exact sequence $0 \to \mathcal{O}_{\mathbb{P}^1} \to E(k_0) \to E' \to 0$ and prove that E' is torsion-free;
3. Arguing by induction on the rank of E, conclude that $E(k_0)$ has the required form.

Exercise 1.3. Give an example of a vector bundle E over $\mathbb{P}^2$ such that $H^0(\mathbb{P}^2, E) = 0$ but $H^0(L, E|_L) \neq 0$ for any line $L \simeq \mathbb{P}^1 \subset \mathbb{P}^2$.

Exercise 1.4*. Assume that E is a vector bundle over $\mathbb{P}^2$ such that $E|_L$ is trivial for any line $L \simeq \mathbb{P}^1 \subset \mathbb{P}^2$. Prove that E is trivial.

Exercise 1.5. Assume that X is a projective manifold and $0 \to F' \to F \to F'' \to 0$ is a short exact sequence of coherent sheaves over X. Is it always possible to construct $\mathcal{G} \in Coh(X \times \mathbb{A}^1)$ such that $\mathcal{G}$ is flat over $\mathbb{A}^1$, $\mathcal{G}_0 \simeq F$ and $\mathcal{G}_t \simeq F' \oplus F''$ for $t \neq 0$, where $\mathcal{G}_t$ is the fibre of $\mathcal{G}$ over the point $t \in \mathbb{A}^1$?

Exercise 1.6. Give an example of a coherent sheaf $\mathcal{F} \in Coh(X \times \mathbb{A}^1)$ with the following property: $H^0(X, \mathcal{F}_t) \neq 0$ for $t \neq 0$ and $H^0(X, \mathcal{F}_0) = 0$. *Hint:* $\mathcal{F}$ should not be flat over $\mathbb{A}^1$.