Prof. Dr. Daniel Huybrechts
Dr. Andrey Soldatenkov

Winter term 2016/17

Intersection theory and pure motives, Exercises – Week 2

Exercise 5. *Zero cycles on curves.*
Let C be a curve over a field k and $U \subset C$ a dense open subset.

(i) Assume that C is smooth. Show that every closed point $x \in C$ is rationally equivalent to a cycle supported in U. More generally, for any $\alpha \in \mathrm{CH}_0(C)$ there exists a zero-cycle $\sum n_i[x_i]$ with $x_i \in U$ such that $\alpha = \sum n_i[x_i] \in \mathrm{CH}_0(C)$.

(ii) Show that (i) is true without the smoothness assumption on C if k is assumed to be algebraically closed.

(iii) Show that in (i) one still has $\alpha = \sum n_i[x_i] \in \mathrm{CH}_0(C) \otimes \mathbb{Q}$ without the smoothness assumption on C.

Exercise 6. *Zero cycles supported on open subsets.*
Generalize the previous exercise to higher dimensions. More precisely, let X be a quasi-projective variety over an algebraically closed field k and let $U \subset X$ be a dense open subset. Show that for any $\alpha \in \mathrm{CH}_0(X)$ one can find a zero-cycle $\sum n_i x_i$ with $x_i \in U$ such that $\alpha = \sum n_i[x_i]$ in $\mathrm{CH}_0(X)$. One could also drop the assumption on k and work in $\mathrm{CH}_0(X) \otimes \mathbb{Q}$.
Observe that the analogue of this assertion is in general not true for $\mathrm{CH}_k(X)$, $k \geq 1$.

Exercise 7. *Chow groups of generic fibres.*
Let $f \colon X \to S$ be a morphism of varieties, $\eta \in S$ the generic point, $X_\eta = f^{-1}(\eta)$ the generic fibre. For an open subset $U \subset S$ let $X_U = f^{-1}(U)$. Prove that $\mathrm{CH}^k(X_\eta) = \varinjlim_{U \subset S} \mathrm{CH}^k(X_U)$, where the limit is over all nonempty open subsets of S.

Exercise 8. *Chow groups of reducible curves.*
Compute all Chow groups $\mathrm{CH}_k(X)$ of $X := V(x_1 x_2) \subset \mathbb{A}^2_k$. In particular, compare $\mathrm{CH}_0(X) = \mathrm{CH}^1(X)$ with $\mathrm{Pic}(X)$.

Exercise 9. *Chow groups of varieties with affine stratification.*
Compute the Chow groups of the blow-up of $\mathbb{P}^2$ in a point and of $\mathbb{P}^3$ in a line.
Compute the Chow groups of $\mathbb{P}^1 \times \mathbb{P}^1 \setminus (F_1 \cup F_2)$, where F_i is a fibre of the i-th projection.

Further recommended exercises: Example 1.6.4 and 1.6.5. in [Fulton].

Due Wednesday 2 November, 2016.