Graduate Seminar on Advanced Geometry S4D3
Topics in Complex Geometry and Hodge Theory

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Winter term 2017/18, Thursday 16-18, room 1.008

Preliminary meeting: 31.07.17, 10.00-12.00, room 0.003 (Endenicher Allee 60)

The goal of the seminar is to discuss various topics related to the study of algebraic cycles by Hodge-theoretic methods. This is a vast field of research, its origins going back to Poincaré. We will discuss the theory in its modern form, as it appears in the works of Deligne and Griffiths. We will start by discussing intermediate Jacobians and Abel-Jacobi maps, Deligne cohomology, normal functions and Hodge loci. In the end we will discuss more recent developments (due to Voisin and others) related to decomposition of the diagonal and integral Hodge classes.

Below is a tentative list of topics for the talks. Each topic should be covered in 2-3 talks. We can decide how much time to spend on certain topics depending on the preferences of the audience.

For further information or if you want to give a talk in the seminar, please contact aosoldat@.

I: Intermediate Jacobians

1. A brief reminder on Hodge theory: Hodge decomposition, Hodge to de Rham spectral sequence, pure Hodge structures (follow [Gre] lecture 1 or [V1] chapters 6, 7, 8).

2. Intermediate Jacobians. Explain the construction ([V1], section 12.1), define the Abel-Jacobi map and prove that it is holomorphic ([V1], theorem 12.4). Compute the infinitesimal AbelJacobi map ([V1], Lemma 12.6). View Albanese and Picard variety as special cases, prove [V1] Lemma 12.11. Explain the universality property of the Albanese. Construction in the algebraic category (Serre).

3. The algebraic part of the intermediate Jacobian. Consider homological modulo algebraic equivalence and define the Griffiths group ([V1], section 12.2). State Griffithss result for the quintic threefold saying that there are non-torsion points in the Griffiths group. Mention the results of Clemens and Voisin that the Griffiths group is not necessarily finitely generated ([V1], theorem 12.21). State the conjecture asserting the surjectivity of the Abel-Jacobi map onto the algebraic part and show that it would follow from the Hodge conjecture. Prove it for uniruled threefolds, see ([V1], exercises to chapter 12).

II: Deligne cohomology

1. Define Deligne cohomology ([V1], section 12.3 and [EV] §1). Explain the relation to Hodge cycles and to the intermediate Jacobian ([V1], proposition 12.26). Discuss low degree cases in ([EV], section 1.4, 1.5, Lemma 1.6]). Define the cup product for Deligne cohomology classes.

2. Sketch the construction of fundamental classes in Deligne cohomology, see ([EV], §7, [V1], 12.3.2, 12.3.3). Explain why it factors through Chow groups ([EV], Prop. 7.6, Cor. 7.7). Compare it to the Abel-Jacobi map. View the intermediate Jacobians as an ideal of square zero ([EV], Prop. 7.10).

3. Chern classes in Deligne cohomology via splitting principle (following [EV] section 8 or [Gri]).

III: Hodge locus

1. Brief overview of variations of Hodge structures (following [V2] chapter 5 or [Gre] lecture 3): Kodaira-Spencer map, Gauss-Manin connection, Griffiths transversality.

2. Define the Hodge locus ([V2], section 5.3) and derive the local description ([V2] proposition 5.14). First order description of the Hodge locus ([V2], Lemma 5.16). The components of the Noether-Lefschetz are the Hodge loci for degree two classes. Discuss the case of curves in surfaces ([V2] proposition 5.19). In interesting cases the Noether-Lefschetz locus is dense ([V2], proposition 5.20).

3. The Hodge locus is algebraic: overview of [CDK].

IV: Normal functions and infinitesimal invariants

1. Define Jacobian fibration associated to an integral variation of Hodge struvtures ([V2] section 7.1.1). Define normal functions and infinitesimal invariants ([V2] sections 7.1.2, 7.1.3). Abel-Jacobi map in familie, Griffiths theorem ([V2] theorem 7.9).

2. Normal functions via Deligne cohomology ([Gre] page 70). Geometric interpretation of the infinitesimal invariant ([V2] section 7.2.2)

3. We may include the discussion of the case of hypersurfaces in $\mathbb{P}^n$ (section 7.3 in [V2]). The goal will be to prove theorem 7.19 in [V2]. Preliminary discussion of Hodge filtration on hypersurfaces and monodromy action may be necessary.

V: Bloch-Srinivas construction and Mumford's theorem

1. Briefly recall necessary facts about Chow groups ([V2] chapter 9). Define when the group of zero-cycles is called representable ([V2] section 10.1.1.), prove proposition 10.10 in [V2]. Prove Roitman's theorem 10.11.

2. Mumford's theorem (follow section 10.1.3 of [V2]). Bloch-Srinivas construction: prove theorem 10.19 in [V2]. Deduce Mumford's theorem (section 10.2.2 in [V2]).

VI: Integral Hodge classes, birational invariants and decomposition of the diagonal

1. Counterexamples to the integral Hodge conjecture ([V3] section 6.1). Explain the approach of Totaro to Atiyah-Hirzebruch examples: define the complex cobordism ring and explain its main properties ([T1] section 1 or [V3] section 6.1.1), explain that the cycle class map factors through complex cobordism ([T1] section 3 or [V3] section 6.1.1). Explain the construction of Godeaux-Serre varieties ([AH] proposition 6.6, see also [T2] section 5), if possible give an idea why the cycle class map has non-trivial cokernel for these varieties. Explain Kollár's example in [V3] section 6.1.2.

2. Integral Hodge classes and rationality problem (follow section 6.2 in [V3]). Give a brief introduction to rationality problem and explain some relevant notions (rational, unirational, rationally connected varieties; this can be found e.g. in [Ko]). Explain the relation of integral Hodge classes to rationality problem (Lemma 6.3 in [V3]). Sketch the proofs of Voisin's results on $Z^{2n-2}(X)$ ([V3] theorems 6.5, 6.9, 6.10). If time permits, explain the relation of $Z^4(X)$ to unramified cohomology ([V3] section 6.2.2).

3. Decomposition of the diagonal and structure of the Abel-Jacobi map (following [V3] section 6.3). Define the integral decomposition of the diagonal and explain its consequences; explain the natural questions related to the structure of Abel-Jacobi map ([V3] qestions 6.26, 6.29, 6.30). Prove theorems 6.31, 6.38, 6.41 in [V3].

References

[AH] M. Atiyah, F. Hirzebruch, *Analytic cycles on complex manifolds*, Topology 1, 25-45, 1962.

[CDK] E. Cattani, P. Deligne, A. Kaplan, *On the locus of Hodge classes*, JAMS. 8, (1995), 483-506.

[EV] H. Esnault, E. Viehweg, *Deligne-Beilinson cohomology*, in [RSS].

[Gri] J. Grivaux, *Chern classes in Deligne cohomology for coherent analytic sheaves*, arXiv:0712.2207.

[Gre] M. Green, *Infinitesimal Methods in Hodge theory*, CIME Notes. Springer 1994.

[Ko] J. Kollár, *Rational Curves on Algebraic Varieties*, Springer 1999.

[RSS] M. Rapoport, N. Schappacher, P. Schneider, *Beilinsons conjectures on special values of L-functions*, Perspectives in Math. Academic Press 1988.

[T1] B. Totaro, *Torsion algebraic cycles and complex cobordism*, J. Amer. Math. Soc. 10 467-493, 1997.

[T2] B. Totaro, *The Chow ring of a classifying space*. Algebraic K-theory (Seattle, WA, 1997), 249-281, Proc. Sympos. Pure Math., 67, Amer. Math. Soc., Providence, RI, 1999

[V1] C. Voisin, *Hodge theory and complex algebraic geometry I*, CUP 2002.

[V2] C. Voisin, *Hodge theory and complex algebraic geometry II*, CUP 2003.

[V3] C. Voisin, *Chow Rings, Decomposition of the Diagonal, and the Topology of Families*, Princeton University Press 2014.