Teichmüller spaces of hyperkähler manifolds and rigid currents

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Hyperkähler manifolds

Definition

A hyperkähler structure on a C^∞-manifold X is a tuple (g, I, J, K), where:

- *g is Riemannian metric;*
- *I, J and K are complex structures s.t. $IJ = -JI = K$;*
- *g is Kähler w.r.t. I, J and K.*

We have two-forms ω_I, ω_J and ω_K:

$$\omega_I(u, v) = g(Iu, v),$$
$$\omega_J(u, v) = g(Ju, v),$$
$$\omega_K(u, v) = g(Ku, v).$$

These forms are closed:

$$d\omega_I = d\omega_J = d\omega_K = 0.$$

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A Riemannian metric g as above is called hyperkähler.

Equivalently: g is hyperkähler if $\mathrm{Hol}(\nabla^g) \subset Sp(n)$,

∇^g is the Levi-Civita connection for g.

$Sp(n)$ = group of quaternionic-linear transformations of $\mathbb{H}^n$ that preserve the quaternionic-Hermitian scalar product.

Consider the 2-form $\sigma_I = \omega_J + i\omega_K$.

σ_I is a non-degenerate closed (2,0)-form on X_I, i.e. a holomorphic symplectic form.

Today we assume: a hyperkähler manifold X is compact and of maximal holonomy, i.e. $\mathrm{Hol}(\nabla^g) = Sp(n)$.

This implies: $\pi_1(X) = 1$ and $H^0(X_I, \Omega^2_{X_I}) = \mathbb{C}\sigma_I$.

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Since σ_I is symplectic, we have:

- $\dim_{\mathbb{C}}(X_I) = 2n$,
- σ_I^n is a nowhere vanishing section of $K_{X_I} = \Omega^{2n}_{X_I}$.

Theorem (Beauville, Bogomolov, Fujiki)

There exists $c_X \in \mathbb{Q}$ such that for all $a \in H^2(X, \mathbb{Q})$

$$\int_X a^{2n} = c_X q(a)^n,$$

where q is a quadratic form on $H^2(X, \mathbb{Q})$,
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Known deformation types of hyperkähler manifolds:

- $K3^{[n]}$-type. Let S be a complex projective K3 surface, i.e. a surface with $\pi_1(S) = 1$ and $K_S = \mathcal{O}_S$. $S^{[n]}$ = the Hilbert scheme of length n subschemes of S. The manifold $S^{[n]}$ is hyperkähler with $b_2 = 23$ for $n > 1$ and $b_2 = 22$ for $n = 1$.
- Kum^n-type. Let $T = \mathbb{C}^2/\mathbb{Z}^4$. The Albanese morphism:

$$a \colon T^{[n+1]} \to T, \quad (x_0, \ldots, x_n) \mapsto \sum x_i$$

  $K^n T = a^{-1}(0)$ = the generalized Kummer variety, it is hyperkähler with $b_2 = 7$ for $n > 1$.
- $OG6$ and $OG10$-types. O'Grady's exceptional hyperkähler manifolds of dimensions 6 and 10 with $b_2 = 8$ and $b_2 = 24$ respectively.

All other known examples of hyperkähler manifolds are obtained from the above by deforming the complex structure.

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The Teichmüller space

Denote:

- $\mathscr{D}\!\mathit{iff}(X)$ the Fréchet Lie group of diffeomorphisms of X;
- $\mathscr{D}\!\mathit{iff}^\circ(X) \subset \mathscr{D}\!\mathit{iff}(X)$ the connected component of the identity.

Definition

- $\mathcal{MCG}(X) = \mathscr{D}\!\mathit{iff}(X)/\mathscr{D}\!\mathit{iff}^\circ(X)$ *the mapping class group;*
- *A complex structure I on X is of hyperkähler type if I is part of a hyperkähler structure;*
- $\mathscr{I}(X)$ *= all complex structures of hyperkähler type on* X*;*
- *The Teichmüller space:* $\mathcal{T}(X) = \mathscr{I}(X)/\mathscr{D}\!\mathit{iff}^\circ(X)$. *It is a non-Hausdorff complex manifold.*

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Denote:

- $\mathscr{D}\!\mathit{iff}(X)$ the Fréchet Lie group of diffeomorphisms of X;
- $\mathscr{D}\!\mathit{iff}^\circ(X) \subset \mathscr{D}\!\mathit{iff}(X)$ the connected component of the identity.

Definition

- $\mathcal{MCG}(X) = \mathscr{D}\!\mathit{iff}(X)/\mathscr{D}\!\mathit{iff}^\circ(X)$ *the mapping class group;*
- *A complex structure I on X is of hyperkähler type if I is part of a hyperkähler structure;*
- $\mathscr{I}(X)$ *= all complex structures of hyperkähler type on* X;
- *The Teichmüller space:* $\mathcal{T}(X) = \mathscr{I}(X)/\mathscr{D}\!\mathit{iff}^\circ(X)$. *It is a non-Hausdorff complex manifold.*

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The Teichmüller space and the period domain

Denote:

$$V = H^2(X, \mathbb{Q}), \quad V_{\mathbb{R}} = V \otimes \mathbb{R}, \quad V_{\mathbb{C}} = V \otimes \mathbb{C}.$$

q is the BBF form on V of signature $(3, d-3)$, $d = \dim(V)$.

The period domain for $\mathcal{T}$:

$$\begin{aligned} \mathcal{D} &= \{L \subset V_{\mathbb{R}} \mid \dim(L) = 2, L \text{ is oriented and positive}\} \\ &\simeq O(3, d-3)/SO(2) \times O(1, d-3). \end{aligned}$$

The period map $\rho\colon \mathcal{T} \to \mathcal{D}$,

$$I \mapsto L = \langle \mathrm{Re}[\sigma_I], \mathrm{Im}[\sigma_I] \rangle$$

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The Torelli theorems

Theorem (The local Torelli — Beauville, Bogomolov)

The period map $\rho\colon \mathcal{T} \to \mathcal{D}$ is a local isomorphism of complex manifolds.

Identify the non-separated points of $\mathcal{T}$, get a Hausdorff complex manifold $\widetilde{\mathcal{T}}$.

$$\begin{tikzcd} \mathcal{T} \arrow[d] \arrow[dr, "\rho"] & \\ \widetilde{\mathcal{T}} \arrow[r, "\tilde{\rho}"] & \mathcal{D} \end{tikzcd}$$

Fix a connected component $\mathcal{T}^\circ$ of the Teichmüller space.

Theorem (The global Torelli — Verbitsky)

The period map $\tilde{\rho}^\circ\colon \widetilde{\mathcal{T}}^\circ \to \mathcal{D}$ is an isomorphism of complex manifolds.

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How to describe the non-separated points of $\mathcal{T}$,
i.e. the fibre of $\rho^\circ\colon \mathcal{T}^\circ \to \mathcal{D}$?

Theorem (Huybrechts)

The non-separated points of $\mathcal{T}$ correspond to bimeromorphic hyperkähler manifolds.

Fix $L \subset V_\mathbb{R}$ corresponding to a point $[L] = \rho^\circ(I) \in \mathcal{D}$.
The restriction of q to $V_\mathbb{R}^{1,1} = L^\perp \subset V_\mathbb{R}$ has signature $(1, d-3)$.

Definition

The positive cone $\mathcal{C}^+ \subset V_\mathbb{R}^{1,1}$ is the connected component of the set $\{x \in V_\mathbb{R}^{1,1} \mid q(x) > 0\}$ that contains the Kähler classes.

The points of the fibre $\rho^{\circ -1}[L]$ are distinguished by their Kähler cones $\mathcal{K}_{X_I} \subset \mathcal{C}^+$.

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Amerik, Verbitsky: there is a collection of integral classes

$$\mathrm{MBM} \subset H^2(X, \mathbb{Z})$$

called monodromy birationally minimal or MBM classes with the following properties.

Let $\mathrm{MBM}^{1,1} = \mathrm{MBM} \cap V_{\mathbb{R}}^{1,1}$. Then the hyperplanes $x^\perp$, where $x \in \mathrm{MBM}^{1,1}$, cut $\mathcal{C}^+$ into open chambers.

The points of the fibre $\rho^{\circ -1}[L]$ are in 1-1 correspondence with the chambers: each chamber is the Kähler cone for a manifolld X_I, where $I \in \rho^{\circ -1}[L]$.

Theorem (Amerik, Verbitsky)

There exists a constant $M > 0$ such that for all $x \in \mathrm{MBM}$ we have $-M \leqslant q(x) < 0$.

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Let $\mathcal{MCG}^\circ \subset \mathcal{MCG}$ be the stabilizer of the connected component $\mathcal{T}^\circ$.

There is a representation $r\colon \mathcal{MCG}^\circ \to O(V, q)$.

Theorem (Verbitsky, Markman)

$\Gamma = \mathrm{Im}(r)$ *is a finite index subgroup in* $O(H^2(X, \mathbb{Z}), q)$ *called the monodromy group.*

The period map $\rho^\circ\colon \mathcal{T}^\circ \to \mathcal{D}$ is $\mathcal{MCG}^\circ$-equivariant.
We want to understand the $\mathcal{MCG}^\circ$-orbits.

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The action of Γ is not properly discontinuous.

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Ratner's theory

Setting:

- $\mathbf{G} \subset SL(V)$ a connected semisimple algebraic $\mathbb{Q}$-group;
- $G = \mathbf{G}(\mathbb{R})^\circ$ the identity component of $\mathbf{G}(\mathbb{R})$;
- $\Gamma \subset \mathbf{G}(\mathbb{Q}) \cap G$ an arithmetic lattice;
- $S \subset G$ a Lie subgroup generated by unipotents.

Theorem (Ratner)

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Positive $(1,1)$-currents

Let X be Kähler, $\dim(X) = k$. Denote:

- $\mathcal{E} = H^0(X, \Lambda^{k-1,k-1}X)$ the space of C^∞ $(k-1, k-1)$-forms.
- $\mathcal{E}'$ the space of $(1,1)$-currents, i.e. continuous linear functionals on $\mathcal{E}$.

For $T \in \mathcal{E}'$ write $T \geqslant 0$ if $T = \overline{T}$ and for any $(1,0)$-forms $\eta_1, \ldots, \eta_{k-1}$ we have $\langle T, \prod_j \sqrt{-1}\eta_j \wedge \bar{\eta}_j \rangle \geqslant 0$.

Let $\alpha \in H^{1,1}_{\mathbb{R}}(X)$. Define

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- α is pseudo-effective if $\mathcal{C}_\alpha \neq \emptyset$;
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- α is pseudo-effective if $\mathcal{C}_\alpha \neq \emptyset$;
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Denote: $\mathcal{P}_X \subset H^{1,1}_{\mathbb{R}}(X)$ the set of pseudo-effective classes.

Positive $(1,1)$-currents

Let X be Kähler, $\dim(X) = k$. Denote:

- $\mathcal{E} = H^0(X, \Lambda^{k-1,k-1}X)$ the space of C^∞ $(k-1,k-1)$-forms.
- $\mathcal{E}'$ the space of $(1,1)$-currents, i.e. continuous linear functionals on $\mathcal{E}$.

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Rigid pseudo-effective classes

Example: S a projective surface and $C \subset S$ an irreducible curve with $C^2 < 0$. Then the cohomology class $[C]$ is rigid: the unique positive closed current in $[C]$ is the current of integration over C.

Demailly–Peternell–Schneider: an example of a surface S with a nef irreducible curve $C \subset X$ such that $[C]$ is rigid.

Cantat: Let X be Kähler. Call $\alpha \in \mathcal{P}_X$ dynamical if there exists an automorphism $f \colon X \to X$ and a volume form Vol on X such that $f^*\alpha = \lambda\alpha$ with $\lambda > 1$ and $f^*\mathrm{Vol} = \mathrm{Vol}$. One can show: a dynamical class α is rigid.

Note: a dynamical class satisfies $\alpha^k = 0$, where $k = \dim(X)$.

Definition

A cohomology class $\alpha \in H^{1,1}_{\mathbb{R}}(X)$ is called parabolic if it is nef and $\alpha^k = 0$.

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Main results

Theorem (Sibony–S.–Verbitsky)

Assume that X is a hyperkähler manifold with $b_2(X) \geqslant 7$ and $u \in H^{1,1}_{\mathbb{R}}(X)$ is a parabolic class, i.e. a nef class with $q(u) = 0$, where q is the BBF form. The class u is rigid in the following cases:

- *if $u^\perp \cap H^2(X, \mathbb{Q}) = 0$;*
- *if $u^\perp \cap H^2(X, \mathbb{Q})$ is spanned by $v \in H^{2,0}(X) \oplus H^{0,2}(X)$.*

Corollary

Assume that X is a hyperkähler manifold with $b_2(X) \geqslant 7$ and non-maximal Picard group, i.e. the rank of $\mathrm{Pic}(X)$ is less than $b_2(X) - 2$. Then there exists a non-empty open subset U of the boundary of the Kähler cone $\partial\mathcal{K}_X$ such that the rigid parabolic classes are dense in U.

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Diameters of pseudo-effective classes

Idea of the proof: any hyperkähler manifold with $b_2 \geqslant 5$ has a deformation that admits a hyperbolic automorphism and a dynamical rigid class, as shown by Amerik–Verbitsky.

Use Ratner's theory and semicontinuity of the diameter of pseudo-effective classes.

Let $\eta \in \Lambda^{1,1}_{\mathbb{R}} X$ with $[\eta] = \alpha \in \mathcal{P}_X$.

Define $\Phi_\eta = \{\varphi \in L^1(X) \mid \sup(\varphi) = 0,\, \eta + dd^c\varphi \geqslant 0\}$.
Φ_η is a compact subset of $L^1(X)$.

$$\mathrm{diam}(\Phi_\eta) = \sup_{\varphi,\psi \in \Phi_\eta} \left\{ \int_X |\varphi - \psi| \mathrm{Vol} \right\} < +\infty.$$

Definition

The diameter function $\delta\colon \mathcal{P}_X \to \mathbb{R}$:

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Semicontinuity of the diameter

Consider $\pi\colon \mathcal{X} \to B$, a proper submersion of complex manifolds.

Assume that we have:

- a smooth 2-form $\tilde{\omega} \in \Lambda^{1,1}\mathcal{X}$ s.t. for any $t \in B$ the form $\omega_t = \tilde{\omega}|_{\mathcal{X}_t}$ is Kähler;
- a fibrewise volume form $\mathcal{V}\mathrm{ol}$ on $\mathcal{X}$, s.t. $\mathrm{Vol}_t = \mathcal{V}\mathrm{ol}|_{\mathcal{X}_t}$ is a volume form on the fibre.

The spaces $H^{1,1}_{\mathbb{R}}(\mathcal{X}_t)$ form a C^∞-vector bundle over B.
The pseudo-effective cone $\mathcal{P}_{\mathcal{X}/B}$ is a subset of the total space of this bundle.
As above, we have the diameter function: $\delta\colon \mathcal{P}_{\mathcal{X}/B} \to \mathbb{R}$.

Theorem (Sibony–S.–Verbitsky)

The function $\delta\colon \mathcal{P}_{\mathcal{X}/B} \to \mathbb{R}$ is upper semi-continuous.

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The spaces $H^{1,1}_{\mathbb{R}}(\mathcal{X}_t)$ form a C^∞-vector bundle over B.
The pseudo-effective cone $\mathcal{P}_{\mathcal{X}/B}$ is a subset of the total space of this bundle.
As above, we have the diameter function: $\delta\colon \mathcal{P}_{\mathcal{X}/B} \to \mathbb{R}$.

Theorem (Sibony–S.–Verbitsky)

The function $\delta\colon \mathcal{P}_{\mathcal{X}/B} \to \mathbb{R}$ is upper semi-continuous.

Semicontinuity of the diameter

Consider $\pi\colon \mathcal{X} \to B$, a proper submersion of complex manifolds.

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The parabolic Teichmüller space

X is a hyperkähler manifold, I a complex structure of hyperkähler type, $V = H^2(X, \mathbb{Q})$.

Recall: $\alpha \in V^{1,1}_{I,\mathbb{R}}$ is parabolic if $q(\alpha) = 0$ and α is nef.

Definition

- *The parabolic Teichmüller space:*

$$\mathcal{T}_p(X) = \{(I, \alpha) \in \mathcal{T} \times V_{\mathbb{R}} \mid 0 \neq \alpha \text{ is parabolic for } I\};$$

- *the parabolic period domain:*

$$\mathcal{D}_p = \{(L, u) \in \mathcal{D} \times V_{\mathbb{R}} \mid 0 \neq u \in L^{\perp},\, q(u) = 0\};$$

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Orbits in $\mathcal{D}_p$ and $\mathcal{T}_p$

For the fixed connected component $\mathcal{T}^\circ$ we will also denote

$$\mathcal{T}_p^\circ = \mathcal{T}_p \cap (\mathcal{T}^\circ \times V_{\mathbb{R}}).$$

It is clear that $\mathcal{MCG}^\circ$ acts on $\mathcal{T}_p^\circ$ and Γ acts on $\mathcal{D}_p$.

Theorem (Sibony–S.–Verbitsky)

Assume that $d \geqslant 7$.

- *Let $(L, u) \in \mathcal{D}_p$ s.t. $u^\perp$ does not contain non-zero rational vectors. Then the Γ-orbit of (L, u) is dense in $\mathcal{D}_p$;*
- *Let $(I, u) \in \mathcal{T}_p^\circ$ s.t. $u^\perp$ does not contain non-zero rational vectors. Then the $\mathcal{MCG}^\circ$-orbit of (I, u) is dense in $\mathcal{T}_p^\circ$.*

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Sketch of proof of the main theorem

Theorem (Sibony–S.–Verbitsky, simplified)

Assume: X_I *is a compact hyperkähler manifold with* $b_2 \geqslant 7$ *and* $u \in H^{1,1}_{\mathbb{R}}(X_I)$ *is a parabolic class s.t.* $u^\perp \cap H^2(X, \mathbb{Q}) = 0$. *Then* u *is rigid.*

Proof.

The $\mathcal{MCG}^\circ$-orbit of $(I, u) \in \mathcal{T}_p^\circ$ is dense in $\mathcal{T}_p^\circ$.

Amerik–Verbitsky: there exists $(I_0, u_0) \in \mathcal{T}_p^\circ$, where u_0 is a dynamical rigid class.

Consider the universal deformation $\pi\colon \mathcal{X} \to B$ of X_{I_0}, There exist: $t_i \in B$, $\mu_i \in \mathcal{MCG}^\circ$ and $u_i \in H^2(X, \mathbb{R})$ such that $t_i \to 0$, $\mathcal{X}_{t_i} \simeq X_{\mu_i^* I}$, $u_i = \mu_i^* u$, $u_i \to u_0$ when $i \to +\infty$.

The action of $\mathcal{MCG}^\circ$ preserves the diameter, so $\delta(u_i) = \delta(u)$.

We conclude: $0 = \delta(u_0) \geqslant \limsup_{i\to\infty} \delta(u_i) = \delta(u)$, hence u is rigid. $\square$

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