Prof. Dr. Daniel Huybrechts
Dr. Andrey Soldatenkov

Summer term 2015

Exercises, Algebra I (Commutative Algebra) – Week 3

Exercise 14. (5=2+3 points)

i) Let k be a field and $(a_1, \ldots, a_n) \in k^n$. Show that the kernel of the evaluation map
$$k[x_1, \ldots, x_n] \to k, \quad f \mapsto f(a_1, \ldots, a_n)$$
is a maximal ideal with residue field k.

ii) Consider a compact topological space X and let $A = C(X)$ be the ring of continuous functions $f\colon X \to \mathbb{C}$. Let $\mathfrak{m}_x \subset A$ for a point $x \in X$ be the kernel of the evaluation map $f \mapsto f(x)$. Show that $\mathfrak{m}_x$ is a maximal ideal and that the induced map $X \to \mathrm{MaxSpec}(A)$ is surjective. (In fact, for a normal space X the map is also injective and in fact a homeomorphism. The proof of this uses the lemma of Uryson).

Exercise 15. (6 points)
Let k be a field and $A = k[X, Y]/(X^2, Y^2, XY)$. Determine all units in A, all prime ideals and all ideals.

Exercise 16. (6 points)
Describe $\mathrm{Spec}(A)$ and $\mathrm{MaxSpec}(A)$ of the following rings:

i) $\mathbb{F}_p[X]$;

ii) $k[X]/(X^3)$;

iii) $k[[X]]$.

Exercise 17. (4 points)
Let A be a ring and $\mathfrak{N} \subset A$ be its nilradical. Show the equivalence of the following conditions:

i) A has exactly one prime ideal.

ii) Every element in A is either a unit or nilpotent.

iii) $A/\mathfrak{N}$ is a field.

Exercise 18. (6 points)
A topological space X is called *irreducible* if X is non-empty and the intersection $U \cap V$ of any two non-empty open subsets $U, V \subset X$ is again non-empty. Show that $\mathrm{Spec}(A)$ with the Zariski topology is irreducible if and only if the nilradical $\mathfrak{N} \subset A$ is a prime ideal

Exercise 19. (6 points)
A topological space X is called *connected* if X is not the disjoint union of non-empty open subsets (or, equivalently, of non-empty closed subsets). Show that $\mathrm{Spec}(A)$ with the Zariski topology is not connected if and only if there exists an element $0, 1 \neq e \in A$ with $e^2 = e$. (Such an element is called idempotent.)

Due Monday April 27.