Retry Exam, Algebraic Geometry II — Solutions

Problem A. (2+1 points)
Let X be the product $\prod C_i$ of smooth projective curves C_i over a field k. Decide under which condition the canonical bundle $\omega_{X/k}$ is trivial or ample. Is in the remaining cases $\omega^*_{X/k}$ automatically ample?

Solution. We may assume k algebraically closed. The canonical bundle is $\omega_{X/k} = \prod \pi_i^* \omega_{C_i/k}$ where π_i is the projection to the i-th factor. This is ample iff all of $\omega_{C_i/k}$ are ample (iff all curves have positive genus). If all $\omega_{C_i/k}$ are ample, then some power of $\omega_{X/k}$ is the restriction of $\mathcal{O}(1,\dots,1)$ under an embedding of X into $\prod \mathbb{P}^{n_i}$ for some n_i, and $\mathcal{O}(1,\dots,1)$ is very ample (Segre embedding). In the other direction: if $\omega_{X/k}$ is ample, consider C_i embedded into X as a fibre of the projection $X \to \prod_{i\neq j} C_j$. Restriction of $\omega_{X/k}$ to C_i is $\omega_{C_i/k}$, so it must be ample.

Analogously $\omega_{X/k}$ is trivial iff all C_i are of genus zero. In the remaining cases one of $\omega_{C_i/k}$ may be ample, the other anti-ample, so neither $\omega_{X/k}$ nor $\omega^*_{X/k}$ will be ample (take the product of a curve of positive genus and $\mathbb{P}^1$).

Problem B. (3+3 points)
Discuss the notion of flatness for morphisms and its properties, give examples. Explain the relation of flatness to the Hilbert polynomial along the fibres of a proper morphism (with the idea of proof).

Solution. 3 point for the definition, properties and examples, 3 points for the theorem about the Hilbert polynomial

Problem C. (4 points)
Let $X \subset \mathbb{P}^3_k$ be a smooth hypersurface of degree four. Compute the Hodge numbers $h^{p,q}(X)$.

Solution. From the exact triple

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-4) \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_X \to 0$$

we find $h^{0,0} = 1$, $h^{0,1} = 0$ and $h^{0,2} = 1$. By adjunction $\omega_X \simeq \mathcal{O}_X$, so this also implies $h^{2,0} = 1$, $h^{2,1} = 0$ and $h^{2,2} = 1$ (alternatively use Serre duality). To compute $h^i(X, \Omega_X)$ use the conormal sequence

$$0 \to \mathcal{O}_X(-4) \to \Omega_{\mathbb{P}^3}|_X \to \Omega_X \to 0.$$

From

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-8) \to \mathcal{O}_{\mathbb{P}^3}(-4) \to \mathcal{O}_X(-4) \to 0$$

we find $h^0(X, \mathcal{O}_X(-4)) = h^1(X, \mathcal{O}_X(-4)) = 0$, $h^2(X, \mathcal{O}_X(-4)) = 34$, and from

$$0 \to \Omega_{\mathbb{P}^3}(-4) \to \Omega_{\mathbb{P}^3} \to \Omega_{\mathbb{P}^3}|_X \to 0$$

we find $h^0(X, \Omega_{\mathbb{P}^3}|_X) = 0$, $h^1(X, \Omega_{\mathbb{P}^3}|_X) = 1$, $h^2(X, \Omega_{\mathbb{P}^3}|_X) = 15$ (use the Euler sequence). Using these computations we find from the conormal sequence that $h^{1,0} = h^0(X, \Omega_X) = 0$, and by Serre duality $h^{1,2} = h^2(X, \Omega_X) = 0$. Then again use the conormal sequence to find that $h^{1,1} = h^1(X, \Omega_X) = 20$.

Problem D. (1 + 1 + 1 points)
Let $\sigma\colon X \to \mathbb{P}^2_k$ be the blow up in the two points $p_1 := [0:0:1]$ and $p_2 := [0:1:1]$ with the exceptional divisor $E = E_1 \sqcup E_2$. Compute the degree of $\mathcal{O}(E)|_{\widetilde{C}}$ for the strict transform $\widetilde{C}$ (i.e. the closure of $\sigma^{-1}(C \setminus \{p_1, p_2\})$) of the three curves $C = C_1, C_2$, and C_3, where C_1 is the line through p_1, p_2, $C_2 = Z(x_0 - x_1)$, and $C_3 = Z(x_0^2 - x_1^2)$.

Solution. C_1 passes through both points, so the total transform meets the exceptional divisor in 2 points transversally and the degree is 2. Analogously for C_2 which passes through one point the degree is 1. For C_3 the strict transform is the union of two lines intersecting only E_1 transversally, so the degree is 2.

Problem E. (3 points)
Discuss the notion of smoothness for morphisms of schemes, give examples of smooth and non-smooth morphisms.

Solution. The definition, properties, examples.

Problem F. (2+2+2 points)
Let $f\colon C \to D$ be a morphism between smooth projective curves over a field k and let $\mathcal{L}$ be an invertible sheaf on C.

(i) Show that for f finite the direct image sheaf $f_*\mathcal{L}$ is a locally free sheaf of rank $\deg(f)$.

(ii) What can be said about the higher direct images $R^i f_* \mathcal{L}$?

(iii) Describe an example that shows that the smoothness of D is essential in (i). What about the smoothness of C?

Solution. (i) Since the question is local, we may assume that $\mathcal{L} = \mathcal{O}_C$. Assume also that $C = \operatorname{Spec}(A)$ and $D = \operatorname{Spec}(B)$. Then B is a finite A-module. We need to prove that B is projective. Localizing at a closed point of C we may assume that A is local, hence a DVR (use smoothness here). Let's prove that B is flat. All ideals in A are generated by powers of a uniformizing element $t \in A$, so all we need to prove is that t does not annihilate any element of B. But if this was the case, we would have a zero-divisor in B, such that the subscheme defined by this zero divisor would map into a closed point of C which contradicts finiteness. So B is flat, hence free (follows from Nakayama's lemma). The rank of the corresponding vector bundle can be computed at the generic point, so it equals the degree of the corresponding field extension, that is $\deg(f)$.

(ii) If f is finite then it is affine, so all higher direct images vanish. If f is not finite, then it is constant, its image is a point $x \in D$. Then $R^i f_*\mathcal{L} \simeq H^i(C, \mathcal{L}) \otimes \mathcal{O}_x$.

(iii) Let X be the spectrum of $k[x,y]/(xy)$. If we take $D = X$ and C its normalization, then the direct image will have torsion at $x = y = 0$, so not locally free. Smoothness of C is not essential. As an example take $C = X$ and $D = \operatorname{Spec}(k[t])$ with $t \mapsto x + y$. Here $k[x,y]/(xy)$ is a free module over $k[t]$ with a basis consisting e.g. of 1 and x.

Problem G. (2+3 points)
Consider the morphism $\varphi\colon \mathbb{P}^1_k \to \mathbb{P}^2_k$, $[x_0 : x_1] \mapsto [x_0^2 : x_0 x_1 : x_1^2]$. Describe the normal bundle of its graph $\Gamma_\varphi \subset \mathbb{P}^2_k \times \mathbb{P}^2_k$ as a sum of line bundles.

Solution. Let us first check that for any morphism $f\colon X \to Y$ the normal bundle N of Γ_f is isomorphic to f^*T_Y. We have $T_{X\times Y} \simeq \pi_X^* T_X \oplus \pi_Y^* T_Y$ and so on $\Gamma_f \simeq X$ we have $0 \to T_X \to T_X \oplus f^*T_Y \to N \to 0$, where the components of the first morphism are id and the canonical morphism $\gamma\colon T_X \to f^*T_Y$. Consider the automorphism of $T_X \oplus f^*T_Y$ given by $\alpha = \begin{pmatrix} \mathrm{id} & 0 \\ -\gamma & \mathrm{id} \end{pmatrix}$ and include it into the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_X & \xrightarrow{\binom{\mathrm{id}}{\gamma}} & T_X \oplus f^*T_Y & \longrightarrow & N & \longrightarrow & 0 \\ & & \downarrow \mathrm{id} & & \downarrow \alpha & & \downarrow & & \\ 0 & \longrightarrow & T_X & \longrightarrow & T_X \oplus f^*T_Y & \longrightarrow & f^*T_Y & \longrightarrow & 0 \end{array}$$

where in the lower line the first map is the inclusion of the direct summand. Here all vertical arrows are isomorphisms, so $N \simeq f^*T_Y$.

Let us compute $\varphi^*T_{\mathbb{P}^2}$ in our case. The image of φ is a smooth conic in $\mathbb{P}^2$, so pulling back the Euler sequence we get $0 \to \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}(2)^{\oplus 3} \to \varphi^*T_{\mathbb{P}^2} \to 0$. We find that $\det(\varphi^*T_{\mathbb{P}^2}) = \mathcal{O}(6)$, which implies that $\varphi^*T_{\mathbb{P}^2} \simeq \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)$, with $a+b=6$. Let us prove that $a=b=3$. Note that $\varphi^*\Omega_{\mathbb{P}^2} \simeq \mathcal{O}_{\mathbb{P}^1}(-a) \oplus \mathcal{O}_{\mathbb{P}^1}(-b)$, and so it is enough to show that all cohomology groups of the bundle $(\varphi^*\Omega_{\mathbb{P}^2}) \otimes \mathcal{O}_{\mathbb{P}^1}(2)$ vanish. Note that $(\varphi^*\Omega_{\mathbb{P}^2}) \otimes \mathcal{O}_{\mathbb{P}^1}(2) \simeq \varphi^*(\Omega_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(1))$. Pulling back the Euler sequence from $\mathbb{P}^2$ we get $0 \to \varphi^*(\Omega_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(1)) \to \mathcal{O}_{\mathbb{P}^1}^{\oplus 3} \to \mathcal{O}_{\mathbb{P}^1}(2) \to 0$, and we get the necessary cohomology vanishing since the map $\mathcal{O}_{\mathbb{P}^1}^{\oplus 3} \to \mathcal{O}_{\mathbb{P}^1}(2)$ induces isomorphism on global sections by the definition of φ.