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Introduction to Kähler geometry

Exercise sheet 4

Exercise 4.1. Check the following properties of the Hodge $*$-operator on an n-dimensional Riemannian manifold M:

1. $*^2 = (-1)^{k(n-k)}\mathrm{Id}$ on $\Lambda^k M$;
2. $*1 = \mathrm{Vol}$;
3. $g(*\alpha, *\beta) = g(\alpha, \beta)$;
4. $\alpha \wedge *\beta = \beta \wedge *\alpha$.

Exercise 4.2. Prove that on a complex manifold M of complex dimension n with Hermitian metric g the $\mathbb{C}$-linear extension of the $*$-operator maps $\Lambda^{p,q}M$ to $\Lambda^{n-q,n-p}M$.

Exercise 4.3. Show that for a compact Kähler manifold M with Kähler metric g, Kähler form ω and $\dim_{\mathbb{C}} M = n$ we have

$$\int_M \omega^n = n!\,\mathrm{Vol}(M),$$

where $\mathrm{Vol}(M)$ is the integral of the Riemannian volume form corresponding to the metric g.

Exercise 4.4. Prove that on any Kähler manifold we have $0 = dd^c + d^c d = dd^{c*} + d^{c*}d = d^*d^{c*} + d^{c*}d^* = d^*d^c + d^c d^*$. Also, check that $dd^c = 2\sqrt{-1}\partial\bar{\partial}$.

Exercise 4.5. Let M be a Kähler manifold with complex structure I and Kähler form ω. Prove that the Laplace operator Δ commutes with I, L_ω and Λ_ω.

Exercise 4.6. Let M be a compact Kähler manifold and $\alpha \in \Gamma(M, \Lambda^k M)$ a holomorphic k-form. Prove that $d\alpha = 0$.

Exercise 4.7. Prove that on any Kähler manifold the Kähler form is harmonic.