Exercise 70. (2 points) *Codimension.*
Let $Y \subset X$ be an integral subscheme and $\eta_Y \in Y$ be its generic point. Show that

$$\dim \mathcal{O}_{X,\eta_Y} = \operatorname{codim}(Y).$$

Solution. Recall that $\operatorname{codim}(Y)$ is the supremum of length of chains $Y = Y_0 \subsetneq Y_1 \subsetneq \ldots \subsetneq Y_n \subset X$ of irreducible closed subsets containing Y. Pick an affine open subset $U = \operatorname{Spec}(A)$ containing η_Y. Generic points η_i of Y_i contain η_Y in their closure, hence $\eta_i \in U$ and we reduce to the case when X is affine. In this case η_i are identified with prime ideals in A and we consider chains of prime ideals contained in $\eta_0 = \eta_Y$. This gives the dimension of the localization of A at η_Y which equals $\dim \mathcal{O}_{X,\eta_Y}$.

Exercise 71. (4 points) *Ample invertible sheaves.*
Let X be a noetherian scheme.

i) Show that if $\mathcal{L}$ and $\mathcal{M}$ are two invertible sheaves on X such that $\mathcal{L}$ is ample, then $\mathcal{L}^n \otimes \mathcal{M}$ is ample for $n \gg 0$. Conclude that any invertible sheaf $\mathcal{M}$ is isomorphic to some $\mathcal{L}_1 \otimes \mathcal{L}_2^*$ with $\mathcal{L}_1$ and $\mathcal{L}_2$ ample if there exists an ample invertible sheaf at all.

ii) Is the tensor product $\mathcal{L}_1 \otimes \mathcal{L}_2$ of two ample (resp. very ample) invertible sheaves again ample (resp. very ample)?

Solutions. i) Let us first prove that if two invertible sheaves $\mathcal{L}_1$ and $\mathcal{L}_2$ are globally generated then the same is true about their tensor product. For any point $x \in X$ there exist sections s_1 and s_2 of $\mathcal{L}_1$ and $\mathcal{L}_2$ that do not vanish at x, that is $s_{i,x} \notin \mathfrak{m}_x \mathcal{O}_{X,x}$. Then the product $s_1 s_2$ — which is a section of $\mathcal{L}_1 \otimes \mathcal{L}_2$ — also does not vanish at x.

Let's prove the first statement. We need to check that for any coherent sheaf F the sheaf $(\mathcal{L}^n \otimes \mathcal{M})^m \otimes F$ is globally generated for m big enough. By definition of ampleness there exist n_0, n_1 such that the sheaf $\mathcal{L}^n \otimes \mathcal{M}$ is globally generated for $n \geq n_0$ and $\mathcal{L}^n \otimes F$ is globally generated for $n \geq n_1$. Then $(\mathcal{L}^n \otimes \mathcal{M})^m \otimes F = (\mathcal{L}^{n-1} \otimes \mathcal{M})^m \otimes (\mathcal{L}^m \otimes F)$. This sheaf is globally generated for $m \geq n_1$ and $n \geq n_0 + 1$. This shows that $\mathcal{L}^n \otimes \mathcal{M}$ is ample for $n \geq n_0 + 1$.

ii) True in both cases. For very ample $\mathcal{L}_1$ and $\mathcal{L}_2$ this is Exercise 60 from sheet 11. If $\mathcal{L}_1$ and $\mathcal{L}_2$ are ample, for any coherent sheaf F choose n_0 so that $\mathcal{L}_1^n$ and $\mathcal{L}_2^n \otimes F$ are globally generated for $n \geq n_0$. Then $(\mathcal{L}_1 \otimes \mathcal{L}_2)^n \otimes F = \mathcal{L}_1^n \otimes (\mathcal{L}_2^n \otimes F)$ is globally generated for $n \geq n_0$.

Exercise 72. (4 points) *Ample invertible sheaves on the quadric.*
Consider the quadric $Q = \mathbb{P}_k^1 \times \mathbb{P}_k^1$ and use that $\operatorname{Pic}(Q) \cong \mathbb{Z} \oplus \mathbb{Z}$, i.e. every invertible sheaf on Q is isomorphic to a unique $\mathcal{O}(a,b) := p_1^*\mathcal{O}(a) \otimes p_2^*\mathcal{O}(b)$.

i) Determine all ample invertible sheaves on Q. Are they all very ample?

ii) Compute the cohomology groups $H^1(Q, \mathcal{O}(a,b))$.

Solutions. i) The product of two very ample invertible sheaves pulled back from the two factors is very ample (this was exercise 60). So for $a, b \geq 1$ the sheaf $\mathcal{O}(a,b)$ is very ample (hence also ample). On the other hand, if a line bundle is ample then its restriction to any subvariety is also ample. Denote by X_1 and X_2 the fibers of p_1 and p_2 over some k-points in $\mathbb{P}_k^1$. The restriction of $\mathcal{O}(a,b)$ to $X_1 \simeq \mathbb{P}_k^1$ is $\mathcal{O}(b)$, and the restriction to $X_2 \simeq \mathbb{P}_k^1$ is $\mathcal{O}(a)$.

These restrictions are ample if and only if $a, b \geq 1$. So $\mathcal{O}(a,b)$ is ample if and only if $a, b \geq 1$.

ii) *Step 1.* Recall the Segre embedding $Q \hookrightarrow \mathbb{P}^3_k$ given by identifying Q with

$$\operatorname{Proj}(k[x_1y_1, x_1y_2, x_2y_1, x_2y_2]) = \operatorname{Proj}(k[a,b,c,d]/(ad - bc))$$

. The image is a quadric, hence the exact triple

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2) \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_Q \to 0.$$

Twisting by $\mathcal{O}_{\mathbb{P}^3}(n)$ we get

$$0 \to \mathcal{O}_{\mathbb{P}^3}(n-2) \to \mathcal{O}_{\mathbb{P}^3}(n) \to \mathcal{O}_Q(n,n) \to 0.$$

We know that $H^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(n)) = H^2(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(n)) = 0$ for all n, so from the long exact cohomology sequence we see that $H^1(Q, \mathcal{O}(n,n)) = 0$ for all n.

Step 2. Consider the restriction of $\mathcal{O}(a,b)$ to the fiber of p_2. The ideal sheaf of this fiber is $\mathcal{O}(0,-1)$, so we have

$$0 \to \mathcal{O}(a, b-1) \to \mathcal{O}(a,b) \to \mathcal{O}_{\mathbb{P}^1}(a) \to 0.$$

First consider the case $a, b \geq 0$. In this case we have a surjective map of global sections $H^0(Q, \mathcal{O}(a,b)) \twoheadrightarrow H^0(\mathbb{P}^1, \mathcal{O}(a))$ and we have $H^1(\mathbb{P}^1, \mathcal{O}(a)) = 0$, so we get an isomorphism

$$H^1(Q, \mathcal{O}(a, b-1)) \simeq H^1(Q, \mathcal{O}(a,b))$$

for all $a, b \geq 0$. By symetry we also have

$$H^1(Q, \mathcal{O}(a-1, b)) \simeq H^1(Q, \mathcal{O}(a,b))$$

for all $a, b \geq 0$. Combining this with step 1 and using induction we get $H^1(Q, \mathcal{O}(a,b)) = 0$ for all $a, b \geq -1$.

Step 3. Note that if one of a, b is less than zero then $H^0(Q, \mathcal{O}(a,b)) = 0$, otherwise we could restrict to a fiber of p_1 or p_2 and get a non-zero section of $\mathcal{O}_{\mathbb{P}^1}(n)$ for $n < 0$.

Assume that $a \geq 0$ and consider as in step 2

$$0 \to \mathcal{O}(a, -2) \to \mathcal{O}(a, -1) \to \mathcal{O}_{\mathbb{P}^1}(a) \to 0.$$

From step 2 we know that $H^1(Q, \mathcal{O}(a,-1)) = 0$. So we get an isomorphism $H^0(\mathbb{P}^1, \mathcal{O}(a)) \simeq H^1(Q, \mathcal{O}(a,-2))$. We also have $H^1(\mathbb{P}^1, \mathcal{O}(a)) = 0$, so analogously for $b \leq -2$ we get

$$0 \to H^0(\mathbb{P}^1, \mathcal{O}(a)) \to H^1(Q, \mathcal{O}(a, b-1)) \to H^1(Q, \mathcal{O}(a,b)) \to 0.$$

Arguing by induction and using $H^0(\mathbb{P}^1, \mathcal{O}(a)) \simeq \mathbb{C}^{a+1}$ we get that

$$H^1(Q, \mathcal{O}(a,b)) \simeq \mathbb{C}^{(a+1)(-b-1)}$$

for $a \geq 0$, $b \leq -2$. By symmetry

$$H^1(Q, \mathcal{O}(a,b)) \simeq \mathbb{C}^{(-a-1)(b+1)}$$

for $a \leq -2$, $b \geq 0$.

Step 4. It remains to treat the case $a, b \leq -1$. In this case from

$$0 \to \mathcal{O}(a, b-1) \to \mathcal{O}(a,b) \to \mathcal{O}_{\mathbb{P}^1}(a) \to 0$$

we get $H^1(Q, \mathcal{O}(a, b-1)) \hookrightarrow H^1(Q, \mathcal{O}(a,b))$. By induction on b we get $H^1(Q, \mathcal{O}(a, b-1)) \hookrightarrow H^1(Q, \mathcal{O}(a,-1))$. By analogous induction on a we get $H^1(Q, \mathcal{O}(a,-1)) \hookrightarrow H^1(Q, \mathcal{O}(-1,-1))$, but $H^1(Q, \mathcal{O}(-1,-1)) = 0$ by step 1. So we get $H^1(Q, \mathcal{O}(a,b)) = 0$ for $a, b \leq -1$.

Exercise 73. (4 points). *Trivial and torsion invertible sheaves.*
Let X be an integral projective scheme over an algebraically closed field k.

i) Assume $H^0(X, \mathcal{L}) \neq 0$ and $H^0(X, \mathcal{L}^*) \neq 0$ for some invertible sheaf $\mathcal{L}$. Show that then $\mathcal{L} \cong \mathcal{O}_X$.

ii) Let $\mathcal{L} \in \mathrm{Pic}(X)$ be of order n. Show $H^0(X, \mathcal{L}^m) = k$ for $n|m$ and $= 0$ otherwise.

Solutions. i) Pick non-zero sections $s \in H^0(X, \mathcal{L})$ and $t \in H^0(X, \mathcal{L}^*)$. The product st is a section of $\mathcal{L} \otimes \mathcal{L} = \mathcal{O}_X$ and this section is non-zero since X is integral. Non-zero sections of $\mathcal{O}$ do not vanish at any point of X, hense s and t also do not vanish at any point. This implies that $\mathcal{L}$ and $\mathcal{L}^*$ are trivial.

ii) It is clear that for $n|m$ we have $\mathcal{L}^m = \mathcal{O}_X$, so $H^0(X, \mathcal{L}^m) = k$. Conversely, assume that s is a non-zero section of $\mathcal{L}^m$ for some m. Then s^n is a non-zero section of $\mathcal{L}^{mn}$. Non-zero sections of $\mathcal{L}^{mn} = \mathcal{O}_X$ do not vanish at any point, so the same is true about s. Hence $\mathcal{L}^m = \mathcal{O}_X$ and the order of $\mathcal{L}$ divides m.

Exercise 74. (6 points) *Base locus.*
Let X be a projective integral scheme over $k = \bar{k}$ and $\mathcal{L}$ an invertible sheaf on X. Let V be a subspace in $H^0(X, \mathcal{L})$. A point $x \in X$ is a base point of the linear system $\mathbb{P}(V) \subset |\mathcal{L}|$ if $s_x \in \mathfrak{m}_x\mathcal{L}$ for all $s \in V$. Thus, $\mathcal{L}$ is globally generated if and only if $|\mathcal{L}|$ has no base points.

i) Prove that the base locus $\mathrm{Bs} \subset X$, i.e. the set of all base points, is closed.

ii) Show that for any $\mathcal{L}$ there exists an effective (Cartier) divisor D such that the base locus of the complete linear system given by $\mathcal{L}(-D) := \mathcal{L} \otimes \mathcal{O}(-D)$ is of codimension ≥ 2.

Solutions. i) Pick a basis $s_1, \ldots, s_n$ of V. A point $x \in X$ lies in Bs if and only if s_i vanish at x. So Bs is an intersection of finitely many closed subsets.

ii) *The statement is unfortunately false unless we assume that Cartier divisors = Weil divisors, e.g. X is locally factorial.*

We will assume that X is locally factorial. Let $V = H^0(X, \mathcal{L})$ and $Y = \mathrm{Bs}|\mathcal{L}|$.

The structure sheaf of Y can be defined via

$$V \otimes \mathcal{L}^* \to \mathcal{O}_X \to \mathcal{O}_Y \to 0.$$

The first map in the sequence above is obtained from the natural evaluatin map $V \otimes \mathcal{O}_X \to \mathcal{L}$ by tensoring with $\mathcal{L}^*$.

Pick a basis $s_1, \ldots, s_n \in V$. Suppose we have a divisor D such that $s_i|_D = 0$. This means the following: if we represent D as the zero locus of a section t of the line bundle $\mathcal{L}' = \mathcal{O}(D)$, then for any affine subset U over which $\mathcal{L}$ and $\mathcal{L}'$ trivialize, we have $s_i|_U = t|_U \tilde{s}_i$ where $\tilde{s}_i$ are some regular functions. This means that s_i/t are well-defined sections of the bundle $\mathcal{L} \otimes (\mathcal{L}')^* = \mathcal{L}(-D)$. Note that by construction $\mathrm{Bs}|\mathcal{L}(-D)| \subset \mathrm{Bs}|\mathcal{L}|$.

The claim is obtained by taking D as above to be "maximal". Let's make this precise. Let $D_1, \ldots, D_m$ be codimension one irreducible components of Y. They are divisors on X and are given as zero loci of sections t_j of $\mathcal{L}_j = \mathcal{O}(D_j)$. Restricting to small enough affine open subsets U_j containing D_j we find that $s_i|_{U_j} = t_j^{n_{ij}}|_{U_j}\tilde{s}_{ij}$ where n_{ij} are orders of vanishing of s_i along D_j, that is $\tilde{s}_{ij}$ are not divisible by t_j. Set $n_j = \min_i\{n_{ij}\}$ and $D = \sum_j n_j D_j$. Then D satisfies the conditions as above (every section of $\mathcal{L}$ vanishes along D), so we get regular sections s_i/t of $\mathcal{L}(-D)$. Note that by construction for any D_j there is some section of $\mathcal{L}(-D)$ that does not vanish along D_j. So the base locus of $|\mathcal{L}(-D)|$ does not contain any of D_j. Hence the base locus does not have any components of codimension one.