1 Differential Calculus

Module Overview

1.1 Derivative of a Function

1.1.1 Introduction

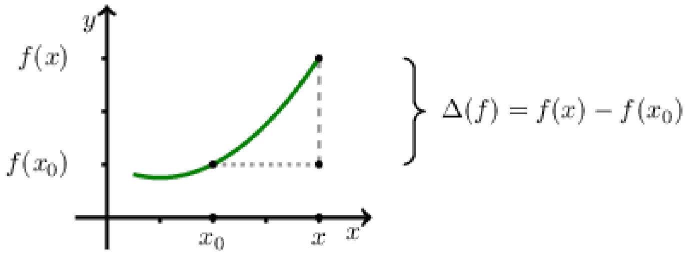
A family is travelling on holiday by car. The car is moving with a velocity of 60 km/h through a roadworks site. The sign at the end of the roadworks site says that the speed limit is as of now 120 km/h. Even though the car driver puts the pedal to the metal, the velocity of the car will not jump up immediately but increase as a function of time. If the velocity increases from 60 km/h to 120 km/h in 5 seconds at a constant rate of change, then the *acceleration* (= change of velocity per time) equals in the present case this constant rate of velocity change: The acceleration is the quotient of the velocity change and the time required for this change. Thus, its value is here 12 kilometre per hour per second. In reality, the velocity of the car will not increase at a constant rate of change but at a *time-dependent* rate of change. If the velocity v is described as a function of time t, then the acceleration is the slope of this function. This does not depend on the fact whether this slope is constant (in time) or not. On other words: The acceleration is the *derivative* of the velocity*function* v with respect to the time t.

Similar relations can also be found in other technical fields, as for example, in the calculation of internal forces acting in steel frames of buildings, in the forecast of atmospheric and ocean currents, or also in the nowadays so relevant modelling of financial markets.

This chapter reviews the basic ideas underlying these calculations, i.e. it deals with **differential calculus**. In other words: We will take derivatives of functions to find their slopes or rates of change, respectively. Even thought these calculations will be carried out here in a strictly mathematical way, their motivation is not purely mathematical. Derivatives, interpreted as rates of change of different functions, play an important role in many scientific fields and are often investigated as outstanding quantities.

1.1.2 Relative Rate of Change of a Function

Consider a function $f : [a; b] \to \mathbb{R}$, $x \mapsto f(x)$ and a sketch of the graph of f shown in the figure below. We like to describe the rate of change of this function at an arbitrary point x_0 between a and b. This will lead us to the notion of a derivative of a function. Generally, calculation rules are to be applied that are as simple as possible.



If x_0 and the corresponding function value $f(x_0)$ are fixed and another arbitrary, but variable point x between a and b as well as the corresponding function value $f(x)$ are chosen, then through these two points, i.e. the points $(x_0; f(x_0))$ and $(x; f(x))$, a line can be drawn that is characterised by its slope and its y-intercept. For the slope of this line one obtains the so called **difference quotient**

$$\frac{\Delta(f)}{\Delta(x)} = \frac{f(x) - f(x_0)}{x - x_0}$$

that describes how the function values of f between x_0 and x change **in average**. Thus, an average rate of change of the function f on the interval $[x_0; x]$ is found. This quotient is also called **relative change**.

If we let the variable point x approach the point x_0, then we see that the line that intersects the graph of the function in the points $(x_0; f(x_0))$ and $(x; f(x))$ gradually becomes a tangent line to the graph in the point $(x_0; f(x_0))$. In this way, the rate of change of the function f – or the **slope** of the graph of f – at the point x_0 **itself** can be determined. If the approaching process of x to x_0 described above leads to, figuratively speaking, a unique tangent line (i.e. a line with a unique slope that, in particular, must not be infinity), then, in mathematical terms, one says, that the **limit** of the difference quotient does **exist**. This limiting process, i.e. if x approaches x_0, is described here and in the following by the symbol

$$\lim_{x \to x_0} ,$$

where lim is an abbreviation for the Latin word *limes*, meaning "border" or "boundary". If the limit of the difference quotient exists, then

$$f'(x_0) = \lim_{x \to x_0} \frac{\Delta(f)}{\Delta(x)} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

denotes the value of the **derivative** of f at x_0. The function f is then said to be **differentiable** at the point x_0.

Example 1.1.1

For the function $f(x) = \sqrt{x}$ the relative change at the point $x_0 = 1$ is given by

$$\frac{f(x) - f(x_0)}{x - x_0} = \frac{\sqrt{x} - \sqrt{1}}{x - 1} = \frac{\sqrt{x} - 1}{(\sqrt{x} - 1)(\sqrt{x} + 1)} = \frac{1}{\sqrt{x} + 1} .$$

If x approaches $x_0 = 1$, this results in the limit

$$\lim_{x \to x_0} \frac{\Delta(f)}{\Delta(x)} = \frac{1}{2} .$$

The value of the derivative of the function f at the point $x_0 = 1$ is denoted by $f'(1) = \frac{1}{2}$.

Exercise 1.1.1
Let the function $f : \mathbb{R} \to \mathbb{R}$ with $x \mapsto f(x) = x^2$ and a point $x_0 = 1$ be given. At this point, the relative change for a real value of x equals $\dfrac{f(x) - f(1)}{x - 1} = \boxed{}$.

If x approaches $x_0 = 1$, this results in the slope $\boxed{}$ of the graph of the function f at the point $x_0 = 1$.

<u>Solution:</u>

For $f(x) = x^2$, the relative change at the point $x_0 = 1$ is given by

$$\frac{f(x) - f(1)}{x - 1} = \frac{x^2 - 1}{x - 1} = \frac{(x-1)(x+1)}{x - 1} = x + 1 \,.$$

Then, if x approaches x_0, this results in the limit

$$\lim_{x \to 1} \frac{\Delta(f)}{\Delta(x)} = 2 \,.$$

This is the slope of the tangent line to the graph of f at the point $(x_0; f(x_0)) = (1; 1)$. The value of the derivative of f at the point $x_0 = 1$ is denoted by $f'(1) = 2$.

Using the formula for the relative rate of change, the derivative can only be calculated very cumbersome and also only for very simple functions. Typically, the derivative is determined by applying calculation rules and inserting known derivatives for the individual terms.

1.1.3 Derivative

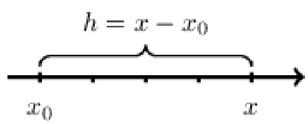
Notation of the Derivative 1

In mathematics as well as in the sciences and engineering, different notations for derivatives are used equivalently:

$$f'(x_0) = \frac{df}{dx}(x_0) = \frac{d}{dx} f(x_0) \,.$$

These different notations all denote the derivative of the function f at the point x_0.

If the derivative is to be calculated using the difference quotient $\frac{f(x) - f(x_0)}{x - x_0}$, then it is often convenient to rewrite the difference quotient in another way. Denoting the difference of x and x_0 by $h := x - x_0$ (see figure below),



the difference quotient can be rewritten as

$$\frac{f(x) - f(x_0)}{x - x_0} = \frac{f(x_0 + h) - f(x_0)}{h} \,,$$

where $x = x_0 + h$. There is no statement whether x has to be greater or less than x_0. Hence, the quantity h can take positive or negative values. To determine the derivative of the function f, the limit for $h \to 0$ has to be calculated:

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \,.$$

If this limit exists **for all** points x_0 in a functions domain, then the function is said to be **differentiable** (everywhere). Many of the common functions are differentiable. However, a simple example for a function that is not differentiable everywhere is the absolute value function $f : \mathbb{R} \to \mathbb{R}$ with $x \mapsto f(x) := |x|$.

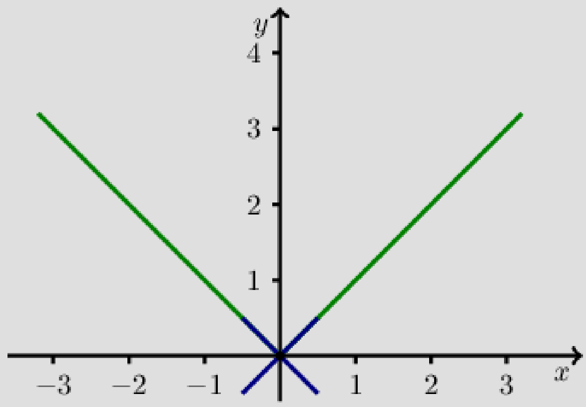
Example 1.1.2

The absolute value function (see Module **??**, Section **??** on page ??) is not differentiable at the point $x_0 = 0$. The difference quotient of f at the point $x_0 = 0$ is:

$$\frac{f(0+h) - f(0)}{h} = \frac{|h| - |0|}{h} = \frac{|h|}{h} .$$

Since h can be greater or less than 0, two cases are to be distinguished: For $h > 0$, we have $\frac{|h|}{h} = \frac{h}{h} = 1$, and for $h < 0$, we have $\frac{|h|}{h} = \frac{-h}{h} = -1$. In these two cases, the limiting process, i.e. if h approaches 0, results in two different values (1 and -1). Thus, **the** limit of the difference quotient at the point $x_0 = 0$ does not exist. Hence, the absolute value function is not differentiable at the point $x_0 = 0$.

The graph changes its direction at the point $(0; 0)$ abruptly: Casually speaking, one says that the graph of the function has a kink at the point $(0; 0)$.



Likewise, also in the case that a function has a jump at a certain point, a unique tangent line to the graph at this point does not exist and thus, the function has no derivative at this point.

1.1.4 Exercises

Exercise 1.1.2
Using the difference quotient, calculate the derivative of $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto f(x) := 4 - x^2$ at the points $x_1 = -2$ and $x_2 = 1$.

Answer:

a. The difference quotient of f at the point $x_1 = -2$ is $\boxed{}$ and has for $x \to -2$ the limit $f'(-2) = \boxed{}$.

b. The difference quotient of f at the point $x_2 = 1$ is $\boxed{}$ and has for $x \to 1$ the limit $f'(1) = \boxed{}$.

Solution:

a. At the point $x_1 = -2$, we have for the difference quotient

$$\frac{\Delta(f)}{\Delta(x)} = \frac{f(x) - f(x_1)}{x - x_1} = \frac{f(x) - f(-2)}{x - (-2)} = \frac{4 - x^2 - 0}{x + 2} = \frac{(2 - x)(2 + x)}{2 + x} = 2 - x \; .$$

For $x \to x_1$, i.e. for $x \to -2$, this difference quotient tends to $2 - (-2) = 4$; hence, $f'(-2) = 4$.

b. At the point $x_2 = 1$, we have for the difference quotient

$$\frac{\Delta(f)}{\Delta(x)} = \frac{f(x) - f(x_2)}{x - x_2} = \frac{f(x) - f(1)}{x - 1} = \frac{4 - x^2 - 3}{x - 1} = \frac{1 - x^2}{x - 1} = -\frac{(x - 1)(x + 1)}{x - 1} = -x - 1 \; .$$

For $x \to x_2$, i.e. for $x \to 1$, this difference quotient has the limit $-1-1 = -2$; hence, $f'(1) = -2$.

Exercise 1.1.3
Explain why the functions

a. $f : [-3; \infty[\to \mathbb{R}$ with $f(x) := \sqrt{x + 3}$ at $x_0 = -3$ and

b. $g : \mathbb{R} \to \mathbb{R}$ with $g(x) := 6 \cdot |2x - 10|$ at $x_0 = 5$

are not differentiable.

Answer:

a. The derivative of the function f at the point $x_0 = -3$ does not exist since the difference quotient $\boxed{}$ does not converge for $h \to 0$.

b. The derivative of the function g at the point $x_0 = 5$ does not exist since the difference quotient for $h < 0$ has the value $\boxed{}$ and for $h > 0$ has the value $\boxed{}$. Thus, the limit for $h \to 0$ does not exist.

Solution:

a. The difference quotient of the function f at the point $x_0 = -3$ is

$$\frac{\Delta(f)}{\Delta(x)} = \frac{f(x_0+h)-f(x_0)}{h} = \frac{\sqrt{-3+h+3}-\sqrt{-3+3}}{h} = \frac{\sqrt{h}-0}{h} = \frac{1}{\sqrt{h}}\ .$$

For $h \to 0$ ($h > 0$), this difference quotient increases infinitely, i.e. the limit of the difference quotient does not exist.

b. The difference quotient of the function g at the point $x_0 = 5$ is

$$\frac{\Delta(g)}{\Delta(x)} = \frac{g(x_0+h)-g(x_0)}{h} = \frac{6\cdot|2(5+h)-10|-6\cdot|2\cdot 5-10|}{h} = \frac{12|h|-0}{h} = \frac{12|h|}{h}\ .$$

For $h < 0$, since $|h| = -h$, the difference quotient has the value -12. In contrast, for $h > 0$, since $|h| = h$, it has the value 12. Thus, the limit of the difference quotient does not exist. (The limit has always to be unique.)

1.2 Standard Derivatives

1.2.1 Introduction

Most of the common functions, as for example, polynomials, trigonometric functions, and exponential functions (see Module **??**) are differentiable. In the following, the differentiation rules for these functions are repeated.

1.2.2 Derivatives of Power Functions

In the last section, the derivative was introduced as the limit of the difference quotient. Accordingly, for a linear affine function (see Module **??**, Section **??** on page ??) $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto f(x) = mx + b$, where m and b are given numbers, we obtain for the derivative at the point x_0 the value $f'(x_0) = m$. (Readers are invited to verify that fact themselves.)

For monomials x^n with $n \geq 1$, it is easiest to determine the derivative using the difference quotient. Without any detailed calculation or any proof we state the following rules:

Derivative of x^n 2

Let a natural number n and a real number r be given.

The constant function $f : \mathbb{R} \to \mathbb{R}$ with $x \mapsto f(x) := r = r \cdot x^0$ has the derivative $f' : \mathbb{R} \to \mathbb{R}$ with $x \mapsto f'(x) = 0$.

The function $f : \mathbb{R} \to \mathbb{R}$ with $x \mapsto f(x) := r \cdot x^n$ has the derivative

$$f' : \mathbb{R} \to \mathbb{R} \;\text{ with }\; x \mapsto f'(x) = r \cdot n \cdot x^{n-1} \,.$$

This differentiation rule is also true for $n \in \mathbb{R} \setminus \{0\}$.

Again, we leave the verification of these statements to the reader.

Example 1.2.1

Let us consider the function $f : \mathbb{R} \to \mathbb{R}$ with $x \mapsto f(x) = 5x^3$. According to the notation above, this is a function with $r = 5$ and $n = 3$. Thus, we have for the value of the derivative at the point x

$$f'(x) = 5 \cdot 3x^{3-1} = 15x^2 \,.$$

For root functions, an equivalent statement holds. However, it should be noted that root functions are only differentiable for $x > 0$ since the tangent line to the graph of the function at the point $(0; 0)$ is parallel to the y-axis and thus, it is not a graph of a function.

Derivative of $x^{\frac{1}{n}}$ 3

For $n \in \mathbb{Z}$ with $n \neq 0$, the function $f : [0; \infty[\to \mathbb{R}$, $x \mapsto f(x) := x^{\frac{1}{n}}$ is differentiable for $x > 0$, and we have

$$f' :]0; \infty[\to \mathbb{R}\,, \quad x \mapsto f'(x) = \frac{1}{n} \cdot x^{\frac{1}{n}-1}\,.$$

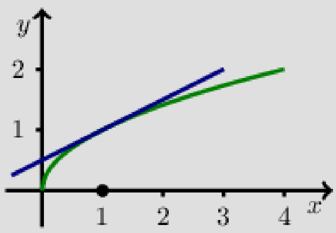
For $n \in \mathbb{N}$, root functions are described by $f(x) = x^{\frac{1}{n}}$. Of course, the differentiation rule given here also holds for $n = 1$ or $n = -1$.

Example 1.2.2

The root function $f : [0; \infty[\to \mathbb{R}$ with $x \mapsto f(x) := \sqrt{x} = x^{\frac{1}{2}}$ is differentiable for $x > 0$. The value of the derivative at an arbitrary point $x > 0$ is given by

$$f'(x) = \frac{1}{2} \cdot x^{\frac{1}{2}-1} = \frac{1}{2} \cdot x^{-\frac{1}{2}} = \frac{1}{2 \cdot \sqrt{x}}\,.$$

The derivative at the point $x_0 = 0$ does not exist since the slope of the tangent line to the graph of f would be infinite there.



The tangent line to the graph of the given root function at the point $(1; 1)$ has the slope $\frac{1}{2\sqrt{1}} = \frac{1}{2}$.

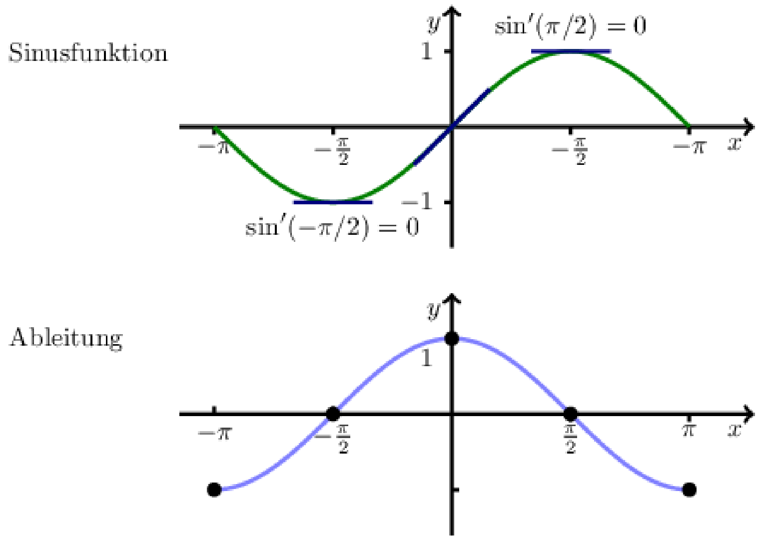
For $x > 0$, the statements above can be extended to exponents $p \in \mathbb{R}$ with $p \neq 0$: The value $f'(x)$ of the derivative of the function f with the mapping rule $f(x) = x^p$ is for $x > 0$

$$f'(x) = p \cdot x^{p-1}\,.$$

1.2.3 Derivatives of Special Functions

Derivatives of Trigonometric Functions

The sine function $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto f(x) = \sin(x)$ is periodic with period 2π. Thus, it is sufficient to consider the function on an interval of length 2π. A section of the graph for $-\pi \leq x \leq \pi$ is shown in the figure below:



As we see from the figure above, the slope of the sine function at $x_0 = \pm\frac{\pi}{2}$ is $f'(\pm\frac{\pi}{2}) = 0$. The tangent line to the graph of the sine function at $x_0 = 0$ has the slope $f'(0) = 1$. At $x_0 = \pm\pi$, the tangent line has the same slope as the tangent line at $x_0 = 0$, but the sign is opposite. Hence, the slope at $x_0 = \pm\pi$ is $f'(\pm\pi) = -1$. Thus, the derivative of the sine function is a function that exhibits exactly these properties. A detailed investigation of the regions between these specially chosen points shows that the derivative of the sine function is the cosine function:

Derivatives of Trigonometric Functions 4

For the sine function $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto f(x) := \sin(x)$, we have

$$f' : \mathbb{R} \to \mathbb{R}\ ,\quad x \mapsto f'(x) = \cos(x)\ .$$

For the cosine function $g : \mathbb{R} \to \mathbb{R}$, $x \mapsto g(x) := \cos(x)$, we have

$$g' : \mathbb{R} \to \mathbb{R}\ ,\quad x \mapsto g'(x) = -\sin(x)\ .$$

For the tangent function $h : \mathbb{R} \setminus \{\frac{\pi}{2} + k\pi\ :\ k \in \mathbb{Z}\} \to \mathbb{R}$, $x \mapsto h(x) := \tan(x)$, we have

$$h' : \mathbb{R} \setminus \{\frac{\pi}{2} + k\pi\ :\ k \in \mathbb{Z}\} \to \mathbb{R}\ ,\quad x \mapsto h'(x) = 1 + (\tan(x))^2 = \frac{1}{\cos^2(x)}\ .$$

The latter also results form the calculation rules explained in the following and the definition of the tangent function as the quotient of the sine function and the cosine function.

Derivative of the Exponential Function

The exponential function $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto f(x) := e^x = \exp(x)$ has the special property that its derivative f' is in turn the exponential function, i.e. $f'(x) = e^x = \exp(x)$.

Derivative of the Logarithmic Function

The derivative of the logarithmic function is given here without proof. For $f :]0;\infty[\to \mathbb{R}$ with $x \mapsto f(x) = \ln(x)$ one obtains $f' :]0;\infty[\to \mathbb{R}$, $x \mapsto f'(x) = \frac{1}{x}$.

1.2.4 Exercises

Exercise 1.2.1
Find the following derivatives by simplifying the terms of the functions and then applying your knowledge on the differentiation of common functions ($x > 0$):

a. $f(x) := x^6 \cdot x^{\frac{7}{2}} = \boxed{}$.

b. $g(x) := \frac{x^{-\frac{3}{2}}}{\sqrt{x}} = \boxed{}$.

Thus, we have:

a. $f'(x) = \boxed{}$.

b. $g'(x) = \boxed{}$.

Solution:

a. It is $f(x) = x^6 \cdot x^{\frac{7}{2}} = x^{6+\frac{7}{2}} = x^{\frac{19}{2}}$. Thus, we have $f'(x) = \frac{19}{2} x^{\frac{19}{2}-1} = \frac{19}{2} x^{\frac{17}{2}}$.

b. It is $g(x) = \frac{x^{-\frac{3}{2}}}{\sqrt{x}} = x^{-\frac{3}{2}} \cdot x^{-\frac{1}{2}} = x^{-\frac{3}{2}-\frac{1}{2}} = x^{-2}$. Thus, we have $g'(x) = (-2) \cdot x^{-2-1} = -2 \cdot x^{-3} = -\frac{2}{x^3}$.

Exercise 1.2.2
Simplify the terms of the functions and find their derivatives:

a. $f(x) := 2\sin\left(\frac{x}{2}\right) \cdot \cos\left(\frac{x}{2}\right) = \boxed{}$.

b. $g(x) := \cos^2(3x) + \sin^2(3x) = \boxed{}$.

Thus, we have:

a. $f'(x) = \boxed{}$.

b. $g'(x) = \boxed{}$.

Solution:

a. Generally, we have
$$\sin(u) \cdot \cos(v) = \frac{1}{2}\left(\sin(u-v) + \sin(u+v)\right) \ .$$
Thus, in the present case we have $f(x) = 2 \cdot \frac{1}{2}(\sin(0) + \sin(x)) = \sin(x)$, and hence $f'(x) = \cos(x)$.

b. Since $\sin^2(u) + \cos^2(u) = 1$, we have $g(x) = 1$, and thus $g'(x) = 0$.

Exercise 1.2.3
Simplify the terms of the functions and find the derivatives (for $x > 0$ in the first part of this exercise):

a. $f(x) := 3\ln(x) + \ln\left(\frac{1}{x}\right) = \boxed{}$.

b. $g(x) := (e^x)^2 \cdot e^{-x} =$ [ ] .

Thus, we have:

a. $f'(x) =$ [ ] .

b. $g'(x) =$ [ ] .

Solution:

a. We have

$$f(x) = 3\ln(x) + \ln\left(\frac{1}{x}\right) = \ln(x^3) + \ln\left(\frac{1}{x}\right) = \ln\left(x^3 \cdot \frac{1}{x}\right) = \ln\left(x^2\right) .$$

For the value of the derivative at the point x ($x > 0$), it follows from the chain rule $f'(x) = \frac{1}{x^2} \cdot 2x = \frac{2}{x}$. (The chain rule is explained in detail in Section 1.3.4 on page 16.)

b. We have

$$g(x) = (e^x)^2 \cdot e^{-x} = e^x \cdot e^x \cdot e^{-x} = e^{x+x-x} = e^x .$$

Hence, it follows $g'(x) = e^x$.

1.3 Calculation Rules

1.3.1 Introduction

Using a few calculation rules and the derivatives presented in the last section, a variety of functions can be differentiated.

1.3.2 Multiples and Sums of Functions

In the following, $u, v : D \to \mathbb{R}$ will denote two arbitrary differentiable functions, and r denotes an arbitrary real number.

Sum Rule and Constant Factor Rule 5

Let two differentiable functions u and v be given. Then, the sum $f := u + v$ with $f(x) = (u+v)(x) := u(x) + v(x)$ is also differentiable, and we have

$$f'(x) = u'(x) + v'(x) \ .$$

Likewise, a function multiplied by a factor r, i.e. $f := r \cdot u$ with $f(x) = (r \cdot u)(x) := r \cdot u(x)$, is also differentiable, and we have

$$f'(x) = r \cdot u'(x) \ .$$

Using these two rules together with the differentiation rules for monomials x^n, any arbitrary polynomial can be differentiated. Here are some examples.

Example 1.3.1

The polynomial f with the mapping rule $f(x) = \frac{1}{4}x^3 - 2x^2 + 5$ is differentiable, and we have

$$f'(x) = \frac{3}{4}x^2 - 4x \ .$$

The derivative of the function $g :]0; \infty[\to \mathbb{R}$ with $g(x) = x^3 + \ln(x)$ is

$$g' :]0; \infty[\to \mathbb{R} \ \text{ with } \ g'(x) = 3x^2 + \frac{1}{x} = \frac{3x^3 + 1}{x} \ .$$

Differentiating the function $h : [0; \infty[\to \mathbb{R}$ with $h(x) = 4^{-1} \cdot x^2 - \sqrt{x} = \frac{1}{4}x^2 + (-1) \cdot x^{\frac{1}{2}}$ results for $x > 0$ in

$$h'(x) = \frac{1}{2}x - \frac{1}{2}x^{-\frac{1}{2}} = \frac{x^{\frac{3}{2}} - 1}{2\sqrt{x}} \ .$$

1.3.3 Product and Quotient of Functions

Product and Quotient Rule 6

Likewise, the product of functions, i.e. $f := u \cdot v$ with $f(x) = (u \cdot v)(x) := u(x) \cdot v(x)$, is differentiable, and the following **product rule** applies:

$$f'(x) = u'(x) \cdot v(x) + u(x) \cdot v'(x) \; .$$

The quotient of functions, i.e. $f := \frac{u}{v}$ with $f(x) = \left(\frac{u}{v}\right)(x) := \frac{u(x)}{v(x)}$, is defined and differentiable for all x with $v(x) \neq 0$, and the following **quotient rule** applies:

$$f'(x) = \frac{u'(x) \cdot v(x) - u(x) \cdot v'(x)}{(v(x))^2} \; .$$

These calculation rules shall be illustrated by means of a few examples.

Example 1.3.2

Find the derivative of $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2 \cdot e^x$. The product rule can be applied choosing, for example, $u(x) = x^2$ and $v(x) = e^x$. The corresponding derivatives are $u'(x) = 2x$ and $v'(x) = e^x$. Combining these terms according to the product rule results in the derivative of the function f:

$$f' : \mathbb{R} \to \mathbb{R} \; , \quad x \mapsto f'(x) = 2xe^x + x^2 e^x = (x^2 + 2x)e^x \; .$$

Next, we investigate the tangent function g with $g(x) = \tan(x) = \frac{\sin(x)}{\cos(x)}$ $(\cos(x) \neq 0)$. With respect to the quotient rule we set $u(x) = \sin(x)$ and $v(x) = \cos(x)$. The corresponding derivatives are $u'(x) = \cos(x)$ and $v'(x) = -\sin(x)$. Combining these terms according to the quotient rule results in the derivative of the function g:

$$g'(x) = \frac{\cos(x) \cdot \cos(x) - \sin(x) \cdot (-\sin(x))}{\cos^2(x)} \; .$$

This result can be transformed into one of the following expressions:

$$g'(x) = 1 + \left(\frac{\sin(x)}{\cos(x)}\right)^2 = 1 + \tan^2(x) = \frac{1}{\cos^2(x)} \; .$$

For the last transformation, the relation $\sin^2(x) + \cos^2(x) = 1$ was used, which was described in Module **??** (see Section **??** on page ??).

Exercise 1.3.1

Calculate the derivative of $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = \sin(x) \cdot x^3$ by factorising the product into two factors, taking the derivatives of each single factor, and finally combining the results according to the product rule.

a. The derivative of the left factor $u(x) = \square$ is $u'(x) = \square$.

b. The derivative of the right factor $v(x) = \square$ is $v'(x) = \square$.

c. Thus, applying the product rule to f results in $f'(x) = \square$.

Solution:
The four terms are

$$u(x) = \sin(x)\ ,\ \ u'(x) = \cos(x)\ ,\ \ v(x) = x^3\ ,\ \ v'(x) = 3x^2\ ,$$

and combining them according to the product rule results in

$$f'(x) = \cos(x) \cdot x^3 + \sin(x) \cdot 3x^2\ .$$

Exercise 1.3.2
Calculate the derivative of $f :]0; \infty[\to \mathbb{R}$ with $f(x) = \frac{\ln(x)}{x^2}$ by splitting the quotient up into numerator and denominator, taking the derivatives of both, and combining them according to the quotient rule.

a. The derivative of the numerator $u(x) = \square$ is $u'(x) = \square$.

b. The derivative of the denominator $v(x) = \square$ is $v'(x) = \square$.

c. Thus, applying the quotient rule to f results in $f'(x) = \square$.

Solution:
The four terms are

$$u(x) = \ln(x)\ ,\ \ u'(x) = \frac{1}{x}\ ,\ \ v(x) = x^2\ ,\ \ v'(x) = 2x\ ,$$

and combining them according to the quotient rule results in

$$f'(x) = \frac{\frac{1}{x} \cdot x^2 - \ln(x) \cdot 2x}{x^4} = \frac{1 - 2\ln(x)}{x^3}\ ,$$

where the last transformation step (cancelling x) is only for simplifying the expression.

1.3.4 Composition of Functions

Last, we investigate the composition (see Module **??**, Section **??** on page ??) of functions: What happens if a function u (the inner function) is substituted into another function v (the outer function)? In mathematics, such a composition is denoted by $f := v \circ u$ with $f(x) = (v \circ u)(x) := v(u(x))$. That is, first the value of a function u is determined depending on the variable x. The value $u(x)$ calculated this way is then used as an argument of the function v. This results in the final function value $v(u(x))$.

Chain Rule 7

The derivative of the function $f := v \circ u$ with $f(x) = (v \circ u)(x) := v(u(x))$ can be calculated applying the **chain rule**; we have:

$$f'(x) = v'(u(x)) \cdot u'(x) \ .$$

Here, the expression $v'(u(x))$ is considered in such a way that v is a function of u and thus, the derivative is taken with respect to u; then $v'(u)$ is evaluated for $u = u(x)$.
The following mnemonic phrase is useful: The derivative of a composite function is the product of the outer derivative and the inner derivative.

This differentiation rule shall be illustrated by a few examples.

Example 1.3.3

Find the derivative of the function $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = (3 - 2x)^5$. For applying the chain rule, inner and outer functions are to be identified. If we take for the inner function u the function $u(x) = 3 - 2x$, then the outer function v is given by $v(u) = u^5$. With this, we have $v(u(x)) = f(x)$, as required.

Taking the derivative of the inner function u with respect to x results in $u'(x) = -2$. For the outer derivative, the function v is differentiated with respect to u, which results in $v'(u) = 5u^4$. Inserting these terms into the chain rule results in the derivative f' of the function f with

$$f'(x) = 5(u(x))^4 \cdot (-2) = 5(3 - 2x)^4 \cdot (-2) = -10(3 - 2x)^4 \ .$$

As a second example, the derivative of $g : \mathbb{R} \to \mathbb{R}$ with $g(x) = e^{x^3}$ is calculated. For the inner function u the assignment $x \mapsto u(x) = x^3$ and for the outer function v the assignment $u \mapsto v(u) = e^u$ is appropriate. Taking the inner and the outer derivative results in $u'(x) = 3x^2$ and $v'(u) = e^u$. Inserting these terms into the chain rule results in the derivative of the function g:

$$g' : \mathbb{R} \to \mathbb{R} \ , \ \ x \mapsto g'(x) = e^{u(x)} \cdot 3x^2 = e^{x^3} \cdot 3x^2 = 3x^2 e^{x^3} \ .$$

1.3.5 Exercises

Exercise 1.3.3

Calculate the derivatives of the functions f, g, and h defined by the following mapping rules:

a. The derivative of $f(x) := 3 + 5x$ is $f'(x) =$ ☐ .

b. The derivative of $g(x) := \frac{1}{4x} - x^3$ is $g'(x) =$ ☐ .

c. The derivative of $h(x) := 2\sqrt{x} + 4x^{-3}$ is $h'(x) =$ ☐ .

Solution:

a. We have $f'(x) = 0 + 5 \cdot 1 \cdot x^0 = 0 + 5 = 5$.

b. Since $g(x) = \frac{1}{4x} - x^3 = \frac{1}{4}x^{-1} - x^3$, we have $g'(x) = \frac{1}{4} \cdot (-1) \cdot x^{-2} - 3 \cdot x^2 = -\frac{1}{4x^2} - 3x^2$.

c. Since $h(x) = 2\sqrt{x} + 4x^{-3} = 2x^{\frac{1}{2}} + 4x^{-3}$, we have $h'(x) = 2 \cdot \frac{1}{2} \cdot x^{-\frac{1}{2}} + 4 \cdot (-3) \cdot x^{-4} = \frac{1}{\sqrt{x}} - \frac{12}{x^4}$.

Exercise 1.3.4

Calculate the derivatives of the functions f, g, and h described by the following mapping rules, and simplify the results.

a. The derivative of $f(x) := \cot x = \frac{\cos x}{\sin x}$ is $f'(x) =$ ☐ .

b. The derivative of $g(x) := \sin(3x) \cdot \cos(3x)$ is $g'(x) =$ ☐ .

c. The derivative of $h(x) := \frac{\sin(3x)}{\sin(6x)}$ is $h'(x) =$ ☐ .

Solution:

a. From the quotient rule, we find

$$f'(x) = \frac{(-\sin(x)) \cdot \sin(x) - \cos(x) \cdot \cos(x)}{(\sin(x))^2} = -\frac{\sin^2(x) + \cos^2(x)}{\sin^2(x)} = -\frac{1}{\sin^2(x)} .$$

b. Form the product rule and the chain rule, we find

$$g'(x) = \cos(3x) \cdot 3 \cdot \cos(3x) + \sin(3x) \cdot (-\sin(3x)) \cdot 3 = 3\left(\cos^2(3x) - \sin^2(3x)\right) .$$

Since generally $\cos^2(u) - \sin^2(u) = \cos(2u)$, we have $g'(x) = 3\cos(6x)$.

c. According to the general relation $\sin(2u) = 2\sin(u)\cos(u)$, we have

$$h(x) = \frac{\sin(3x)}{\sin(6x)} = \frac{\sin(3x)}{2\sin(3x)\cos(3x)} = \frac{1}{2\cos(3x)} = \frac{1}{2} \cdot (\cos(3x))^{-1} .$$

Applying the chain rule several times results in

$$h'(x) = \frac{1}{2} \cdot (-1) \cdot (\cos(3x))^{-2} \cdot (-\sin(3x)) \cdot 3 = \frac{3\sin(3x)}{2\cos^2(3x)} = \frac{3\tan(3x)}{2\cos(3x)} .$$

Exercise 1.3.5
Calculate the derivatives of the functions f, g, and h defined by the following mapping rules:

a. The derivative of $f(x) := e^{5x}$ is $f'(x) =$ $\boxed{}$.

b. The derivative of $g(x) := x \cdot e^{6x}$ is $g'(x) =$ $\boxed{}$.

c. The derivative of $h(x) := (x^2 - x) \cdot e^{-2x}$ is $h'(x) =$ $\boxed{}$.

Solution:

a. From the chain rule, we immediately find $f'(x) = 5e^{5x}$.

b. From the product rule and the chain rule, we find $g'(x) = 1 \cdot e^{6x} + x \cdot e^{6x} \cdot 6 = e^{6x}(1 + 6x)$.

c. From the product rule and the chain rule, we find $h'(x) = (2x-1) \cdot e^{-2x} + (x^2 - x) \cdot e^{-2x} \cdot (-2) = -(2x^2 - 4x + 1)e^{-2x}$.

Exercise 1.3.6
Calculate the first four derivatives of $f : \mathbb{R} \to \mathbb{R}$ with $f(x) := \sin(1 - 2x)$.

Answer: The kth derivative of f is denoted by $f^{(k)}$. Here, $f^{(1)} = f'$, $f^{(2)}$ is the derivative of $f^{(1)}$, $f^{(3)}$ is the derivative of $f^{(2)}$, etc. Thus, we have:

- $f^{(1)}(x) =$ $\boxed{}$.
- $f^{(2)}(x) =$ $\boxed{}$.
- $f^{(3)}(x) =$ $\boxed{}$.
- $f^{(4)}(x) =$ $\boxed{}$.

Solution:
From the chain rule, we find successively:

$$\begin{aligned}
f^{(1)}(x) &= \cos(1-2x) \cdot (-2) = -2\cos(1-2x) \;, \\
f^{(2)}(x) &= -2 \cdot (-\sin(1-2x)) \cdot (-2) = -4\sin(1-2x) \;, \\
f^{(3)}(x) &= -4 \cdot \cos(1-2x) \cdot (-2) = 8\cos(1-2x) \;, \\
f^{(4)}(x) &= 8 \cdot (-\sin(1-2x)) \cdot (-2) = 16\sin(1-2x) \;.
\end{aligned}$$

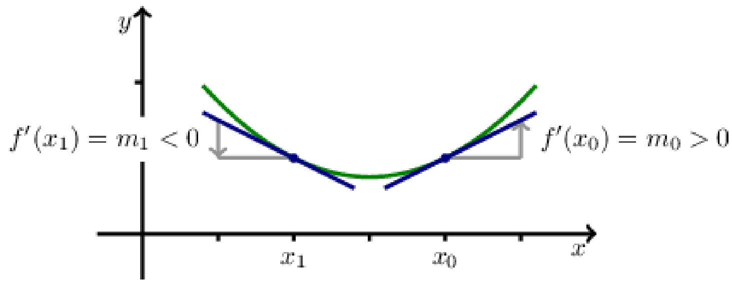
1.4 Properties of Functions

1.4.1 Introduction

The derivative was introduced above by means of a tangent line to a graph of a function. This tangent line describes the given function "approximately" in a certain region. The properties of this tangent line give also information on the properties of the approximated function in this region.

1.4.2 Monotony

The derivative of a function can be used to study the growth behaviour, i.e. whether for increasing values of x the corresponding function values increase or decrease. For this purpose, we consider a function $f : D \to \mathbb{R}$ that is differentiable on $]a; b[\subseteq D$:



If $f'(x) \leq 0$ for all x between a and b, then f is monotonically decreasing on the interval $]a; b[$.

If $f'(x) \geq 0$ for all x between a and b, then f is monotonically increasing on the interval $]a; b[$.

Thus, it is sufficient to determine the sign of the derivative f' to decide whether a function is monotonically increasing or decreasing on the interval $]a; b[$.

Example 1.4.1

The function $f : \mathbb{R} \to \mathbb{R}, x \mapsto x^3$ is differentiable with $f'(x) = 3x^2$. Since $x^2 \geq 0$ for all $x \in \mathbb{R}$, we have $f'(x) \geq 0$ and hence, f is monotonically increasing.

For $g : \mathbb{R} \to \mathbb{R}$ with $g(x) = 2x^3+6x^2-18x+10$, the function $g'(x) = 6x^2+12x-18 = 6(x+3)(x-1)$ has the roots $x_1 = -3$ and $x_2 = 1$. If the monotony of the function g is investigated, then three regions are to be distinguished in which g' has a different sign.

The following table is used to determine in which region the derivative of g is positive or negative. These regions correspond to the monotony regions of g. The entry "+" says that the considered term is positive on the given interval. If the term is negative, then "−" is entered.

x	$x < -3$	$-3 < x < 1$	$1 < x$
$x + 3$	−	+	+
$x - 1$	−	−	+
$g'(x)$	+	−	+
g is monotonically	increasing	decreasing	increasing

For the function $h : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ with $h(x) = \frac{1}{x}$, we have $h'(x) = -\frac{1}{x^2}$, that is $h'(x) < 0$ for all $x \neq 0$.

Even though the function h exhibits the same monotony behaviour for the two subregions $x < 0$ and $x > 0$, it is not monotonically decreasing on the entire region. As a counterexample let us consider the function values $h(-2) = -\frac{1}{2}$ and $h(1) = 1$. Here, we have $-2 < 1$ but also $h(-2) < h(1)$. This corresponds to an increasing growth behaviour if we change from one subregion to the other. The statement that the function h is monotonically decreasing on $]-\infty; 0[$ thus means that the restriction of h on this interval is monotonically decreasing. Moreover, the function h is also monotonically decreasing for all $x > 0$.

1.4.3 Second Derivative and Bending Properties (Curvature)

Let us consider a function $f : D \to \mathbb{R}$ that is differentiable on the interval $]a; b[\subseteq D$. If its derivative f' is also differentiable on the interval $]a; b[\subseteq D$, then f is called **twice-differentiable**. If the derivative of the first derivative of f is taken, then $(f')' = f''$ is called the **second derivative** of the function f.

The second derivative of the function f can be used to investigate the bending behaviour (curvature) of the function:

Bending Properties (Curvature) 8

If $f''(x) \geq 0$ for all x between a and b, then f is called **convex** (**left curved** or **concave up**) on the interval $]a; b[$.

If $f''(x) \leq 0$ for all x between a and b, then f is called **concave** (**right curved** or **concave down**) on the interval $]a; b[$.

Thus, it is sufficient to determine the sign of the second derivative f'' to decide whether a function is convex (left curved) or concave (right curved).

Comment on the Notation 9

The second derivative and further “higher” derivatives are often denoted using superscript natural numbers in round brackets: $f^{(k)}$ then denotes the kth derivative of f. In particular, this notation is used in generally written formulas also for the (first) derivative ($k = 1$) and for the function f itself ($k = 0$).

Hence,

- $f^{(0)} = f$ denotes the function f,

- $f^{(1)} = f'$ denotes the (first) derivative,
- $f^{(2)} = f''$ the second derivative,
- $f^{(3)}$ the third derivative, and
- $f^{(4)}$ the fourth derivative of f.

This list can be continued as long as the derivatives of f exist.

The following example shows that a monotonically increasing function can be convex on one region and concave on another.

Example 1.4.2

Certainly, the function $f : \mathbb{R} \to \mathbb{R}, x \mapsto x^3$ is at least twice-differentiable. Since $f'(x) = 3x^2 \geq 0$ for all $x \in \mathbb{R}$, the function f is monotonically increasing on its entire domain. Moreover, we have $f''(x) = 6x$. Thus, for all $x < 0$, we also have $f''(x) < 0$ and hence, the function f is concave (right curved) on this region. For $x > 0$, we have $f''(x) > 0$. Hence, for $x > 0$, the function f is convex (left curved).

1.4.4 Exercises

Exercise 1.4.1
Specify the (maximum) open intervals on which the function f with $f(x) := \frac{x^2-1}{x^2+1}$ is monotonically increasing or decreasing?

Answer:

- f is monotonically [ ] on $]-\infty; 0[$.
- f is monotonically [ ] on $]0; \infty[$.

Solution:
The derivative f' of the function f is given by

$$f'(x) = \frac{2x \cdot (x^2+1) - (x^2-1) \cdot 2x}{(x^2+1)^2} = \frac{2x\left(x^2+1-x^2+1\right)}{(x^2+1)^2} = \frac{4x}{(x^2+1)^2} \ .$$

Since the denominator of $f'(x)$ is always positive, the sign of $f'(x)$ is determined only by the numerator: For all negative $x \in \mathbb{R}$, we have $f'(x) < 0$, and hence, the function f is monotonically decreasing on this region. In contrast, for all positive $x \in \mathbb{R}$, we have $f'(x) > 0$, and hence, the function f is monotonically increasing on this region.

Exercise 1.4.2
Specify the (maximum) open intervals $]c; d[$ on which the function f with $f(x) := \frac{x^2-1}{x^2+1}$ for $x > 0$ is convex or concave? Answer:

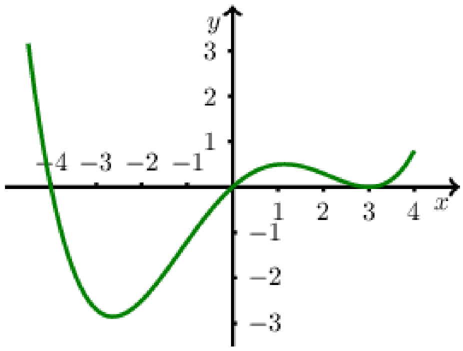
- The function f is convex on [ ] .
- The function f is concave on [ ] .

Solution:
From the quotient rule, we have for the first and the second derivative of f

$$\begin{aligned} f'(x) &= \frac{4x}{(x^2+1)^2} \ , \\ f''(x) &= \frac{4 \cdot (x^2+1)^2 - 4x \cdot 2(x^2+1) \cdot 2x}{(x^2+1)^4} = \frac{4x^2+4-16x^2}{(x^2+1)^3} = \frac{4(1-3x^2)}{(x^2+1)^3} \ . \end{aligned}$$

Since $4/(x^2+1)^3$ is always positive, the sign of $f''(x)$ is only determined by the factor $(1-3x^2)$. The simple roots of $f''(x)$ are at $x_0 = \pm\frac{1}{\sqrt{3}}$. Thus, for $x > 0$, the derivative $f''(x)$ is greater than 0 on the open interval $\left]0; \frac{1}{\sqrt{3}}\right[$, and hence, the function f is convex (left curved) on this region. On the interval $\left]\frac{1}{\sqrt{3}}; \infty\right[$, we have $f''(x) < 0$; hence, the function f is concave (right curved) on this region.

Exercise 1.4.3
Let a function $f : [-4.5; 4] \to \mathbb{R}$ with $f(0) := 2$ be given; its derivative f' has the graph shown in the figure below:



a. Where is the function f monotonically increasing and where it is monotonically decreasing? Find the maximum open intervals $]c;d[$ on which f has this property.

b. What can you say about the maximum and minimum points of the function f?

Answer:

- The function f is monotonically ______ on $]-4.5;$ ______ $[$.
- The function f is monotonically ______ on $]$ ______ $;0[$.
- The function f is monotonically ______ on ______ .
- The function f is monotonically ______ on $]3;4[$.

The maximum point of f is at ______ . The minimum point of f is at ______ .

Solution:
The monotony behaviour is determined by the derivative f' of the function f. Since the graph of the derivative f' is given in the exercise, we only have to read off on which intervals the graph lies above (or below) the x-axis: On the intervals $[-4.5;-4[$, $]0;3[$, and $]3;4[$, we have $f'(x) > 0$, and hence, the function f is monotonically increasing there. However, on the interval $]-4;0[$, we have $f'(x) < 0$, and hence, the function f is monotonically decreasing there.
At an extremum point x_e (maximum or minimum point) of a function f (which does not lie on the boundary of the domain) the first derivative is zero: $f'(x_e) = 0$. Graphically this means, that the tangent line to the graph of f is a horizontal line. According to the exercise text, the zeros of $f'(x)$ are at $x_1 = -4$, $x_2 = 0$, and $x_3 = 3$. Since f is monotonically increasing on $]-4.5;-4[$ and monotonically decreasing on $]-4;0[$, $x_1 = -4$ is a maximum point. Accordingly, it is reasoned that the function has a minimum point at $x_2 = 0$. (At $x_3 = 3$ the function has a saddle point.)

1.5 Applications

1.5.1 Curve Sketching

Let a differentiable function $f :]a; b[\to \mathbb{R}$ with the mapping rule $x \mapsto y = f(x)$ for $x \in]a; b[$ be given. In this course, a complete curve sketching of f includes the following information:

- maximum domain
- x- and y-intercepts of the graph
- symmetry of the graph
- limiting behaviour/asymptotes
- first derivatives
- extremum values
- monotony behaviour
- inflexion points
- bending behaviour (curvature)
- sketch of the graph

Many of these points were already discussed in Module ?? on page ??. Therefore, in this section we shall only briefly repeat what is meant by the different steps of curve sketching. Subsequently, we will discuss one example of a curve sketching in detail.

The first part of the curve sketching involves the algebraic and geometric aspects of f:

Maximum Domain: All real numbers x are determined for which $f(x)$ exists. The set D of these numbers is called maximum domain.

x- and y-Intercepts:

- x-axis: All zeros of f are determined.
- y-axis: The function value $f(0)$ (if $0 \in D$) is calculated.

Symmetry of the Graph: The graph of the function is symmetrical with respect to the y-axis if $f(-x) = f(x)$ for all $x \in D$. Then the function f is also called **even**. If $f(-x) = -f(x)$ for all $x \in D$, the graph is centrally symmetric with respect to the origin $(0; 0)$ of the coordinate system. In this case, the function is also called **odd**.

Asymptotic Behaviour at the Domains Boundary: The limits of the function f at the boundaries of its domain are investigated.

In the second part, the function is investigated analytically by means of conclusions from the first derivatives. Of course, the first and the second derivative have to be calculated first, provided they exist.

Derivatives: Calculation of the first and the second derivative (if they exist).

Extremum Values and Monotony: The necessary condition, for x to be an extremum point (if $x \in D$

is not a boundary point of D), is $f'(x) = 0$.
Thus, we calculate the points x_0 at which the derivative f' takes the value zero. If at these points also the second derivative exists, then we have:

- $f''(x_0) > 0$: x_0 is a minimum point of f.
- $f''(x_0) < 0$: x_0 is a maximum point of f.

The function f is monotonically increasing on that intervals of the domain on which we have $f'(x) \geq 0$. It is monotonically decreasing on that intervals where $f'(x) \leq 0$.

Inflexion Points and Curvature Properties: The necessary condition, for x to be an inflexion point (if the second derivative f'' exists), is $f''(x) = 0$
If $f''(w_0) = 0$ and $f^{(3)}(w_0) \neq 0$, then w_0 is an inflexion point, i.e. the bending behaviour of f changes at this point.
The function f is convex (left curved) on that intervals of the domain in which we have $f^{(2)}(x) \geq 0$. It is concave (right curved) on that intervals on which we have $f^{(2)}(x) \leq 0$.

Sketch of the Graph: A sketch of the graph is drawn based on the information gained during the previous steps.

1.5.2 Detailed Example

We investigate a function f defined by the mapping rule

$$f(x) \ = \ \frac{4x}{x^2+2} \ .$$

Maximum Domain
The maximum domain of this function is $D_f = \mathbb{R}$ since the denominator of the function is $x^2 + 2 \geq 2$, i.e. it is always non-zero, and hence no points have to be excluded.

x- and y-Intercepts
The zeros of the function are the zeros of the numerator. Hence, the graph of f intersects the x-axis only in the origin $(0; 0)$ since the numerator is only zero for $x = 0$. At the same time, this is the only intersection point with the y-axis since there we have $f(0) = 0$.

Symmetry
To investigate the symmetry we replace the argument x by $(-x)$. We have

$$f(-x) \ = \ \frac{4 \cdot (-x)}{(-x)^2+2} \ = \ -\frac{4x}{x^2+2} \ = \ -f(x)$$

for all $x \in \mathbb{R}$. Hence, the graph of the function f is centrally symmetric with respect to the origin.

Limiting Behaviour
The function is defined on the entire set of real numbers $\mathbb{R}$, and thus, only the limiting behaviour for $x \to \infty$ and $x \to -\infty$ has to be investigated. Since $f(x)$ is a fraction of two polynomials and the denominator has a greater power than the numerator, the x-axis is a horizontal asymptote in both directions:

$$\lim_{x \to \pm\infty} f(x) = 0 \ .$$

Derivatives
The first two derivatives of the function are calculated from the quotient rule. For the first derivative, we have:

$$f'(x) \;=\; 4\cdot\frac{1\cdot(x^2+2)-x\cdot 2x}{(x^2+2)^2} \;=\; 4\cdot\frac{-x^2+2}{(x^2+2)^2}\,.$$

Taking the derivative again and simplifying the terms results in:

$$\begin{aligned} f''(x) &= 4\cdot\frac{-2x(x^2+2)^2-(-x^2+2)\cdot 2(x^2+2)\cdot 2x}{(x^2+2)^4} \\ &= 4\cdot\frac{-2x(x^2+2)-(-x^2+2)\cdot 4x}{(x^2+2)^3} \\ &= 4\cdot\frac{-2x^3-4x+4x^3-8x}{(x^2+2)^3} \\ &= 4\cdot\frac{2x^3-12x}{(x^2+2)^3} \\ &= 8\cdot\frac{x(x^2-6)}{(x^2+2)^3}\,. \end{aligned}$$

Extremum Values
The necessary condition for x to be an extremum point, is $f'(x)=0$, in this case $-x^2+2=0$. Thus, we obtain $x_1=\sqrt{2}$ and $x_2=-\sqrt{2}$. In addition, we have to investigate the behaviour of the second derivative at these points:

$$f''(x_1) \;=\; 8\frac{\sqrt{2}\cdot(2-6)}{(2+2)^3}<0\,,\quad f''(x_2) \;=\; 8\frac{(-\sqrt{2})\cdot(2-6)}{(2+2)^3}>0\,.$$

Hence, x_1 is a maximum point and x_2 is a minimum point of f. Inserting these values in f results in the maximum point $\left(\sqrt{2};\sqrt{2}\right)$ and the minimum point $\left(-\sqrt{2};-\sqrt{2}\right)$ of f.

Monotony Behaviour
Since f is defined on the entire set of real numbers $\mathbb{R}$, the monotony behaviour can be derived from the position of the extremum points and their types: f is monotonically decreasing on $\left]-\infty;-\sqrt{2}\right[$, monotonically increasing on $\left]-\sqrt{2};\sqrt{2}\right[$, and monotonically decreasing on $\left]\sqrt{2};\infty\right[$. Monotony intervals are always given as open intervals.

Inflexion Points
From the necessary condition $f''(x)=0$ for x to be an inflexion point, we have the equation $8x(x^2-6)=0$. Thus, $w_0=0$, $w_1=\sqrt{6}$, and $w_2=-\sqrt{6}$ are the only solutions. The polynomial in the denominator of f'' is always greater than zero. Since the polynomial in the numerator has only single roots, the second derivative $f''(x)$ changes its sign at all these points. Hence, these points are inflexion points of f. The coordinates of the inflexion points $(0;0)$, $\left(\sqrt{6};\frac{1}{2}\sqrt{6}\right)$, $\left(-\sqrt{6};-\frac{1}{2}\sqrt{6}\right)$ are determined by inserting the corresponding values for x in f.

Bending Behaviour
The twice-differentiable function f is convex if the second derivative is greater or equal to zero. It is concave if the second derivative is less or equal to zero. Since the polynomial in the denominator of $f''(x)$ is always positive, it is sufficient to examine the sign of the polynomial $p(x)=8x(x-\sqrt{6})(x+\sqrt{6})$

in the numerator. For $0 < x < \sqrt{6}$, it is negative (f is concave there). For $x > \sqrt{6}$ it is positive (f is convex there). Since f is centrally symmetric, it follows that f is convex on the intervals $\left]-\sqrt{6};0\right[$ and $\left]\sqrt{6};\infty\right[$ and concave on the intervals $\left]-\infty;-\sqrt{6}\right[$ and $\left]0;\sqrt{6}\right[$.

Sketch of the Graph

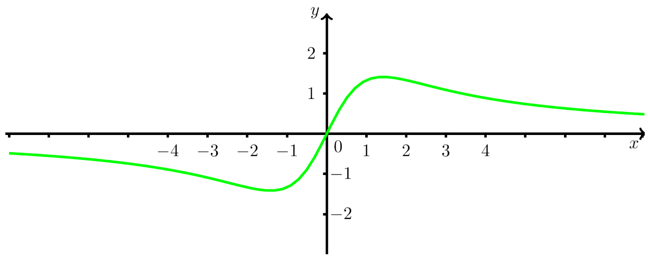


Figure 1.1: The graph of the function f, sketched on the interval $[-8;8]$.

1.5.3 Exercises

With the following exercise the elements of the curve sketching method can be trained:

Exercise 1.5.1
Carry out a complete curve sketching for a function f with $f(x) = (2x - x^2)e^x$ and enter your results into the input fields:

Maximum domain: $\boxed{}$.

Set of intersection points with the x-axis (zeros of $f(x)$): $\boxed{}$.

The y-intercept is at y = $\boxed{}$.

Symmetry: The function is

☐ axially symmetric with respect to the y-axis,

☐ centrally symmetric with respect to the origin.

Limiting behaviour: For $x \to \infty$, the functions values $f(x)$ tend to $\boxed{}$, and for $x \to -\infty$, they tend to $\boxed{}$.

Derivatives: We have $f'(x) =$ $\boxed{}$ and $f''(x) =$ $\boxed{}$.

Monotony behaviour: The function is monotonically increasing on the interval $\boxed{}$ and monotonically decreasing otherwise.

Extremum values: The point $x_1 =$ $\boxed{}$ is a minimum point and the point $x_2 =$ $\boxed{}$ is a maximum point.

Inflexion points: The set of inflexion points consists of $\boxed{}$.

Sketch the graph, and subsequently, compare your result to the sample solution.

Solution:

Maximum Domain
We have $f(x) = -x(x-2)e^x$ and $e^x > 0$ for all $x \in \mathbb{R}$; thus, $D_f = \mathbb{R} =]-\infty; \infty[$ is the maximum domain.

x- and y-Intercepts
The intersection points with the x-axis (roots of the function) lie at $x_1 = 0$ and $x_2 = 2$, i.e. the coordinates of the points are $(0; 0)$ and $(2; 0)$. The y-intercept is the point $(0; 0)$.

Symmetry
The function f is neither even nor odd, and hence, the graph of f is neither axially symmetric with respect to the y-axis nor centrally symmetric with respect to the origin.

Limiting Behaviour

Since the function is defined for all real numbers, only the asymptotes for $\pm\infty$ have to be investigated:

$$\lim_{x\to\infty} -x(x-2)e^x \;=\; -\infty \text{ and } \lim_{x\to-\infty} -x(x-2)e^x \;=\; 0\,.$$

Hence, $y = 0$ is an asymptote for $x \to -\infty$.

Derivatives
The first two derivatives of f are

$$\begin{aligned} f'(x) &= (2-2x)e^x + (2x-x^2)e^x = (2-x^2)e^x = -(x^2-2)e^x\,, \\ f''(x) &= -2xe^x + (2-x^2)e^x = -(x^2+2x-2)e^x\,. \end{aligned}$$

Monotony Behaviour and Extremum Values
The solutions of $f'(x) = 0$ are $x_1 = -\sqrt{2}$ and $x_2 = \sqrt{2}$. Furthermore, we have $x_1 < x_2$ and

$$f'(x) \;=\; -(x+\sqrt{2})(x-\sqrt{2})e^x\,.$$

On $\left]-\infty;-\sqrt{2}\right[$ the first derivative f' is negative and hence, f is monotonically decreasing there. On $\left]-\sqrt{2};\sqrt{2}\right[$ the first derivative f' is positive and hence, f is monotonically increasing there. On $\left]\sqrt{2};\infty\right[$ the first derivative f' is negative and hence, f is monotonically decreasing there. Thus, $x_1 = -\sqrt{2}$ is a minimum point, and $x_2 = \sqrt{2}$ is a maximum point.

Inflexion Points
The necessary condition $f''(x) = 0$ for x to be an inflexion point results in the quadratic equation $x^2 + 2x - 2 = 0$. It has the solutions $w_1 = \frac{-2-\sqrt{4+8}}{2} = -1-\sqrt{3}$ and $w_2 = \frac{-2+\sqrt{4+8}}{2} = -1+\sqrt{3}$.

Sketch of the Graph

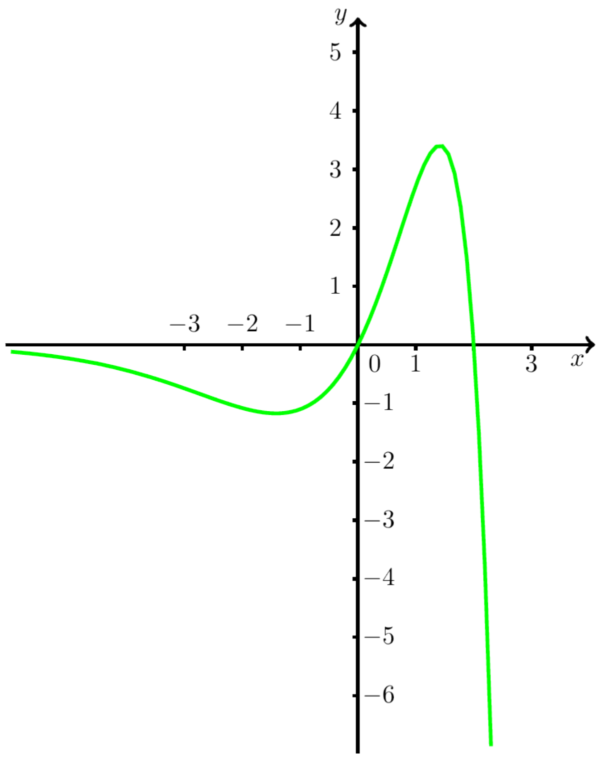


Figure 1.2: Graph of the function f, sketched on the interval $]-6.2;3[$.

1.5.4 Optimisation Problems

In many applications in engineering and business, problem solutions can be found that are not unique. Often they depend on variable conditions. To find an ideal solution, additional properties (constraints) are defined that are to be satisfied by the solution. This very often results in so called **optimisation problems**, in which from a family of solutions that one solution has to be selected that satisfies a previously specified property the best.

As an example, we consider the problem of constructing a cylindrical can. This can shall satisfy the additional condition to have a capacity (volume) V of one litre, i.e. one cubic decimetre ($1\,\mathrm{dm}^3$). Thus, if V is specified in dm^3 and r is the radius and h the height of the can, each measured in decimetre (dm), then the volume shall be $V = \pi r^2 \cdot h = 1$. The can with the least surface area $O = 2 \cdot \pi r^2 + 2\pi r h$ is required in order to save material. Here, the surface area O, measured in square decimetres (dm^2), is a function of the radius r and the height h of the can.

In mathematical terms, our question results in the problem of finding a minimum of the surface function O, where the minimum has to be found for values of r and h that also satisfy the additional condition for the volume: $V = \pi r^2 \cdot h = 1$. In the context of finding extremum points, such an additional condition is also called a **constraint**.

Optimisation Problem 10

In an **optimisation problem**, we search for an extremum point x_{ext} of a function f satisfying a given equation $g(x_{\text{ext}}) = b$.

If we search for a minimum point, this problem is also called a **minimisation problem**. If we search for a maximum point, this problem is also called a **maximisation problem**.

The function f is called **target function**, and the equation $g(x) = b$ is also called **constraint** of the optimisation problem.

1.5.5 Example

Let us consider the example above in more detail. Obviously, the problem is to minimise the surface area of a cylindrical can with a given volume (base multiplied by height):

$$V = \pi r^2 h = 1 \ ,$$

where r is the radius of the base and h is the height of the can. The surface area consists of the lid and the base (both with an area of πr^2) and the lateral surface (with an area of $2\pi r h$), which results in the equation $O = 2\pi r^2 + 2\pi r h$ for the surface area of the can. The surface area of the can is a function of the radius r and the height h. In contrast, a fixed volume (constraint) is assigned to the volume. Thus, it can be written:

$$O\,(r, h) = 2\pi r^2 + 2\pi r h \ .$$

Due to the constraint $V = \pi r^2 h = 1$, this problem that initially involves two variables (r and h) can be reduced to a problem that only involves one variable. Solving the constraint for the height of the can results in:

$$\begin{aligned} \pi r^2 h &= 1 \\ \Leftrightarrow \quad h &= \frac{1}{\pi r^2} \ . \end{aligned}$$

Substituting this formula into the function $O(r,h)$ results in a function that only depends on one variable. This function is also called O **for simplicity**:

$$O\left(r,h\right) = 2\pi r^2 + 2\pi r h = 2\pi r^2 + 2\pi r \frac{1}{\pi r^2} = 2\left(\pi r^2 + \frac{1}{r}\right) = O\left(r\right) \ .$$

After this manipulation, the problem of finding the cans minimal surface area can be solved analogously to normal extremum value problems of functions. Thus, we take the first derivative of the function O with respect to the variable r and let this derivative be equal to zero:

$$\begin{aligned} O'\left(r\right) = 2\left(2\pi r - \frac{1}{r^2}\right) &= 0 \\ \Leftrightarrow \quad 2\pi r &= \frac{1}{r^2} \\ \Leftrightarrow \quad 2\pi r^3 &= 1 \\ \Leftrightarrow \quad r^3 &= \frac{1}{2\pi} \\ \Leftrightarrow \quad r &= \sqrt[3]{\frac{1}{2\pi}} \ . \end{aligned}$$

The last equivalent transformation used the fact that the radius r cannot take negative values. Substituting this result into the second derivative of O shows whether actually a minimum was found ($O''(r) = 4\pi + 4/r^3$):

$$O''\left(\sqrt[3]{\frac{1}{2\pi}}\right) = 4\pi + \frac{4}{\left(\sqrt[3]{\frac{1}{2\pi}}\right)^3} = 12\pi > 0 \ .$$

For the radius $r = \sqrt[3]{\frac{1}{2\pi}}$, the surface area of the cylindrical can with the given volume $V = 1$ is a minimum. The corresponding height of the can is $h = \frac{1}{\pi\left(\sqrt[3]{\frac{1}{2\pi}}\right)^2} = \sqrt[3]{\frac{4}{\pi}}$. If a can of these dimensions is manufactured, the material usage for the given volume is minimised.

1.6 Final Test

1.6.1 Final Test Module 6

Exercise 1.6.1
In a container at 9 a.m. a temperature of $-10°\mathrm{C}$ is measured. At 3 p.m. the measured temperature is $-58°\mathrm{C}$. After a period of 14 hours, the temperature is fallen to $-140°\mathrm{C}$.

a. What is the average rate of temperature change due to the first and the second measurement?

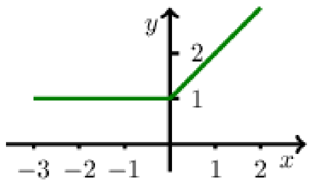
   Answer: ☐

b. In the (average) rate of change, the property of a falling temperature shows in the fact that the rate of change is ☐ .
   Hint:
   Enter an adjective.

c. Calculate the average rate of temperature change for the whole measuring period.

   Answer: ☐

Exercise 1.6.2
A function $f : [-3; 2] \to \mathbb{R}$, $x \mapsto f(x)$ has a first derivative f' whose graph is shown in the figure below.



The function values of f between -3 and 0

- ☐ are constant,
- ☐ increase by 3,
- ☐ decrease.

At the point $x = 0$ the function f has

- ☐ a jump,
- ☐ no derivative,
- ☐ a derivative of 1.

Exercise 1.6.3
Calculate for the function

a. $f : \{x \in \mathbb{R} \,:\, x > 0\} \to \mathbb{R}$ with $f(x) := \ln\left(x^3 + x^2\right)$ the value of the first derivative f' at x:
   $f'(x) =$ ☐ .

b. $g : \mathbb{R} \to \mathbb{R}$ with $g(x) := x \cdot e^{-x}$ the value of the second derivative g'' at x:
   $g''(x) =$ ☐ .

Exercise 1.6.4
Consider the function $f :]0;\infty[\to \mathbb{R}$, $x \mapsto f(x)$ with $f'(x) = x \cdot \ln x$. On what regions is f monotonically decreasing, on what regions is f concave? Specify the regions as open intervals $]c;d[$ that are as large as possible:

a. f is monotonically decreasing on $\boxed{}$.

b. f is concave on $\boxed{}$.

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