Problem A. (1+1+1 points)
Describe examples of smooth projective varieties X over a field k of dimension at least two for which i) the canonical bundle $\omega_{X/k}$ is ample, ii) for which $\omega^*_{X/k}$ is ample, and iii) for which neither of the two holds.

Solution. Consider a smooth hypersurface $X \subset \mathbb{P}^n$ of degree d with $n \geq 3$, then $\omega_X = \mathcal{O}(d-n-1)$. For i) take $d \geq n+2$, for ii) take $d \leq n$ and for iii) take $d = n+1$.

Problem B. (2+2+2 points)
Let k be an algebraically closed field with $\mathrm{char}(k) \neq 2$. Discuss degree two morphisms $X \to \mathbb{P}^1_k$ with X a smooth projective curve of genus g over k.

Solution. The following things should be part of it: Hurwitz formula, the word hyperelliptic should appear, if $g \leq 2$, then all curves are hyperelliptic, can prescribe ramification points. Points should be given for the statement and an explanation but also for indicating the proofs.

Problem C. (3 points)
Let C_1, C_2 be two smooth projective irreducible curves over a field k of genus g_1 and g_2, respectively. Compute all Hodge numbers of $C_1 \times C_2$.

Solution. Let $X = C_1 \times C_2$, then $h^{p,q}(X) = \dim H^q(X, \Omega^p_{X/k})$ can be computed by Künneth formula:

$$h^{p,q}(X) = \sum_{\substack{p_1+p_2=p \\ q_1+q_2=q}} h^{p_1,q_1}(C_1)h^{p_2,q_2}(C_2).$$

Compute: $h^{0,0} = h^{2,2} = 1$, $h^{1,0} = h^{0,1} = h^{1,2} = h^{2,1} = g_1 + g_2$, $h^{2,0} = h^{0,2} = g_1g_2$, $h^{1,1} = 2 + 2g_1g_2$.

Problem D. (5 points)
Let $\pi\colon X \to \mathbb{P}^1 \times \mathbb{P}^1$ be the blow-up of a closed point $y = ([0:1],[s:t]) \in \mathbb{P}^1 \times \mathbb{P}^1$. Consider the exceptional divisor $E := \pi^{-1}(y)$ and the strict transform $\tilde{\Delta} \subset X$ of the diagonal $\Delta \subset \mathbb{P}^1 \times \mathbb{P}^1$ (i.e. the closure of $\pi^{-1}(\Delta \setminus y)$).

Compute $\chi(X_t, \mathcal{O}(\tilde{\Delta}+2E)|_{X_t})$ and $h^0(X_t, \mathcal{O}(\tilde{\Delta}+2E)|_{X_t})$ for the fibres X_t of the composition $f = \mathrm{pr}_1 \circ \pi\colon X \to \mathbb{P}^1$.

Solution. Note that $\mathcal{O}(\Delta) = \mathcal{O}(1,1)$. First consider a point $t \in \mathbb{P}^1$, $t \neq [0:1]$. In this case $X_t \simeq \mathbb{P}^1$ is isomorphic to the fibre of the projection pr_1, so $\mathcal{O}(2E)|_{X_t} = \mathcal{O}_{X_t}$ and $\mathcal{O}(\tilde{\Delta})|_{X_t} = \mathcal{O}_{X_t}(1)$. We compute: $\chi(X_t, \mathcal{O}(\tilde{\Delta}+2E)|_{X_t}) = h^0(X_t, \mathcal{O}(\tilde{\Delta}+2E)|_{X_t}) = 2$. Note that the variety X is irreducible, so the map $f\colon X \to \mathbb{P}^1$ is flat. In particular the locally free sheaf $\mathcal{O}(\tilde{\Delta}+2E)$ is flat over $\mathbb{P}^1$. This implies that the Euler characterictic χ is constant, and we conclude that $\chi = 2$ for all fibres.

To compute h^0 at the point $t = [0:1]$ note that the fibre X_t has two irreducible components (both isomorphic to $\mathbb{P}^1$): the exceptional curve E and the strict transform of the fibre of pr_1 which we will denote by F. The two components meet at an ordinary double point p. Let $\eta\colon E \coprod F \to X_t$ be the normalization. Denote $\mathcal{L} = \mathcal{O}(\tilde{\Delta}+2E)|_{X_t}$. Then we have

$$0 \to \mathcal{L} \to \eta_*\eta^*\mathcal{L} \to \mathcal{O}_p \to 0.$$

We have to consider two cases.

1) The point y does not lie on the diagonal Δ. In this case $\mathcal{O}(\tilde{\Delta}) = \pi^*\mathcal{O}(\Delta)$ and $\mathcal{L}|_E = \mathcal{O}(2E)|_E \simeq \mathcal{O}_{\mathbb{P}^1}(-2)$, $\mathcal{L}|_F = \mathcal{O}(\Delta+2E)|_F \simeq \mathcal{O}_{\mathbb{P}^1}(3)$. We find that $h^0(\eta^*\mathcal{L}) = 4$. Since any

section of $\mathcal{L}$ has to vanish at p, we get $h^0(\mathcal{L}) = 3$. One can also check from the exact sequence above, that $h^1(\mathcal{L}) = 1$ (as it should be from the Euler characteristic).

2) The point y lies on the diagonal Δ. In this case $\mathcal{O}(\tilde{\Delta}) = \mathcal{O}(\pi^*\Delta - E)$ and $\mathcal{L}|_E = \mathcal{O}(-E + 2E)|_E \simeq \mathcal{O}_{\mathbb{P}^1}(-1)$, $\mathcal{L}|_F = \mathcal{O}(\Delta + 2E)|_F \simeq \mathcal{O}_{\mathbb{P}^1}(2)$. Again, any section of $\mathcal{L}$ has to vanish at p, so in this case $h^0(\mathcal{L}) = 2$ and $h^1(\mathcal{L}) = 0$.

Problem E. (2+2+2 points)
Compare the notion étale and unramified for a morphism $f\colon X \to Y$.
Solution. One needs to define both, give examples of morphisms that are unramified and not etale and that are not even unramified. Explain the equivalences: etale $\Leftrightarrow$ flat and unramified $\Leftrightarrow$ flat and $\Omega = 0 \Leftrightarrow$ smooth of relative dimension zero.

Problem F. (2+2+1 points)
Let $f\colon X \to Y$ be a projective morphism of Noetherian schemes and let $\mathcal{F}$ be a coherent sheaf on X which is flat over Y.

(i) Show that $f_*\mathcal{F}$ is torsion free.

(ii) What can be said about the higher direct images $R^i f_*\mathcal{F}$?

(iii) Find examples that show that the flatness in (i) is essential.

Solution. i) We may assume that $Y = \mathrm{Spec}(A)$ for some Noetherian ring A. Higher direct images of f can be computed as cohomology modules of a complex $0 \to K_0 \to \ldots \to K_n \to 0$ of projective A-modules, so $f_*\mathcal{F}$ is a submodule in K_0 which is torsion free.

ii) Let E be an elliptic curve, $e \in E$ a closed point. Consider the divisor $D = \Delta - \{e\} \times E$ in $E \times E$, where Δ is the diagonal. Let $\mathcal{F} = \mathcal{O}(D)$, $X = E \times E$, and f the projection to the first factor. Then $R^1 f_*\mathcal{F}$ is supported at the point e: when $t \in E$ and $t \neq e$ then $\mathcal{F}_t \simeq \mathcal{O}(t - e)$ and $H^1(E, \mathcal{F}_t) = 0$; when $t = e$ then $\mathcal{F}_e \simeq \mathcal{O}$ and $H^1(E, \mathcal{F}_e) = k$.

iii) Take X to be the normalization of $\mathrm{Spec}(k[x, y]/(xy))$ and $\mathcal{F} = \mathcal{O}_X$.

Problem G. (5 points)
Let $\varphi\colon C \to C$ be an endomorphism of a smooth projective connected curve C over a field k. Consider the graph and the diagonal $\Gamma_\varphi, \Delta \subset C \times C$. Decide under which conditions the invertible sheaf $\mathcal{O}(\Delta - \Gamma_\varphi)$ is of the form $\mathrm{pr}_1^*\mathcal{L}$.

Solution. If $g(C) = 0$ then $\mathcal{O}(\Delta) = \mathcal{O}(1, d)$ and $\mathcal{O}(\Gamma_\phi) = \mathcal{O}(d, 1)$, so $\mathcal{O}(\Delta - \Gamma_\varphi) = \mathcal{O}(1 - d, 0) = \mathrm{pr}_1^*\mathcal{O}(1 - d)$.

Consider the case $g(C) > 0$ and assume that $\mathcal{O}(\Delta - \Gamma_\varphi) = \mathrm{pr}_1^*\mathcal{L}$. Consider the fibre C_t of pr_1 over some point $t \in C$. By assumption $\mathcal{O}(\Delta - \Gamma_\varphi)|_{C_t} \simeq \mathcal{O}_{C_t}$. But we also have $\mathcal{O}(\Delta - \Gamma_\varphi)|_{C_t} \simeq \mathcal{O}_{C_t}(t - \varphi(t))$, so we see that the effective divisors t and $\varphi(t)$ are linearly equivalent, which implies that $t = \varphi(t)$. So our condition is equivalent to $\Delta = \Gamma_\varphi$.