Exam, Algebraic Geometry I — Solutions

Problem A. (2+2 points)
Let $\varphi\colon \mathcal{L} \to \mathcal{M}$ be a homomorphism of invertible sheaves on a scheme (or, more generally, a locally ringed space). Show that φ is an isomorphism if φ is surjective and give an example where φ is injective but not an isomorphism.

Solution. We need to show that φ is injective when it is surjective. Let $x \in X$ be a point of the locally ringed space and $A = \mathcal{O}_{X,x}$. The sheaves $\mathcal{L}$, $\mathcal{M}$ are locally free of rank one, hence $\mathcal{L}_x = \mathcal{M}_x = A$ as A-modules. Then $\varphi_x : A \to A$ is surjective by assumption. This means that $\varphi_x(1) = a$ is invertible in A, so $\varphi_x(b) = ab$ is non-zero when $b \neq 0$. This shows that φ_x is injective for any point $x \in X$, hence φ is injective.

For the example one can consider $\mathbb{P}^1$ and the sequence $0 \to \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}(1) \to \mathcal{O}_x \to 0$, where the map $\mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}(1)$ is defined by a non-zero section of $\mathcal{O}_{\mathbb{P}^1}(1)$ and x is its zero locus. The morphism $\mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}(1)$ is injective but not surjective.

Problem B. (2 points + 1 extra point if they don't use Riemann-Roch)
Let X be a projective regular curve over an algebraically closed field k and let $x_1, x_2 \in X$ be two closed points. Show that $\chi(X, \mathcal{O}(x_1 - x_2))$ is independent of the chosen points x_1, x_2.

Solution. Let $\mathcal{L} = \mathcal{O}(x_1 - x_2)$. By Riemann-Roch formula we have $\chi(\mathcal{L}) = \deg(\mathcal{L}) + 1 - g$ where g is the genus of the curve. The degree of $\mathcal{L}$ is zero and does not depend on x_1, x_2.

Alternative solution. We have two exact sequences: $0 \to \mathcal{O}_X(-x_2) \to \mathcal{O}_X \to \mathcal{O}_{x_2} \to 0$ and $0 \to \mathcal{O}_X(-x_1) \to \mathcal{O}_X \to \mathcal{O}_{x_1} \to 0$. Tensoring both of them by $\mathcal{O}(x_1)$ we get:

$$0 \to \mathcal{O}_X(x_1 - x_2) \to \mathcal{O}_X(x_1) \to \mathcal{O}_{x_2} \to 0,$$

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(x_1) \to \mathcal{O}_{x_1} \to 0,$$

where we used that $\mathcal{O}_{x_i}$ are skyscraper sheaves and are not changed when tensoring by a line bundle. Then we can use additivity of the Euler characteristic: $\chi(\mathcal{O}_X(x_1 - x_2)) = \chi(\mathcal{O}_X(x_1)) - \chi(\mathcal{O}_{x_2}) = \chi(\mathcal{O}_X(x_1)) - 1$ and $\chi(\mathcal{O}_X(x_1)) = \chi(\mathcal{O}_X) + \chi(\mathcal{O}_{x_1}) = \chi(\mathcal{O}_X) + 1$. So we conclude that $\chi(\mathcal{O}_X(x_1 - x_2)) = \chi(\mathcal{O}_X)$.

Problem C. (2+2 points)
Consider the standard open set $D(x) \subset \mathbb{A}^1_k = \operatorname{Spec}(k[x])$ and the coherent sheaf $\mathcal{F} = \tilde{M}$, where M is the $k[x, x^{-1}]$-module $k[x]/(x-1) \oplus k[x, x^{-1}]$. Describe a coherent extension of $\mathcal{F}$ to $\mathbb{A}^1_k$, i.e. a coherent sheaf $\mathcal{G}$ on $\mathbb{A}^1_k$ with $\mathcal{G}|_{D(x)} \cong \mathcal{F}$. Is this extension unique?

Solution. Consider the $k[x]$-module $N = k[x]/(x-1) \oplus k[x]$. This module defines a coherent sheaf on $\mathbb{A}^1_k$. Passage to the open subset $D(x)$ is given by localization with respect to the multiplicative system S generated by x. The first summand of N stays the same under localization, since x acts on it identically. The second summand localizes to $k[x, x^{-1}]$. We see that M is the localization of N, so $\mathcal{G} = \tilde{N}$ gives an extension of $\tilde{M}$. The extension is not unique. To see this, note that there exist $k[x]$-modules that have trivial localization with respect to S, for example $k[x]/(x)$. So $N' = N \oplus k[x]/(x)$ also defines an extension.

Problem D. (1 + 1 + 2 + 2 points + 1 extra point if they treat the geometric generic fibre) Let k be an algebraically closed field, $\mathrm{char}(k) \neq 3$. Consider the scheme $X \subset \mathbb{P}^1_k \times \mathbb{P}^2_k$ for which the fibres of the first projection $X \to \mathbb{P}^1_k$ over closed points $[t_0 : t_1]$ are the curves $X_{[t_0:t_1]} \subset \mathbb{P}^2_k$ given by the equation $t_0(x_0^3 + x_1^3 + x_2^3) + t_1 x_0 x_1 x_2 = 0$. Describe X as the zero locus of a section of a line bundle. (The scheme X is the total space of the Hesse pencil.) Find a closed point for which the fibre is irreducible and a closed point for which the fibre is reducible. Is the generic fibre integral? (You will get extra points if you consider last question for the geometric generic fibre.)

Solution. Note that t_0 and t_1 are sections of $\mathcal{O}_{\mathbb{P}^1}(1)$, while $x_0^3 + x_1^3 + x_2^3$ and $x_0x_1x_2$ are sections of $\mathcal{O}_{\mathbb{P}^2}(3)$. So the scheme X is the zero locus of the section $t_0(x_0^3+x_1^3+x_2^3)+t_1x_0x_1x_2$ of $\pi_1^*\mathcal{O}_{\mathbb{P}^1}(1) \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^2}(3)$ where π_1 and π_2 are projections to the two factors.

For the point $[t_0 : t_1] = [0 : 1]$ the fibre over it is reducible, being the union of three lines $x_0 = 0$, $x_1 = 0$, $x_2 = 0$.

Consider the point $[t_0 : t_1] = [1 : 0]$ and the fibre over it, given by $p(x_0, x_1, x_2) = x_0^3+x_1^3+x_2^3 = 0$. Let us prove that the fibre is irreducible. Assume the contrary, then $p = p_1p_2$ for some non-constant polynomials p_1 and p_2. Let us denote by ∂_{x_i} the partial derivative with respect to x_i. We have $\partial_{x_i}p = p_1\partial_{x_i}p_2 + p_2\partial_{x_i}p_1$. We see that p and all its partial derivatives vanish at those closed points $[x_0 : x_1 : x_2]$ where $p_1(x_0, x_1, x_2) = p_2(x_0, x_1, x_2) = 0$. Since both p_1 and p_2 are non-constant, there exists at least one such point (to see this, note that one of the two polynomials, say p_1, is linear and defines a line in $\mathbb{P}^2_k$, then p_2 is a quadratic polynomial, and its restriction to the line has a zero because k is algebraically closed). This leads to a contradiction: compute $\partial_{x_0}p = 3x_0^2$, $\partial_{x_1}p = 3x_1^2$, $\partial_{x_2}p = 3x_2^2$, and observe that these polynomials have no common zeros.

The geometric generic fibre is the scheme obtained by base change to $\mathrm{Spec}(K)$, where K is the algebraic closure of the residue field $k(t_0, t_1)$ at the generic point. The geometric generic fibre is given by the equation $x_0^3 + x_1^3 + x_2^3 + (t_1/t_0)x_0x_1x_2 = 0$ where $\gamma = t_1/t_0$ is viewed as an element of K. Let us prove that the geometric generic fibre is integral, that is the polynomial $x_0^3 + x_1^3 + x_2^3 + \gamma x_0x_1x_2 \in K[x_0, x_1, x_2]$ is irreducible. Arguing as above, we have the condition of vanishing of partial derivatives: $3x_0^2 = \gamma x_1x_2$, $3x_1^2 = \gamma x_0x_2$, $3x_2^2 = \gamma x_0x_1$. These conditions imply $x_0^3 = x_1^3 = x_2^3$, which implies that $[x_0 : x_1 : x_2] = [a : b : c]$ where $a, b, c \in k$ are cubic roots of unity. None of such points satisfies the equation $3x_0^2 = \gamma x_1x_2$, so we see that the vanishing locus of partial derivatives is empty. This proves that the geometric generic fibre is integral. This also implies integrality of the generic fibre (which is a scheme over $k(t_0, t_1)$).

Problem E. (4 points)
Let $\varphi\colon \mathbb{P}^n_k \to X$ be a morphism of projective k-schemes. Show that either the image of φ consists of a single point or that φ is quasi-finite.

Solution. Decompose φ as $\mathbb{P}^n_k \hookrightarrow \mathbb{P}^n_X \to X$, where $\mathbb{P}^n_X = \mathbb{P}^n_k \times_{\mathrm{Spec}(k)} X$. Denote by $i\colon \mathbb{P}^n_k \hookrightarrow \mathbb{P}^n_X$ the closed embedding and by $p_1 : \mathbb{P}^n_X \to \mathbb{P}^n_k$, $p_2 : \mathbb{P}^n_X \to X$ the two projections. Then $\varphi = p_2 \circ i$. Let $\mathcal{L}$ be a very ample bundle on X, then $\varphi^*\mathcal{L} = \mathcal{O}_{\mathbb{P}^n_k}(d)$ for some d. The corresponding linear system should be non-empty, so $d \geq 0$. If $d = 0$ then the map φ is constant and the image of φ is a pont. Assuming that $d > 0$, let us prove that the map φ is quasi-finite, which means that the fibres of φ are finite. Pick a point $x \in X$ and denote by $K = k(x)$ the residue field at x. The scheme-theoretic fibre $F = \varphi^{-1}(x)$ is the fibre product $\mathbb{P}^n_k \times_X \mathrm{Spec}(K)$. Consider the disagram obtained by base change:

$$\begin{tikzcd}
F \arrow[r, hook, "j"] \arrow[d, "\alpha"] & \mathbb{P}^n_K \arrow[r, "\pi"] \arrow[d, "\beta"] & \mathrm{Spec}(K) \arrow[d, "\gamma"] \\
\mathbb{P}^n_k \arrow[r, hook, "i"] & \mathbb{P}^n_X \arrow[r, "p_2"] & X
\end{tikzcd}$$

Both squares are Cartesian, i and j are closed embeddings. Note that $\alpha^*\varphi^*\mathcal{L} = j^*\pi^*\gamma^*\mathcal{L} = j^*\pi^*\mathcal{O}_{\mathrm{Spec}(K)} = \mathcal{O}_F$ is the trivial bundle. On the oter hand,

$$\alpha^*\varphi^*\mathcal{L} = \alpha^*\mathcal{O}_{\mathbb{P}^n_k}(d) = \alpha^*i^*p_1^*\mathcal{O}_{\mathbb{P}^n_k}(d) = j^*\beta^*p_1^*\mathcal{O}_{\mathbb{P}^n_k}(d) = j^*\mathcal{O}_{\mathbb{P}^n_K}(d).$$

This means that the restriction to F of the very ample line bundle $\mathcal{O}_{\mathbb{P}^n_K}(d)$ is trivial. This restriction should be a very ample line bundle on F, which implies that the fibre F consists of a finite number of points.

Problem F. (4+3 points)
Let $X = \mathbb{P}^1_k$ over a field k. Consider the short exact sequences $0 \to \mathcal{O}_X \to \mathcal{K}_X \to \mathcal{K}_X/\mathcal{O}_X \to 0$ and $0 \to \mathcal{O}_X^* \to \mathcal{K}_X^* \to \mathcal{K}_X^*/\mathcal{O}_X^* \to 0$. Do they define flasque resolutions of $\mathcal{O}_X$ and $\mathcal{O}_X^*$, respectively? What can you conclude for the cohomology H^i, $i > 1$ of $\mathcal{O}_X$ and $\mathcal{O}_X^*$?

Solution. 1) Consider the sequence $0 \to \mathcal{O}_X \to \mathcal{K}_X \to \mathcal{K}_X/\mathcal{O}_X \to 0$. We will prove that it is a flasque resolution of $\mathcal{O}_X$. The sheaf $\mathcal{K}_X$ is a constant sheaf, so it is flasque because X is an irreducible topological space. Indeed, every open subset $U \subset X$ is connected. So any section of $\mathcal{K}_X$ over U is given by one rational function on X, and the same rational function can be considered as a section of $\mathcal{K}_X$ over X.

Let us prove that the sheaf $\mathcal{K}_X/\mathcal{O}_X$ is flasque. Denote by η the generic point of X and note that $\mathcal{K}_{X,\eta}/\mathcal{O}_{X.\eta} = 0$. Let $U \subset X$ be an open subset and $s \in \Gamma(U, \mathcal{K}_X/\mathcal{O}_X)$ be a section. The support of s (i.e. the set of points $x \in X$ where $s_x \neq 0$ as an element of $\mathcal{K}_{X,x}/\mathcal{O}_{X.x}$) is closed. It does not contain the generic point η, because the stalk at this point is trivial. So the support of s consists of a finite number of points $p_1, \ldots, p_n$. By the sheaf axiom the section s can be glued with the zero section over $X \backslash \{p_1, \ldots, p_n\}$. This gives an extension $s' \in \Gamma(X, \mathcal{K}_X/\mathcal{O}_X)$ of s and we see that the sheaf $\mathcal{K}_X/\mathcal{O}_X$ is flasque.

We get the long exact cohomology sequence

$$0 \to k \to K_X \to H^0(X, \mathcal{K}_X/\mathcal{O}_X) \to H^1(X, \mathcal{O}_X) \to 0 \to 0 \to H^2(X, \mathcal{O}_X) \to 0 \to \ldots$$

From this we see immediately that $H^i(X, \mathcal{O}_X) = 0$ for $i > 1$.

2) Consider the sequence $0 \to \mathcal{O}_X^* \to \mathcal{K}_X^* \to \mathcal{K}_X^*/\mathcal{O}_X^* \to 0$. As before, the sheaf $\mathcal{K}_X^*$ is flasque because it is a constant sheaf on an irreducible space. The sheaf $\mathcal{K}_X^*/\mathcal{O}_X^*$ is also flasque. To see this recall that for an open subset $U \subset X$ a section s of $\mathcal{K}_X^*/\mathcal{O}_X^*$ is a Cartier (= Weil) divisor on U. A divisor on U (which can be represented as a finite linear combination of closed points) can be extended to X.

We get the long exact cohomology sequence

$$0 \to k^* \to K_X^* \to H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*) \to H^1(X, \mathcal{O}_X^*) \to 0 \to 0 \to H^2(X, \mathcal{O}_X^*) \to 0 \to \ldots$$

From this sequence we conclude that $H^i(X, \mathcal{O}_X^*) = 0$ for $i > 1$.

Problem G. (2+1 points + 2 extra points if they come up with an example that disproves the last claim)
Let X be a projective scheme over $k = \bar{k}$ and let $\mathcal{L} \in \mathrm{Pic}(X)$. Show that the base locus $\mathrm{Bs}(\mathcal{L})$ of $\mathcal{L}$ contains the base locus $\mathrm{Bs}(\mathcal{L}^n)$ of any power $\mathcal{L}^n$, $n > 0$. Do you think it is true that for $n > m$ one has $\mathrm{Bs}(\mathcal{L}^n) \subset \mathrm{Bs}(\mathcal{L}^m)$?

Solution. The base locus of $\mathcal{L}$ is the intersection of zero loci of all sections of $\mathcal{L}$. If $x \in X$ is not in $\mathrm{Bs}(\mathcal{L})$, then there exists a section $s \in \Gamma(X, \mathcal{L})$ that does not vanish at x, meaning that $s_x \notin \mathfrak{m}_x \subset \mathcal{O}_{X,x}$. Then $s^n \in \Gamma(X, \mathcal{L}^n)$ does not vanish at x also, so $x \notin \mathrm{Bs}(\mathcal{L}^n)$.

The answer to the last question is negative. As an example consider an elliptic curve X and a non-trivial line bundle $\mathcal{L}$ on X of degree zero, and such that $\mathcal{L}^2 = \mathcal{O}_X$ (a 2-torsion element in the Picard group). Then $\mathcal{L}$ has no non-zero sections (a non-zero section would vanish at some points, so the degree of $\mathcal{L}$ would be positive). So $\mathrm{Bs}(\mathcal{L}^3) = \mathrm{Bs}(\mathcal{L}) = X$. But $\mathrm{Bs}(\mathcal{L}^2) = \emptyset$, because $\mathcal{L}^2 = \mathcal{O}_X$ has a nowhere vanishing section. We see that $\mathrm{Bs}(\mathcal{L}^3) \not\subseteq \mathrm{Bs}(\mathcal{L}^2)$.