MA 493 Project 2: Sensitivities Analysis

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Part 1

Clear the workspace

```
clear all
close all
```

Set the final time

```
tFinal = 8;
```

Set the step size in the forward difference approximation (5)

```
h = 0.1;
```

Create the time vector with interval size h

```
t = [0:h:tFinal];
```

Set the nominal values of the parameters

```
mNom = 1;
kNom = 1;
cNom = 2.3;
ANom = 0.1;
%thetaNom = [mNom,kNom,cNom,ANom];

p(1) = mNom;
p(2) = cNom;
p(3) = kNom;
```

Set the value of the perturbation for computing scaled sensitivities

```
alpha = 0.1;

options = odeset('AbsTol',1e-9, 'RelTol', 1e-8);
```

Specify the model to be solved - see below for function definition

```
modelRHS = @(t,y)(Damped(t,y,p));
```

Solve the ODE system, numerically, using "ode45"

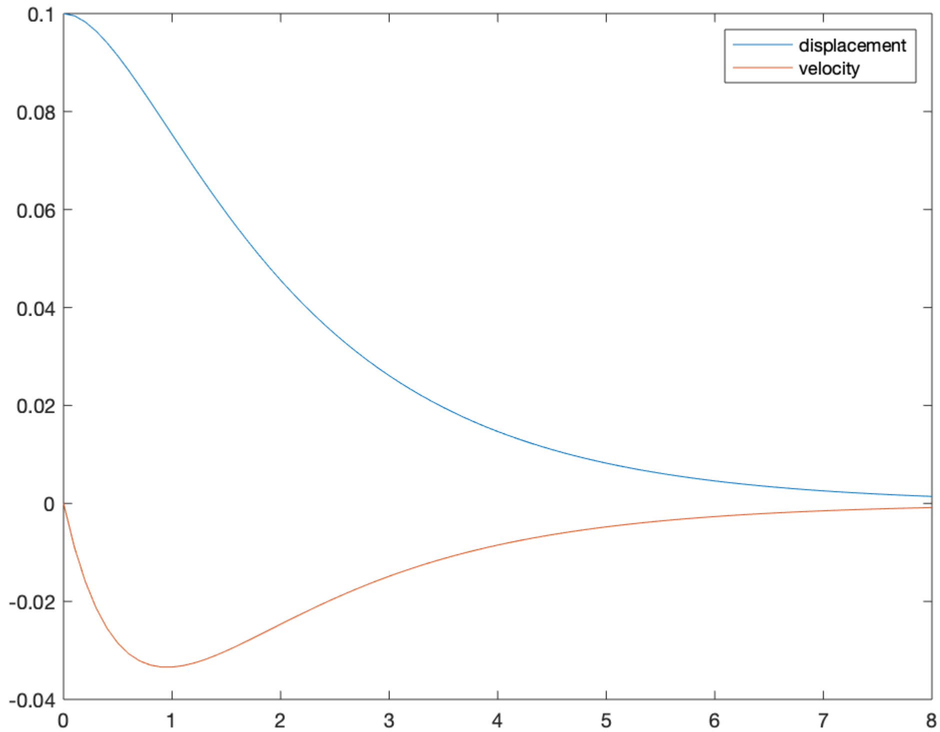
```
[t,y] = ode45(modelRHS, t, [ANom 0], options);
```

Extract the relevant solutions

```
disp = y(:,1);
vel = y(:,2);
```

Evaluate and plot the response at the nominal values

```
figure(1)
plot(t,y);
legend("displacement", "velocity")
```



Calculating Normal Sensitivities and Forward Distance Sensitivities

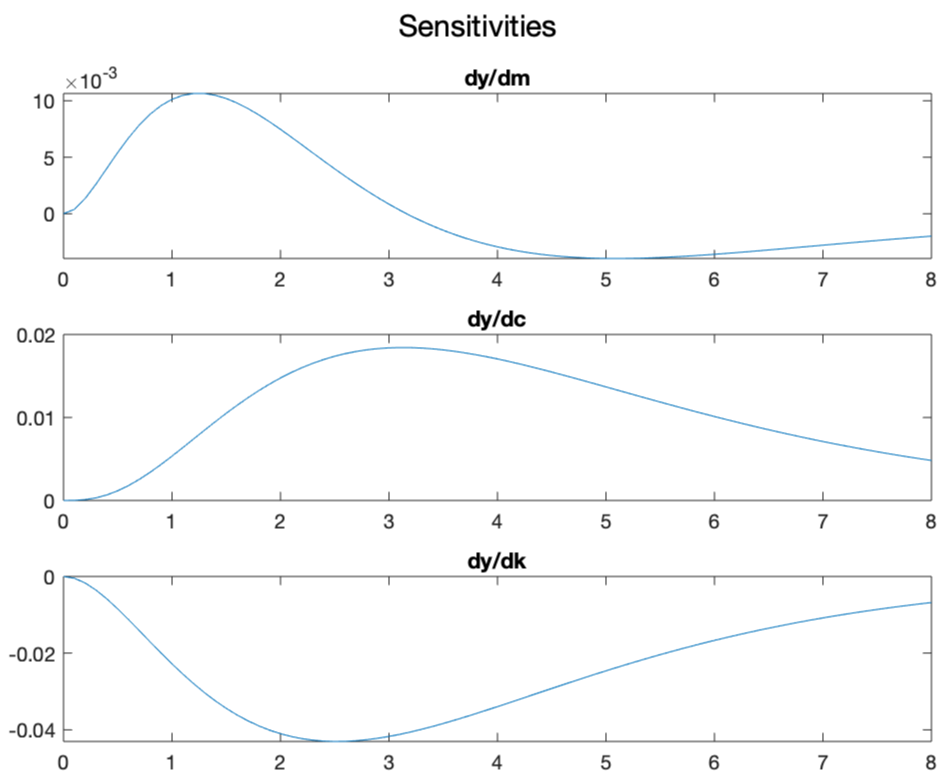
Calculate the sensitivities via the forward difference rule (5)

```
modelRHS1 = @(t,y)(Damped(t,y,p+[h,0,0]));
[t1,y1] = ode45(modelRHS1, t, [ANom 0], options);
s1 = (y1(:,1) - disp)/h;
scaleds1 = 2*alpha*mNom*(y1(:,1) - disp)/h;


modelRHS2 = @(t,y)(Damped(t,y,p+[0,h,0]));
[t2,y2] = ode45(modelRHS2, t, [ANom 0], options);
s2 = (y2(:,1) - disp)/h;
scaleds2 = 2*alpha*kNom*(y2(:,1) - disp)/h;


modelRHS3 = @(t,y)(Damped(t,y,p+[0,0,h]));
```

```
[t3,y3] = ode45(modelRHS3, t, [ANom 0], options);
s3 = (y3(:,1) - disp)/h;
scaleds3 = 2*alpha*cNom*(y3(:,1) - disp)/h;
```

Plot the normal sensitivities

```
figure(2)
subplot(3,1,1)
plot(t1,s1);
title('dy/dm')
hold on
subplot(3,1,2);
plot(t2,s2);
title('dy/dc')
hold on
subplot(3,1,3);
plot(t3,s3);
title('dy/dk')
sgtitle("Sensitivities")
```



Plot the scaled sensitivities

```
figure(3)
subplot(3,1,1)
plot(t1,scaleds1);
title('dy/dm')
hold on
subplot(3,1,2);
```

```
plot(t2,scaleds2);
title('dy/dc')
hold on
subplot(3,1,3);
plot(t3,scaleds3);
title('dy/dk')
sgtitle("Scaled Sensitivities - forward difference approximation")
```

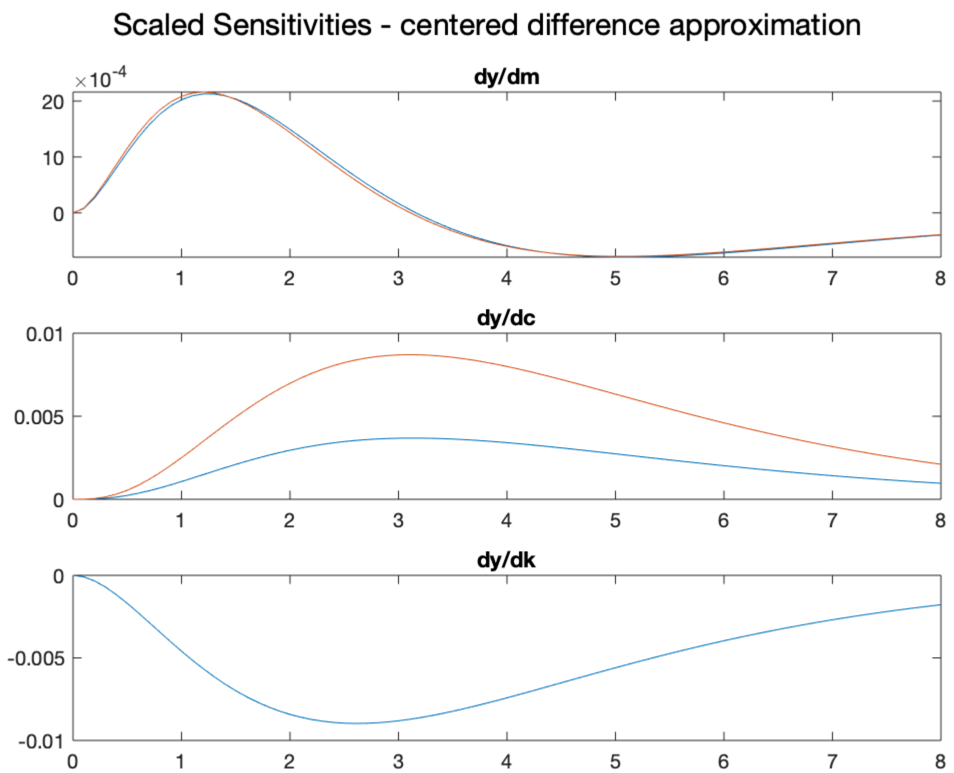
Calculating Centered Difference Sensitivities

Calculate the sensitivities via the centered difference rule (5)

```
modelRHS1 = @(t,y)(Damped(t,y,p+[h,0,0]));
[t1,y11] = ode45(modelRHS1, t, [ANom 0], options);
modelRHS1 = @(t,y)(Damped(t,y,p-[h,0,0]));
[t1,y12] = ode45(modelRHS1, t, [ANom 0], options);
s1 = (y11(:,1) - y12(:,1))/(2*h);
scaleds1 = 2*alpha*mNom*(y11(:,1) - y12(:,1))/(2*h);


modelRHS2 = @(t,y)(Damped(t,y,p+[0,h,0]));
[t2,y21] = ode45(modelRHS2, t, [ANom 0], options);
modelRHS2 = @(t,y)(Damped(t,y,p-[0,h,0]));
[t2,y22] = ode45(modelRHS2, t, [ANom 0], options);
s2 = (y21(:,1) - y22(:,1))/(2*h);
scaleds2 = 2*alpha*cNom*(y21(:,1) - y22(:,1))/(2*h);


modelRHS3 = @(t,y)(Damped(t,y,p+[0,0,h]));
[t3,y31] = ode45(modelRHS3, t, [ANom 0], options);
modelRHS3 = @(t,y)(Damped(t,y,p-[0,0,h]));
[t3,y32] = ode45(modelRHS3, t, [ANom 0], options);
s3 = (y31(:,1) - y32(:,1))/(2*h);
scaleds3 = 2*alpha*kNom*(y31(:,1) - y32(:,1))/(2*h);
```

Plot the centered difference scaled sensitivities

```
figure(3)
subplot(3,1,1)
plot(t1,scaleds1);
title('dy/dm')
hold on
subplot(3,1,2);
plot(t2,scaleds2);
title('dy/dc')
hold on
subplot(3,1,3);
plot(t3,scaleds3);
title('dy/dk')
sgtitle("Scaled Sensitivities - centered difference approximation")
```



Lastly, compute the sensitivity matrix at the 4 indicated data points

Normal sensitivity matrix

Here, we have 4 data points (rows) and 3 parameters (columns)

```
S = zeros(4,3);
```

Store the time values where data is collected

```
tData = [2 4 6 8];
```

Compute the senstivity matrix based on forward differences

```
dyData = @(t,y)(Damped(tData,y,p));
[t,yData] = ode45(dyData, tData, [ANom 0], options);

S1 = @(t,y)(Damped(tData,y,p+[h,0,0]));
[t1,y1Data] = ode45(S1, tData, [ANom 0], options);
S(:,1) = (y1Data(:,1) - yData(:,1))/h;
% scaled - S(:,1) = 2*alpha*mNom*(fOD(tData,thetaNom+[h,0,0,0])-yData)/h;

S2 = @(t,y)(Damped(tData,y,p+[0,h,0]));
[t2,y2Data] = ode45(S2, tData, [ANom 0], options);
S(:,2) = (y2Data(:,1) - yData(:,1))/h;
```

```
% scaled - S(:,2) = 2*alpha*kNom*(fOD(tData,thetaNom+[0,h,0,0])-yData)/h;

S3 = @(t,y)(Damped(tData,y,p+[0,0,h]));
[t1,y3Data] = ode45(S3, tData, [ANom 0], options);
S(:,3) = (y3Data(:,1) - yData(:,1))/h;
% scaled - S(:,3) = 2*alpha*cNom*(fOD(tData,thetaNom+[0,0,h,0])-yData)/h;

S
```

```
S = 4×3
         0         0         0
    0.0075    0.0148   -0.0410
   -0.0029    0.0171   -0.0339
   -0.0036    0.0101   -0.0167
```

Forward Distance Scaled Sensitivity Matrix

Here, we have 4 data points (rows) and 3 parameters (columns)

```
SScaled = zeros(4,3);
```

Store the time values where data is collected

```
tData = [2 4 6 8];
```

Compute the senstivity matrix based on forward differences

```
dyData = @(t,y)(Damped(tData,y,p));
[t,yData] = ode45(dyData, tData, [ANom 0], options);

S1 = @(t,y)(Damped(tData,y,p+[h,0,0]));
[t1,y1Data] = ode45(S1, tData, [ANom 0], options);
SScaled(:,1) = 2*alpha*mNom*(y1Data(:,1) - yData(:,1))/h;
% scaled - S(:,1) = 2*alpha*mNom*(fOD(tData,thetaNom+[h,0,0,0])-yData)/h;

S2 = @(t,y)(Damped(tData,y,p+[0,h,0]));
[t2,y2Data] = ode45(S2, tData, [ANom 0], options);
SScaled(:,2) = 2*alpha*cNom*(y2Data(:,1) - yData(:,1))/h;
% scaled - S(:,2) = 2*alpha*kNom*(fOD(tData,thetaNom+[0,h,0,0])-yData)/h;

S3 = @(t,y)(Damped(tData,y,p+[0,0,h]));
[t1,y3Data] = ode45(S3, tData, [ANom 0], options);
SScaled(:,3) = 2*alpha*kNom*(y3Data(:,1) - yData(:,1))/h;
% scaled - S(:,3) = 2*alpha*cNom*(fOD(tData,thetaNom+[0,0,h,0])-yData)/h;

SScaledForward = SScaled
```

```
SScaledForward = 4×3
         0         0         0
    0.0015    0.0068   -0.0082
   -0.0006    0.0078   -0.0068
   -0.0007    0.0047   -0.0033
```

Centered Difference Scaled Sensitivity Matrix

```
SScaled = zeros(4,3);
```

Store the time values where data is collected

```
tData = [2 4 6 8];
```

Compute the senstivity matrix based on forward differences

```
dyData = @(t,y)(Damped(tData,y,p));
[t,yData] = ode45(dyData, tData, [ANom 0], options);


S11 = @(t,y)(Damped(tData,y,p+[h,0,0]));
[t1,y11] = ode45(S11, tData, [ANom 0], options);

S12 = @(t,y)(Damped(tData,y,p-[h,0,0]));
[t1,y12] = ode45(S12, tData, [ANom 0], options);

SScaled(:,1) = 2*alpha*mNom*(y11(:,1) - y12(:,1))/(2*h);



S21 = @(t,y)(Damped(tData,y,p+[0,h,0]));
[t2,y21] = ode45(S21, tData, [ANom 0], options);

S22 = @(t,y)(Damped(tData,y,p-[0,h,0]));
[t2,y22] = ode45(S22, tData, [ANom 0], options);

SScaled(:,2) = 2*alpha*cNom*(y21(:,1) - y22(:,1))/(2*h);



S31 = @(t,y)(Damped(tData,y,p+[0,0,h]));
[t3,y31] = ode45(S31, tData, [ANom 0], options);

S32 = @(t,y)(Damped(tData,y,p-[0,0,h]));
[t3,y32] = ode45(S32, tData, [ANom 0], options);

SScaled(:,3) = 2*alpha*kNom*(y31(:,1) - y32(:,1))/(2*h);




SScaledCentered = SScaled
```

```
SScaledCentered = 4×3
         0         0         0
    0.0014    0.0070   -0.0084
   -0.0006    0.0080   -0.0074
```

```
-0.0007    0.0046   -0.0040
```

Part 2: SEIR Model

Part A

Explain the significance of each term on the right hand side of equations (3-6)

k_1 = infection rate, k_2 = exposure rate, k_3 = recovery rate

Equation 3: The right-hand side of this equation models infections due to interaction between susceptibility and infection. As infections increase, the number of susceptible people will decrease, which is why the right-hand side is negative.

Equation 4: The right-hand side of this equations models the difference between infections due to interaction between susceptibility and infection, and exposure that moves people from exposure to infection. As infections increase, the number of people exposed will increase to an extent, and then begin decreasing.

Equation 5: The right-hand side of this equations models the difference between exposure that moves people from exposure to infection, and recovery that moves people from infection to recovery. As exposures increase, the number of people infected will also increase, and then will begin to level off.

Equation 6: The right-hand side of this equations models the recovery that moves people from infection to recovery. Since the recovery rate is positive, the rate of change of recovery will always be positive.

Initial conditions

We are modeling a situation where the disease has just been introduced, so no one has recovered from the disease ($R(0) = 0$). However, there are S_0 people susceptible, E_0 people already exposed, and I_0 people already infected.

Part B

Clear the workspace

```
clear all
close all
```

Set the final time

```
tFinal =100;
```

Set the step size in the forward difference approximation (5)

```
h = 0.1;
```

Create the time vector with interval size h

```
t = [0:h:tFinal];
```

Set the nominal values of the parameters

```
k1 = 1e-5;
k2 = 0.03;
k3 = 0.01;


p(1) = k1;
p(2) = k2;
p(3) = k3;
```

Set initial conditions

```
S0 = 1e6; % 1 million susceptible individuals
E0 = 0;
I0 = 1000; % 0.1% of the population is 1000 individuals
R0 = 0;
ic = [S0 E0 I0 R0];
```

Set the value of the perturbation for computing scaled sensitivities

```
alpha = 0.1;

options = odeset('AbsTol',1e-9, 'RelTol', 1e-8);
```

Specify the model to be solved - see below for function definition

```
modelRHS = @(t,y)(seir(t,y,p));
```

Solve the ODE system, numerically, using "ode45"

```
[t,y] = ode45(modelRHS, t, ic, options);
```

Extract the relevant solutions

```
s = y(:,1);
e = y(:,2);
i = y(:,3);
r =y(:,4);
```

Evaluate and plot the response at the nominal values

```
figure(1)
hold on
plot(t,s)
plot(t,e)
plot(t,i)
plot(t,r)
```

```
hold off
title("SEIR Model for the Population")
legend("susceptible", "exposed", "infected", "recovered")
```

Part C

Forward Difference Rule

Calculate the sensitivities via the forward difference rule (5)

```
figure(2)

modelRHS1 = @(t,y)(seir(t,y,p+[h,0,0]));
[t1,y1] = ode45(modelRHS1, t, ic, options);
scaleds1s = 2*alpha*k1*(y1(:,1) - s) /h;
scaleds1e = 2*alpha*k1*(y1(:,2) - e) /h;
scaleds1i = 2*alpha*k1*(y1(:,3) - i) /h;
scaleds1r = 2*alpha*k1*(y1(:,4) - r) /h;
subplot(3, 1, 1)
hold on
plot(t1, scaleds1s)
plot(t1, scaleds1e)
plot(t1, scaleds1i)
plot(t1, scaleds1r)
hold off
title('dy/dk1')


modelRHS2 = @(t,y)(seir(t,y,p+[0,h,0]));
[t2,y2] = ode45(modelRHS2, t, ic, options);
scaleds2s = 2*alpha*k2*(y2(:,1) - s) /h;
scaleds2e = 2*alpha*k2*(y2(:,2) - e) /h;
scaleds2i = 2*alpha*k2*(y2(:,3) - i) /h;
scaleds2r = 2*alpha*k2*(y2(:,4) - r) /h;
subplot(3, 1, 2)
hold on
plot(t2, scaleds2s)
plot(t2, scaleds2e)
plot(t2, scaleds2i)
plot(t2, scaleds2r)
hold off
title('dy/dk2')


modelRHS3 = @(t,y)(seir(t,y,p+[0,0,h]));
[t3,y3] = ode45(modelRHS3, t, ic, options);
scaleds3s = 2*alpha*k3*(y3(:,1) - s) /h;
scaleds3e = 2*alpha*k3*(y3(:,2) - e) /h;
scaleds3i = 2*alpha*k3*(y3(:,3) - i) /h;
scaleds3r = 2*alpha*k3*(y3(:,4) - r) /h;
subplot(3, 1, 3)
hold on
```

```
plot(t3, scaleds3s)
plot(t3, scaleds3e)
plot(t3, scaleds3i)
plot(t3, scaleds3r)
hold off
title('dy/dk3')

sgtitle("Scaled Sensitivities - Forward Difference Approximation")
```

Scaled Sensitivity Matrix

Lastly, compute the SCALED sensitivity matrix at the 4 indicated data points

Here, we have 4 data points (rows) and 3 parameters (columns)

```
SSScaled = zeros(4,3);
SEScaled = zeros(4,3);
SIScaled = zeros(4,3);
SRScaled = zeros(4,3);
```

Store the time values where data is collected

```
tData = [20 40 60 80];
```

Compute the senstivity matrix based on forward differences

```
dyData = @(t,y)(seir(tData,y,p));
[t,yData] = ode45(dyData, tData, ic, options);
s = yData(:,1);
e = yData(:,2);
i  = yData(:,3);
r = yData(:,4);


S1 = @(t,y)(seir(tData,y,p+[h,0,0]));
[t1,y1Data] = ode45(S1, tData, ic, options);
SSScaled(:,1) = 2*alpha*k1*(y1Data(:,1) - s) /h;
SEScaled(:,1) = 2*alpha*k1*(y1Data(:,2) - e) /h;
SIScaled(:,1) = 2*alpha*k1*(y1Data(:,3) - i) /h;
SRScaled(:,1) = 2*alpha*k1*(y1Data(:,4) - r) /h;



S2 = @(t,y)(seir(tData,y,p+[0,h,0]));
[t2,y2Data] = ode45(S2, tData, ic, options);
SSScaled(:,2) = 2*alpha*k2*(y2Data(:,1) - s) /h;
SEScaled(:,2) = 2*alpha*k2*(y2Data(:,2) - e) /h;
SIScaled(:,2) = 2*alpha*k2*(y2Data(:,3) - i) /h;
SRScaled(:,2) = 2*alpha*k2*(y2Data(:,4) - r) /h;



S3 = @(t,y)(seir(tData,y,p+[0,0,h]));
```

```
[t1,y3Data] = ode45(S3, tData, ic, options);
SSScaled(:,3) = 2*alpha*k3*(y3Data(:,1) - s) /h;
SEScaled(:,3) = 2*alpha*k3*(y3Data(:,2) - e) /h;
SIScaled(:,3) = 2*alpha*k3*(y3Data(:,3) - i) /h;
SRScaled(:,3) = 2*alpha*k3*(y3Data(:,4) - r) /h;



SSScaled
```

```
SSScaled = 4×3
         0         0         0
   -0.0000   -0.0003    0.1884
   -0.0000   -0.0000   -0.0000
    0.0000    0.0000   -0.0000
```

```
SEScaled
```

```
SEScaled = 4×3
10^4 ×
         0         0         0
   -0.0003   -3.3842    0.0381
   -0.0002   -2.2472    0.0209
   -0.0001   -1.2622    0.0115
```

```
SIScaled
```

```
SIScaled = 4×3
10^4 ×
         0         0         0
    0.0002    3.0018   -0.2546
    0.0001    1.4174   -0.7571
   -0.0000    0.2751   -0.9931
```

```
SRScaled
```

```
SRScaled = 4×3
10^3 ×
         0         0         0
    0.0005    3.8245    2.1645
    0.0009    8.2977    7.3615
    0.0009    9.8708    9.8162
```

Part D

We are going to promote herd immunity, which will ultimately increase the rate of exposed people more quickly, and then lead to a sharp decline. This leads to increasing the rate of recovered people much faster. We will model this by perturbing k_1, the rate of infection.

Set k_1 to be even higher for its nominal value

```
k1 = 1e-3;

p(1) = k1;
```

Re-set initial conditions

```
S0 = 1e6; % 1 million susceptible individuals
E0 = 0;
I0 = 1000; % 0.1% of the population is 1000 individuals
R0 = 0;
ic = [S0 E0 I0 R0];
```

Re-specify the model to be solved - see below for function definition

```
modelRHS = @(t,y)(seir(t,y,p));
```

Re-solve the ODE system, numerically, using "ode45"

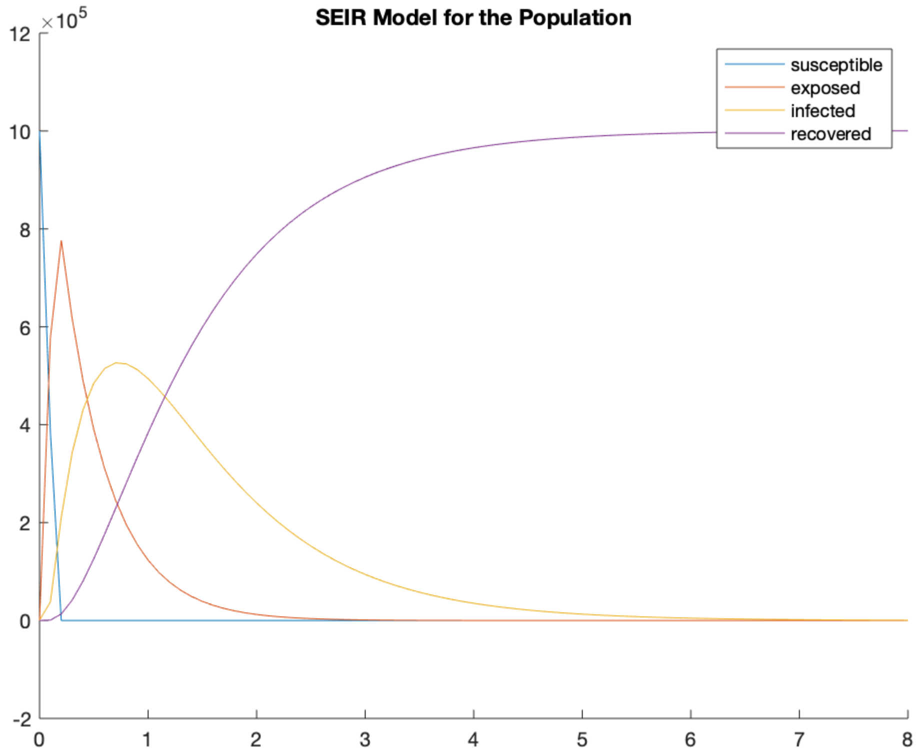
```
[t,y] = ode45(modelRHS, t, ic, options);
```

Extract the relevant solutions

```
s = y(:,1);
e = y(:,2);
i = y(:,3);
r =y(:,4);
```

Evaluate and plot the response at the nominal values

```
figure;
hold on
plot(t,s)
plot(t,e)
plot(t,i)
plot(t,r)
hold off
title("SEIR Model for the Population")
legend("susceptible", "exposed", "infected", "recovered")
```



Bonus

Below we will implements the complex step first derivative approximation using equation (7) from Topic 4.2

Notes: $f'(x) \approx \dfrac{I(f(x+i*h))}{h}$.

When perturbing the m, c, and k parameters by h we will also multiply by i after which we will extract the imaginary component.

The sublplot shown below shows that the convergence of the displacement seems to be slower than the convergence found via forward and centered difference approximations.

Clear the workspace

```
clear all
close all
```

Set the final time

```
tFinal = 8;
```

Set the step size in the forward difference approximation (5)

```
h = 0.1;
```

Create the time vector with interval size h

```
t = [0:h:tFinal];
```

Set the nominal values of the parameters

```
mNom = 1;
kNom = 1;
cNom = 2.3;
ANom = 0.1;
%thetaNom = [mNom,kNom,cNom,ANom];

p(1) = mNom;
p(2) = cNom;
p(3) = kNom;
```

Set the value of the perturbation for computing scaled sensitivities

```
alpha = 0.1;

options = odeset('AbsTol',1e-9, 'RelTol', 1e-8);
```

Specify the model to be solved - see below for function definition

```
modelRHS = @(t,y)(Damped(t,y,p));
```

Solve the ODE system, numerically, using "ode45"

```
[t,y] = ode45(modelRHS, t, [ANom 0], options);
```

Extract the relevant solutions

```
disp = y(:,1);
vel = y(:,2);
```

Calculating Complex Sensitivities

Calculate the sensitivities via Equation 7 in Section IV.2

```
modelRHS1 = @(t,y)(Damped(t,y,p+[i*h,0,0]));
[t1,y1] = ode45(modelRHS1, t, [ANom 0], options);
complexs1 = imag(y1(:,1))/h
```

```
complexs1 = 81×1
         0
    0.0004
    0.0015
    0.0028
    0.0043
    0.0057
    0.0071
    0.0082
    0.0092
```

```
    0.0099
      ⋮
```

```
modelRHS2 = @(t,y)(Damped(t,y,p+[0,i*h,0]));
[t2,y2] = ode45(modelRHS2, t, [ANom 0], options);
complexs2 = imag(y2(:,1))/h
```

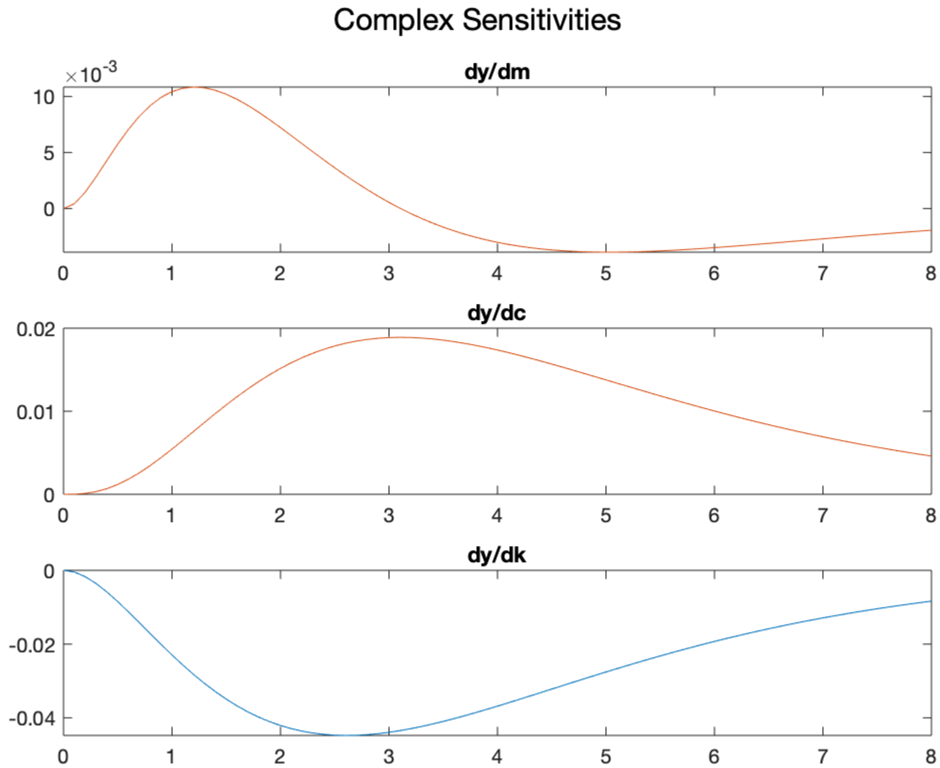
```
complexs2 = 81×1
         0
    0.0000
    0.0001
    0.0003
    0.0007
    0.0012
    0.0018
    0.0026
    0.0035
    0.0044
      ⋮
```

```
modelRHS3 = @(t,y)(Damped(t,y,p+[0,0,i*h]));
[t3,y3] = ode45(modelRHS3, t, [ANom 0], options);
complexs3 = imag(y3(:,1))/h
```

```
complexs3 = 81×1
         0
   -0.0005
   -0.0017
   -0.0036
   -0.0059
   -0.0085
   -0.0113
   -0.0142
   -0.0172
   -0.0201
      ⋮
```

Plot the complex sensitivities

```
figure(2)
subplot(3,1,1)
plot(t1,complexs1);
title('dy/dm')
hold on
subplot(3,1,2);
plot(t2,complexs2);
title('dy/dc')
hold on
subplot(3,1,3);
plot(t3,complexs3);
title('dy/dk')
sgtitle("Complex Sensitivities")
```



Functions

```
% Functon "Damped"
%
% Implements ODE model for damped oscillatory motion of a
% mass-spring-dashpot system

% Inputs:
%   t - time
%   y - state variables (displacement and velocity)
%   p - vector of model parameters [m c k]
% Output:
%   dy/dt
```

```
function dy = Damped(t, y, p)
  dy = [0; 0];
```

```
  m = p(1);
  c = p(2);
  k = p(3);

  dy(1) = y(2);
  dy(2) = -k/m * y(1) - c/m * y(2);

end
```

```
% Functon "seir"
%
% Implements ODE model for damped oscillatory motion of a
% mass-spring-dashpot system

% Inputs:
%   t - time
%   y - [S(t) ; E(t); I(t); R(t)]
%   p - vector of model parameters [k1 k2 k3]
% Output:
%   dy/dt

function dy = seir(t, y, p)
  dy = [0; 0; 0; 0];
  k1 = p(1);
  k2 = p(2);
  k3 = p(3);

  dy(1) = -k1 * y(1)* y(3);
  dy(2) = (k1 * y(1)* y(3)) - (k2*y(2));
  dy(3) = (k2*y(2)) - (k3*y(3));
  dy(4) = k3*y(3) ;

end
```