Prof. Dr. Daniel Huybrechts
Dr. Andrey Soldatenkov

Summer term 2016

Exercises, Algebraic Geometry II – Week 5

Exercise 21. (3 points) *Flatness of morphisms.*
Recall the definition of flatness of a morphism. Try to find examples and non-examples of flat morphisms. In particular, answer the following questions:

1. Let $f\colon X \to \mathbb{A}^n_k$ be the blow-up of the origin. Is f flat?
2. Consider $X = V(3x^2 + 6y^2) \subset \mathbb{A}^2_{\mathbb{Z}}$ and the natural projection $f\colon X \to \mathrm{Spec}(\mathbb{Z})$. Is f flat?
3. Let $X \subset \mathbb{A}^2_{\mathbb{Z}}$ be the closure (with reduced scheme structure) of $V(3x^2 + 6y^2) \subset \mathbb{A}^2_{\mathbb{Q}}$ in $\mathbb{A}^2_{\mathbb{Z}}$. Is the projection $f\colon X \to \mathrm{Spec}(\mathbb{Z})$ flat?

Exercise 22. (3 points) *Smoothness of morphisms.*
Decide whether the following morphisms are smooth.

1. $f\colon \mathbb{A}^1_k \to \mathbb{A}^1_k$ given by $k[x] \to k[y]$, $x \mapsto y^2$.
2. The natural projection $f\colon X \to \mathrm{Spec}(\mathbb{Z})$, where $X = V(y^2z - x^3 + 5xz^2) \subset \mathbb{P}^2_{\mathbb{Z}}$.
3. The natural projection $f\colon X \to \mathbb{A}^1_k$, $(x, y, t) \mapsto t$, where $X \subset \mathbb{A}^2_k \times \mathbb{A}^1_k$ is defined by $x^2 + y^2 = t$.

Exercise 23. (3 points) *Locally free = fibres of constant dimension.*
Let $\mathcal{F}$ be a coherent sheaf on a reduced, locally Noetherian scheme X. Show that $\mathcal{F}$ is locally free of rank r if and only if $x \mapsto \dim_{k(x)}(\mathcal{F} \otimes k(x))$ is the constant function $\equiv r$. Show that if X is in addition of finite type over a field then it is enough to check that this function is constant on the set of closed points.

Exercise 24. (5 points) *Koszul complex.*
Let A be a commutative ring and $a_1, \ldots, a_n \in A$ a sequence of elements. This sequence is called *regular*, if for any $1 \leq i \leq n$ the element a_i is not a zero-divisor in $A/(a_1, \ldots, a_{i-1})$. Consider the complex

$$K_i^\bullet = (\ldots \to 0 \to A \xrightarrow{a_i} A \to 0 \to \ldots)$$

concentrated in degrees -1 an 0, where the only non-zero differential is multiplication by a_i. Let $K^\bullet$ be the tensor product $K^\bullet = K_1^\bullet \otimes_A \ldots \otimes_A K_n^\bullet$, so that $K^\bullet$ is concentrated in degrees $-n, \ldots, 0$. Assuming that the sequence $a_1, \ldots, a_n$ is regular, prove that $H^i(K^\bullet) = 0$ for $i \neq 0$ and $H^0(K^\bullet) = A/(a_1, \ldots, a_n)$. The complex $K^\bullet$ is called *Koszul complex* and it gives a free resolution for the A-module $A/(a_1, \ldots, a_n)$.

Now let X be a variety, $\mathcal{E}$ a locally free sheaf of finite rank r on X and $s \in H^0(X, \mathcal{E})$ a section which is locally given by a regular sequence of elements in the corresponding ring (this is true

Due Monday 23 May, 2016. Before the lecture.

if the zero locus of s has codimension equal to the rank of $\mathcal{E}$). Let $Y \subset X$ be the zero locus of s. Observe that the local Koszul complexes glue into a global locally free resolution of $\mathcal{O}_Y$ (which is also called Koszul complex):

$$0 \to \Lambda^r \mathcal{E}^\vee \to \Lambda^{r-1}\mathcal{E}^\vee \to \ldots \to \Lambda^2 \mathcal{E}^\vee \to \mathcal{E}^\vee \to \mathcal{O}_X \to \mathcal{O}_Y \to 0.$$

The differentials are given by convolution with s.

Exercise 25. (4 points) *Universal family.*
Consider the 'incidence variety' $X \subset \mathbb{P}^n_k \times \mathbb{P}^N_k$ of all degree d hypersurfaces given by the equation $\sum_I a_I x^I = 0$, where $x^I = x_0^{i_0} \ldots x_n^{i_n}$ and the sum is over all $I = (i_0, \ldots, i_n)$ with $\sum i_k = d$. (So $N = \dim |\mathcal{O}_{\mathbb{P}^n_k}(d)|$ or, in other words, $\mathbb{P}^N_k = |\mathcal{O}_{\mathbb{P}^n_k}(d)|$.) The second projection $f\colon X \to \mathbb{P}^N_k$ is called the universal family of hypersurfaces of degree d. Show that f is flat.

Consider the fibres $X_{(a_I)}$, $(a_I) \in \mathbb{P}^N_k$. Is for all i the dimension of $H^i(X_{(a_I)}, \mathcal{O})$ constant? What about $\chi(X_{(a_I)}, \mathcal{O})$ or $\chi(X_{(a_I)}, \mathcal{O}(k))$?