Hyperkähler manifolds and Hodge theory

Andrey Soldatenkov

IMECC – UNICAMP

21.03.2025

Reminder: Kähler manifolds

Let X be a C^∞-manifold (without boundary)

Definition

A complex structure on X is an endomorphism

$$I\colon TX \to TX,$$

such that $I^2 = -Id$, and I is integrable, i.e.

$$[T^{1,0}X, T^{1,0}X] \subset T^{1,0}X,$$

where $T^{1,0}X \subset TX \otimes \mathbb{C}$ is the eigenbundle for the eigenvalue $\sqrt{-1}$ of I.

Equivalently: a complex structure on X is an atlas with charts in $\mathbb{C}^n$, such that transition functions are holomorphic.

Reminder: Kähler manifolds

Let X be a C^∞-manifold (without boundary)

Definition

A complex structure on X is an endomorphism

$$I\colon TX \to TX,$$

such that $I^2 = -Id$, and I is integrable, i.e.

$$[T^{1,0}X, T^{1,0}X] \subset T^{1,0}X,$$

where $T^{1,0}X \subset TX \otimes \mathbb{C}$ is the eigenbundle for the eigenvalue $\sqrt{-1}$ of I.

Equivalently: a complex structure on X is an atlas with charts in $\mathbb{C}^n$, such that transition functions are holomorphic.

Reminder: Kähler manifolds

Let X be a C^∞-manifold (without boundary)

Definition

A complex structure on X is an endomorphism

$$I\colon TX \to TX,$$

such that $I^2 = -Id$, and I is integrable, i.e.

$$[T^{1,0}X, T^{1,0}X] \subset T^{1,0}X,$$

where $T^{1,0}X \subset TX \otimes \mathbb{C}$ is the eigenbundle for the eigenvalue $\sqrt{-1}$ of I.

Equivalently: a complex structure on X is an atlas with charts in $\mathbb{C}^n$, such that transition functions are holomorphic.

Reminder: Kähler manifolds

Let X be a C^∞-manifold (without boundary)

Definition

A *complex structure* *on* X *is an endomorphism*

$$I\colon TX \to TX,$$

such that $I^2 = -Id$, *and* I *is* *integrable*, *i.e.*

$$[T^{1,0}X, T^{1,0}X] \subset T^{1,0}X,$$

where $T^{1,0}X \subset TX \otimes \mathbb{C}$ *is the eigenbundle for the eigenvalue* $\sqrt{-1}$ *of* I*.*

Equivalently: a complex structure on X is an atlas with charts in $\mathbb{C}^n$, such that transition functions are holomorphic.

Reminder: Kähler manifolds

Let X be a C^∞-manifold (without boundary)

Definition

A *complex structure* *on* X *is an endomorphism*

$$I \colon TX \to TX,$$

such that $I^2 = -Id$, *and* I *is* *integrable*, *i.e.*

$$[T^{1,0}X, T^{1,0}X] \subset T^{1,0}X,$$

where $T^{1,0}X \subset TX \otimes \mathbb{C}$ *is the eigenbundle for the eigenvalue* $\sqrt{-1}$ *of* I.

Equivalently: a complex structure on X is an atlas with charts in $\mathbb{C}^n$, such that transition functions are holomorphic.

Reminder: Kähler manifolds

Let (X, I) be a complex manifold.

Definition

A Hermitian metric on (X, I) is a Riemannian metric $g \in S^2T^*X$, such that

$$g(Iu, Iv) = g(u, v)$$

for all $u, v \in TX$.

For a Hermitian metric g define $\omega(u, v) = g(Iu, v)$. Then $\omega \in \Lambda^2 X$.

Definition

A Hermitian metric is called Kähler if $d\omega = 0$

Reminder: Kähler manifolds

Let (X, I) be a complex manifold.

Definition

A Hermitian metric on (X, I) is a Riemannian metric $g \in S^2T^*X$, such that

$$g(Iu, Iv) = g(u, v)$$

for all $u, v \in TX$.

For a Hermitian metric g define $\omega(u, v) = g(Iu, v)$. Then $\omega \in \Lambda^2 X$.

Definition

A Hermitian metric is called Kähler if $d\omega = 0$

Reminder: Kähler manifolds

Let (X, I) be a complex manifold.

Definition

A Hermitian metric on (X, I) is a Riemannian metric $g \in S^2 T^* X$, such that

$$g(Iu, Iv) = g(u, v)$$

for all $u, v \in TX$.

For a Hermitian metric g define $\omega(u, v) = g(Iu, v)$. Then $\omega \in \Lambda^2 X$.

Definition

A Hermitian metric is called Kähler if $d\omega = 0$

Reminder: Kähler manifolds

Let (X, I) be a complex manifold.

Definition

A ***Hermitian metric*** *on* (X, I) *is a Riemannian metric* $g \in S^2T^*X$*, such that*

$$g(Iu, Iv) = g(u, v)$$

for all $u, v \in TX$*.*

For a Hermitian metric g define $\omega(u, v) = g(Iu, v)$. Then $\omega \in \Lambda^2X$.

Definition

A Hermitian metric is called ***Kähler*** *if* $d\omega = 0$

Hyperkähler manifolds

Definition

A hyperkähler manifold is a C^∞-manifold X with complex structures I, J, K and a Riemannian metric g, such that:

- $IJ = -JI = K$;
- *g is Kähler with respect to I, J and K.*

We have three Kähler forms: ω_I, ω_J and ω_K.

Consider the 2-form

$$\sigma_I = \omega_J + \sqrt{-1}\omega_K$$

It is a non-degenerate holomorphic 2-form on (X, I), i.e. a holomorphic symplectic form.

Hyperkähler manifolds

Definition

A hyperkähler manifold is a C^∞-manifold X with complex structures I, J, K and a Riemannian metric g, such that:

- $IJ = -JI = K$;
- *g is Kähler with respect to I, J and K.*

We have three Kähler forms: ω_I, ω_J and ω_K.

Consider the 2-form

$$\sigma_I = \omega_J + \sqrt{-1}\omega_K$$

It is a non-degenerate holomorphic 2-form on (X, I), i.e. a holomorphic symplectic form.

Hyperkähler manifolds

Definition

A *hyperkähler manifold* *is a* C^∞*-manifold* X *with complex structures* I, J, K *and a Riemannian metric* g, *such that:*

- $IJ = -JI = K$;
- g *is Kähler with respect to* I, J *and* K.

We have three Kähler forms: ω_I, ω_J and ω_K.

Consider the 2-form

$$\sigma_I = \omega_J + \sqrt{-1}\omega_K$$

It is a non-degenerate holomorphic 2-form on (X, I), i.e. a holomorphic symplectic form.

Hyperkähler manifolds

Definition

A hyperkähler manifold is a C^∞-manifold X with complex structures I, J, K and a Riemannian metric g, such that:

- $IJ = -JI = K$;
- *g is Kähler with respect to I, J and K.*

We have three Kähler forms: ω_I, ω_J and ω_K.

Consider the 2-form

$$\sigma_I = \omega_J + \sqrt{-1}\omega_K$$

It is a non-degenerate holomorphic 2-form on (X, I), i.e. a holomorphic symplectic form.

Hyperkähler manifolds

Definition

A hyperkähler manifold is a C^∞-manifold X with complex structures I, J, K and a Riemannian metric g, such that:

- $IJ = -JI = K$;
- *g is Kähler with respect to I, J and K.*

We have three Kähler forms: ω_I, ω_J and ω_K.

Consider the 2-form

$$\sigma_I = \omega_J + \sqrt{-1}\omega_K$$

It is a non-degenerate holomorphic 2-form on (X, I), i.e. a holomorphic symplectic form.

Hyperkähler manifolds

Definition

A hyperkähler manifold is a C^∞-manifold X with complex structures I, J, K and a Riemannian metric g, such that:

- $IJ = -JI = K$;
- *g is Kähler with respect to I, J and K.*

We have three Kähler forms: ω_I, ω_J and ω_K.

Consider the 2-form

$$\sigma_I = \omega_J + \sqrt{-1}\omega_K$$

It is a non-degenerate holomorphic 2-form on (X, I), i.e. a holomorphic symplectic form.

Hyperkähler manifolds

Definition

A hyperkähler manifold is a C^∞-manifold X with complex structures I, J, K and a Riemannian metric g, such that:

- $IJ = -JI = K$;
- *g is Kähler with respect to I, J and K.*

We have three Kähler forms: ω_I, ω_J and ω_K.

Consider the 2-form

$$\sigma_I = \omega_J + \sqrt{-1}\omega_K$$

It is a non-degenerate holomorphic 2-form on (X, I), i.e. a holomorphic symplectic form.

Hyperkähler manifolds

Let $\Omega^k_{X,I}$ be the bundle of I-holomorphic k-forms on X

Then

$$\sigma_I \in H^0(X, \Omega^2_{X,I}).$$

Since σ_I is symplectic, $\dim_{\mathbb{C}}(X, I) = 2n$, and σ_I^n is a nowhere vanishing section of the canonical bundle $K_{X,I} = \Omega^{2n}_{X,I}$.
The canonical bundle of (X, I) is trivial.

Definition

A compact hyperkähler manifold (X, I, J, K, g) is called irreducible holomorphic symplectic (IHS), if

- $\pi_1(X) = 1$
- $H^0(X, \Omega^2_{X,I})$ *is spanned by* σ_I

Hyperkähler manifolds

Let $\Omega^k_{X,I}$ be the bundle of I-holomorphic k-forms on X

Then

$$\sigma_I \in H^0(X, \Omega^2_{X,I}).$$

Since σ_I is symplectic, $\dim_{\mathbb{C}}(X, I) = 2n$, and σ_I^n is a nowhere vanishing section of the canonical bundle $K_{X,I} = \Omega^{2n}_{X,I}$.
The canonical bundle of (X, I) is trivial.

Definition

A compact hyperkähler manifold (X, I, J, K, g) is called irreducible holomorphic symplectic (IHS), if

- $\pi_1(X) = 1$
- $H^0(X, \Omega^2_{X,I})$ *is spanned by* σ_I

Hyperkähler manifolds

Let $\Omega^k_{X,I}$ be the bundle of I-holomorphic k-forms on X

Then

$$\sigma_I \in H^0(X, \Omega^2_{X,I}).$$

Since σ_I is symplectic, $\dim_{\mathbb{C}}(X, I) = 2n$, and σ_I^n is a nowhere vanishing section of the canonical bundle $K_{X,I} = \Omega^{2n}_{X,I}$.

The canonical bundle of (X, I) is trivial.

Definition

A compact hyperkähler manifold (X, I, J, K, g) is called irreducible holomorphic symplectic (IHS), if

- $\pi_1(X) = 1$
- $H^0(X, \Omega^2_{X,I})$ *is spanned by* σ_I

Hyperkähler manifolds

Let $\Omega^k_{X,I}$ be the bundle of I-holomorphic k-forms on X

Then

$$\sigma_I \in H^0(X, \Omega^2_{X,I}).$$

Since σ_I is symplectic, $\dim_{\mathbb{C}}(X, I) = 2n$, and σ_I^n is a nowhere vanishing section of the canonical bundle $K_{X,I} = \Omega^{2n}_{X,I}$.
The canonical bundle of (X, I) is trivial.

Definition

A compact hyperkähler manifold (X, I, J, K, g) is called irreducible holomorphic symplectic (IHS), if

- $\pi_1(X) = 1$
- $H^0(X, \Omega^2_{X,I})$ *is spanned by* σ_I

Hyperkähler manifolds

Let $\Omega^k_{X,I}$ be the bundle of I-holomorphic k-forms on X

Then

$$\sigma_I \in H^0(X, \Omega^2_{X,I}).$$

Since σ_I is symplectic, $\dim_{\mathbb{C}}(X, I) = 2n$, and σ_I^n is a nowhere vanishing section of the canonical bundle $K_{X,I} = \Omega^{2n}_{X,I}$.
The canonical bundle of (X, I) is trivial.

Definition

A compact hyperkähler manifold (X, I, J, K, g) is called irreducible holomorphic symplectic (IHS), if

- $\pi_1(X) = 1$
- $H^0(X, \Omega^2_{X,I})$ *is spanned by* σ_I

Hyperkähler manifolds

Let $\Omega^k_{X,I}$ be the bundle of I-holomorphic k-forms on X

Then

$$\sigma_I \in H^0(X, \Omega^2_{X,I}).$$

Since σ_I is symplectic, $\dim_{\mathbb{C}}(X, I) = 2n$, and σ_I^n is a nowhere vanishing section of the canonical bundle $K_{X,I} = \Omega^{2n}_{X,I}$.
The canonical bundle of (X, I) is trivial.

Definition

A compact hyperkähler manifold (X, I, J, K, g) *is called* *irreducible holomorphic symplectic (IHS), if*

- $\pi_1(X) = 1$
- $H^0(X, \Omega^2_{X,I})$ *is spanned by* σ_I

Hyperkähler manifolds

Let $\Omega^k_{X,I}$ be the bundle of I-holomorphic k-forms on X

Then

$$\sigma_I \in H^0(X, \Omega^2_{X,I}).$$

Since σ_I is symplectic, $\dim_{\mathbb{C}}(X, I) = 2n$, and σ_I^n is a nowhere vanishing section of the canonical bundle $K_{X,I} = \Omega^{2n}_{X,I}$.
The canonical bundle of (X, I) is trivial.

Definition

A compact hyperkähler manifold (X, I, J, K, g) *is called* *irreducible holomorphic symplectic (IHS)*, *if*

- $\pi_1(X) = 1$
- $H^0(X, \Omega^2_{X,I})$ *is spanned by* σ_I

Examples of hyperkähler manifolds

- $\mathbb{H}$ = the algebra of quaternions, $\mathbb{Z}^4 \subset \mathbb{H}$ a lattice.
  Then $T = \mathbb{H}/\mathbb{Z}^4$ is hyperkähler:
  I, J, K — multiplication by imaginary quaternions,
  g is the standard flat metric.
- Let S be a complex K3 surface, for example

  $$S = \{(x_0 : \ldots : x_3) \mid x_0^4 + \ldots + x_3^4 = 0\} \subset \mathbb{C}P^3.$$

  $S = (X, I)$, where X is the underlying real 4-fold and I is induced by the complex structure on $\mathbb{C}P^3$.
  Calabi-Yau theorem $\Rightarrow$ there exists a hyperkähler structure on S, i.e. there exist J, K and g as in the definition.
  S is an IHS manifold.

Examples of hyperkähler manifolds

- $\mathbb{H}$ = the algebra of quaternions, $\mathbb{Z}^4 \subset \mathbb{H}$ a lattice.
  Then $T = \mathbb{H}/\mathbb{Z}^4$ is hyperkähler:
  I, J, K — multiplication by imaginary quaternions,
  g is the standard flat metric.
- Let S be a complex K3 surface, for example

$$S = \{(x_0 : \ldots : x_3) \mid x_0^4 + \ldots + x_3^4 = 0\} \subset \mathbb{C}P^3.$$

  $S = (X, I)$, where X is the underlying real 4-fold and I is induced by the complex structure on $\mathbb{C}P^3$.
  Calabi-Yau theorem $\Rightarrow$ there exists a hyperkähler structure on S, i.e. there exist J, K and g as in the definition.
  S is an IHS manifold.

Examples of hyperkähler manifolds

- $\mathbb{H}$ = the algebra of quaternions, $\mathbb{Z}^4 \subset \mathbb{H}$ a lattice.
  Then $T = \mathbb{H}/\mathbb{Z}^4$ is hyperkähler:
  I, J, K — multiplication by imaginary quaternions,
  g is the standard flat metric.
- Let S be a complex K3 surface, for example

$$S = \{(x_0 : \ldots : x_3) \mid x_0^4 + \ldots + x_3^4 = 0\} \subset \mathbb{C}P^3.$$

  $S = (X, I)$, where X is the underlying real 4-fold and I is induced by the complex structure on $\mathbb{C}P^3$.
  Calabi-Yau theorem $\Rightarrow$ there exists a hyperkähler structure on S, i.e. there exist J, K and g as in the definition.
  S is an IHS manifold.

Examples of hyperkähler manifolds

- $\mathbb{H}$ = the algebra of quaternions, $\mathbb{Z}^4 \subset \mathbb{H}$ a lattice.
  Then $T = \mathbb{H}/\mathbb{Z}^4$ is hyperkähler:
  I, J, K — multiplication by imaginary quaternions,
  g is the standard flat metric.
- Let S be a complex K3 surface, for example

$$S = \{(x_0 : \ldots : x_3) \mid x_0^4 + \ldots + x_3^4 = 0\} \subset \mathbb{C}P^3.$$

$S = (X, I)$, where X is the underlying real 4-fold and I is induced by the complex structure on $\mathbb{C}P^3$.
Calabi-Yau theorem $\Rightarrow$ there exists a hyperkähler structure on S, i.e. there exist J, K and g as in the definition.
S is an IHS manifold.

Examples of hyperkähler manifolds

- $\mathbb{H}$ = the algebra of quaternions, $\mathbb{Z}^4 \subset \mathbb{H}$ a lattice.
  Then $T = \mathbb{H}/\mathbb{Z}^4$ is hyperkähler:
  I, J, K — multiplication by imaginary quaternions,
  g is the standard flat metric.
- Let S be a complex K3 surface, for example

  $$S = \{(x_0 : \ldots : x_3) \mid x_0^4 + \ldots + x_3^4 = 0\} \subset \mathbb{C}P^3.$$

  $S = (X, I)$, where X is the underlying real 4-fold and I is induced by the complex structure on $\mathbb{C}P^3$.
  Calabi-Yau theorem $\Rightarrow$ there exists a hyperkähler structure on S, i.e. there exist J, K and g as in the definition.
  S is an IHS manifold.

Examples of hyperkähler manifolds

- $\mathbb{H}$ = the algebra of quaternions, $\mathbb{Z}^4 \subset \mathbb{H}$ a lattice.
  Then $T = \mathbb{H}/\mathbb{Z}^4$ is hyperkähler:
  I, J, K — multiplication by imaginary quaternions,
  g is the standard flat metric.
- Let S be a complex K3 surface, for example

$$S = \{(x_0 : \ldots : x_3) \mid x_0^4 + \ldots + x_3^4 = 0\} \subset \mathbb{C}P^3.$$

  $S = (X, I)$, where X is the underlying real 4-fold and I is induced by the complex structure on $\mathbb{C}P^3$.
  Calabi-Yau theorem $\Rightarrow$ there exists a hyperkähler structure on S, i.e. there exist J, K and g as in the definition.
  S is an IHS manifold.

Examples of hyperkähler manifolds

- $\mathbb{H}$ = the algebra of quaternions, $\mathbb{Z}^4 \subset \mathbb{H}$ a lattice.
  Then $T = \mathbb{H}/\mathbb{Z}^4$ is hyperkähler:
  I, J, K — multiplication by imaginary quaternions,
  g is the standard flat metric.
- Let S be a complex K3 surface, for example

$$S = \{(x_0 : \ldots : x_3) \mid x_0^4 + \ldots + x_3^4 = 0\} \subset \mathbb{C}P^3.$$

  $S = (X, I)$, where X is the underlying real 4-fold and I is induced by the complex structure on $\mathbb{C}P^3$.
  Calabi-Yau theorem $\Rightarrow$ there exists a hyperkähler structure on S, i.e. there exist J, K and g as in the definition.
  S is an IHS manifold.

Examples of hyperkähler manifolds

- Let S be a complex K3 surface. The symmetric power $S^{(n)}$ parametrizes n-tuples of points in S. It is singular, but it admits a natural resolution of singularities

$$r\colon S^{[n]} \to S^{(n)},$$

where $S^{[n]}$ is the Hilbert scheme of length n subschemes of S. The manifold $S^{[n]}$ admits a hyperkähler structure, and $S^{[n]}$ is IHS.

- Let $T = \mathbb{C}^2/\mathbb{Z}^4$ be a 2-dimensional complex torus. The Albanese morphism:

$$a\colon T^{[n+1]} \to T, \quad (x_0, \ldots, x_n) \mapsto \sum x_i$$

$K^nT = a^{-1}(0)$ — generalized Kummer variety, it is IHS

- O'Grady's exceptional IHS manifolds of dimension 6 and 10.

Examples of hyperkähler manifolds

- Let S be a complex K3 surface. The symmetric power $S^{(n)}$ parametrizes n-tuples of points in S. It is singular, but it admits a natural resolution of singularities

$$r\colon S^{[n]} \to S^{(n)},$$

where $S^{[n]}$ is the Hilbert scheme of length n subschemes of S. The manifold $S^{[n]}$ admits a hyperkähler structure, and $S^{[n]}$ is IHS.

- Let $T = \mathbb{C}^2/\mathbb{Z}^4$ be a 2-dimensional complex torus. The Albanese morphism:

$$a\colon T^{[n+1]} \to T, \quad (x_0, \ldots, x_n) \mapsto \sum x_i$$

$K^nT = a^{-1}(0)$ — generalized Kummer variety, it is IHS

- O'Grady's exceptional IHS manifolds of dimension 6 and 10.

Examples of hyperkähler manifolds

- Let S be a complex K3 surface. The symmetric power $S^{(n)}$ parametrizes n-tuples of points in S. It is singular, but it admits a natural resolution of singularities

$$r\colon S^{[n]} \to S^{(n)},$$

where $S^{[n]}$ is the Hilbert scheme of length n subschemes of S. The manifold $S^{[n]}$ admits a hyperkähler structure, and $S^{[n]}$ is IHS.

- Let $T = \mathbb{C}^2/\mathbb{Z}^4$ be a 2-dimensional complex torus. The Albanese morphism:

$$a\colon T^{[n+1]} \to T, \quad (x_0, \ldots, x_n) \mapsto \sum x_i$$

$K^nT = a^{-1}(0)$ — generalized Kummer variety, it is IHS

- O'Grady's exceptional IHS manifolds of dimension 6 and 10.

Examples of hyperkähler manifolds

- Let S be a complex K3 surface. The symmetric power $S^{(n)}$ parametrizes n-tuples of points in S. It is singular, but it admits a natural resolution of singularities

$$r\colon S^{[n]} \to S^{(n)},$$

  where $S^{[n]}$ is the Hilbert scheme of length n subschemes of S. The manifold $S^{[n]}$ admits a hyperkähler structure, and $S^{[n]}$ is IHS.
- Let $T = \mathbb{C}^2/\mathbb{Z}^4$ be a 2-dimensional complex torus.
  The Albanese morphism:

$$a\colon T^{[n+1]} \to T, \quad (x_0, \ldots, x_n) \mapsto \sum x_i$$

  $K^n T = a^{-1}(0)$ — generalized Kummer variety, it is IHS
- O'Grady's exceptional IHS manifolds of dimension 6 and 10.

Examples of hyperkähler manifolds

- Let S be a complex K3 surface. The symmetric power $S^{(n)}$ parametrizes n-tuples of points in S. It is singular, but it admits a natural resolution of singularities

$$r\colon S^{[n]} \to S^{(n)},$$

  where $S^{[n]}$ is the Hilbert scheme of length n subschemes of S. The manifold $S^{[n]}$ admits a hyperkähler structure, and $S^{[n]}$ is IHS.
- Let $T = \mathbb{C}^2/\mathbb{Z}^4$ be a 2-dimensional complex torus. The Albanese morphism:

$$a\colon T^{[n+1]} \to T, \quad (x_0, \ldots, x_n) \mapsto \sum x_i$$

  $K^nT = a^{-1}(0)$ — generalized Kummer variety, it is IHS
- O'Grady's exceptional IHS manifolds of dimension 6 and 10.

Beauville-Bogomolov decomposition

Le M be a compact Kähler manifold, $\dim_{\mathbb{C}} M = n$, and $K_M = \Omega^n_M$ its canonical bundle.

Theorem (Beauville-Bogomolov)

Assume that $K_M \simeq \mathcal{O}_M$. Then there exists a finite étale covering $\pi \colon \tilde{M} \to M$ and a unique decomposition

$$\tilde{M} = T \times \prod_i Y_i \times \prod_j Z_j,$$

where

- *T is a complex torus*
- *Y_i are IHS manifolds*
- *Z_j are Calabi-Yau manifolds, i.e. $\pi_1(Z_j) = 1$, $K_{Z_j} = \mathcal{O}_{Z_j}$ and $H^0(Z_j, \Omega^2_{Z_j}) = 0$.*

From now on consider only hyperkähler manifolds that are IHS.

Beauville-Bogomolov decomposition

Le M be a compact Kähler manifold, $\dim_{\mathbb{C}} M = n$, and $K_M = \Omega^n_M$ its canonical bundle.

Theorem (Beauville-Bogomolov)

Assume that $K_M \simeq \mathcal{O}_M$. Then there exists a finite étale covering $\pi \colon \tilde{M} \to M$ and a unique decomposition

$$\tilde{M} = T \times \prod_i Y_i \times \prod_j Z_j,$$

where

- *T is a complex torus*
- *Y_i are IHS manifolds*
- *Z_j are Calabi-Yau manifolds, i.e. $\pi_1(Z_j) = 1$, $K_{Z_j} = \mathcal{O}_{Z_j}$ and $H^0(Z_j, \Omega^2_{Z_j}) = 0$.*

From now on consider only hyperkähler manifolds that are IHS.

Beauville-Bogomolov decomposition

Le M be a compact Kähler manifold, $\dim_{\mathbb{C}} M = n$, and $K_M = \Omega^n_M$ its canonical bundle.

Theorem (Beauville-Bogomolov)

Assume that $K_M \simeq \mathcal{O}_M$. Then there exists a finite étale covering $\pi\colon \tilde{M} \to M$ and a unique decomposition

$$\tilde{M} = T \times \prod_i Y_i \times \prod_j Z_j,$$

where

- *T is a complex torus*
- *Y_i are IHS manifolds*
- *Z_j are Calabi-Yau manifolds, i.e. $\pi_1(Z_j) = 1$, $K_{Z_j} = \mathcal{O}_{Z_j}$ and $H^0(Z_j, \Omega^2_{Z_j}) = 0$.*

From now on consider only hyperkähler manifolds that are IHS.

Beauville-Bogomolov decomposition

Le M be a compact Kähler manifold, $\dim_{\mathbb{C}} M = n$, and $K_M = \Omega^n_M$ its canonical bundle.

Theorem (Beauville-Bogomolov)

Assume that $K_M \simeq \mathcal{O}_M$. Then there exists a finite étale covering $\pi\colon \tilde{M} \to M$ and a unique decomposition

$$\tilde{M} = T \times \prod_i Y_i \times \prod_j Z_j,$$

where

- *T is a complex torus*
- *Y_i are IHS manifolds*
- *Z_j are Calabi-Yau manifolds, i.e. $\pi_1(Z_j) = 1$, $K_{Z_j} = \mathcal{O}_{Z_j}$ and $H^0(Z_j, \Omega^2_{Z_j}) = 0$.*

From now on consider only hyperkähler manifolds that are IHS.

Beauville-Bogomolov decomposition

Le M be a compact Kähler manifold, $\dim_{\mathbb{C}} M = n$, and $K_M = \Omega^n_M$ its canonical bundle.

Theorem (Beauville-Bogomolov)

Assume that $K_M \simeq \mathcal{O}_M$. Then there exists a finite étale covering $\pi\colon \tilde{M} \to M$ and a unique decomposition

$$\tilde{M} = T \times \prod_i Y_i \times \prod_j Z_j,$$

where

- *T is a complex torus*
- *Y_i are IHS manifolds*
- *Z_j are Calabi-Yau manifolds, i.e. $\pi_1(Z_j) = 1$, $K_{Z_j} = \mathcal{O}_{Z_j}$ and $H^0(Z_j, \Omega^2_{Z_j}) = 0$.*

From now on consider only hyperkähler manifolds that are IHS.

Beauville-Bogomolov decomposition

Le M be a compact Kähler manifold, $\dim_{\mathbb{C}} M = n$, and $K_M = \Omega^n_M$ its canonical bundle.

Theorem (Beauville-Bogomolov)

Assume that $K_M \simeq \mathcal{O}_M$*. Then there exists a finite étale covering* $\pi\colon \tilde{M} \to M$ *and a unique decomposition*

$$\tilde{M} = T \times \prod_i Y_i \times \prod_j Z_j,$$

where

- T *is a complex torus*
- Y_i *are IHS manifolds*
- Z_j *are Calabi-Yau manifolds, i.e.* $\pi_1(Z_j) = 1$, $K_{Z_j} = \mathcal{O}_{Z_j}$ *and* $H^0(Z_j, \Omega^2_{Z_j}) = 0$.

From now on consider only hyperkähler manifolds that are IHS.

Beauville-Bogomolov decomposition

Le M be a compact Kähler manifold, $\dim_{\mathbb{C}} M = n$, and $K_M = \Omega^n_M$ its canonical bundle.

Theorem (Beauville-Bogomolov)

Assume that $K_M \simeq \mathcal{O}_M$. Then there exists a finite étale covering $\pi\colon \tilde{M} \to M$ and a unique decomposition

$$\tilde{M} = T \times \prod_i Y_i \times \prod_j Z_j,$$

where

- T *is a complex torus*
- Y_i *are IHS manifolds*
- Z_j *are Calabi-Yau manifolds, i.e.* $\pi_1(Z_j) = 1$, $K_{Z_j} = \mathcal{O}_{Z_j}$ *and* $H^0(Z_j, \Omega^2_{Z_j}) = 0$.

From now on consider only hyperkähler manifolds that are IHS.

Reminder: cohomology of Kähler manifolds

Let M be a compact Kähler manifold, $\dim_{\mathbb{C}} M = n$, with Kähler form ω.
Hodge decomposition:

$$H^k(M,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(M), \qquad H^{p,q}(M) \simeq H^q(M,\Omega^p_M).$$

Hodge filtration: $F^p H^k(X,\mathbb{C}) = \oplus_{j\geqslant p} H^{j,k-j}(X)$.
We have $[\omega] \in H^{1,1}(M) \cap H^2(M,\mathbb{R})$.

Definition

Define two operators $\theta, L \in \mathrm{End}(H^\bullet(M,\mathbb{R}))$

- $\theta|_{H^k(M,\mathbb{R})} = (k-n)\, Id$
- *The Lefschetz operator* $L\colon H^k(M,\mathbb{R}) \to H^{k+2}(M,\mathbb{R})$ *is the cup product with* $[\omega]$.

Reminder: cohomology of Kähler manifolds

Let M be a compact Kähler manifold, $\dim_{\mathbb{C}} M = n$, with Kähler form ω.

Hodge decomposition:

$$H^k(M,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(M), \qquad H^{p,q}(M) \simeq H^q(M,\Omega^p_M).$$

Hodge filtration: $F^pH^k(X,\mathbb{C}) = \oplus_{j\geqslant p} H^{j,k-j}(X)$.
We have $[\omega] \in H^{1,1}(M) \cap H^2(M,\mathbb{R})$.

Definition

Define two operators $\theta, L \in \mathrm{End}(H^\bullet(M,\mathbb{R}))$

- $\theta|_{H^k(M,\mathbb{R})} = (k-n)\,Id$
- *The Lefschetz operator* $L\colon H^k(M,\mathbb{R}) \to H^{k+2}(M,\mathbb{R})$ *is the cup product with* $[\omega]$.

Reminder: cohomology of Kähler manifolds

Let M be a compact Kähler manifold, $\dim_{\mathbb{C}} M = n$, with Kähler form ω.
Hodge decomposition:

$$H^k(M,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(M), \qquad H^{p,q}(M) \simeq H^q(M,\Omega^p_M).$$

Hodge filtration: $F^pH^k(X,\mathbb{C}) = \oplus_{j\geqslant p} H^{j,k-j}(X)$.
We have $[\omega] \in H^{1,1}(M) \cap H^2(M,\mathbb{R})$.

Definition

Define two operators $\theta, L \in \mathrm{End}(H^\bullet(M,\mathbb{R}))$

- $\theta|_{H^k(M,\mathbb{R})} = (k-n)\,Id$
- *The Lefschetz operator* $L\colon H^k(M,\mathbb{R}) \to H^{k+2}(M,\mathbb{R})$ *is the cup product with* $[\omega]$.

Reminder: cohomology of Kähler manifolds

Let M be a compact Kähler manifold, $\dim_{\mathbb{C}} M = n$, with Kähler form ω.
Hodge decomposition:

$$H^k(M,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(M), \qquad H^{p,q}(M) \simeq H^q(M,\Omega^p_M).$$

Hodge filtration: $F^pH^k(X,\mathbb{C}) = \oplus_{j\geqslant p} H^{j,k-j}(X)$.
We have $[\omega] \in H^{1,1}(M) \cap H^2(M,\mathbb{R})$.

Definition

Define two operators $\theta, L \in \mathrm{End}(H^\bullet(M,\mathbb{R}))$

- $\theta|_{H^k(M,\mathbb{R})} = (k-n)\,Id$
- *The Lefschetz operator $L\colon H^k(M,\mathbb{R}) \to H^{k+2}(M,\mathbb{R})$ is the cup product with $[\omega]$.*

Reminder: cohomology of Kähler manifolds

Let M be a compact Kähler manifold, $\dim_{\mathbb{C}} M = n$, with Kähler form ω.
Hodge decomposition:

$$H^k(M,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(M), \qquad H^{p,q}(M) \simeq H^q(M,\Omega^p_M).$$

Hodge filtration: $F^pH^k(X,\mathbb{C}) = \oplus_{j\geqslant p} H^{j,k-j}(X)$.
We have $[\omega] \in H^{1,1}(M) \cap H^2(M,\mathbb{R})$.

Definition

Define two operators $\theta, L \in \mathrm{End}(H^\bullet(M,\mathbb{R}))$

- $\theta|_{H^k(M,\mathbb{R})} = (k-n)\, Id$
- *The Lefschetz operator $L\colon H^k(M,\mathbb{R}) \to H^{k+2}(M,\mathbb{R})$ is the cup product with $[\omega]$.*

Reminder: cohomology of Kähler manifolds

Let M be a compact Kähler manifold, $\dim_{\mathbb{C}} M = n$, with Kähler form ω.
Hodge decomposition:

$$H^k(M, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(M), \qquad H^{p,q}(M) \simeq H^q(M, \Omega^p_M).$$

Hodge filtration: $F^p H^k(X, \mathbb{C}) = \oplus_{j \geqslant p} H^{j,k-j}(X)$.
We have $[\omega] \in H^{1,1}(M) \cap H^2(M, \mathbb{R})$.

Definition

Define two operators $\theta, L \in \mathrm{End}(H^\bullet(M, \mathbb{R}))$

- $\theta|_{H^k(M,\mathbb{R})} = (k-n)\, Id$
- *The Lefschetz operator* $L \colon H^k(M, \mathbb{R}) \to H^{k+2}(M, \mathbb{R})$ *is the cup product with* $[\omega]$.

Reminder: cohomology of Kähler manifolds

Let M be a compact Kähler manifold, $\dim_{\mathbb{C}} M = n$, with Kähler form ω.
Hodge decomposition:

$$H^k(M,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(M), \qquad H^{p,q}(M) \simeq H^q(M,\Omega^p_M).$$

Hodge filtration: $F^p H^k(X,\mathbb{C}) = \oplus_{j\geqslant p} H^{j,k-j}(X)$.
We have $[\omega] \in H^{1,1}(M) \cap H^2(M,\mathbb{R})$.

Definition

Define two operators $\theta, L \in \mathrm{End}(H^\bullet(M,\mathbb{R}))$

- $\theta|_{H^k(M,\mathbb{R})} = (k-n)\, Id$
- *The Lefschetz operator* $L\colon H^k(M,\mathbb{R}) \to H^{k+2}(M,\mathbb{R})$ *is the cup product with* $[\omega]$.

Reminder: cohomology of Kähler manifolds

Let M be a compact Kähler manifold, $\dim_{\mathbb{C}} M = n$, with Kähler form ω.
Hodge decomposition:

$$H^k(M,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(M), \qquad H^{p,q}(M) \simeq H^q(M,\Omega^p_M).$$

Hodge filtration: $F^pH^k(X,\mathbb{C}) = \oplus_{j\geqslant p} H^{j,k-j}(X)$.
We have $[\omega] \in H^{1,1}(M) \cap H^2(M,\mathbb{R})$.

Definition

Define two operators $\theta, L \in \mathrm{End}(H^\bullet(M,\mathbb{R}))$

- $\theta|_{H^k(M,\mathbb{R})} = (k-n)\, Id$
- *The Lefschetz operator* $L\colon H^k(M,\mathbb{R}) \to H^{k+2}(M,\mathbb{R})$ *is the cup product with* $[\omega]$.

Reminder: hard Lefschetz theorem

Theorem (Lefschetz)

There exists a unique operator (the dual Lefschetz operator)

$$\Lambda\colon H^k(M,\mathbb{R}) \to H^{k-2}(M,\mathbb{R}),$$

such that L, θ *and* Λ *form an* $\mathfrak{sl}_2$*-triple:*

$$[L,\Lambda] = \theta, \qquad [\theta, L] = 2L, \qquad [\theta,\Lambda] = -2\Lambda.$$

For any $k = 0,\ldots,n$ *we have isomorphisms*

$$L^k\colon H^{n-k}(M,\mathbb{R}) \xrightarrow{\sim} H^{n+k}(M,\mathbb{R}).$$

One can decompose $H^\bullet(M,\mathbb{C})$ *into a direct sum of irreducible* $\mathfrak{sl}_2$*-modules (Lefschetz decomposition).*

Reminder: hard Lefschetz theorem

Theorem (Lefschetz)

There exists a unique operator (the dual Lefschetz operator)

$$\Lambda\colon H^k(M,\mathbb{R}) \to H^{k-2}(M,\mathbb{R}),$$

such that L, θ *and* Λ *form an* $\mathfrak{sl}_2$*-triple:*

$$[L,\Lambda] = \theta, \qquad [\theta, L] = 2L, \qquad [\theta,\Lambda] = -2\Lambda.$$

For any $k = 0,\ldots,n$ *we have isomorphisms*

$$L^k\colon H^{n-k}(M,\mathbb{R}) \xrightarrow{\sim} H^{n+k}(M,\mathbb{R}).$$

One can decompose $H^\bullet(M,\mathbb{C})$ *into a direct sum of irreducible* $\mathfrak{sl}_2$*-modules (Lefschetz decomposition).*

Reminder: hard Lefschetz theorem

Theorem (Lefschetz)

There exists a unique operator (the dual Lefschetz operator)

$$\Lambda\colon H^k(M,\mathbb{R}) \to H^{k-2}(M,\mathbb{R}),$$

such that L, θ *and* Λ *form an* $\mathfrak{sl}_2$*-triple:*

$$[L,\Lambda] = \theta, \qquad [\theta, L] = 2L, \qquad [\theta,\Lambda] = -2\Lambda.$$

For any $k = 0,\dots,n$ *we have isomorphisms*

$$L^k\colon H^{n-k}(M,\mathbb{R}) \xrightarrow{\sim} H^{n+k}(M,\mathbb{R}).$$

One can decompose $H^\bullet(M,\mathbb{C})$ *into a direct sum of irreducible* $\mathfrak{sl}_2$*-modules (Lefschetz decomposition).*

Cohomology of compact hyperkähler manifolds

Let (X, I, J, K, g) be a compact hyperkähler manifold.

We have:

- the Kähler forms: $\omega_I, \quad \omega_J, \quad \omega_K,$
- the Lefschetz operators: $L_I, \quad L_J, \quad L_K,$
- the dual Lefschetz operators: $\Lambda_I, \quad \Lambda_J, \quad \Lambda_K.$

Theorem (Verbitsky)

The Lie subalgebra of $\mathrm{End}(H^\bullet(X, \mathbb{R}))$ *generated by* L_I, L_J, L_K, Λ_I, Λ_J *and* Λ_K *is isomorphic to* $\mathfrak{so}(4,1)$. *We have*

$$[\Lambda_I, \Lambda_J] = 0, \quad [\Lambda_J, \Lambda_K] = 0, \quad [\Lambda_K, \Lambda_I] = 0.$$

Moreover $[\Lambda_J, L_K] = W_I$ *is the Weil operator, i.e.*

$$W_I|_{H_I^{p,q}(X)} = \sqrt{-1}(p-q)Id.$$

Cohomology of compact hyperkähler manifolds

Let (X, I, J, K, g) be a compact hyperkähler manifold.

We have:

- the Kähler forms: $\omega_I, \quad \omega_J, \quad \omega_K,$
- the Lefschetz operators: $L_I, \quad L_J, \quad L_K,$
- the dual Lefschetz operators: $\Lambda_I, \quad \Lambda_J, \quad \Lambda_K.$

Theorem (Verbitsky)

The Lie subalgebra of $\mathrm{End}(H^\bullet(X,\mathbb{R}))$ *generated by* L_I, L_J, L_K, Λ_I, Λ_J *and* Λ_K *is isomorphic to* $\mathfrak{so}(4,1)$. *We have*

$$[\Lambda_I, \Lambda_J] = 0, \quad [\Lambda_J, \Lambda_K] = 0, \quad [\Lambda_K, \Lambda_I] = 0.$$

Moreover $[\Lambda_J, L_K] = W_I$ *is the Weil operator, i.e.*

$$W_I|_{H_I^{p,q}(X)} = \sqrt{-1}(p-q)Id.$$

Cohomology of compact hyperkähler manifolds

Let (X, I, J, K, g) be a compact hyperkähler manifold.

We have:

- the Kähler forms: $\omega_I, \quad \omega_J, \quad \omega_K,$
- the Lefschetz operators: $L_I, \quad L_J, \quad L_K,$
- the dual Lefschetz operators: $\Lambda_I, \quad \Lambda_J, \quad \Lambda_K.$

Theorem (Verbitsky)

The Lie subalgebra of $\mathrm{End}(H^\bullet(X, \mathbb{R}))$ *generated by* L_I, L_J, L_K, Λ_I, Λ_J *and* Λ_K *is isomorphic to* $\mathfrak{so}(4,1)$. *We have*

$$[\Lambda_I, \Lambda_J] = 0, \quad [\Lambda_J, \Lambda_K] = 0, \quad [\Lambda_K, \Lambda_I] = 0.$$

Moreover $[\Lambda_J, L_K] = W_I$ *is the Weil operator, i.e.*

$$W_I|_{H_I^{p,q}(X)} = \sqrt{-1}(p-q)Id.$$

Cohomology of compact hyperkähler manifolds

Let (X, I, J, K, g) be a compact hyperkähler manifold.

We have:

- the Kähler forms: $\omega_I, \quad \omega_J, \quad \omega_K,$
- the Lefschetz operators: $L_I, \quad L_J, \quad L_K,$
- the dual Lefschetz operators: $\Lambda_I, \quad \Lambda_J, \quad \Lambda_K.$

Theorem (Verbitsky)

The Lie subalgebra of $\mathrm{End}(H^\bullet(X,\mathbb{R}))$ *generated by* L_I, L_J, L_K, Λ_I, Λ_J *and* Λ_K *is isomorphic to* $\mathfrak{so}(4,1)$. *We have*

$$[\Lambda_I, \Lambda_J] = 0, \quad [\Lambda_J, \Lambda_K] = 0, \quad [\Lambda_K, \Lambda_I] = 0.$$

Moreover $[\Lambda_J, L_K] = W_I$ *is the Weil operator, i.e.*

$$W_I|_{H_I^{p,q}(X)} = \sqrt{-1}(p-q)Id.$$

Cohomology of compact hyperkähler manifolds

Let (X, I, J, K, g) be a compact hyperkähler manifold.

We have:

- the Kähler forms: $\omega_I, \quad \omega_J, \quad \omega_K,$
- the Lefschetz operators: $L_I, \quad L_J, \quad L_K,$
- the dual Lefschetz operators: $\Lambda_I, \quad \Lambda_J, \quad \Lambda_K.$

Theorem (Verbitsky)

The Lie subalgebra of $\mathrm{End}(H^\bullet(X, \mathbb{R}))$ *generated by* L_I, L_J, L_K, Λ_I, Λ_J *and* Λ_K *is isomorphic to* $\mathfrak{so}(4,1)$. *We have*

$$[\Lambda_I, \Lambda_J] = 0, \quad [\Lambda_J, \Lambda_K] = 0, \quad [\Lambda_K, \Lambda_I] = 0.$$

Moreover $[\Lambda_J, L_K] = W_I$ *is the Weil operator, i.e.*

$$W_I|_{H_I^{p,q}(X)} = \sqrt{-1}(p-q)Id.$$

Cohomology of compact hyperkähler manifolds

Let (X, I, J, K, g) be a compact hyperkähler manifold.

We have:

- the Kähler forms: $\omega_I, \quad \omega_J, \quad \omega_K,$
- the Lefschetz operators: $L_I, \quad L_J, \quad L_K,$
- the dual Lefschetz operators: $\Lambda_I, \quad \Lambda_J, \quad \Lambda_K.$

Theorem (Verbitsky)

The Lie subalgebra of $\mathrm{End}(H^\bullet(X, \mathbb{R}))$ *generated by* L_I, L_J, L_K, Λ_I, Λ_J *and* Λ_K *is isomorphic to* $\mathfrak{so}(4,1)$. *We have*

$$[\Lambda_I, \Lambda_J] = 0, \quad [\Lambda_J, \Lambda_K] = 0, \quad [\Lambda_K, \Lambda_I] = 0.$$

Moreover $[\Lambda_J, L_K] = W_I$ *is the Weil operator, i.e.*

$$W_I|_{H_I^{p,q}(X)} = \sqrt{-1}(p-q)Id.$$

Cohomology of compact hyperkähler manifolds

Let (X, I, J, K, g) be a compact hyperkähler manifold.

We have:

- the Kähler forms: $\omega_I, \quad \omega_J, \quad \omega_K,$
- the Lefschetz operators: $L_I, \quad L_J, \quad L_K,$
- the dual Lefschetz operators: $\Lambda_I, \quad \Lambda_J, \quad \Lambda_K.$

Theorem (Verbitsky)

The Lie subalgebra of $\mathrm{End}(H^\bullet(X, \mathbb{R}))$ *generated by* L_I, L_J, L_K, Λ_I, Λ_J *and* Λ_K *is isomorphic to* $\mathfrak{so}(4,1)$. *We have*

$$[\Lambda_I, \Lambda_J] = 0, \quad [\Lambda_J, \Lambda_K] = 0, \quad [\Lambda_K, \Lambda_I] = 0.$$

Moreover $[\Lambda_J, L_K] = W_I$ *is the Weil operator, i.e.*

$$W_I|_{H_I^{p,q}(X)} = \sqrt{-1}(p-q)Id.$$

Cohomology of compact hyperkähler manifolds

Let (X, I, J, K, g) be compact hyperkähler, $\dim_{\mathbb{R}} X = 4n$.

Definition

An element $a \in H^2(X,\mathbb{R})$ has Lefschetz property if for all $k = 0, \ldots, 2n$ we have isomorphisms

$$(L_a)^k \colon H^{2n-k}(X,\mathbb{R}) \xrightarrow{\sim} H^{2n+k}(X,\mathbb{R}),$$

where L_a is the operator of cup product with a.

If $a \in H^2(X,\mathbb{R})$ has Lefschetz property, then there exists the dual Lefschetz operator Λ_a, such that L_a, θ and Λ_a span $\mathfrak{sl}_2$.

Cohomology of compact hyperkähler manifolds

Let (X, I, J, K, g) be compact hyperkähler, $\dim_{\mathbb{R}} X = 4n$.

Definition

An element $a \in H^2(X, \mathbb{R})$ has Lefschetz property if for all $k = 0, \ldots, 2n$ we have isomorphisms

$$(L_a)^k \colon H^{2n-k}(X, \mathbb{R}) \xrightarrow{\sim} H^{2n+k}(X, \mathbb{R}),$$

where L_a is the operator of cup product with a.

If $a \in H^2(X, \mathbb{R})$ has Lefschetz property, then there exists the dual Lefschetz operator Λ_a, such that L_a, θ and Λ_a span $\mathfrak{sl}_2$.

Cohomology of compact hyperkähler manifolds

Let (X, I, J, K, g) be compact hyperkähler, $\dim_{\mathbb{R}} X = 4n$.

Definition

An element $a \in H^2(X, \mathbb{R})$ has Lefschetz property if for all $k = 0, \ldots, 2n$ we have isomorphisms

$$(L_a)^k \colon H^{2n-k}(X, \mathbb{R}) \xrightarrow{\sim} H^{2n+k}(X, \mathbb{R}),$$

where L_a is the operator of cup product with a.

If $a \in H^2(X, \mathbb{R})$ has Lefschetz property, then there exists the dual Lefschetz operator Λ_a, such that L_a, θ and Λ_a span $\mathfrak{sl}_2$.

Cohomology of compact hyperkähler manifolds

Theorem (Looijenga-Lunts, Verbitsky)

- *The Lie subalgebra* $\mathfrak{g}_{\mathrm{tot}}(X) \subset \mathrm{End}(H^\bullet(X,\mathbb{R}))$ *generated by* L_a *and* Λ_a *for all* $a \in H^2(X,\mathbb{R})$ *with the Lefschetz property is isomorphic to* $\mathfrak{so}(4, b_2(X)-2)$.
- *The grading on* $\mathfrak{g}_{\mathrm{tot}}(X)$*:*

$$\mathfrak{g}_{\mathrm{tot}}(X) = \mathfrak{g}_{\mathrm{tot}}^{-2}(X) \oplus \mathfrak{g}_{\mathrm{tot}}^{0}(X) \oplus \mathfrak{g}_{\mathrm{tot}}^{2}(X).$$

  The semisimple part of $\mathfrak{g}_{\mathrm{tot}}^{0}(X)$ *is isomorphic to* $\mathfrak{so}(V,q)$*, where* $V = H^2(X,\mathbb{R})$*, and* $q \in S^2V^*$ *is the Beauville-Bogomolov-Fujiki form of signature* $(3, b_2(X)-3)$.
- *The algebra* $\mathfrak{g}_{\mathrm{tot}}^{0}(X)$ *acts on* $H^\bullet(X,\mathbb{R})$ *by derivations. The Hodge structures on cohomology groups are induced by this action.*

Cohomology of compact hyperkähler manifolds

Theorem (Looijenga-Lunts, Verbitsky)

- *The Lie subalgebra* $\mathfrak{g}_{\mathrm{tot}}(X) \subset \mathrm{End}(H^\bullet(X,\mathbb{R}))$ *generated by* L_a *and* Λ_a *for all* $a \in H^2(X,\mathbb{R})$ *with the Lefschetz property is isomorphic to* $\mathfrak{so}(4, b_2(X)-2)$.
- *The grading on* $\mathfrak{g}_{\mathrm{tot}}(X)$*:*

$$\mathfrak{g}_{\mathrm{tot}}(X) = \mathfrak{g}_{\mathrm{tot}}^{-2}(X) \oplus \mathfrak{g}_{\mathrm{tot}}^{0}(X) \oplus \mathfrak{g}_{\mathrm{tot}}^{2}(X).$$

The semisimple part of $\mathfrak{g}_{\mathrm{tot}}^{0}(X)$ *is isomorphic to* $\mathfrak{so}(V,q)$*, where* $V = H^2(X,\mathbb{R})$*, and* $q \in S^2V^*$ *is the Beauville-Bogomolov-Fujiki form of signature* $(3, b_2(X)-3)$*.*

- *The algebra* $\mathfrak{g}_{\mathrm{tot}}^{0}(X)$ *acts on* $H^\bullet(X,\mathbb{R})$ *by derivations. The Hodge structures on cohomology groups are induced by this action.*

Cohomology of compact hyperkähler manifolds

Theorem (Looijenga-Lunts, Verbitsky)

- *The Lie subalgebra* $\mathfrak{g}_{\mathrm{tot}}(X) \subset \mathrm{End}(H^\bullet(X,\mathbb{R}))$ *generated by* L_a *and* Λ_a *for all* $a \in H^2(X,\mathbb{R})$ *with the Lefschetz property is isomorphic to* $\mathfrak{so}(4, b_2(X) - 2)$.
- *The grading on* $\mathfrak{g}_{\mathrm{tot}}(X)$*:*

$$\mathfrak{g}_{\mathrm{tot}}(X) = \mathfrak{g}_{\mathrm{tot}}^{-2}(X) \oplus \mathfrak{g}_{\mathrm{tot}}^{0}(X) \oplus \mathfrak{g}_{\mathrm{tot}}^{2}(X).$$

  The semisimple part of $\mathfrak{g}_{\mathrm{tot}}^{0}(X)$ *is isomorphic to* $\mathfrak{so}(V,q)$*, where* $V = H^2(X,\mathbb{R})$*, and* $q \in S^2V^*$ *is the Beauville-Bogomolov-Fujiki form of signature* $(3, b_2(X) - 3)$.
- *The algebra* $\mathfrak{g}_{\mathrm{tot}}^{0}(X)$ *acts on* $H^\bullet(X,\mathbb{R})$ *by derivations. The Hodge structures on cohomology groups are induced by this action.*

Cohomology of compact hyperkähler manifolds

Theorem (Looijenga-Lunts, Verbitsky)

- *The Lie subalgebra* $\mathfrak{g}_{\mathrm{tot}}(X) \subset \mathrm{End}(H^\bullet(X,\mathbb{R}))$ *generated by* L_a *and* Λ_a *for all* $a \in H^2(X,\mathbb{R})$ *with the Lefschetz property is isomorphic to* $\mathfrak{so}(4, b_2(X)-2)$.
- *The grading on* $\mathfrak{g}_{\mathrm{tot}}(X)$*:*

$$\mathfrak{g}_{\mathrm{tot}}(X) = \mathfrak{g}_{\mathrm{tot}}^{-2}(X) \oplus \mathfrak{g}_{\mathrm{tot}}^{0}(X) \oplus \mathfrak{g}_{\mathrm{tot}}^{2}(X).$$

  The semisimple part of $\mathfrak{g}_{\mathrm{tot}}^{0}(X)$ *is isomorphic to* $\mathfrak{so}(V,q)$*, where* $V = H^2(X,\mathbb{R})$*, and* $q \in S^2V^*$ *is the Beauville-Bogomolov-Fujiki form of signature* $(3, b_2(X)-3)$.
- *The algebra* $\mathfrak{g}_{\mathrm{tot}}^{0}(X)$ *acts on* $H^\bullet(X,\mathbb{R})$ *by derivations. The Hodge structures on cohomology groups are induced by this action.*

H^2 of a compact hyperkähler manifold

- X an IHS manifold, $\dim_{\mathbb{C}}(X) = 2n$. Then $V = H^2(X, \mathbb{Q})$ carries a Hodge structure of K3 type, i.e. $\dim V^{2,0} = 1$
- There exists a constant $c_X \in \mathbb{Q}$ and a quadratic form $q \in S^2V^*$ – the Beauville-Bogomolov-Fujiki (BBF) form – such that:
$$q(a)^n = c_X \int_X a^{2n}$$
for any $a \in H^2(X, \mathbb{Q})$.
- The form q has signature $(3, b_2(X) - 3)$.

H^2 of a compact hyperkähler manifold

- X an IHS manifold, $\dim_{\mathbb{C}}(X) = 2n$. Then $V = H^2(X, \mathbb{Q})$ carries a Hodge structure of K3 type, i.e. $\dim V^{2,0} = 1$
- There exists a constant $c_X \in \mathbb{Q}$ and a quadratic form $q \in S^2V^*$ – the Beauville-Bogomolov-Fujiki (BBF) form – such that:
$$q(a)^n = c_X \int_X a^{2n}$$
for any $a \in H^2(X, \mathbb{Q})$.
- The form q has signature $(3, b_2(X) - 3)$.

H^2 of a compact hyperkähler manifold

- X an IHS manifold, $\dim_{\mathbb{C}}(X) = 2n$. Then $V = H^2(X, \mathbb{Q})$ carries a Hodge structure of K3 type, i.e. $\dim V^{2,0} = 1$
- There exists a constant $c_X \in \mathbb{Q}$ and a quadratic form $q \in S^2V^*$ – the Beauville-Bogomolov-Fujiki (BBF) form – such that:

$$q(a)^n = c_X \int_X a^{2n}$$

  for any $a \in H^2(X, \mathbb{Q})$.
- The form q has signature $(3, b_2(X) - 3)$.

H^2 of a compact hyperkähler manifold

- X an IHS manifold, $\dim_{\mathbb{C}}(X) = 2n$. Then $V = H^2(X, \mathbb{Q})$ carries a Hodge structure of K3 type, i.e. $\dim V^{2,0} = 1$
- There exists a constant $c_X \in \mathbb{Q}$ and a quadratic form $q \in S^2V^*$ – the Beauville-Bogomolov-Fujiki (BBF) form – such that:
$$q(a)^n = c_X \int_X a^{2n}$$
for any $a \in H^2(X, \mathbb{Q})$.
- The form q has signature $(3, b_2(X) - 3)$.

The Kuga-Satake construction

- Kuga and Satake (1967): attach to V a Hodge structure W of weight 1
- $W = \mathcal{C}l(V, q)$ the Clifford algebra.
  Define:
  $$W^{1,0} = V^{2,0} \cdot W_{\mathbb{C}}, \quad W^{0,1} = \overline{W^{1,0}}$$
  This gives a rational Hodge structure on W, polarizable when X is projective.

Theorem (Kurnosov, S., Verbitsky)

Let X be a hyperkähler manifold of dimension $2n$. For some integer $m > 0$ there exist embeddings of Hodge structures

$$\nu_i \colon H^{i+2n}(X, \mathbb{Q}(n)) \hookrightarrow \Lambda^{i+2d}(W^{\oplus m})(d),$$

where $i = -2n, \ldots, 2n$ and $d = \frac{1}{4} m \dim_{\mathbb{Q}}(W)$

The Kuga-Satake construction

- Kuga and Satake (1967): attach to V a Hodge structure W of weight 1
- $W = \mathcal{C}l(V,q)$ the Clifford algebra.
  Define:
  $$W^{1,0} = V^{2,0} \cdot W_{\mathbb{C}}, \quad W^{0,1} = \overline{W^{1,0}}$$
  This gives a rational Hodge structure on W, polarizable when X is projective.

Theorem (Kurnosov, S., Verbitsky)

Let X be a hyperkähler manifold of dimension $2n$. For some integer $m > 0$ there exist embeddings of Hodge structures

$$\nu_i \colon H^{i+2n}(X, \mathbb{Q}(n)) \hookrightarrow \Lambda^{i+2d}(W^{\oplus m})(d),$$

where $i = -2n, \ldots, 2n$ and $d = \frac{1}{4} m \dim_{\mathbb{Q}}(W)$

The Kuga-Satake construction

- Kuga and Satake (1967): attach to V a Hodge structure W of weight 1
- $W = \mathcal{C}l(V, q)$ the Clifford algebra.
  Define:
  $$W^{1,0} = V^{2,0} \cdot W_{\mathbb{C}}, \quad W^{0,1} = \overline{W^{1,0}}$$
  This gives a rational Hodge structure on W, polarizable when X is projective.

Theorem (Kurnosov, S., Verbitsky)

Let X be a hyperkähler manifold of dimension $2n$. For some integer $m > 0$ there exist embeddings of Hodge structures

$$\nu_i \colon H^{i+2n}(X, \mathbb{Q}(n)) \hookrightarrow \Lambda^{i+2d}(W^{\oplus m})(d),$$

where $i = -2n, \ldots, 2n$ and $d = \frac{1}{4} m \dim_{\mathbb{Q}}(W)$

Application: absolute Hodge classes

- $X/\mathbb{C}$ — non-singular projective variety,
  X^{an} — the corresponding complex manifold.
- De Rham cohomology:

$$H^\bullet_{dR}(X) \simeq H^\bullet(X, \Omega^\bullet_{X/\mathbb{C}})$$

  $H^k_{dR}(X)$ — a $\mathbb{C}$-vector space + Hodge filtration $F^\bullet$
- Singular cohomology:
  $H^k(X^{an}, \mathbb{C})$ — a $\mathbb{C}$-vector space + a $\mathbb{Q}$-structure $H^k(X^{an}, \mathbb{Q})$
- Comparison: $\Omega^\bullet_{X^{an}} \simeq \mathbb{C}$ implies $H^k_{dR}(X) \simeq H^k(X^{an}, \mathbb{C})$
- $\alpha \in F^p H^{2p}_{dR}(X)$ is a Hodge class, if under the isomorphism above

$$\alpha \in (2\pi i)^p H^{2p}(X^{an}, \mathbb{Q})$$

Application: absolute Hodge classes

- $X/\mathbb{C}$ — non-singular projective variety,
  X^{an} — the corresponding complex manifold.

- De Rham cohomology:

$$H^\bullet_{dR}(X) \simeq H^\bullet(X, \Omega^\bullet_{X/\mathbb{C}})$$

  $H^k_{dR}(X)$ — a $\mathbb{C}$-vector space + Hodge filtration $F^\bullet$

- Singular cohomology:
  $H^k(X^{an}, \mathbb{C})$ — a $\mathbb{C}$-vector space + a $\mathbb{Q}$-structure $H^k(X^{an}, \mathbb{Q})$

- Comparison: $\Omega^\bullet_{X^{an}} \simeq \mathbb{C}$ implies $H^k_{dR}(X) \simeq H^k(X^{an}, \mathbb{C})$

- $\alpha \in F^p H^{2p}_{dR}(X)$ is a Hodge class, if under the isomorphism above

$$\alpha \in (2\pi i)^p H^{2p}(X^{an}, \mathbb{Q})$$

Application: absolute Hodge classes

- $X/\mathbb{C}$ — non-singular projective variety,
  X^{an} — the corresponding complex manifold.

- De Rham cohomology:

$$H^\bullet_{dR}(X) \simeq H^\bullet(X, \Omega^\bullet_{X/\mathbb{C}})$$

  $H^k_{dR}(X)$ — a $\mathbb{C}$-vector space + Hodge filtration $F^\bullet$

- Singular cohomology:
  $H^k(X^{an}, \mathbb{C})$ — a $\mathbb{C}$-vector space + a $\mathbb{Q}$-structure $H^k(X^{an}, \mathbb{Q})$

- Comparison: $\Omega^\bullet_{X^{an}} \simeq \mathbb{C}$ implies $H^k_{dR}(X) \simeq H^k(X^{an}, \mathbb{C})$

- $\alpha \in F^p H^{2p}_{dR}(X)$ is a Hodge class, if under the isomorphism above

$$\alpha \in (2\pi i)^p H^{2p}(X^{an}, \mathbb{Q})$$

Application: absolute Hodge classes

- $X/\mathbb{C}$ — non-singular projective variety,
  X^{an} — the corresponding complex manifold.

- De Rham cohomology:

$$H^\bullet_{dR}(X) \simeq H^\bullet(X, \Omega^\bullet_{X/\mathbb{C}})$$

  $H^k_{dR}(X)$ — a $\mathbb{C}$-vector space + Hodge filtration $F^\bullet$

- Singular cohomology:
  $H^k(X^{an}, \mathbb{C})$ — a $\mathbb{C}$-vector space + a $\mathbb{Q}$-structure $H^k(X^{an}, \mathbb{Q})$

- Comparison: $\Omega^\bullet_{X^{an}} \simeq \mathbb{C}$ implies $H^k_{dR}(X) \simeq H^k(X^{an}, \mathbb{C})$

- $\alpha \in F^p H^{2p}_{dR}(X)$ is a Hodge class, if under the isomorphism above

$$\alpha \in (2\pi i)^p H^{2p}(X^{an}, \mathbb{Q})$$

Application: absolute Hodge classes

- $X/\mathbb{C}$ — non-singular projective variety,
  X^{an} — the corresponding complex manifold.

- De Rham cohomology:

$$H^\bullet_{dR}(X) \simeq H^\bullet(X, \Omega^\bullet_{X/\mathbb{C}})$$

  $H^k_{dR}(X)$ — a $\mathbb{C}$-vector space + Hodge filtration $F^\bullet$

- Singular cohomology:
  $H^k(X^{an}, \mathbb{C})$ — a $\mathbb{C}$-vector space + a $\mathbb{Q}$-structure $H^k(X^{an}, \mathbb{Q})$

- Comparison: $\Omega^\bullet_{X^{an}} \simeq \mathbb{C}$ implies $H^k_{dR}(X) \simeq H^k(X^{an}, \mathbb{C})$

- $\alpha \in F^p H^{2p}_{dR}(X)$ is a Hodge class, if under the isomorphism above

$$\alpha \in (2\pi i)^p H^{2p}(X^{an}, \mathbb{Q})$$

Application: absolute Hodge classes

- $X/\mathbb{C}$ — non-singular projective variety,
  X^{an} — the corresponding complex manifold.

- De Rham cohomology:

$$H^\bullet_{dR}(X) \simeq H^\bullet(X, \Omega^\bullet_{X/\mathbb{C}})$$

  $H^k_{dR}(X)$ — a $\mathbb{C}$-vector space + Hodge filtration $F^\bullet$

- Singular cohomology:
  $H^k(X^{an}, \mathbb{C})$ — a $\mathbb{C}$-vector space + a $\mathbb{Q}$-structure $H^k(X^{an}, \mathbb{Q})$

- Comparison: $\Omega^\bullet_{X^{an}} \simeq \mathbb{C}$ implies $H^k_{dR}(X) \simeq H^k(X^{an}, \mathbb{C})$

- $\alpha \in F^p H^{2p}_{dR}(X)$ is a Hodge class, if under the isomorphism above

$$\alpha \in (2\pi i)^p H^{2p}(X^{an}, \mathbb{Q})$$

Application: absolute Hodge classes

- Let $\tau \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$. We have:

$$H^k_{dR}(X) \simeq H^k_{dR}(X) \otimes_{\mathbb{C},\tau} \mathbb{C} \simeq H^k_{dR}(X_\tau) \simeq H^k(X^{an}_\tau, \mathbb{C})$$

  where X_τ — base change of X via τ
- $\alpha \in F^p H^{2p}_{dR}(X)$ is an absolute Hodge class, if

$$\alpha \in (2\pi i)^p H^{2p}(X^{an}_\tau, \mathbb{Q})$$

  for any $\tau \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$
- If α is an algebraic class, then it is absolute Hodge. The Hodge conjecture: every Hodge class is algebraic, therefore absolute Hodge.
- Deligne (1982): any Hodge class on an abelian variety A is absolute Hodge.

Theorem (S.)

Let X be a projective hyperkähler manifold of $\mathrm{K3}^{[n]}$, generalized Kummer, or OG6 or OG10 deformation type. Then all Hodge classes on X are absolute.

Application: absolute Hodge classes

- Let $\tau \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$. We have:

$$H^k_{dR}(X) \simeq H^k_{dR}(X) \otimes_{\mathbb{C},\tau} \mathbb{C} \simeq H^k_{dR}(X_\tau) \simeq H^k(X^{an}_\tau, \mathbb{C})$$

  where X_τ — base change of X via τ

- $\alpha \in F^p H^{2p}_{dR}(X)$ is an absolute Hodge class, if

$$\alpha \in (2\pi i)^p H^{2p}(X^{an}_\tau, \mathbb{Q})$$

  for any $\tau \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$

- If α is an algebraic class, then it is absolute Hodge. The Hodge conjecture: every Hodge class is algebraic, therefore absolute Hodge.

- Deligne (1982): any Hodge class on an abelian variety A is absolute Hodge.

Theorem (S.)

Let X be a projective hyperkähler manifold of $\mathrm{K3}^{[n]}$, generalized Kummer, or OG6 or OG10 deformation type. Then all Hodge classes on X are absolute.

Application: absolute Hodge classes

- Let $\tau \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$. We have:

$$H^k_{dR}(X) \simeq H^k_{dR}(X) \otimes_{\mathbb{C},\tau} \mathbb{C} \simeq H^k_{dR}(X_\tau) \simeq H^k(X_\tau^{an}, \mathbb{C})$$

  where X_τ — base change of X via τ

- $\alpha \in F^p H^{2p}_{dR}(X)$ is an absolute Hodge class, if

$$\alpha \in (2\pi i)^p H^{2p}(X_\tau^{an}, \mathbb{Q})$$

  for any $\tau \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$

- If α is an algebraic class, then it is absolute Hodge.
  The Hodge conjecture: every Hodge class is algebraic, therefore absolute Hodge.

- Deligne (1982): any Hodge class on an abelian variety A is absolute Hodge.

Theorem (S.)

Let X be a projective hyperkähler manifold of $\mathrm{K3}^{[n]}$, generalized Kummer, or OG6 or OG10 deformation type. Then all Hodge classes on X are absolute.

Application: absolute Hodge classes

- Let $\tau \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$. We have:
  $$H^k_{dR}(X) \simeq H^k_{dR}(X) \otimes_{\mathbb{C},\tau} \mathbb{C} \simeq H^k_{dR}(X_\tau) \simeq H^k(X^{an}_\tau, \mathbb{C})$$
  where X_τ — base change of X via τ
- $\alpha \in F^p H^{2p}_{dR}(X)$ is an absolute Hodge class, if
  $$\alpha \in (2\pi i)^p H^{2p}(X^{an}_\tau, \mathbb{Q})$$
  for any $\tau \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$
- If α is an algebraic class, then it is absolute Hodge.
  The Hodge conjecture: every Hodge class is algebraic, therefore absolute Hodge.
- Deligne (1982): any Hodge class on an abelian variety A is absolute Hodge.

Theorem (S.)

Let X be a projective hyperkähler manifold of $\mathrm{K3}^{[n]}$, generalized Kummer, or OG6 or OG10 deformation type. Then all Hodge classes on X are absolute.

Application: absolute Hodge classes

- Let $\tau \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$. We have:
$$H^k_{dR}(X) \simeq H^k_{dR}(X) \otimes_{\mathbb{C},\tau} \mathbb{C} \simeq H^k_{dR}(X_\tau) \simeq H^k(X^{an}_\tau, \mathbb{C})$$
where X_τ — base change of X via τ
- $\alpha \in F^p H^{2p}_{dR}(X)$ is an absolute Hodge class, if
$$\alpha \in (2\pi i)^p H^{2p}(X^{an}_\tau, \mathbb{Q})$$
for any $\tau \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$
- If α is an algebraic class, then it is absolute Hodge.
The Hodge conjecture: every Hodge class is algebraic, therefore absolute Hodge.
- Deligne (1982): any Hodge class on an abelian variety A is absolute Hodge.

Theorem (S.)

Let X be a projective hyperkähler manifold of $\mathrm{K3}^{[n]}$, generalized Kummer, or OG6 or OG10 deformation type. Then all Hodge classes on X are absolute.

Degenerations of compact hyperkähler manifolds

Let $\pi\colon \mathcal{X} \to \Delta$ be a flat projective morphism, such that

- $\mathcal{X}$ is a smooth complex manifold, $\Delta = \{t \in \mathbb{C} \mid |t| < 1\}$
- π is smooth over $\Delta^* = \Delta \setminus \{0\}$
- for any $t \in \Delta^*$ the fibre $\mathcal{X}_t = X$ is an IHS manifold.

Definition

The limit mixed Hodge structure (MHS) on $H^k(X, \mathbb{Q})$ is given by two filtrations:

- *the increasing weight filtration (defined over $\mathbb{Q}$)*

$$\ldots \subset W_m H^k(X, \mathbb{Q}) \subset W_{m+1} H^k(X, \mathbb{Q}) \subset \ldots$$

- *the decreasing limit Hodge filtration (defined over $\mathbb{C}$)*

$$\ldots \subset F^p_{lim} H^k(X, \mathbb{C}) \subset F^{p-1}_{lim} H^k(X, \mathbb{C}) \subset \ldots$$

Degenerations of compact hyperkähler manifolds

Let $\pi\colon \mathcal{X} \to \Delta$ be a flat projective morphism, such that

- $\mathcal{X}$ is a smooth complex manifold, $\Delta = \{t \in \mathbb{C} \mid |t| < 1\}$
- π is smooth over $\Delta^* = \Delta \setminus \{0\}$
- for any $t \in \Delta^*$ the fibre $\mathcal{X}_t = X$ is an IHS manifold.

Definition

The limit mixed Hodge structure (MHS) on $H^k(X,\mathbb{Q})$ is given by two filtrations:

- *the increasing weight filtration (defined over $\mathbb{Q}$)*

$$\ldots \subset W_m H^k(X,\mathbb{Q}) \subset W_{m+1} H^k(X,\mathbb{Q}) \subset \ldots$$

- *the decreasing limit Hodge filtration (defined over $\mathbb{C}$)*

$$\ldots \subset F^p_{lim} H^k(X,\mathbb{C}) \subset F^{p-1}_{lim} H^k(X,\mathbb{C}) \subset \ldots$$

Degenerations of compact hyperkähler manifolds

Let $\pi\colon \mathcal{X} \to \Delta$ be a flat projective morphism, such that

- $\mathcal{X}$ is a smooth complex manifold, $\Delta = \{t \in \mathbb{C} \mid |t| < 1\}$
- π is smooth over $\Delta^* = \Delta \setminus \{0\}$
- for any $t \in \Delta^*$ the fibre $\mathcal{X}_t = X$ is an IHS manifold.

Definition

The limit mixed Hodge structure (MHS) on $H^k(X, \mathbb{Q})$ is given by two filtrations:

- *the increasing weight filtration (defined over $\mathbb{Q}$)*

$$\ldots \subset W_m H^k(X, \mathbb{Q}) \subset W_{m+1} H^k(X, \mathbb{Q}) \subset \ldots$$

- *the decreasing limit Hodge filtration (defined over $\mathbb{C}$)*

$$\ldots \subset F^p_{lim} H^k(X, \mathbb{C}) \subset F^{p-1}_{lim} H^k(X, \mathbb{C}) \subset \ldots$$

Degenerations of compact hyperkähler manifolds

Let $\pi\colon \mathcal{X} \to \Delta$ be a flat projective morphism, such that

- $\mathcal{X}$ is a smooth complex manifold, $\Delta = \{t \in \mathbb{C} \mid |t| < 1\}$
- π is smooth over $\Delta^* = \Delta \setminus \{0\}$
- for any $t \in \Delta^*$ the fibre $\mathcal{X}_t = X$ is an IHS manifold.

Definition

The limit mixed Hodge structure (MHS) on $H^k(X, \mathbb{Q})$ is given by two filtrations:

- *the increasing weight filtration (defined over $\mathbb{Q}$)*

$$\ldots \subset W_m H^k(X, \mathbb{Q}) \subset W_{m+1} H^k(X, \mathbb{Q}) \subset \ldots$$

- *the decreasing limit Hodge filtration (defined over $\mathbb{C}$)*

$$\ldots \subset F^p_{lim} H^k(X, \mathbb{C}) \subset F^{p-1}_{lim} H^k(X, \mathbb{C}) \subset \ldots$$

Degenerations of compact hyperkähler manifolds

Let $\pi\colon \mathcal{X} \to \Delta$ be a flat projective morphism, such that

- $\mathcal{X}$ is a smooth complex manifold, $\Delta = \{t \in \mathbb{C} \mid |t| < 1\}$
- π is smooth over $\Delta^* = \Delta \setminus \{0\}$
- for any $t \in \Delta^*$ the fibre $\mathcal{X}_t = X$ is an IHS manifold.

Definition

The limit mixed Hodge structure (MHS) on $H^k(X, \mathbb{Q})$ is given by two filtrations:

- *the increasing weight filtration (defined over $\mathbb{Q}$)*

$$\ldots \subset W_m H^k(X, \mathbb{Q}) \subset W_{m+1} H^k(X, \mathbb{Q}) \subset \ldots$$

- *the decreasing limit Hodge filtration (defined over $\mathbb{C}$)*

$$\ldots \subset F^p_{lim} H^k(X, \mathbb{C}) \subset F^{p-1}_{lim} H^k(X, \mathbb{C}) \subset \ldots$$

Degenerations of compact hyperkähler manifolds

Let $\pi \colon \mathcal{X} \to \Delta$ be a flat projective morphism, such that

- $\mathcal{X}$ is a smooth complex manifold, $\Delta = \{t \in \mathbb{C} \mid |t| < 1\}$
- π is smooth over $\Delta^* = \Delta \setminus \{0\}$
- for any $t \in \Delta^*$ the fibre $\mathcal{X}_t = X$ is an IHS manifold.

Definition

The limit mixed Hodge structure (MHS) on $H^k(X, \mathbb{Q})$ is given by two filtrations:

- *the increasing weight filtration (defined over $\mathbb{Q}$)*

$$\ldots \subset W_m H^k(X, \mathbb{Q}) \subset W_{m+1} H^k(X, \mathbb{Q}) \subset \ldots$$

- *the decreasing limit Hodge filtration (defined over $\mathbb{C}$)*

$$\ldots \subset F^p_{lim} H^k(X, \mathbb{C}) \subset F^{p-1}_{lim} H^k(X, \mathbb{C}) \subset \ldots$$

Degenerations of compact hyperkähler manifolds

Let $\pi\colon \mathcal{X} \to \Delta$ be a flat projective morphism, such that

- $\mathcal{X}$ is a smooth complex manifold, $\Delta = \{t \in \mathbb{C} \mid |t| < 1\}$
- π is smooth over $\Delta^* = \Delta \setminus \{0\}$
- for any $t \in \Delta^*$ the fibre $\mathcal{X}_t = X$ is an IHS manifold.

Definition

The limit mixed Hodge structure (MHS) on $H^k(X, \mathbb{Q})$ is given by two filtrations:

- *the increasing weight filtration (defined over $\mathbb{Q}$)*

$$\ldots \subset W_m H^k(X, \mathbb{Q}) \subset W_{m+1} H^k(X, \mathbb{Q}) \subset \ldots$$

- *the decreasing limit Hodge filtration (defined over $\mathbb{C}$)*

$$\ldots \subset F^p_{lim} H^k(X, \mathbb{C}) \subset F^{p-1}_{lim} H^k(X, \mathbb{C}) \subset \ldots$$

Degenerations of compact hyperkähler manifolds

Let $\pi\colon \mathcal{X} \to \Delta$ be a flat projective morphism, such that

- $\mathcal{X}$ is a smooth complex manifold, $\Delta = \{t \in \mathbb{C} \mid |t| < 1\}$
- π is smooth over $\Delta^* = \Delta \setminus \{0\}$
- for any $t \in \Delta^*$ the fibre $\mathcal{X}_t = X$ is an IHS manifold.

Definition

The limit mixed Hodge structure (MHS) on $H^k(X,\mathbb{Q})$ is given by two filtrations:

- *the increasing weight filtration (defined over $\mathbb{Q}$)*

$$\ldots \subset W_m H^k(X,\mathbb{Q}) \subset W_{m+1} H^k(X,\mathbb{Q}) \subset \ldots$$

- *the decreasing limit Hodge filtration (defined over $\mathbb{C}$)*

$$\ldots \subset F^p_{lim} H^k(X,\mathbb{C}) \subset F^{p-1}_{lim} H^k(X,\mathbb{C}) \subset \ldots$$

Monodromy action

$\pi\colon \mathcal{X} \to \Delta, \quad X = \mathcal{X}_t$ for some $t \in \Delta^*$

The weight filtration $W_\bullet H^k(X, \mathbb{Q})$ is determined by the monodromy action on $H^k(X, \mathbb{Q})$.

Recall that the semisimple part of $\mathfrak{g}^0_{\mathrm{tot}}(X)$ is isomorphic to $\mathfrak{so}(V, q)$, where $V = H^2(X, \mathbb{R})$ and q is the BBF form.

Theorem (S.)

There exits a nilpotent element $N \in \mathfrak{so}(V, q)$ and an integer $d \geqslant 1$, such that after a finite base change $\Delta \to \Delta$, $t \mapsto t^d$ the monodromy of the family π acts on $H^k(X, \mathbb{Q})$ for all k as

$$e^N \in \mathrm{Spin}(V, q).$$

Monodromy action

$\pi\colon \mathcal{X} \to \Delta, \quad X = \mathcal{X}_t$ for some $t \in \Delta^*$

The weight filtration $W_\bullet H^k(X,\mathbb{Q})$ is determined by the monodromy action on $H^k(X,\mathbb{Q})$.

Recall that the semisimple part of $\mathfrak{g}_{\mathrm{tot}}^0(X)$ is isomorphic to $\mathfrak{so}(V,q)$, where $V = H^2(X,\mathbb{R})$ and q is the BBF form.

Theorem (S.)

There exits a nilpotent element $N \in \mathfrak{so}(V,q)$ and an integer $d \geqslant 1$, such that after a finite base change $\Delta \to \Delta$, $t \mapsto t^d$ the monodromy of the family π acts on $H^k(X,\mathbb{Q})$ for all k as

$$e^N \in \mathrm{Spin}(V,q).$$

Monodromy action

$\pi\colon \mathcal{X} \to \Delta, \quad X = \mathcal{X}_t$ for some $t \in \Delta^*$

The weight filtration $W_\bullet H^k(X, \mathbb{Q})$ is determined by the monodromy action on $H^k(X, \mathbb{Q})$.

Recall that the semisimple part of $\mathfrak{g}^0_{\mathrm{tot}}(X)$ is isomorphic to $\mathfrak{so}(V, q)$, where $V = H^2(X, \mathbb{R})$ and q is the BBF form.

Theorem (S.)

There exits a nilpotent element $N \in \mathfrak{so}(V, q)$ and an integer $d \geqslant 1$, such that after a finite base change $\Delta \to \Delta$, $t \mapsto t^d$ the monodromy of the family π acts on $H^k(X, \mathbb{Q})$ for all k as

$$e^N \in \mathrm{Spin}(V, q).$$

Monodromy action

$\pi\colon \mathcal{X} \to \Delta, \quad X = \mathcal{X}_t$ for some $t \in \Delta^*$

The weight filtration $W_\bullet H^k(X, \mathbb{Q})$ is determined by the monodromy action on $H^k(X, \mathbb{Q})$.

Recall that the semisimple part of $\mathfrak{g}^0_{\mathrm{tot}}(X)$ is isomorphic to $\mathfrak{so}(V, q)$, where $V = H^2(X, \mathbb{R})$ and q is the BBF form.

Theorem (S.)

There exits a nilpotent element $N \in \mathfrak{so}(V, q)$ and an integer $d \geqslant 1$, such that after a finite base change $\Delta \to \Delta$, $t \mapsto t^d$ the monodromy of the family π acts on $H^k(X, \mathbb{Q})$ for all k as

$$e^N \in \mathrm{Spin}(V, q).$$

Types of degenerations

$N \in \mathfrak{so}(V, q)$ the logarithm of monodromy of $\pi\colon \mathcal{X} \to \Delta$.

The degeneration is

- of type I, if $N = 0$
- of type II, if $N \neq 0$, $N^2 = 0$
- of type III, or maximal, if $N^2 \neq 0$, $N^3 = 0$

Example of a maximal degeneration

$$\mathcal{X}' = \{(x, t) \in \mathbb{C}P^3 \times \mathbb{A}^1 \mid t(x_0^4 + \ldots + x_3^4) + x_0 x_1 x_2 x_3 = 0\}$$

We have $\pi\colon \mathcal{X}' \to \mathbb{A}^1$. For general t the fibre $\pi^{-1}(t)$ is a smooth K3 surface. The fibre over $t = 0$ is the union of 4 planes. The total space $\mathcal{X}'$ is not smooth, but we may take $\mathcal{X}$ to be its resolution of singularities.

Types of degenerations

$N \in \mathfrak{so}(V, q)$ the logarithm of monodromy of $\pi\colon \mathcal{X} \to \Delta$.

The degeneration is

- of type I, if $N = 0$
- of type II, if $N \neq 0$, $N^2 = 0$
- of type III, or maximal, if $N^2 \neq 0$, $N^3 = 0$

Example of a maximal degeneration

$$\mathcal{X}' = \{(x,t) \in \mathbb{C}P^3 \times \mathbb{A}^1 \mid t(x_0^4 + \ldots + x_3^4) + x_0x_1x_2x_3 = 0\}$$

We have $\pi\colon \mathcal{X}' \to \mathbb{A}^1$. For general t the fibre $\pi^{-1}(t)$ is a smooth K3 surface. The fibre over $t = 0$ is the union of 4 planes. The total space $\mathcal{X}'$ is not smooth, but we may take $\mathcal{X}$ to be its resolution of singularities.

Types of degenerations

$N \in \mathfrak{so}(V, q)$ the logarithm of monodromy of $\pi \colon \mathcal{X} \to \Delta$.

The degeneration is

- of type I, if $N = 0$
- of type II, if $N \neq 0$, $N^2 = 0$
- of type III, or maximal, if $N^2 \neq 0$, $N^3 = 0$

Example of a maximal degeneration

$$\mathcal{X}' = \{(x, t) \in \mathbb{C}P^3 \times \mathbb{A}^1 \mid t(x_0^4 + \ldots + x_3^4) + x_0 x_1 x_2 x_3 = 0\}$$

We have $\pi \colon \mathcal{X}' \to \mathbb{A}^1$. For general t the fibre $\pi^{-1}(t)$ is a smooth K3 surface. The fibre over $t = 0$ is the union of 4 planes. The total space $\mathcal{X}'$ is not smooth, but we may take $\mathcal{X}$ to be its resolution of singularities.

Types of degenerations

$N \in \mathfrak{so}(V,q)$ the logarithm of monodromy of $\pi\colon \mathcal{X} \to \Delta$.

The degeneration is

- of type I, if $N = 0$
- of type II, if $N \neq 0$, $N^2 = 0$
- of type III, or maximal, if $N^2 \neq 0$, $N^3 = 0$

Example of a maximal degeneration

$$\mathcal{X}' = \{(x,t) \in \mathbb{C}P^3 \times \mathbb{A}^1 \mid t(x_0^4 + \ldots + x_3^4) + x_0x_1x_2x_3 = 0\}$$

We have $\pi\colon \mathcal{X}' \to \mathbb{A}^1$. For general t the fibre $\pi^{-1}(t)$ is a smooth K3 surface. The fibre over $t = 0$ is the union of 4 planes. The total space $\mathcal{X}'$ is not smooth, but we may take $\mathcal{X}$ to be its resolution of singularities.

Types of degenerations

$N \in \mathfrak{so}(V, q)$ the logarithm of monodromy of $\pi \colon \mathcal{X} \to \Delta$.

The degeneration is

- of type I, if $N = 0$
- of type II, if $N \neq 0$, $N^2 = 0$
- of type III, or maximal, if $N^2 \neq 0$, $N^3 = 0$

Example of a maximal degeneration

$$\mathcal{X}' = \{(x, t) \in \mathbb{C}P^3 \times \mathbb{A}^1 \mid t(x_0^4 + \ldots + x_3^4) + x_0x_1x_2x_3 = 0\}$$

We have $\pi \colon \mathcal{X}' \to \mathbb{A}^1$. For general t the fibre $\pi^{-1}(t)$ is a smooth K3 surface. The fibre over $t = 0$ is the union of 4 planes. The total space $\mathcal{X}'$ is not smooth, but we may take $\mathcal{X}$ to be its resolution of singularities.

Types of degenerations

$N \in \mathfrak{so}(V, q)$ the logarithm of monodromy of $\pi\colon \mathcal{X} \to \Delta$.

The degeneration is

- of type I, if $N = 0$
- of type II, if $N \neq 0$, $N^2 = 0$
- of type III, or maximal, if $N^2 \neq 0$, $N^3 = 0$

Example of a maximal degeneration

$$\mathcal{X}' = \{(x, t) \in \mathbb{C}P^3 \times \mathbb{A}^1 \mid t(x_0^4 + \ldots + x_3^4) + x_0x_1x_2x_3 = 0\}$$

We have $\pi\colon \mathcal{X}' \to \mathbb{A}^1$. For general t the fibre $\pi^{-1}(t)$ is a smooth K3 surface. The fibre over $t = 0$ is the union of 4 planes. The total space $\mathcal{X}'$ is not smooth, but we may take $\mathcal{X}$ to be its resolution of singularities.

Types of degenerations

$N \in \mathfrak{so}(V, q)$ the logarithm of monodromy of $\pi \colon \mathcal{X} \to \Delta$.

The degeneration is

- of type I, if $N = 0$
- of type II, if $N \neq 0$, $N^2 = 0$
- of type III, or maximal, if $N^2 \neq 0$, $N^3 = 0$

Example of a maximal degeneration

$$\mathcal{X}' = \{(x, t) \in \mathbb{C}P^3 \times \mathbb{A}^1 \mid t(x_0^4 + \ldots + x_3^4) + x_0 x_1 x_2 x_3 = 0\}$$

We have $\pi \colon \mathcal{X}' \to \mathbb{A}^1$. For general t the fibre $\pi^{-1}(t)$ is a smooth K3 surface. The fibre over $t = 0$ is the union of 4 planes. The total space $\mathcal{X}'$ is not smooth, but we may take $\mathcal{X}$ to be its resolution of singularities.

Types of degenerations

$N \in \mathfrak{so}(V,q)$ the logarithm of monodromy of $\pi\colon \mathcal{X} \to \Delta$.

The degeneration is

- of type I, if $N = 0$
- of type II, if $N \neq 0$, $N^2 = 0$
- of type III, or maximal, if $N^2 \neq 0$, $N^3 = 0$

Example of a maximal degeneration

$$\mathcal{X}' = \{(x,t) \in \mathbb{C}P^3 \times \mathbb{A}^1 \mid t(x_0^4 + \ldots + x_3^4) + x_0x_1x_2x_3 = 0\}$$

We have $\pi\colon \mathcal{X}' \to \mathbb{A}^1$. For general t the fibre $\pi^{-1}(t)$ is a smooth K3 surface. The fibre over $t = 0$ is the union of 4 planes. The total space $\mathcal{X}'$ is not smooth, but we may take $\mathcal{X}$ to be its resolution of singularities.

Limit MHS for a maximal degeneration

Kulikov, 1977: Description of central fibres and their dual complexes for degenerations of K3 surfaces. Further studied by Persson, Pinkham, Friedman, Scattone, Alexeev and many others.

Kollár–Laza–Saccà–Voisin, 2018: For a maximal degeneration of IHS manifolds the dual complex of the central fibre has rational homology of $\mathbb{C}P^n$. For other types of degenerations the dual complex is contractible.

Theorem (S.)

Assume that π is a maximal degeneration of IHS manifolds. Then for all k the limit mixed Hodge structures on $H^k(X, \mathbb{Q})$ are Hodge-Tate, i.e. $\mathrm{gr}^W_{2j} H^k$ are pure of type (j, j) and $\mathrm{gr}^W_{2j+1} H^k = 0$ for all j.

Limit MHS for a maximal degeneration

Kulikov, 1977: Description of central fibres and their dual complexes for degenerations of K3 surfaces. Further studied by Persson, Pinkham, Friedman, Scattone, Alexeev and many others.

Kollár–Laza–Saccà–Voisin, 2018: For a maximal degeneration of IHS manifolds the dual complex of the central fibre has rational homology of $\mathbb{C}P^n$. For other types of degenerations the dual complex is contractible.

Theorem (S.)

Assume that π is a maximal degeneration of IHS manifolds. Then for all k the limit mixed Hodge structures on $H^k(X, \mathbb{Q})$ are Hodge-Tate, i.e. $\mathrm{gr}^W_{2j} H^k$ are pure of type (j, j) and $\mathrm{gr}^W_{2j+1} H^k = 0$ for all j.

Limit MHS for a maximal degeneration

Kulikov, 1977: Description of central fibres and their dual complexes for degenerations of K3 surfaces. Further studied by Persson, Pinkham, Friedman, Scattone, Alexeev and many others.

Kollár–Laza–Saccà–Voisin, 2018: For a maximal degeneration of IHS manifolds the dual complex of the central fibre has rational homology of $\mathbb{C}P^n$. For other types of degenerations the dual complex is contractible.

Theorem (S.)

Assume that π is a maximal degeneration of IHS manifolds. Then for all k the limit mixed Hodge structures on $H^k(X, \mathbb{Q})$ are Hodge-Tate, i.e. $\mathrm{gr}^W_{2j} H^k$ are pure of type (j, j) and $\mathrm{gr}^W_{2j+1} H^k = 0$ for all j.

Limit MHS for a maximal degeneration

Kulikov, 1977: Description of central fibres and their dual complexes for degenerations of K3 surfaces. Further studied by Persson, Pinkham, Friedman, Scattone, Alexeev and many others.

Kollár–Laza–Saccà–Voisin, 2018: For a maximal degeneration of IHS manifolds the dual complex of the central fibre has rational homology of $\mathbb{C}P^n$. For other types of degenerations the dual complex is contractible.

Theorem (S.)

*Assume that π is a maximal degeneration of IHS manifolds. Then for all k the limit mixed Hodge structures on $H^k(X, \mathbb{Q})$ are **Hodge-Tate**, i.e. $\mathrm{gr}^W_{2j} H^k$ are pure of type (j, j) and $\mathrm{gr}^W_{2j+1} H^k = 0$ for all j.*

Limit MHS for a maximal degeneration

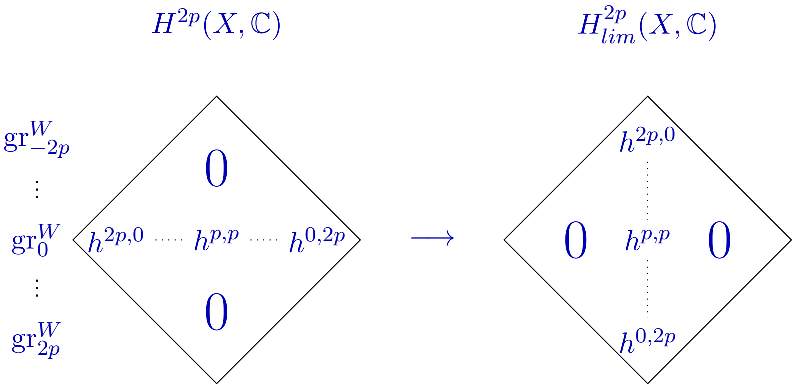
Kulikov, 1977: Description of central fibres and their dual complexes for degenerations of K3 surfaces. Further studied by Persson, Pinkham, Friedman, Scattone, Alexeev and many others.

Kollár–Laza–Saccà–Voisin, 2018: For a maximal degeneration of IHS manifolds the dual complex of the central fibre has rational homology of $\mathbb{C}P^n$. For other types of degenerations the dual complex is contractible.

Theorem (S.)

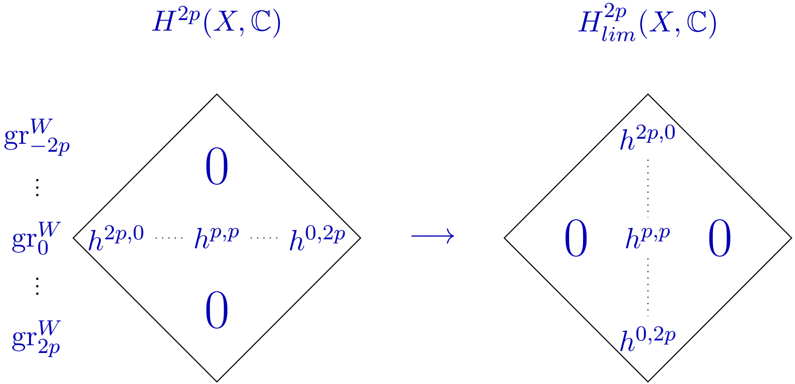
Assume that π is a maximal degeneration of IHS manifolds. Then for all k the limit mixed Hodge structures on $H^k(X, \mathbb{Q})$ are Hodge-Tate, i.e. $\mathrm{gr}^W_{2j} H^k$ are pure of type (j, j) and $\mathrm{gr}^W_{2j+1} H^k = 0$ for all j.

Limit MHS for a maximal degeneration



Expectation: for a maximal degeneration, the central fibre $\mathcal{X}_0$ should be rationally connected

Limit MHS for a maximal degeneration



Expectation: for a maximal degeneration, the central fibre $\mathcal{X}_0$ should be rationally connected

Other research directions

- Teichmüller spaces of hyperkähler manifolds, Torelli-type theorems;
- SYZ conjecture for hyperkähler manifolds;
- Lagrangian fibrations of hyperkähler manifolds, degenerate twistor deformations;
- Dynamics of automorphisms of hyperkähler manifolds, rigid currents;
- Hyperbolic geometry of the ample cone of a hyperkähler manifold;
- ...

Other research directions

- Teichmüller spaces of hyperkähler manifolds, Torelli-type theorems;
- SYZ conjecture for hyperkähler manifolds;
- Lagrangian fibrations of hyperkähler manifolds, degenerate twistor deformations;
- Dynamics of automorphisms of hyperkähler manifolds, rigid currents;
- Hyperbolic geometry of the ample cone of a hyperkähler manifold;
- ...

Other research directions

- Teichmüller spaces of hyperkähler manifolds, Torelli-type theorems;
- SYZ conjecture for hyperkähler manifolds;
- Lagrangian fibrations of hyperkähler manifolds, degenerate twistor deformations;
- Dynamics of automorphisms of hyperkähler manifolds, rigid currents;
- Hyperbolic geometry of the ample cone of a hyperkähler manifold;
- ...

Other research directions

- Teichmüller spaces of hyperkähler manifolds, Torelli-type theorems;
- SYZ conjecture for hyperkähler manifolds;
- Lagrangian fibrations of hyperkähler manifolds, degenerate twistor deformations;
- Dynamics of automorphisms of hyperkähler manifolds, rigid currents;
- Hyperbolic geometry of the ample cone of a hyperkähler manifold;
- ...

Other research directions

- Teichmüller spaces of hyperkähler manifolds, Torelli-type theorems;
- SYZ conjecture for hyperkähler manifolds;
- Lagrangian fibrations of hyperkähler manifolds, degenerate twistor deformations;
- Dynamics of automorphisms of hyperkähler manifolds, rigid currents;
- Hyperbolic geometry of the ample cone of a hyperkähler manifold;
- ...

Other research directions

- Teichmüller spaces of hyperkähler manifolds, Torelli-type theorems;
- SYZ conjecture for hyperkähler manifolds;
- Lagrangian fibrations of hyperkähler manifolds, degenerate twistor deformations;
- Dynamics of automorphisms of hyperkähler manifolds, rigid currents;
- Hyperbolic geometry of the ample cone of a hyperkähler manifold;
- ...

Other research directions

- Teichmüller spaces of hyperkähler manifolds, Torelli-type theorems;
- SYZ conjecture for hyperkähler manifolds;
- Lagrangian fibrations of hyperkähler manifolds, degenerate twistor deformations;
- Dynamics of automorphisms of hyperkähler manifolds, rigid currents;
- Hyperbolic geometry of the ample cone of a hyperkähler manifold;
- ...

Thank you!