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Summer term 2015

Exercises, Algebra I (Commutative Algebra) – Week 2

Aufgabe 7. (4 points)
Consider two A-module homomorphisms $g\colon M_1 \to M_2$ and $f\colon M_2 \to M_3$. Assume that for all A-modules N the induced sequence

$$0 \longrightarrow \mathrm{Hom}(M_3, N) \xrightarrow{\circ f} \mathrm{Hom}(M_2, N) \xrightarrow{\circ g} \mathrm{Hom}(M_1, N)$$

is exact. Show that then $M_1 \to M_2 \to M_3 \to 0$ is exact.

Aufgabe 8. (6 points)
Consider the polynomial ring $A[X]$ for an arbitrary ring and let $f = a_0 + a_1X + \dots a_nX^n \in A[X]$. Prove the following assertions

i) f is a unit if and only if a_0 is a unit and a_i, $i > 0$ are nilpotent.

ii) f is nilpotent if and only if all a_i are nilpotent.

iii) f is a zero divisor if and only if there exists an $0 \neq a \in A$ with $af = 0$.

Aufgabe 9. (6 points)

Consider short exact sequences $0 \longrightarrow M^i \xrightarrow{f_i} N^i \xrightarrow{g_i} P^i \longrightarrow 0$ of A-modules and module homomorphisms $a_i\colon M^i \to M^{i+1}$, $b_i\colon N^i \to N^{i+1}$ and $c_i\colon P^i \to P^{i+1}$ such that $a_{i+1} \circ a_i = b_{i+1} \circ b_i = c_{i+1} \circ c_i = 0$, $b_i \circ f_i = f_{i+1} \circ a_i$ and $c_i \circ g_i = g_{i+1} \circ b_i$. (In short: 'a short exact sequences of complexes' $0 \to M^\bullet \to N^\bullet \to P^\bullet \to 0$.)
Define $H^i(M^\bullet) \coloneqq \mathrm{Ker}(a_i)/\mathrm{Im}(a_{i-1})$ (the 'cohomology of the complex $M^\bullet$') and similarly for $N^\bullet$ and $P^\bullet$. Imitate the proof of the snake lemma in class and prove that there exists a natural exact sequence

$$H^i(M^\bullet) \to H^i(N^\bullet) \to H^i(P^\bullet) \to H^{i+1}(M^\bullet) \to H^{i+1}(N^\bullet) \to H^{i+1}(P^\bullet).$$

Aufgabe 10. (6 points)
For ideals $\mathfrak{a}, \mathfrak{b} \subset A$ one defines the *ideal quotient* as $(\mathfrak{a} : \mathfrak{b}) \coloneqq \{a \in A \mid a\mathfrak{b} \subset \mathfrak{a}\}$. Prove the following assertions.

i) $(\mathfrak{a} : \mathfrak{b})$ is an ideal with $\mathfrak{a} \subset (\mathfrak{a} : \mathfrak{b})$.

ii) $(\mathfrak{a}_1 \cap \mathfrak{a}_2 : \mathfrak{b}) = (\mathfrak{a}_1 : \mathfrak{b}) \cap (\mathfrak{a}_2 : \mathfrak{b})$.

iii) $(\mathfrak{a} : \mathfrak{b}_1 + \mathfrak{b}_2) = (\mathfrak{a} : \mathfrak{b}_1) \cap (\mathfrak{a} : \mathfrak{b}_2)$.

Please turn over.

Due Monday April 20.

Aufgabe 11. (6 points)
For an ideal $\mathfrak{a} \subset A$ one defines the *radical* of $\mathfrak{a}$ as

$$\mathfrak{r}(\mathfrak{a}) := \{a \in A \mid a^n \in \mathfrak{a} \text{ for some } n > 0\}.$$

Prove the following assertions.

i) $\mathfrak{r}(\mathfrak{a})$ is an ideal with $\mathfrak{a} \subset \mathfrak{r}(\mathfrak{a})$.

ii) $\mathfrak{r}(\mathfrak{a}\mathfrak{b}) = \mathfrak{r}(\mathfrak{a} \cap \mathfrak{b}) = \mathfrak{r}(\mathfrak{a}) \cap \mathfrak{r}(\mathfrak{b})$.

iii) $\mathfrak{r}(\mathfrak{a}) = (1)$ if and only if $\mathfrak{a} = (1)$.

Aufgabe 12. (6 points)
Let I be a partially ordered directed set, i.e. for all $i, j \in I$ there exists $k \in I$ with $i, j \leq k$. Consider a family of A-modules M_i, $i \in I$ and homomorphisms $f_{ij} \colon M_i \to M_j$ for all $i \leq j$ such that $f_{ii} = \mathrm{id}$ and $f_{ik} = f_{jk} \circ f_{ij}$ for all $i \leq j \leq k$. (This is called a 'directed system of A-modules'.) Let $\varinjlim M_i$ be the quotient of $\bigoplus M_i$ by the submodule generated by all elements of the form $m_i - f_{ij}(m_i)$, where $m_i \in M_i$ and $f_{ij} \colon M_i \to M_j$. In particular, there exist natural homomorphisms $f_i \colon M_i \to \varinjlim M_i$.

i) Show that every element of $\varinjlim M_i$ is the image of an element of the form $m_i \in M_i \subset \bigoplus M_i$.

ii) Show that $\varinjlim M_i$ has the following universal property: For an A-module N and homomorphisms $g_i \colon M_i \to N$ there exists a unique $g \colon \varinjlim M_i \to N$ with $g \circ f_i = g_i$ for all i if and only if $g_j \circ f_{ij} = g_i$ for all $i \leq j$.