Random Signal Processing: Definition

March 17, 2022

.- Stationarity and Ergodicity

.- Stochastic Modeling

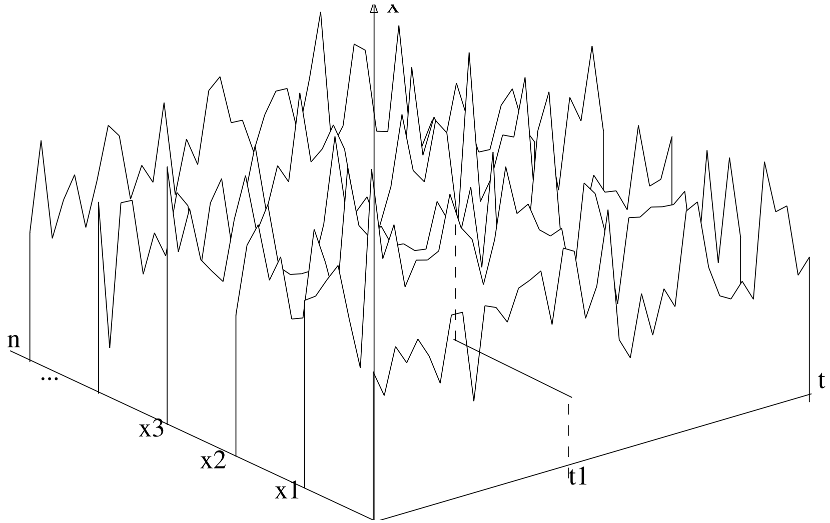
1 Random Signals

Non/stationarity model

A random signal [process] is a variable $\xi(t, s, \ldots, \cdot)$ spanned over domain(s) like time t, space s, etc.

Since time is the most widely considered domain, we will refer only to it hereafter.

$x_m(t)$ is a single (temporal) *trajectory*, while the signal *ensemble* is the set of $\{x_m(t) \in \xi : \forall t \in T, m \in M\}$.



Ensemble of random signals

The random process is termed *nonstationary* if:

$$\boxed{p(\xi, t) = \text{var}, \quad t \in \mathbb{R}}$$

that is, the randomness model varies over the time domain.

On the contrary, a random process becomes *stationary*, if:

$$\boxed{p\left(x_m, t\right) = p\left(x_m, t + \Delta t\right), \ \forall \Delta t}$$

the randomness model remains constant over time. Yet, this assumption is very hard to deal with in practice, and is termed *Narrow-sense Stationarity*.

Let ξ be a *second-order stationary stochastic process*, that is, ξ has finite auto-correlation $R_\xi(t,t')<\infty$ for all indices $t,t'\in T$. This implies that the mean, auto-correlation, and the auto-covariance functions are well defined and finite.

Definition. A random process ξ is said to be *wide sense stationary* if the following two conditions hold for each $m-$ trajectory:

$$\begin{aligned} m_{1m}(t) &= m_{1m} \\ &= \text{const}, \quad \forall t \in T \\ R_m(t,t') &= R_m(t-t') \\ &= R_m(\tau), \forall t,t',\tau \in T, \quad \tau - \text{correlation interval} \end{aligned}$$

Ergodicity: A relaxation assumption over the ensemble holds yields the simplest model of stationarity:

$$\boxed{m_{1m} = m_1; \quad R_m(\tau) = R(\tau), \quad \forall m \in M}$$

Remark. It follows that a wide sense stationary ξ is a second-order stationary stochastic process. However, a second-order process is not necessarily stationary.

Python notebook: TseriesGen

Under the ergodicity condition, the equivalence between moments can be assumed, as follows:

$$\boxed{\text{ensemble-averaged moments}} \equiv \boxed{\text{time-averaged moments}}$$

$$\int_{-\infty}^{\infty} x^n p(x) dx \approx \boxed{\lim_{T\to\infty} \frac{1}{T} \int_T x^n(t)\, dt} \triangleq \overline{x^n(t)}, \quad n \in \mathbb{N} : \text{Initial moment estimates}$$

$$\int_{-\infty}^{\infty} (x - m_{1x})^n p(x) dx \approx \boxed{\lim_{T\to\infty} \frac{1}{T} \int_T \left(x(t) - \overline{x(t)}\right)^n dt,} \quad n \geq 2,\ n \in \mathbb{N} : \text{Centralized moment estimates}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, p(x,y) dx dy \approx \boxed{\lim_{T\to\infty} \frac{1}{T} \int_T x(t) y^*(t+\tau) dt} = R_{xy}(\tau) : \text{(Cross)Correlation function}$$

Correlation function

$$R_x(\tau) = \boxed{\lim_{T\to\infty} \frac{1}{T} \int_T x(t)\, x^*(t+\tau)\, dt} : \text{(Auto)Correlation function. Notation } {}^*\text{stands for conjugate}$$

$$K_x(\tau) = \boxed{\lim_{T\to\infty} \frac{1}{T} \int_T \left(x(t) - \overline{x(t)}\right)\left(x^*(t+\tau) - \overline{x(t)}\right) dt} : \text{Covariance function}$$

$$= R_x(\tau) - \overline{x^2(t)}$$

Properties

(a). *Parity*: $R_x(\tau) = R_x^*(-\tau)$, $K_x(\tau) = K_x(-\tau)$

(b). *Maximal value:* $\max_{\forall\tau} |R_x(\tau)| \le R_x(0)$, $\max_{\forall\tau} |K_x(\tau)| \le K_x(0)$.

$$K_x(0) = \frac{1}{T}\int_0^T \left(x(t) - \overline{x(t)}\right)\left(x^*(t) - \overline{x(t)}\right) dt = \sigma_x^2,$$

Of note, if $\overline{x(t)} = 0$, then, $K_x(0) = \overline{x^2(t)}$

$$R_x(0) = \frac{1}{T}\int_0^T x(t)\,x^*(t)\,dt = \overline{x^2(t)}$$

(c). *Periodicity*. Whenever it holds that $x(t) = x(t-T)$, $\forall t \in T$, then $R_x(\tau) = R_x(\tau - T)$, $\forall\tau \in T$.

(d). *Convergence*. Whenever it holds that $x(t) \neq x(t-T)$, $\forall t \in T$ then

$$\lim_{|\tau|\longrightarrow\infty} R_x(\tau) = \overline{x^2(t)}, \qquad \lim_{|\tau|\longrightarrow\infty} K_x(\tau) = 0,$$

(e). *Shape restriction*. To be an implementable process, its Fourier Transform $\mathscr{F}$ must fulfill the following condition for spectral representation:

$$\mathscr{F}\{R_x(\tau)\} \ge 0, \quad \forall\omega$$

Example:

Find $R_\xi(\tau)$ of $\xi\,(t)=a\cos\,(\omega_c t+\phi)$, for which $a=$const, $\omega_c=$const, and $p(\phi)=1/2\pi$ is the random phase.

$$R_\xi\,(\tau)=\lim_{T\to\infty}\frac{1}{T}\int_0^T a\cos\,(\omega t+\phi)\,a\cos\,(\omega t+\omega\tau+\phi)\,dt$$

$$=\lim_{T\to\infty}\frac{a^2}{T}\int_0^T\frac{1}{2}\left(\cos\,(\omega\tau+\phi)+\cos\,(2\omega t+\omega\tau+2\phi)\right)dt$$

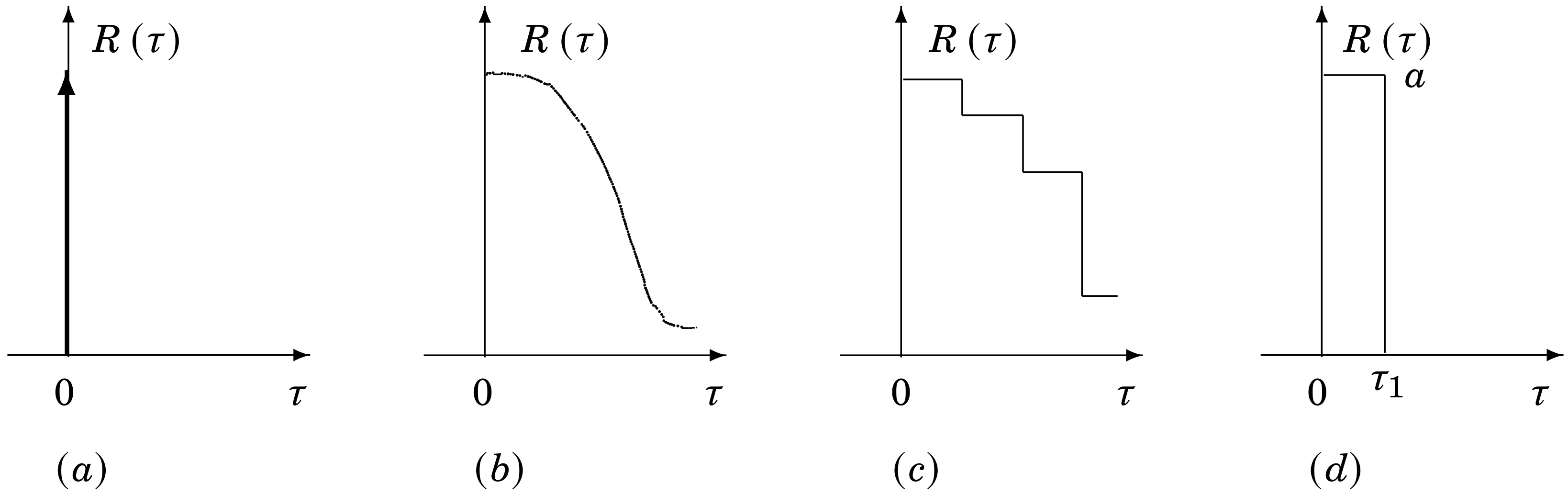
$$\frac{a^2}{2T}\int_0^T\cos\omega\tau dt=\frac{a^2}{2}\cos\omega\tau$$

$$\frac{a^2}{2T}\int_0^T\cos\,(2\omega t+2\phi+\omega\tau)\,dt=\frac{a^2}{2T}\cos\,(2\phi+\omega\tau)\int_0^T\cos 2\omega t dt-\frac{a^2}{2T}\sin\,(2\phi+\omega\tau)\int_0^T\sin 2\omega dt$$

$$\text{since }\lim_{T\to\infty}\int_T\cos k\omega t dt=0,\ \lim_{T\to\infty}\int_T\sin k\omega t dt=0,\forall k\in\mathbb{Z}$$

$$R_\xi\,(\tau)=\frac{a^2}{2}\cos\omega\tau$$

42 Compute $R_\xi(\tau)$ of $\xi\,(t)=k_1\cos(\omega_c t+k_0\phi)+k_2\cos(t)+\eta(t)$ `Python notebook: 05 CorrFunction`



Basic models

Correlation function: Models

(a). $R_x i(\tau) = N_0 \delta(\tau)/2$. Then, $\xi(t)$ is a random process with values totally independent, that is, with the highest uncertainty.

(b). $\lim\limits_{|\tau|\to\infty} R_x(\tau) = 0$. Then, $x(t)$ with fading dependency. The more distant the values, the stronger the independence between them.

(c). $R_x(\tau) = f(R_x(\tau - \tau_1), \ldots, R_x(\tau - \tau_m),)$, m-order Markovian process. $m = 1$ – plain Markovian process

(d). $\lim\limits_{\tau_1\to\infty} R_x(\tau) = \text{const}$. Then, $x(t)$ is a random process with values entirely dependent; that is, there is no uncertainty at all.

(Co)variance Matrix A square matrix that holds the first-order mixed moment between each pair of data elements, for which *variances* appear on the diagonal while *covariances* – on all other elements.

$$\mathrm{cov}_{x,y,z} = \begin{bmatrix} \boxed{\sigma_x^2} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \boxed{\sigma_y^2} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \boxed{\sigma_z^2} \end{bmatrix}$$

where each semipositive-definite scalar value is estimated as:

$$\begin{aligned} \mathrm{cov}_{x,y} &= \mathbb{E}\left\{(X - \mathbb{E}\left\{X\right\})(Y - \mathbb{E}\left\{Y\right\})\right\} \\ &= \frac{\sum_{\forall x_i \in X, y_i \in Y} (x_i - \bar{x})(y_i - \bar{y})}{N - 1} \end{aligned}$$

Notation $\mathbb{E}\left\{\cdot\right\}$ stands for expectation operator.

`Python notebook: 0e CovarianceMatrix`

Wiener-Jinchin Transform Definition of Power Spectral Density:

According to Parseval's Theorem, we have:

$$\frac{1}{2\pi}\int_{\mathbb{R}} X(\omega)X^*(\omega)d\omega = \int_{\mathbb{R}} x^2 dt$$

In cases of random signals, analysis is carried out within a long enough time segment:

$$\begin{aligned} x_T(t) &= \mathrm{rect}_T(t)\, x(t), \\ X_T(\omega) &= \mathscr{F}\{x(t)\,\mathrm{rect}(t/T)\} \end{aligned}$$

$$\begin{aligned} \lim_{T\to\infty}\frac{1}{T}\mathbb{E}\left\{\int_T x^2 dt\right\} &= \lim_{T\to\infty}\frac{1}{2\pi}\int_{\mathbb{R}}\frac{1}{T}\mathbb{E}\left\{X_T(\omega)X_T^*(\omega)\right\}d\omega \\ \text{assuming } m_{1x} = 0, \quad m_{2x} &= \frac{1}{2\pi}\int_{\mathbb{R}} S_x(\omega)d\omega, \\ \boxed{S_x(\omega)} &= \boxed{\lim_{T\to\infty}\frac{|X_T(\omega)X_T^*(-\omega)|}{T} \geq 0,} \quad \text{PSD} \end{aligned}$$

$$\boxed{S_x(\omega) \in \mathbb{R}^+, \qquad S_x(\omega) = S_x(-\omega)}$$

Wiener-Jinchin Transform

$$
\begin{aligned}
\boxed{S_x(\omega)} &= \lim_{T\to\infty} \frac{\mathbb{E}\left\{\left|X_T(\omega)\right|^2\right\}}{T} = \lim_{T\to\infty} \frac{\mathbb{E}\left\{X_T(\omega)\, X_T(-\omega)\right\}}{T} \\
&= \lim_{T\to\infty} \frac{1}{T} \mathbb{E}\left\{ \int_T x_T(t_1)\, e^{(j\omega t_1)} dt_1 \int_T x_T(t_2)\, e^{(-j\omega t_2)} dt_2 \right\} \\
&= \lim_{T\to\infty} \frac{1}{T} \mathbb{E}\left\{ \int_T dt_2 \int_T e^{(-j\omega(t_2-t_1))} x_T(t_1)\, x_T(t_2)\, dt_1 \right\} \\
&= \lim_{T\to\infty} \frac{1}{T} \int_T dt_2 \int_T e^{(-j\omega(t_2-t_1))} \mathbb{E}\left\{x_T(t_1)\, x_T(t_2)\right\} dt_1 \\
&= \lim_{T\to\infty} \frac{1}{T} \int_{-T-t_1}^{T-t_1} d\tau \int_T e^{-j\omega\tau} R_x(t_1, t_1+\tau) dt_1 = \int_{-\infty}^{\infty} \left(\lim_{T\to\infty} \frac{1}{T} \int_T R_x(t_1, t_1+\tau) dt_1 \right) e^{-j\omega\tau} d\tau \\
&= \boxed{\mathscr{F}\left\{\mathbb{E}\left\{R_x(t, t+\tau)\right\}\right\}}
\end{aligned}
$$

$$
\Rightarrow \boxed{S_x(\omega)} = \boxed{\mathscr{F}\left\{R_x(\tau)\right\}}
$$

Likewise, it holds that:

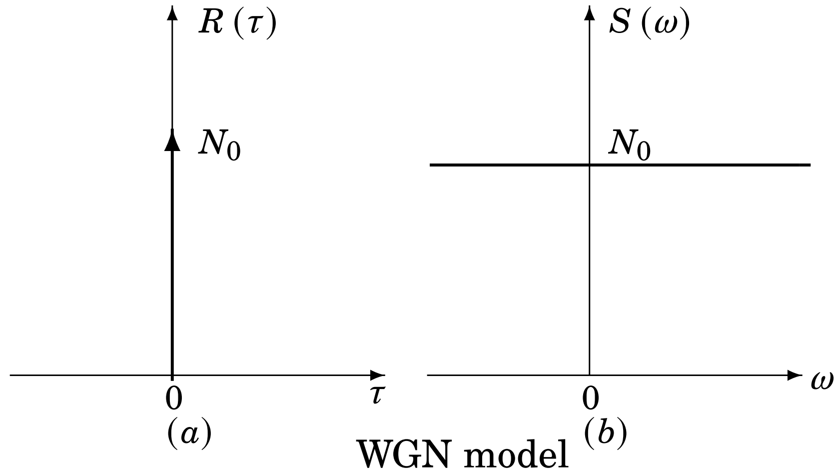
$$
\mathscr{F}^{-1}\left\{S_\xi(\omega)\right\} = \mathscr{F}^{-1}\left\{\mathscr{F}\left\{R_\xi(\tau)\right\}\right\} = R_\xi(\tau)
$$

White Gaussian Noise. An ergodic process that holds all spectral components, each one with the same power in average is called WGN, defined as follows:

$$S(\omega) = N_0, \quad \omega \in (-\infty, \infty).$$

Using the Wiener-Jinchin Transform, we obtain:

$$R(\tau) = \int_{-\infty}^{\infty} \frac{N_0}{2} e^{j2\pi f t} df = \frac{N_0}{2} \delta(\tau)$$



WGN model

Assuming $\Delta\omega<\infty$, then, the following model is termed *colored noise*:

$$\begin{cases} S(\omega) & = N_0, (-\Delta\omega < \omega < \Delta\omega), \\ R(\tau) & = N_0 \Delta\omega \, \mathrm{sinc}(2\Delta\omega\tau), \end{cases}$$

Examples of $S(\omega)$ and $R(\tau)$

Basic Models, Python: 06 ExNoiseColored

Daily temperatures, Python: 07 ExTemperatures

MEG recordings, Python: 07a ExEGG

2 Stochastic Modeling

A real valued (one-dimensional domain) stochastic process is a family of random variables $\{X_t : t \in T\}$ defined on a probabilistic space, X_t : Observation set $\rightarrow \mathbb{R}, t \in T \subseteq \mathbb{R}^+$

Let $\{X(\cdot)\}$ be second-order stationary sequence of random observations recorded at time intervals Δt_n, regularly spaced, as below:

$$\{X(\Delta t_1), X(\Delta t_2), \ldots, X(T)\} = \{X_t \in \mathbb{R} : \forall t\},$$

Within a compact support T, $\{X_t\}$ is a discrete-state process if its values are countable. Otherwise, it is a continuous-state process.

State space – the set $S \subseteq \mathbb{R}$ whose elements are the process values.

A straightforward strategy to model time-dependent randomness is through **Stochastic Differential Equations**, imposing assumptions of relationship between neighboring discretized values in the observation set.

Depending on the statistics provided, modeling by Stochastic Differential Equations can be performed differently:

- An stochastic process is ruled by a probability distribution across-time (**ensemble-based analysis**)
- The changes (evolution) of randomness between neighboring states becomes the stochastic process (**trial-based analysis**)