1 Geometry

Module Overview

The first sections of this chapter will introduce you to elementary geometry, while referring to the previous chapters. As a main issue, we first deal with the properties of triangles before calculating areas of polygons and volumes of simple geometric solids. Advanced problems are solved by means of trigonometric functions. These will give a first outlook to the later modules on calculus and analytic geometry.

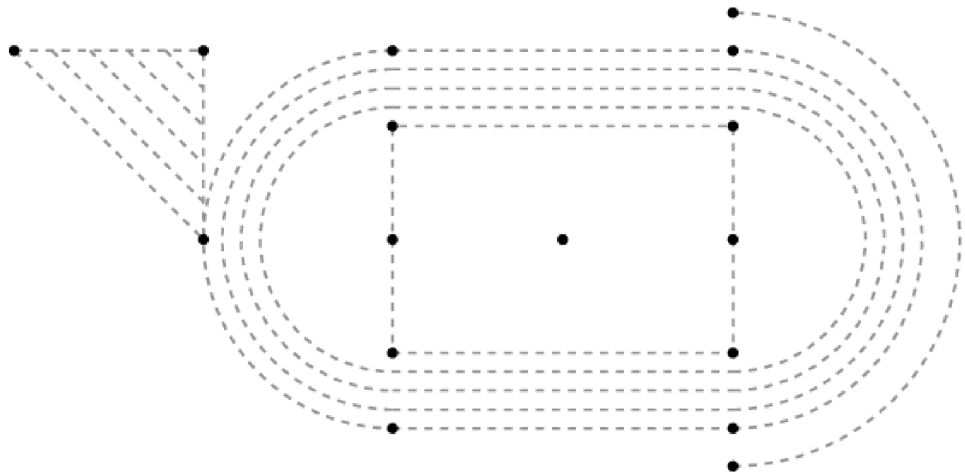
1.1 Elements of Plane Geometry

1.1.1 Introduction

Looking at the stars in a clear, moonless night conveys a vivid impression of the elementary objects of geometry, namely the points. Since time immemorial, people mentally connected the points of light in the night sky by lines they subsequently interpreted as the contours of highly diverse characters. Yet every building, with its vertex corners, edges, and faces, provides evidence of the practical use of this "heavenly" geometry, ...

On the other hand, the invention of resistant pencils, wax tablets, papyrus, or paper enabled people to capture their thoughts and observations "on paper" and to show them to others. For example, the intention to realise the drawing in a real building resulted in the concept of a plan. A plan is a drawing of an idealised image showing, for example, how a stadium shall look like from above.

For the construction of a stadium, significant points are staked out in the terrain. The current status of the project is shown in the following drawing containing the contours and significant points from a plan.



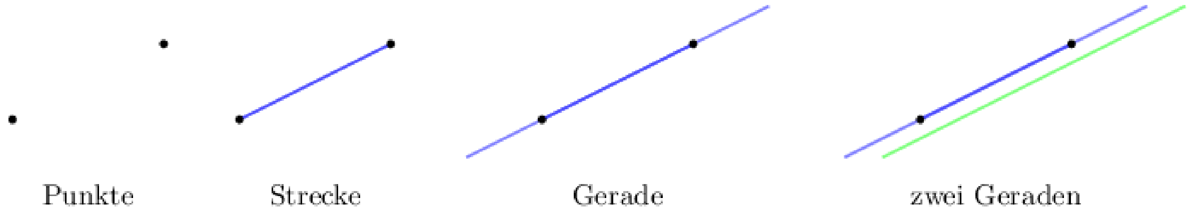
(Measurement) points and lines from a construction plan of a stadium

The drawing can be considered as an idealised image of reality. Along these lines, we will first recapitulate some basic concepts of geometry. Then, applying these concepts, we will construct more complicated figures and geometric solids.

1.1.2 Points and Lines

In geometry, a place or a position in a plane is idealised to the most basic object, namely a point. A single point itself cannot be characterised any further.

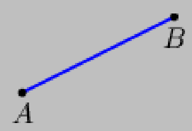
For several points, relations between these points can be considered in different ways — and points can be used to define new objects such as line segments and lines (see figure below). Mathematically, these objects are sets of points.



First, we consider a line segment and the distance between points. To do this, we still need a comparison tool for measuring the distance. In mathematics, this tool is a comparative length with so called unit length. For applications, appropriate length units such as metres or centimetres are chosen, depending on the task in hand.

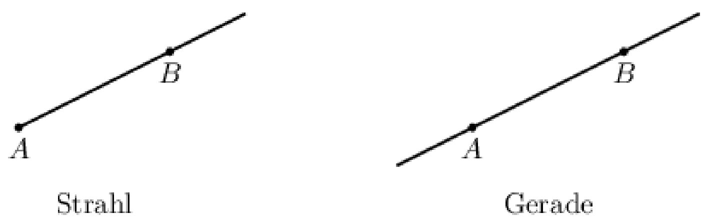
Line Segments and Distances 1.1.1

Given two points A and B, the **line segment** $\overline{AB}$ between A and B is the shortest path between the two points A and B.



The length of the line segment $\overline{AB}$ is denoted by $[\overline{AB}]$. The **line length** equals the distance between the two points A and B.

A ray of light emitted by a distant star or by the sun is an appropriate notion of a **ray** starting at the initial point A and proceeding through a second point B indefinitely. A ray is also called **half-line**.

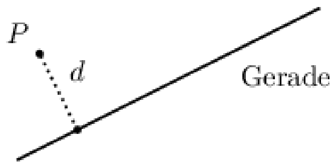


Continuing the path of a line segment $\overline{AB}$ on both ends indefinitely results in a so called line.

Line 1.1.2

Let A and B be two points (i.e. point A is different from point B). Then, A and B define exactly one **line** AB.

Considering, beside A and B, an additional point P, we can ask for the distance d of the point P from the line AB, which is defined as the shortest path between P and one of the points of the line AB.



Given three points P, Q, and S in the plane, the lines SP and SQ can be defined.

The two lines have the point S in common. If the point Q is also on the line SP, then SQ and SP denote one and the same line. If the point Q does not belong to the line SP, the line SQ is different from the line SP. Then, the two lines have only the point S in common. The point S is called **intersection point**.

If any two lines g and h do not have any points in common, the smallest distance between points on g and h, respectively, is called the distance between the lines g and h. Hence, g and h do not have any point in common if they have a distance larger than 0. Two lines are called **parallel** if every point on one of the two lines has the same distance to the other line.

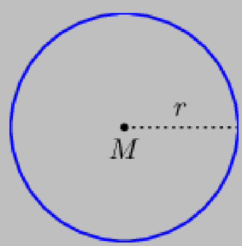
A single line can be described by the distance of two points M and M' as well: The set of all points having the same distance from two points M and M' is a line.

In geometry, it is a typical approach to define new objects by means of certain properties such as the distance. In this way, a circle can also be described very easily.

Circle 1.1.3

Let a point M and a positive real number r be given.

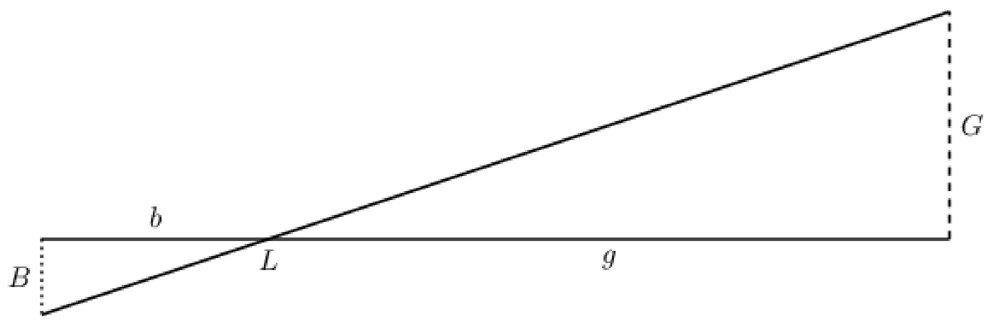
Then, the set of all points at distance r from point M is a **circle** around M with **radius** r.



1.1.3 Intercept Theorems

A pinhole camera provides a small image of the outside space. The ratio of the size of the image B to the size of the object G equals the ratio of the distance b from the pinhole L to the distance g from L:

$$\frac{B}{G} = \frac{b}{g} \,.$$



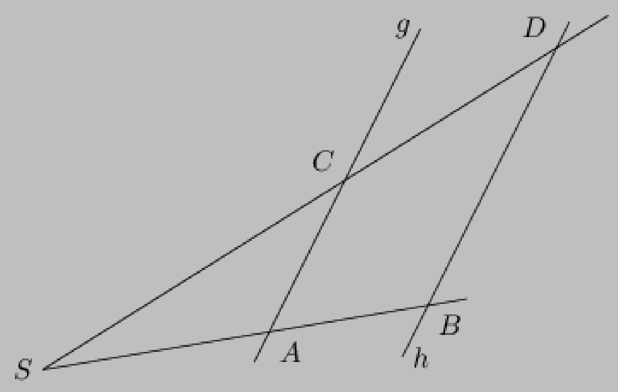
Properties of images arising from uniform scaling can also be described by means of the intercept theorems (see also Figure 1.3.4 on page 24).

What all examples applying the intercept theorems have in common is that rays (or lines) with an intersection point are intersected by parallel lines.

Intercept Theorems 1.1.4

Let S be the common emanating point of the two rays s_1 and s_2 proceeding through the points A and C, respectively. The point B is on the ray s_1 and the point D is on the ray s_2. First, we consider the line segments between the points on the two rays and then the line segments between the rays.

For two points P and Q, $\overline{PQ}$ is the line segment from P to Q and $[\overline{PQ}]$ denotes the length of this line segment.



If the lines g and h are parallel, the following statements hold:

- The ratio of the line segments on one of the two rays equals the corresponding ratio of the line segments on the other:
$$\frac{[\overline{SA}]}{[\overline{SC}]} = \frac{[\overline{AB}]}{[\overline{CD}]} = \frac{[\overline{SB}]}{[\overline{SD}]}\,.$$
This can also be expressed in the form:
$$\frac{[\overline{SA}]}{[\overline{AB}]} = \frac{[\overline{SC}]}{[\overline{CD}]} \quad \text{and} \quad \frac{[\overline{SA}]}{[\overline{SB}]} = \frac{[\overline{SC}]}{[\overline{SD}]}\,.$$
- The ratio of the line segments on the parallel lines equals the ratio of the corresponding line

segments emanating from S on one and the same ray

$$\frac{[\overline{SA}]}{[\overline{SB}]} = \frac{[\overline{AC}]}{[\overline{BD}]} = \frac{[\overline{SC}]}{[\overline{SD}]} \,.$$

This can also be expressed in the form:

$$\frac{[\overline{SA}]}{[\overline{AC}]} = \frac{[\overline{SB}]}{[\overline{BD}]} \quad \text{and} \quad \frac{[\overline{SC}]}{[\overline{CA}]} = \frac{[\overline{SD}]}{[\overline{DB}]} \,,$$

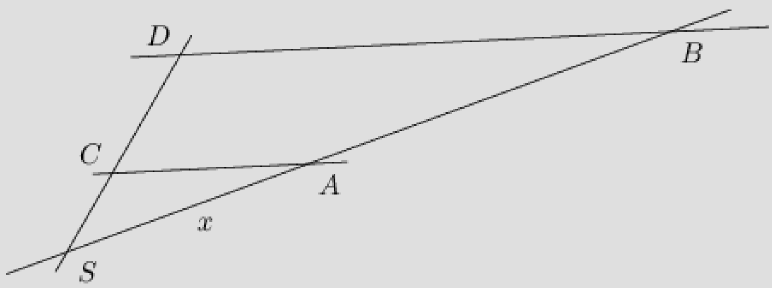
where $[\overline{AC}] = [\overline{CA}]$ and $[\overline{BD}] = [\overline{DB}]$.

The statements of the intercept theorems also hold if two lines intersecting in a point S are considered instead of the two rays. An application example of this case is the pinhole camera mentioned above.

In this way, distances between points can be calculated without measuring the length of the line segments directly.

Example 1.1.5

Let four points A, B, C, and D be given. These points define the two lines AB and CD intersecting in the point S. Furthermore, it is known that the lines AC and BD are parallel. Between the points the following distances were measured: $[\overline{AB}] = 51$, $[\overline{SC}] = 12$, and $[\overline{CD}] = 18$.



From this, the distance between A and S can be calculated. Let x denote the required distance. Then, according to the intercept theorems, we have

$$\frac{x}{[\overline{AB}]} = \frac{[\overline{SC}]}{[\overline{CD}]} \,,$$

from which

$$x = \frac{[\overline{SC}]}{[\overline{CD}]} \cdot [\overline{AB}] = \frac{12}{18} \cdot 51 = \frac{2}{3} \cdot 51 = 34$$

follows.

1.1.4 Exercises

Exercise 1.1.1
The son of the house is looking at the tree on the neighbours property. He observes that the tree is completely covered by the hedge separating the two properties if he only stands close enough to the hedge. Now he is looking for the point at which he just cannot see the tree anymore.

The boy is 1.40 metres tall. If the boy stands 2.50 metres away from the hedge, which is 2.40 metres high, 1 metre wide and clipped into a pointed shape at the top, the tree disappears from his sight.

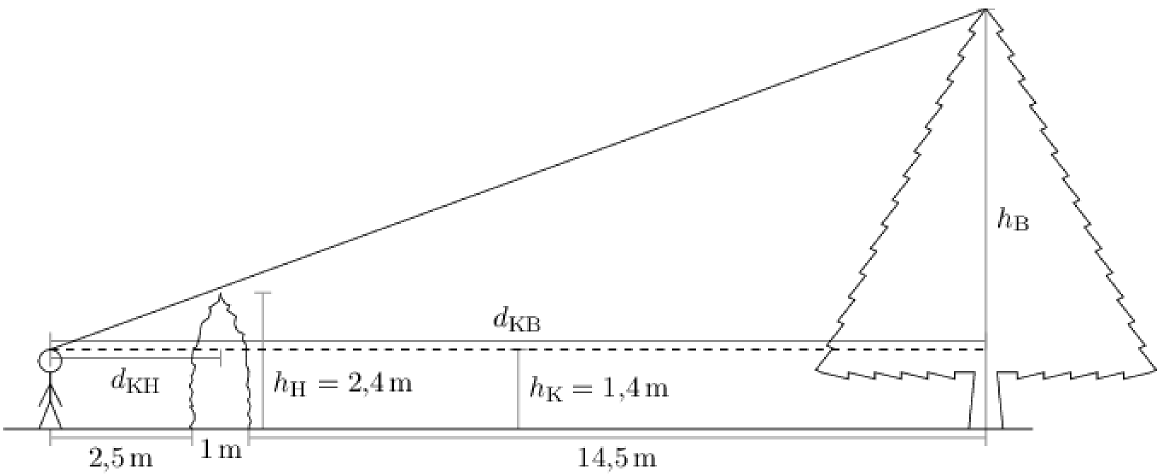
What is the height of the tree if the middle of the trunk is 14.5 metres away from the hedge?

Please carry out the calculation using variables and insert the values only at the end!

Result: ______ m.

Hint:
Take the width of the hedge into account!

Solution:
The line segments are denoted as shown in the figure below.



Applying the second intercept theorem results in „$\frac{\text{full}}{\text{at front}} = \frac{\text{long}}{\text{short}}$“ :

$$\frac{d_{KB}}{d_{KH}} = \frac{h_B - h_K}{h_H - h_K} \qquad \text{or} \qquad h_B = (h_H - h_K) \cdot \frac{d_{KB}}{d_{KH}} + h_K \, .$$

The values are $d_{KH} = 2.5\,\mathrm{m} + \frac{1\,\mathrm{m}}{2} = 3\,\mathrm{m}$ and $d_{KB} = 2.5\,\mathrm{m} + 1\,\mathrm{m} + 14.5\,\mathrm{m} = 18\,\mathrm{m}$. Hence, it follows

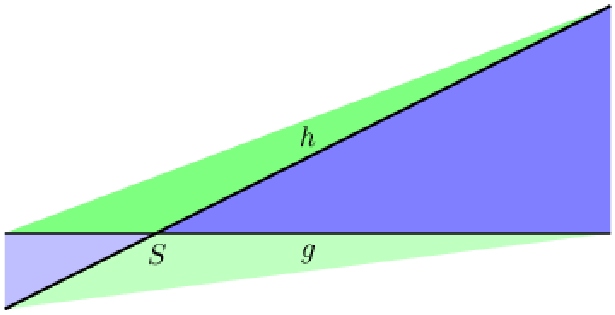
$$h_B = (2.4\,\mathrm{m} - 1.4\,\mathrm{m}) \cdot \frac{18\,\mathrm{m}}{3\,\mathrm{m}} + 1.4\,\mathrm{m} = 1\,\mathrm{m} \cdot 6 + 1.4\,\mathrm{m} = 7.4\,\mathrm{m} \, .$$

1.2 Angles and Angle Measurement

1.2.1 Introduction

Lines intersecting in a point S divide the plane in a characteristic way. To describe this observation the concept of an angle is introduced. The question of how to measure angles can be answered in different ways, which in the end are all based on a subdivision of circles.

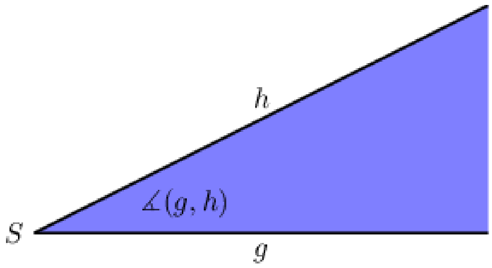
In this module, degree measure and radian measure are described.



Every coloured region represents one of the angles defined by the lines g and h.

1.2.2 Angles

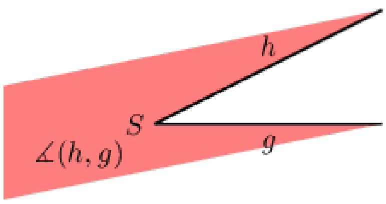
Two rays (half-lines) g and h in the plane emanating from the common point S enclose an **angle** $\measuredangle(g, h)$.



Angle enclosed by the rays g and h.

For the notation of the angle $\measuredangle(g, h)$, the order of g and h is relevant. $\measuredangle(g, h)$ denotes the angle shown in the figure above. It is defined by turning the half-line g counterclockwise to the half-line h.

In contrast, $\measuredangle(h, g)$ denotes the angle from h to g as illustrated by the figure below.



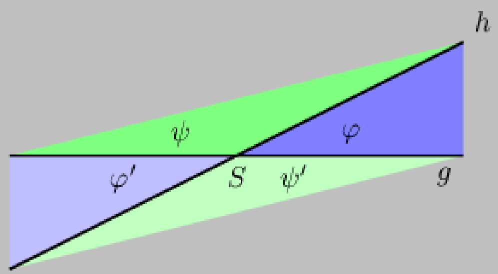
Angle enclosed by the rays h and g

The point S is called **vertex** of the angle, and the two half-lines enclosing the angle are called **arms** of the angle. If A is a point on the line g and B is a point on the line h, then the angle $\measuredangle(g,h)$ can also be denoted by $\measuredangle(ASB)$. In this way, angles between line segments $\overline{SA}$ and $\overline{SB}$ are described.

Angles are often denoted by lowercase Greek letters to distinguish them from variables, which are generally denoted by lowercase Latin letters (see Table **??** on page ?? in module **??**). Further angles can be found if angles formed by intersecting lines are taken into account.

Vertical Angles and Supplementary Angles 1.2.1

Let g and h be two lines intersecting in a point S.



- The angles φ and φ' are called **vertical angles**.
- The angles φ and ψ are called **supplementary angles** with respect to g.

The figure above contains further vertical and supplementary angles.

Exercise 1.2.1
Find all vertical and supplementary angles occurring in the figure above.

Solution:
In addition to φ and φ', ψ and ψ' are also vertical angles. Beside the angles φ and ψ, the angles φ' and ψ' are also supplementary angles of g. Moreover, ψ and φ' as well as ψ' and φ are supplementary angles.

Some special angles have their own dedicated name. For example, the angle bisector w is the half-line whose points have the same distance from the two given half-lines g and h. Then, it can be said that w bisects the angle between g and h.

Names of Special Angles 1.2.2

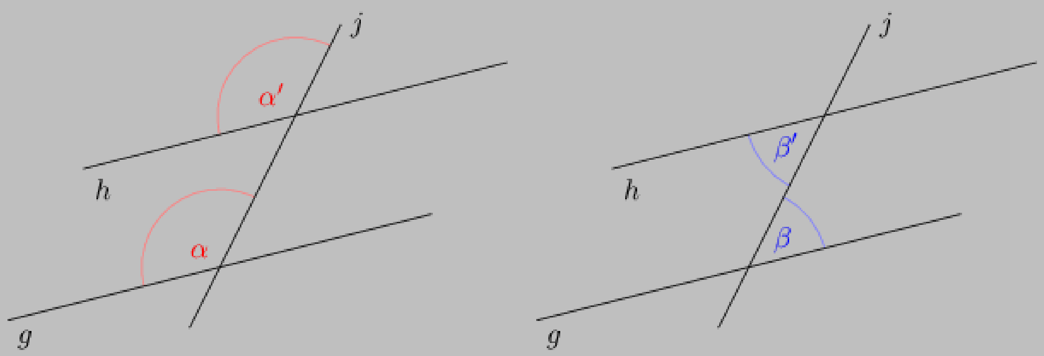
Let g and h be half-lines with the intersection point S.

- The angle covering the entire plane is called **complete angle**.
- If the rays g and h form a line, the angle between g and h is called straight angle.
- The angle between two half-lines bisecting a straight angle is called **right angle**. One also says that g and h **are perpendicular (or orthogonal) to each other**.

Next, three lines are considered. Two of the three lines are parallel, while the third line is not parallel to the others. It is called a transversal. These lines form eight cutting angles. Four of the eight angles are equal.

Angles at Parallel Lines 1.2.3

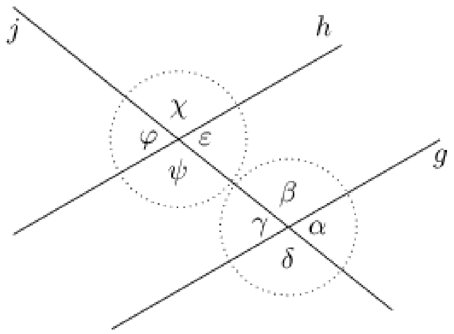
Let two parallel lines g and h be given cut by another transversal line j.



- Then the angle α' is called a **corresponding angle** of α and
- the angle β' is called an **alternate angle** of β.

Since the lines g and h are parallel, the angles α and α' are equal. Likewise, the angles β and β' are equal.

Exercise 1.2.2
The figure shows two parallel lines g and h cut by another line j. Explain which angles are equal and which angles are corresponding angles or alternate angles to each other, respectively.



Solution:

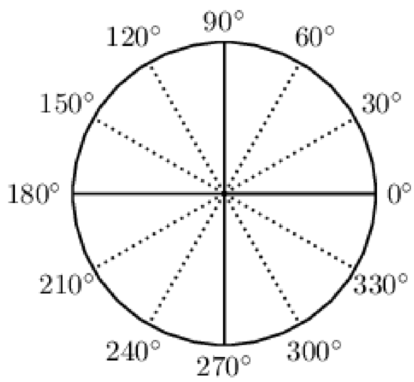
- The angles α, γ, ε, and φ are equal as well as the angles β, δ, χ, and ψ.
- The angles β and ψ as well as the angles γ and ε are alternate angles.
- The angles α and ε are corresponding angles, likewise the angles β and χ, δ and ψ as well as γ and φ.

1.2.3 Angle Measurement

We already explained the notation $\measuredangle(g, h)$ for the angle defined by turning g counterclockwise to h. This explanation provides an idea of how to measure angles, i.e. how to compare angles quantitatively.

Think of the face of an analogue watch with its twelve evenly spaced hour marks. Likewise, the circumference of a circle can be evenly subdivided. In this way, a certain scale for angles is obtained. Depending on the applied scaling, the magnitude of an angle can be specified in different units.

Degree Measure. A disk is subdivided into 360 equal segments. A rotation by one segment defines an angle of 1 degree. This is written as 1°. The figure below shows angles of multiples of 30°.



Radian Measure. Already in the ancient Babylonia, Egypt, and Greece people observed that the ratio of the circumference U of a circle to its diameter D is always the same, and hence circumference and diameter of a circle are proportional to each other. This ratio is called π.

The Number π 1.2.4

Let a circle with circumference U and diameter D be given. Then, the ratio of the circumference U of a circle to its diameter D is

$$\pi = \frac{U}{D} = \frac{U}{2r},$$

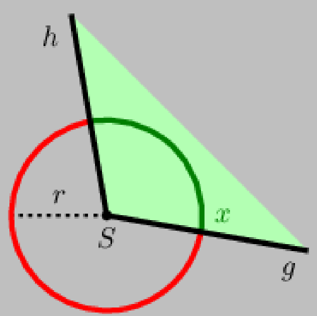
where $r = \frac{1}{2}D$ is the radius of the circle.

The number π is not a rational number. It cannot be expressed as a finite or periodic decimal fraction. From numerical calculations we know that the value of π is approximately $\pi \approx 3.141592653589793$.

If the circle has a radius of exactly 1, the circumference is 2π. Now, for the **radian measure** the circumference of a circle with radius 1 is subdivided. For the radian measure of an angle $\measuredangle(g,h)$ the length of an **arc** "cut" by this angle is used. As a result, the radian measure assigns to each angle a number between 0 and 2π. In science applications, the symbol rad (for radian) is used to express explicitly that the angle is measured in radian measure.

Radian Measure 1.2.5

Let g and h be two half-lines emanating from the common vertex S and enclosing the angle $\measuredangle(g,h)$. If a circle with radius $r = 1$ is drawn around S, the two half-lines cut the circle into two pieces. Now, the angle is described by the one arc x that transforms g into h by a counterclockwise rotation (indicated by a green line in the figure below). In other words, vertex S is always on the left if one moves on the arc x from g towards h.



The length of the arc x is the **radian measure** of the angle $\measuredangle(g,h)$.

By means of an angle measure such as radian measure or degree measure introduced previously, angles can simply be classified into different types and named accordingly.

For repetition and completeness, all names, including ones previously discussed, are listed below.

Names of Different Types of Angles 1.2.6

For angles whose radian measure is in a certain range, the following names are introduced:

- An angle with a radian measure greater than 0 and less than $\frac{\pi}{2}$ is called an **acute angle**.
- An angle with a radian measure of exactly $\frac{\pi}{2}$ is called a **right angle**.
- An angle with a radian measure greater than $\frac{\pi}{2}$ and less than π is called an **obtuse angle**.
- An angle with a radian measure greater than π and less than 2π is called a **reflex angle**.

Two half-lines are said to be **perpendicular to each other** if they form a right angle.

Two half-lines form a line if they enclose an angle of radian measure π.

From the radian measure of the angle $\measuredangle\,(g, h)$, the radian measure of the angle $\measuredangle\,(h, g)$ can also be determined. From definition 1.2.5 on the preceding page it is known that

$$\measuredangle\,(h, g) = 2\pi - \measuredangle\,(g, h)\ .$$

In the figure of definition 1.2.5 on the previous page the radian measure of the angle $\measuredangle(h, g)$ is the length of the red arc of the circle with radius $r = 1$.

The wording in the last sentences might seem awkward. The reason for that lies probably in the fact that we do distinguish precisely between an angle and its measure, e.g. the radian measure in this case.

When it comes to calculate a required value for line segments, often the same notation is used for a segment and its length. Mostly this is clear, and it helps to describe or to illustrate a problem in a simple way. Importantly, the unit of the angle has to be known or explicitly specified. Often, such an agreement – a so called convention – is also used if it is known from the context that a certain angle has to be calculated using a certain angle measure.

Convention 1.2.7

If a calculation does not depend on a certain measure or the unit of the angles is specified in advance, the term angle is used for short denoting both the angle itself and its value in the specified measure.

Hence, for example, we can write $\measuredangle(g, h) = 90°$ and speak about the right angle $\measuredangle(g, h)$ enclosed by the lines g and h at the same time. Accordingly, this applies for the radian measure.

The value of an angle can be converted from radian measure to degree measure (and vice versa) by considering the ratios of its value to the value of the complete angle in the respective angle measure.

The conversion from radian measure to degree measure is described below.

Relation between Radian Measure and Degree Measure 1.2.8

Let g and h be two half-lines enclosing the angle $\measuredangle(g,h)$. The radian measure of the angle is denoted by x and the degree measure of the angle is denoted by α.

Then, the ratio of x to 2π equals the ratio of α to $360°$, and thus:

$$\frac{x}{2\pi} = \frac{\alpha}{360°} \ .$$

Hence,

$$x = \frac{\pi}{180°} \cdot \alpha \quad \text{and} \quad \alpha = \frac{180°}{\pi} \cdot x \ .$$

Therefore, the values in radian measure are proportional to the ones in degree measure. Thus, the conversion using the respective proportionality factors $\frac{\pi}{180°}$ and $\frac{180°}{\pi}$ is very simple.

Exercise 1.2.3
The angle $\measuredangle\,(g,h)$ equals $60°$ in degree measure. Calculate the angle in radian measure:

$\measuredangle\,(g,h) = \boxed{}$.

Solution:
From

$$\frac{\measuredangle\,(g,h)}{2\pi} = \frac{60°}{360°}$$

we have

$$\measuredangle\,(g,h) = \frac{60°}{360°} \cdot 2\pi = \frac{1}{6} \cdot 2\pi = \frac{\pi}{3} \ .$$

Exercise 1.2.4
The angle β equals $\pi/4$ in radian measure. Find its value in degree measure.

$\beta = \boxed{}$ °.

Solution:
From

$$\frac{\pi/4}{2\pi} = \frac{\beta}{360°}$$

we obtain

$$\beta = \frac{\pi/4}{2\pi} \cdot 360° = \frac{1}{8} \cdot 360° = 45° \ .$$

Exercise 1.2.5

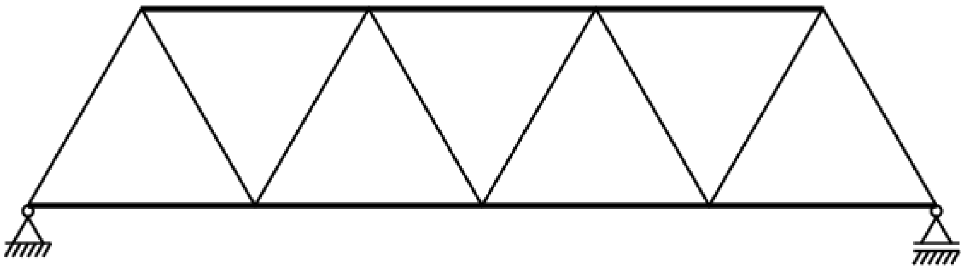
The values of the six angles $\alpha_1, \ldots, \alpha_6$ are specified either in degree measure or in radian measure. Convert their values to the other measure.

	α_1	α_2	α_3	α_4	α_5	α_6
Radian measure	π		$\frac{2\pi}{3}$		$\frac{11\pi}{12}$	
Degree measure		324		270		30

1.3 All about Triangles

1.3.1 Introduction

Technical structures such as trusses and some bridges use triangles as its constructing elements (see figure below).



Conversely, the question arises how an arbitrary surface can be subdivided into triangles. For many geometrical calculations this question is useful. Some examples are given in Section 1.4 on page 27.

Furthermore, the question of how to partition arbitrary surfaces into simple “basic elements” results in constructive answers in applications that are relevant far beyond simple geometric considerations. A first impression of such relevance gives us the integral calculus described in chapter **??** on page ?? together with its application to the calculation of surface areas. There, the first approximation to the integral is a partition of the area into rectangles (consisting of two triangles to stay in the context of triangles). For the three-dimensional computer aided modelling of surfaces, for example in the manufacturing of car bodies, partitions into triangles (triangulations) are the basis of many calculations and deceptively realistic looking virtual animations.

1.3.2 Triangles

Many statements on geometric figures and solids arise from the properties of triangles. A triangle is the “simplest closed figure” which can be determined by three non-collinear points (i.e. the points do not lie on a same straight line).

First, we will represent the important terms. Then, we will answer the questions under which conditions a triangle is uniquely defined and how individual angles and sides can be calculated. Here, the intercept theorems are an important tool since they can also be considered as statements on relations between different triangles.

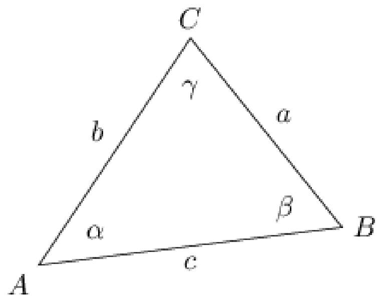
In Section 1.6 on page 47 we will then investigate functional relations between side lengths and angles enabling us to answer advanced questions relevant to applications.

Triangle 1.3.1

A **triangle** is constructed by joining three non-collinear points A, B, and C. The resulting triangle is denoted by ABC.

- The three points are called **vertices** of the triangle, and the three lines are called **sides** of the triangle.
- Each two sides of the triangle form two angles. The smaller angle is called **interior angle** (or simply angle for short) and the greater angle is called **exterior angle**.
- The sum of the three interior angles is always $180°$ or π.

The vertices and sides of a triangle are often denoted as follows: Vertices are denoted by uppercase Latin letters in mathematical positive direction (counterclockwise). The side opposite to a vertex is denoted by its lowercase Latin letter, and the interior angle of the vertex is denoted by the corresponding lowercase Greek letter.
Since exterior angles are far less important than interior angles, the **interior angles** are simply called **angles** of the triangle.



The sum of all (interior) angles is always $180°$ or π. Hence, at most one angle can be equal to or greater than $90°$ or $\frac{\pi}{2}$. Consequently, triangles are classified according to their greatest interior angle into three types:

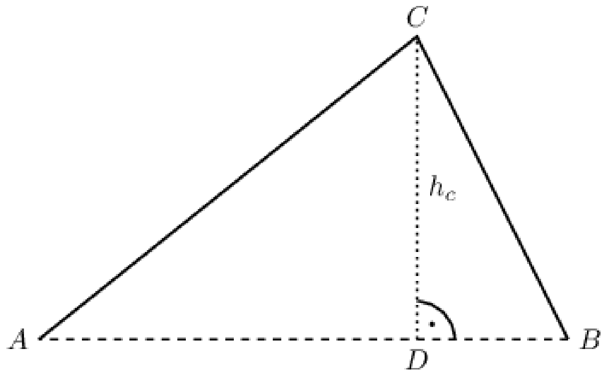
Names of Triangles 1.3.2

Triangles are named according to their angles as follows:

- A triangle that only has angles less than $\frac{\pi}{2}$ is called **acute**.
- A triangle that has a right angle is called **right-angled** triangle or simply right triangle.

  In a right triangle the two sides enclosing the right angle are called **catheti** or **legs**, and the side opposite to the right angle is called **hypotenuse**.
- A triangle that has an angle greater than $\frac{\pi}{2}$ is called **obtuse**.

As an example, let us consider the simple structure of a car jack with the shape of a triangle (see figure below): It consists of two rods connected by a joint. The two other endpoints of the rods can be pulled together. The greater the angle of a rod with respect to the street is, the higher is the joint above the ground.



Thus, in a triangle ABC the shortest line segment between vertex C and the line defined by the side c opposite to C is called **altitude (or height) of the triangle** h_c on the (base) side c. The second endpoint D of the line segment h_c is called **perpendicular foot**. The altitudes h_a and h_b are defined accordingly.

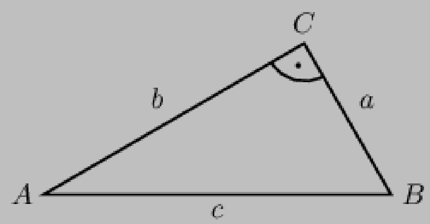
One can also say that altitudes are those line segments that are perpendicular to the line of a side and have the vertex opposite to the relevant side as an endpoint.

1.3.3 Pythagoras' Theorem

One statement relating the lengths of the sides in a right triangle is provided by **Pythagoras' theorem**. The theorem is given here in a frequently used phrasing.

Pythagoras' Theorem 1.3.3

Consider a right triangle with the right angle at vertex C.



Then, the sum of the areas of the squares on the legs a and b equals the area of the square on the hypotenuse c. This statement can be written as an equation (see also the triangle in the figure):

$$a^2 + b^2 = c^2 .$$

If the sides of the triangle are denoted in another way, the equation has to be adapted accordingly!

Example 1.3.4

Let a right triangle with legs of length $a = 6$ and $b = 8$ be given.

The length of the hypotenuse can be calculated by means of Pythagoras' theorem:

$$c = \sqrt{c^2} = \sqrt{a^2 + b^2} = \sqrt{36 + 64} = \sqrt{100} = 10\ .$$

Exercise 1.3.1
Let a right triangle ABC with the right angle at vertex C, hypotenuse $c = \frac{25}{3}$, and altitude (height) $h_c = 4$ be given. The line segment $\overline{DB}$ has the length $q = [\overline{DB}] = 3$. Here, D is the perpendicular foot of the altitude h_c. Calculate the length of the two legs a and b.

Solution:
We apply Pythagoras' theorem to the triangle DBC that has a right angle at the vertex D. Then, we have

$$a = \sqrt{h_c^2 + q^2} = \sqrt{4^2 + 3^2} = \sqrt{25} = 5\ .$$

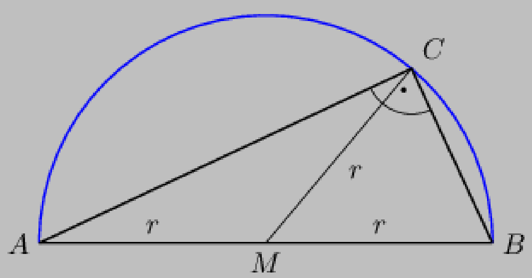
Now, we apply Pythagoras' theorem to the given right triangle ABC, whereof

$$b = \sqrt{c^2 - a^2} = \sqrt{\left(\frac{25}{3}\right)^2 - 5^2} = \sqrt{\frac{400}{9}} = \frac{20}{3}$$

follows.

Thales' theorem is another important theorem that makes a statement on right triangles.

Thales' Theorem 1.3.5



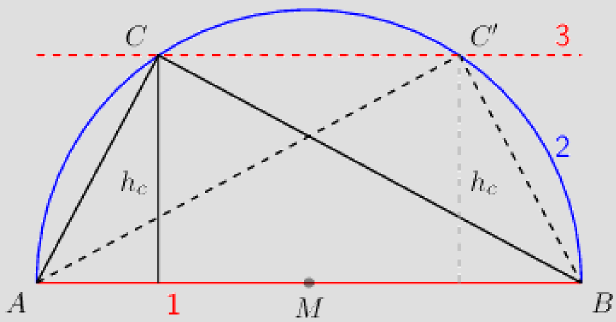
If the triangle ABC has a right angle at the vertex C, then vertex C lies on a circle with radius r whose diameter $2r$ is the hypotenuse $\overline{AB}$.

The converse statement is also true. Construct a half-circle above a line segment $\overline{AB}$. If the points A and B are joint to an arbitrary point C on the half-circle, then the resulting triangle ABC is always right-angled.

Example 1.3.6

Construct a right triangle with a given hypotenuse $c = 6\,\mathrm{cm}$ and altitude $h_c = 2.5\,\mathrm{cm}$.

1. First, draw the hypotenuse
$$c = \overline{AB}\,.$$
2. Let the middle of the hypotenuse be the centre of a circle with radius $r = c/2$.
3. Then draw a parallel to the hypotenuse at distance h_c. This parallel intersects Thales' circle in two points C and C'.



Together with the points A and B these intersections points each form a triangle possessing the required properties, i.e. two solutions exist. Two further solutions are obtained if the construction is repeated drawing a second parallel below the hypotenuse. The constructed triangles are different in position but concerning shape and size these triangles are "congruent" (see also Section 1.3.10 on page 22).

Exercise 1.3.2
Find the maximum altitude (height) h_c of a right triangle with hypotenuse c.

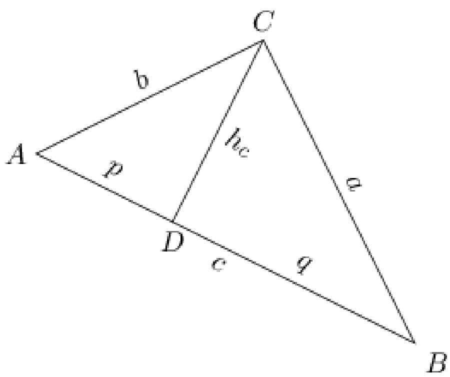
Solution:
The maximum altitude h_c is the radius of the Thales circle on the hypotenuse. Hence, $h_c \leq \frac{c}{2}$.

Advanced topics:

In a right triangle, besides Pythagoras' theorem further statements hold.

For this purpose, we will use the notation illustrated below:

Consider a right triangle with the right angle at the vertex C. The altitude h_c intersects the hypotenuse of the triangle ABC in the point D, called perpendicular foot. Furthermore, let $p = [\overline{AD}]$ and $q = [\overline{BD}]$.



Right Triangle Altitude Theorem 1.3.7

The area of the square on the altitude equals the area of the rectangle created by the two hypotenuse segments:

$$h^2 = p \cdot q \ .$$

Cathetus Theorem 1.3.8

The area of the square on a leg (cathetus) equals the area of the rectangle created by the hypotenuse and the hypotenuse segment adjacent to the leg:

$$a^2 = c \cdot q \ , \ \ b^2 = c \cdot p \ .$$

Example 1.3.9

Let a right triangle with the legs $a = 3$ and $b = 4$ be given.

The length of the hypotenuse can be calculated by means of Pythagoras' theorem:

$$c = \sqrt{a^2 + b^2} = \sqrt{9 + 16} = \sqrt{25} = 5 \ .$$

According to the cathetus theorem the hypotenuse segments p and q are:

$$q = \frac{a^2}{c} = \frac{9}{5} = 1.8 \quad \text{and} \quad p = \frac{b^2}{c} = \frac{16}{5} = 3.2 \ .$$

According to the altitude theorem the altitude h_c is:

$$h_c = \sqrt{p \cdot q} = \sqrt{\frac{9}{5} \cdot \frac{16}{5}} = \sqrt{\frac{144}{25}} = \frac{12}{5} = 2.4 \ .$$

Exercise 1.3.3

Find the length of the two legs of a given right triangle with hypotenuse $c = 10.5$, altitude $h_c = 5.04$, and hypotenuse segment $q = 3.78$.

Solution:

$$\text{Cathetus theorem:} \quad a = \sqrt{c \cdot q} = \sqrt{10.5 \cdot 3.78} = 6.3 \ ;$$

$$\text{Pythagoras' theorem:} \quad b = \sqrt{c^2 - a^2} = \sqrt{10.5^2 - 6.3^2} = 8.4 \ .$$

1.3.4 Congruence and Similar Triangles

Each triangle includes three sides and three angles. The exterior angles are already defined by the interior angles such that the “shape” of a triangle is determined by six characteristics. If two triangles coincide in all these characteristics, they are said to be **congruent**. For that, the position of the triangles is not relevant, i.e. congruent triangles can be transformed into each other by rotation, reflection, and translation.

If four of the six characteristics are known, the triangle is uniquely determined up to rotation or reflection, i.e. its position in the plane. Then, all triangles with these characteristics are congruent. In some cases, only three characteristics are sufficient to determine the triangle uniquely. These cases are described by the following **theorems for congruent triangles**.

Theorems for Congruent Triangles 1.3.10

Up to its position in the plane, a triangle is uniquely defined if one of the following situations is at hand:

- At least four of the six characteristics (three angles and three sides) are known.
- The lengths of all three sides are known.
  (This theorem is usually called “sss” for “side, side, side”.)
- Two angles and the length of the included side are known.
  (This theorem is usually called “asa” for “angle, side, angle”.)
- The lengths of two sides and the included angle are known.
  (This theorem is usually called “sas” for “side, angle, side”.)
- The lengths of two sides and a non-included angle are known such that only one side is a leg of the given angle and the second side is greater than the given leg.

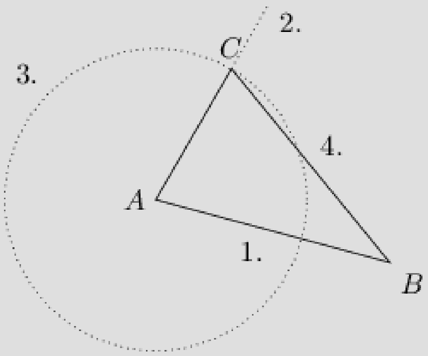
  (This theorem is called “Ssa”, where the uppercase “S” indicates that the side opposite to the given angle is the greater one.)

If for a triangle only two characteristics are known, or three characteristics are known that do not correspond to one of the cases described above, than a number of different triangles with these characteristics do exist that are not congruent.

The next example will illustrate how a triangle can be constructed applying the theorems for congruent triangles. Then, another example will be considered, where only three angles are known and hence none of the theorems described above applies.

Example 1.3.11

Let the sides b, c, and the angle α be given. According to the “sas” theorem the triangle is constructed as follows: 1. Draw a line, in this example side c. 2. Attach the angle α to the corresponding vertex (A). 3. Draw a circle around vertex A with a radius corresponding to the length of the second side (in this case, side b). 4. The intersection point of this circle with the second leg of the angle α is the third vertex (C) of the triangle. (The first leg of α is the side c.)

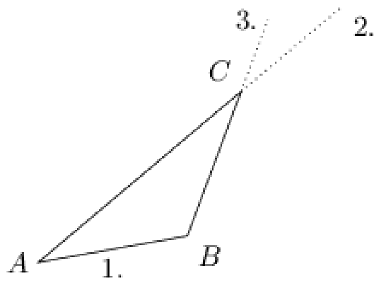


Exercise 1.3.4

Construct a triangle with side $c = 5$ and the two angles $\alpha = 30°$ and $\beta = 120°$ using the notation introduced above.

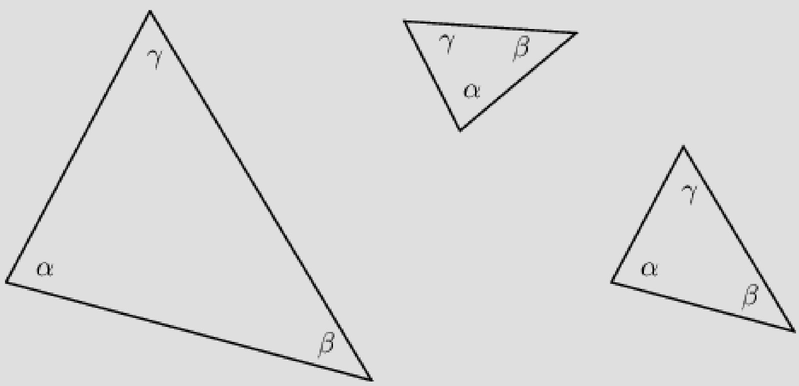
Solution:

1. Draw the given line segment c. 2. Attach to either side of the segment the corresponding angles α and β. 3. The intersection point of the two new legs is the third vertex C of the triangle.



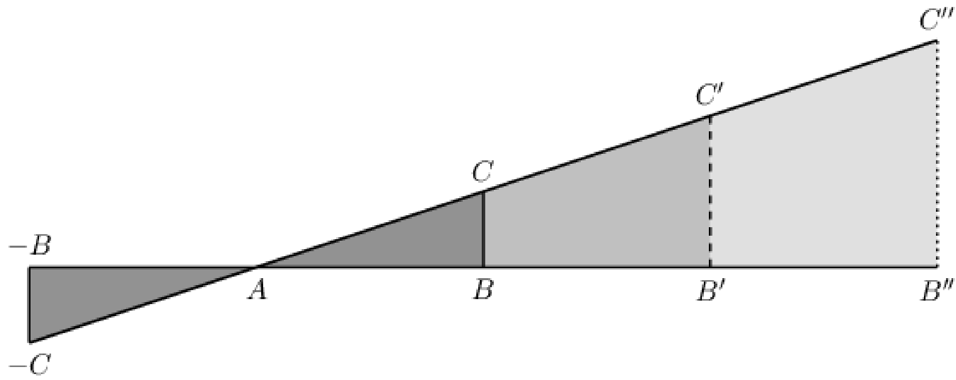
Example 1.3.12

Let three angles $\alpha = 77°$, $\beta = 44°$, and $\gamma = 59°$ be given summing up to 180°. This case does not correspond to one of the cases in the theorems for congruent triangles 1.3.10 on the preceding page. A few examples for triangles with the given angles are shown below.



Actually, an infinite number of triangles with the given angles do exist. They are not congruent to each other, i.e. they cannot be transformed into each other by rotation or reflection.

However, the triangles look similar in a way. Such **similar** triangles are also obtained if, for example, all the side ratios are known. This fact results from the intercept theorems as illustrated by the figure below.

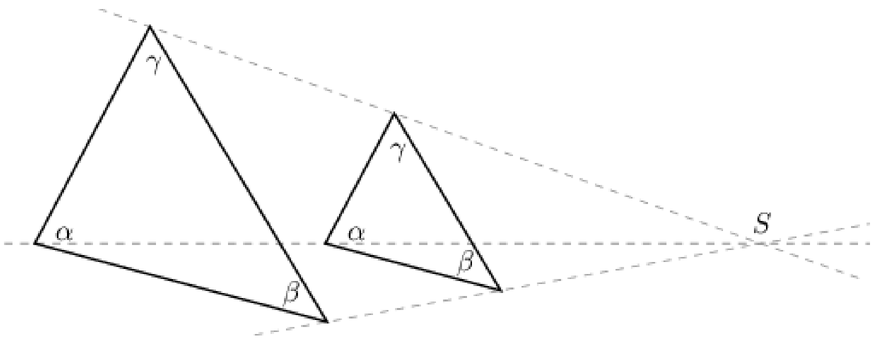


Similarity Theorems for Triangles 1.3.13

Two triangles are called **similar** to each other if they

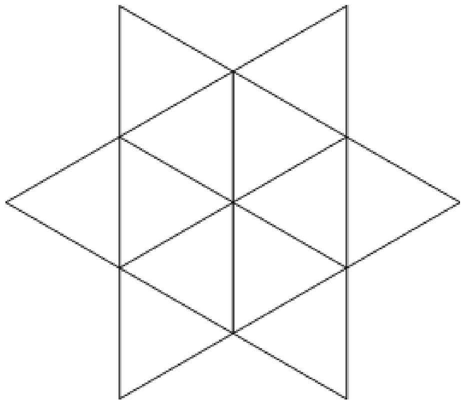
- have two (and because of the triangle postulate also three) congruent angles, or
- have three sides whose lengths have the same **ratio**, or
- have one congruent angle and two adjacent sides whose lengths have the same **ratio**, or
- have two sides whose lengths have the same **ratio** and the angles opposite to the greater side are congruent.

For the right and the left triangle in Example 1.3.12 on the previous page there is a special fact. The left triangle is transformed into the other by uniform scaling with the centre of enlargement S and the scaling factor k.

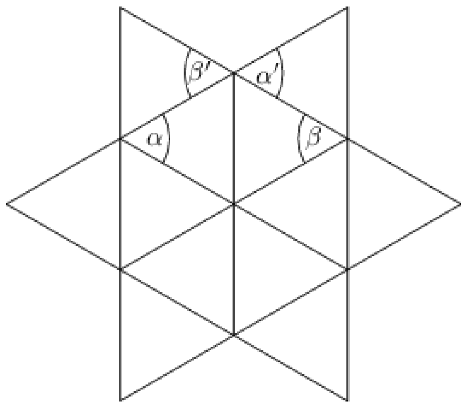


1.3.5 Exercises

Exercise 1.3.5
Find corresponding angles and alternate angles in the figure below.



Solution:

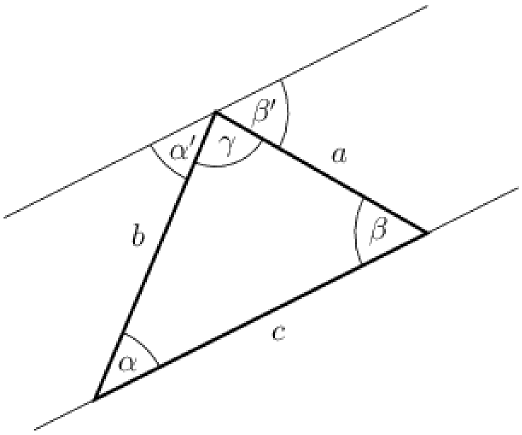


For example, the angles α and α' are corresponding angles. Likewise, angles β and β'.
For example, the angles α' and β are alternate angles. Likewise, angles α and β'.

Exercise 1.3.6
Prove that the sum of interior angles in a triangle is always π or $180°$ using the concept of alternate angles.

Hint:
Draw a parallel to one of the sides of the triangle passing through the third vertex and consider the angles at this vertex.

Solution:



Drawing a parallel to the side c passing through the vertex C one obtains an alternate angle α' to α and an alternate angle β' to β. The angles α', γ, and β' form a straight angle. Therefore,

$$\alpha' + \gamma + \beta' = \pi \ .$$

Furthermore, it is known that $\alpha' = \alpha$ and $\beta' = \beta$. Hence, $\alpha + \gamma + \beta = \pi$.

1.4 Polygons, Area and Circumference

1.4.1 Introduction

In nature, various figures in different shapes can be found. There, rounded shapes are particularly evident. When it comes to partition a surface completely, also some boundaries can be found that can be approximated as line segments. A prominent example are honeycomb structures created by insects. Technical applications are often based on figures bounded by straight line segments.

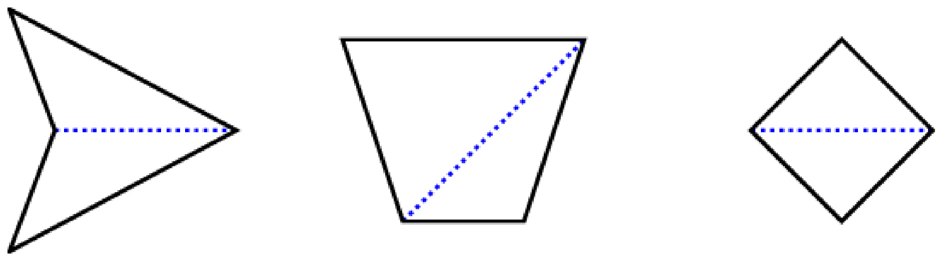
In this section we consider some special cases of polygons which can be used to describe surfaces bounded by straight line segments. To do this, we will first specify some characteristic features. Then, we will address the question how the area of a polygon can be calculated easily.

1.4.2 Quadrilaterals

In the previous section 1.3 on page 16 triangles were considered. They were defined by three non-collinear points. Connecting each two of the three points by a line segment always results in a single closed path in which every point connects exactly two line segments. The line segments in the path have only their endpoints in common. Furthermore, the line segments do not intersect.

For more than three points this is not always true. Even only four points can be connected in such a way that line segments intersect or more than one closed path exists.

In the figure below all given points are to be connected by a single closed path without any intersections.



Obviously, a quadrilateral can be divided into two triangles. Generally, one obtains two triangles if the vertex with the greatest angle is connected to the opposite vertex by a line segment. Such an additional line segment between two vertices of the quadrilateral which are not connected to each other is called a **diagonal** of the quadrilateral. From the fact that the sum of (interior) angles in a triangle equals π or 180° then results that the sum of (interior) angles in a quadrilateral is twice this sum, i.e. equals 2π or 360°.

Quadrilaterals 1.4.1

Consider **quadrilaterals** constructed by connecting the four given points by line segments such that a single, closed and non-intersecting path trough this four points results. There, each three of the four points connected by two line segments must be non-collinear.

As for triangles, the interior angles of quadrilaterals are simply denoted as angles if not otherwise specified in the context.

As the concept of triangles the concept of quadrilaterals is used in technical structures in many ways. Therefore, additional terms are commonly used to specify different types of quadrilaterals.

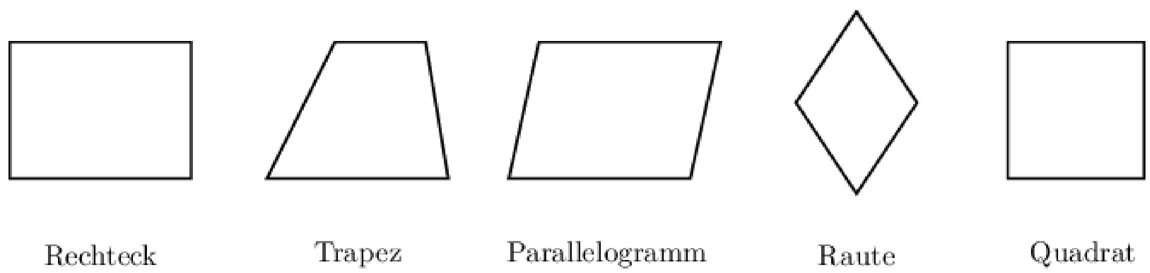
In the same way as triangles also quadrilaterals are classified by the lengths of their sides or by the magnitudes of angles. Compared to the classification of triangles typical differences occur. For example, quadrilaterals can have parallel sides. Furthermore, they can have more than one vertex with a right angle.

Special Types of Quadrilaterals 1.4.2

Quadrilaterals with the following properties have their own terms: A quadrilateral is called

- **trapezoid** if at least one pair of opposite sides is parallel;
- **parallelogram** if two pairs of opposite sides are parallel;
- **rhombus** or **equilateral quadrilateral** or **diamond** if all four sides are of equal length;
- **rectangle** if all four (interior) angles are right angles;
- **square** if it is a rectangle having four sides of equal length;
- **unit square** if it is a square with sides of length 1.

Thus, for the unit square also a measure has to be fixed.



Between the quadrilaterals introduced above a series of relations exist:

Relations between rectangles 1.4.3

Between different quadrilaterals the following relations exist:

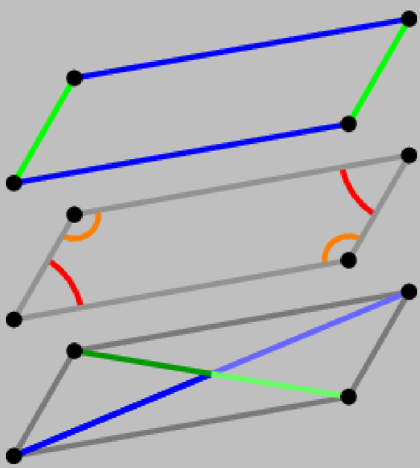
- Every square is a rectangle.
- Every square is a rhombus.
- Every rhombus is a parallelogram.
- Every rectangle is a parallelogram.
- Every parallelogram is a trapezoid.

These quadrilaterals can be characterised by means of the properties of their sides, angles, or diagonals in many ways.

Parallelogram 1.4.4

A quadrilateral is a parallelogram if and only if

- opposite sides are parallel;
- opposite sides are of equal length;
- opposite (interior) angles are equal;
- two adjacent (interior) angles sum up to π or $180°$, respectively;
- diagonals bisect each other.

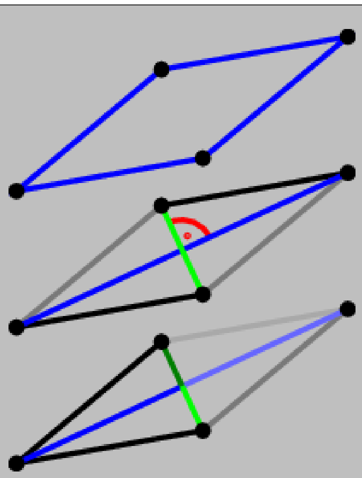


Rhombuses can be described as a special type of parallelograms.

Rhombus 1.4.5

A quadrilateral is a rhombus if and only if

- all sides are of equal length;
- it is a parallelogram in which the diagonals are perpendicular;
- at least two adjacent sides are of equal length and the diagonals bisect each other.

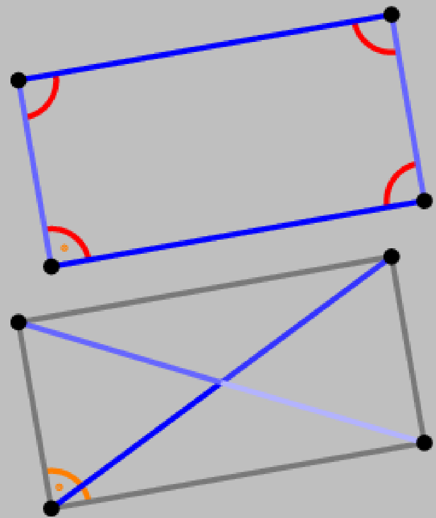


In the case of rectangles one often thinks of right angles since the term rectangle comes from the Latin word rectangulus, which is a combination of rectus (right) and angulus (angle). Apart from that, rectangles can simply be described by means of the properties of their diagonals.

Rectangle 1.4.6

A quadrilateral is a rectangle if and only if

- all (interior) angles are equal;
- it is a parallelogram containing at least one right angle;
- it is a parallelogram whose diagonals are of equal length;
- the diagonals are of equal length and bisect each other;
- the diagonals bisect each other and at least one (interior) angle is a right angle.

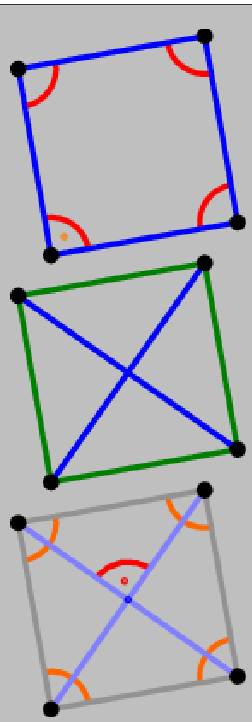


Squares are both special types of rectangles and special types of rhombuses.

Square 1.4.7

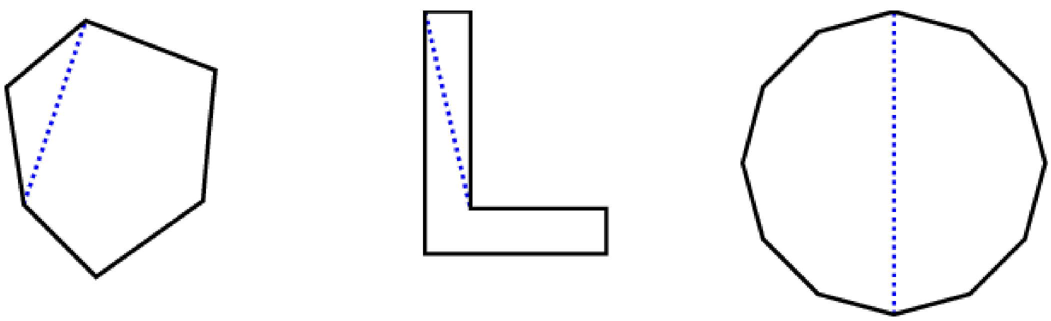
A quadrilateral is a square if and only if

- all sides are of equal length and
  - all (interior) angles are equal or
  - at least one (interior) angle is a right angle;
- the diagonals are of equal length and in addition all sides are of equal length;
- the diagonals are perpendicular and
  - bisect each other and are of equal length or
  - all (interior) angles are equal;
- it is a rhombus whose diagonals are of equal length;
- it is both a rhombus and a rectangle.



1.4.3 Polygons

For triangles, even one vertex or side contributes to the essential properties of the whole triangle, for example, one vertex with a right angle. For quadrilaterals, a vertex isn't such specifying anymore. Instead, there is a greater variety of shapes. If "many" points are connected to a closed figure by line segments, there are many possibilities to create various figures and even to approximate round shapes.



Here, such a detailed classification as for triangles or quadrilaterals is barely possible. The new possibilities such as the approximation of round shapes do also lead to new interesting questions. To this, one does not consider single polygons but construction principles for a series of many polygons. On the other hand, every polygon can be divided into triangles if required, as we have seen already for quadrilaterals. Thus, a property of a single vertex is often considered in terms of what this means for the polygon in the whole.

For classification, the question is convenient whether a certain condition is satisfied by **all** vertices or not, and what this means for the polygons. For example, polygons are classified according to the magnitudes of their angles, e.g. whether all angles of the vertices are less than π or 180°. If so, all diagonals pass through the inside of the polygon. Otherwise, at least one diagonal exists in the outside.

The figure above shows examples of polygons exhibiting different properties. In the polygon to the left all (interior) angles are less than π or 180°. In this case the polygon is said to be convex. In contrast, the polygon in the middle contains a vertex with an angle greater than π or 180°. In the polygon to the right all angles are equal, leading to a very evenly shaped polygon.

Polygons 1.4.8

Let n points in the plane be given, where n is a natural number with $n \geq 3$. Here, we consider **polygons** constructed by connecting points by line segments such that a closed, non-self-intersecting (simple) path is formed, and every point is adjacent to exactly two segments, where every three points connected by successive segments are to be non-collinear.

A polygon is also called **n-gon**.

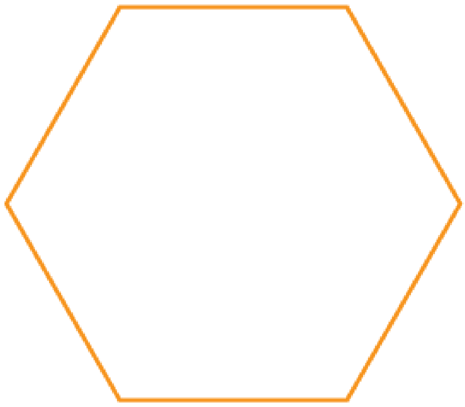
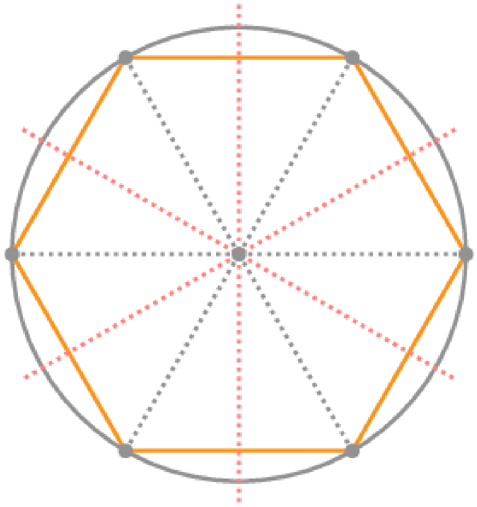
- The n points that are connected are called **vertices** of the polygon, and the n connecting line segments are called **sides** of the polygon.
- Every polygon can be divided into $(n-2)$ non-overlapping triangles. Hence, the sum of the interior angles of a polygon is $(n-2)\cdot\pi$ or $(n-2)\cdot 180°$.
- Line segments connecting two vertices not adjacent to the same side of the polygon are called **diagonals** of the polygon.

Further statements hold for polygons with sides of equal length and equal interior angles. For $n = 3$, these are equilateral triangles, end for quadrilaterals these are squares.

Regular Polygons 1.4.9

A polygon that is equilateral (all sides have the same length) and equiangular (all angles are equal in measure) is called **regular polygon** or **regelar n-gon**.

Honeycombs are – seen from above – approximately regular hexagons.

Regular polygons have different symmetry properties. All lines perpendicular to the sides, passing through the midpoint of the respective side, intersect in a point M. Reflecting a polygon across such a line maps it onto itself.

Furthermore, regular polygons have rotational symmetry, i.e. a n-gon maps onto itself if it is rotated around M by an angle of $\frac{2\pi}{n}$.

The vertices of a regular polygon have all the same distance from M and thus lie all on a circle around M.

1.4.4 Circumference

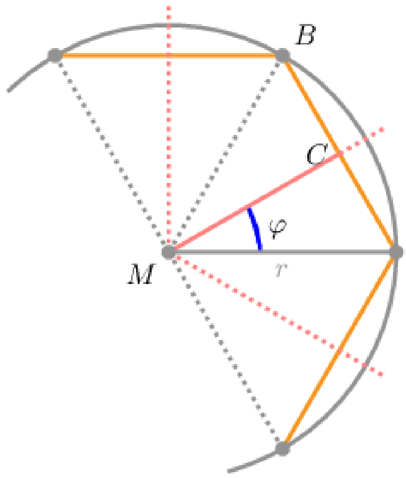
The circumference of a polygon is the sum of the lengths of all its line segments. If a polygon has further properties concerning the side lengths, more statements concerning the circumference can hold.

First, quadrilaterals are considered. If a and b are adjacent sides of a parallelogram, then its circumference is $U = a + b + a + b = 2 \cdot a \cdot b$.

For a rhombus and also for a square, all four sides have the same length a such that its circumference is $U = 4 \cdot a$.

Likewise, for every regular polygon, all sides have the same length. If n is the number of vertices and a is the length of a side, then the circumference U_n can simply be calculated by $U_n = n \cdot a$.

As an outlook to trigonometric functions described in Section 1.6 on page 47 the circumference of a regular polygon shall now be calculated in another way.



The vertices of a regular polygon all lie on a common circle with radius r. The angle φ between the line segments connecting the centre of the circle to the vertices A and B of a side is the n-th part of the complete angle: $\varphi = \frac{2\pi}{n}$. The centre of the circle and the midpoint C of the line segment $\overline{AB}$ form a right triangle MAC with the angle $\measuredangle(AMC) = \frac{1}{2} \cdot \varphi = \frac{\pi}{n}$. If the value of a is calculated by

$$\sin(\measuredangle(AMC)) = \frac{\frac{1}{2}a}{r}$$

and is inserted in $U = n \cdot a$, then we obtain the formula

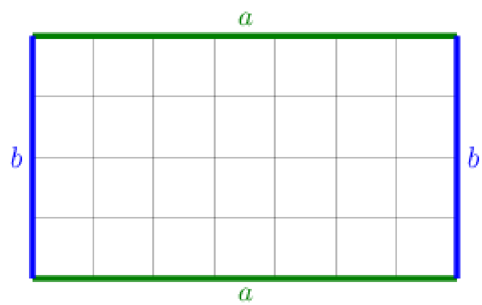
$$U_n = n \cdot a = 2 \cdot r \cdot n \cdot \sin\left(\frac{\pi}{n}\right)$$

for the circumference of a regular polygon. For example, $U_6 = 2 \cdot r \cdot 6 \cdot \frac{1}{2} = 6 \cdot r$. The larger n is, the closer the circumference is to the value $2 \cdot r \cdot \pi \approx 6.283 \cdot r$ describing the circumference of a circle with radius r. This can be shown by means of more advanced methods of calculus. Its basic ideas are introduced in Chapter **??** on page ??. The approach described here is based on the following idea: It is difficult to calculate the value of the circumference of a circle. Therefore, one looks for similar objects, in this case the regular polygons, having two properties: Their circumference can be calculated easily, and if the number of vertices is sufficiently large, then the circumference of the polygon differs from the circumference of a circle less than any given positive number (here, one thinks of "small" numbers). This approach can also be used to calculate the area of surfaces that are not bounded by line segments (see Chapter **??** on page ??). For this purpose, it will be illustrated in the following how to calculate the area of polygons, which is in this respect relatively easy. Further, this can be used as the starting point of an approximation, as the figure above showing a circle inscribed into a hexagon suggests.

1.4.5 Area

The area of a surface equals the number of unit squares required to cover this surface completely.

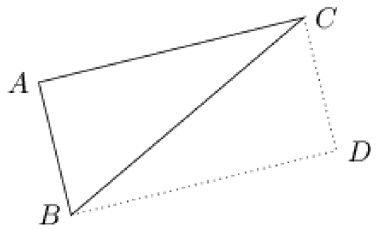
Let us first consider rectangles. If the sides of the rectangle are of lengths a and b, then the rectangle contains b rows with a unit squares, i.e. $b \cdot a$ unit squares.



Area of a Rectangle 1.4.10

The area F of a rectangle with sides of lengths a and b is

$$F = b \cdot a = a \cdot b \ .$$



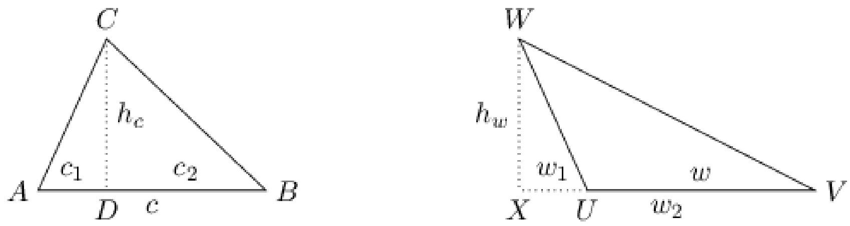
With this, the area of a right triangle can be calculated easily. Let ABC be a right triangle rotated by an angle of 180°. If the original and the rotated triangle are merged along the hypotenuse, one obtains a rectangle.

The area of the right triangle is then half the area of the rectangle, i.e. $F = \frac{1}{2} \cdot a \cdot b$.

And how is the area calculated if the triangle is not right-angled?

Every triangle can be divided into two right triangles by drawing a line from one vertex to the opposite side such that this line is perpendicular to the side. This line is called the **altitude** h_i of a triangle on a specific side i, where i is the index of the side a, b, or c.

Depending on the fact, whether the new line is interior or exterior to the triangle, the area of the triangle equals the sum or the difference of the areas of the two resulting right triangles:



Thus, on the left, we have (if F_Δ is the area of the triangle Δ)

$$F_{ABC} = F_{DBC} + F_{ADC} = \frac{1}{2} \cdot h_c \cdot c_2 + \frac{1}{2} \cdot h_c \cdot c_1 = \frac{1}{2} \cdot h_c \cdot (c_2 + c_1) = \frac{1}{2} \cdot h_c \cdot c \ .$$

On the right, we have

$$F_{UVW} = F_{XVW} - F_{XUW} = \frac{1}{2} \cdot h_w \cdot w_2 - \frac{1}{2} \cdot h_w \cdot w_1 = \frac{1}{2} \cdot h_w \cdot (w_2 - w_1) = \frac{1}{2} \cdot h_w \cdot w \ .$$

Thus, the area can always be calculated from the length of one side and the length of the altitude perpendicular to the corresponding side.

Area of a Triangle 1.4.11

The area F_{ABC} of a triangle equals half the product of the length of a side and the length of the corresponding altitude of the triangle:

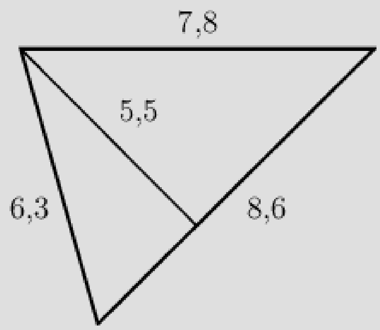
$$F_{ABC} = \frac{1}{2} \cdot a \cdot h_a = \frac{1}{2} \cdot b \cdot h_b = \frac{1}{2} \cdot c \cdot h_c \ .$$

Here, the **altitude of a triangle on a side** denotes the line segment from the vertex opposite the side to the line containing the side itself, perpendicular to this side.

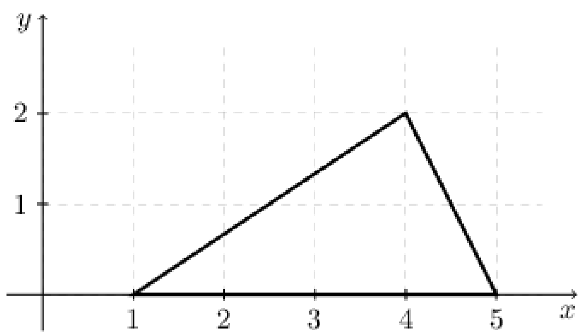
Example 1.4.12

For the triangle to the right, the altitude corresponding to the side of length 8.6 is given. The given values are rounded numerical values. Hence, the area F of the triangle is approximately

$$F = \frac{8.6 \cdot 5.5}{2} = 23.65 \ .$$



Exercise 1.4.1
Calculate the area of the triangle below.

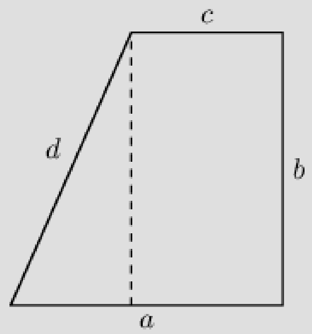


Solution:
For this triangle, one altitude can be read off easily, namely the altitude perpendicular the side on the x-axes. The length h of this altitude is $h = 2$, and the length of the corresponding side is $c = 5 - 1 = 4$. Hence, the area F of the triangle is $F = \frac{1}{2} \cdot c \cdot h = \frac{1}{2} \cdot 4 \cdot 2 = 4$.

Using the formula for the area of triangles also areas of polygons can be calculated. This is due to the fact that every polygon can be divided into triangles by adding diagonals to the polygon until all subareas are triangles. However, the considerations will remain restricted here to a few simple shapes. In the following example, the polygon can be divided into a triangle and a rectangle. As a result, the calculation will be particularly easy.

Example 1.4.13

Consider the polygon to the right, namely a trapezoid. In this example, the polygon can be divided into a right triangle with the legs $(a - c)$ and b and the hypotenuse d as well as a rectangle with sides of length b and c.

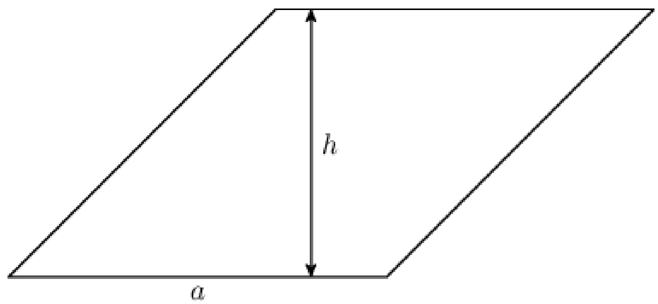


Then, the area of the polygon is:

$$F = F_{\text{triangle}} + F_{\text{rectangle}} = \frac{1}{2}(a-c)\cdot b + b\cdot c = \frac{1}{2}ab - \frac{1}{2}bc + bc = \frac{1}{2}(a+c)\cdot b\ .$$

Exercise 1.4.2

Calculate the area of the **parallelogram** to the right for $a = 4$ and $h = 5$.

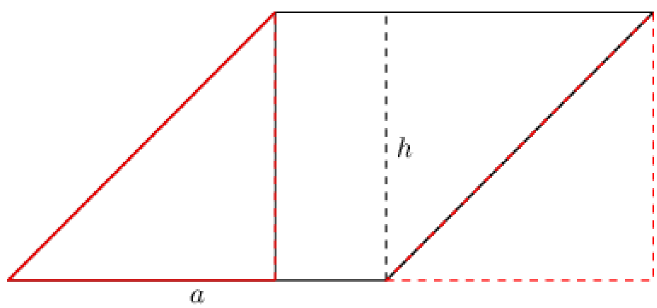


Hint:
Divide the parallelogram appropriately and look at the resulting triangles carefully!

Solution:

The parallelogram can be divided into the left red rectangle, a rectangle, and the right triangle. Shifting the left red triangle to the right one obtains a rectangle with sides of lengths a and h. Then, the area of the parallelogram is

$$F = a \cdot h = 4 \cdot 5 = 20\ .$$



Finally, we will calculate the area of a circle. Info Box 1.2.4 on page 12 introduced the number π describing the ratio of the circumference of a circle to its radius. The formula for the area of the circle also involves π.

Area of a Circle 1.4.14

The area of a circle with radius r is

$$F = \pi \cdot r^2 \ .$$

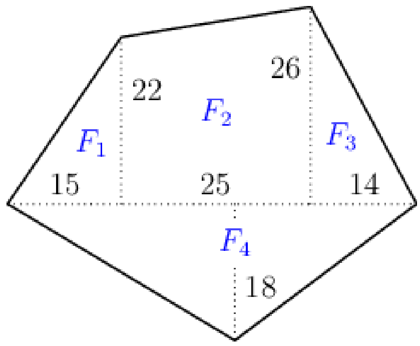
Example 1.4.15

Let the area of a circle with radius $r = 2$ be 12.566. This fact can be used to calculate an approximate value of the number π: We have $F = \pi \cdot r^2$, hence $\pi = \frac{F}{r^2}$. Inserting the given values results in the approximate value

$$\pi = \frac{F}{r^2} \approx \frac{12.566}{4} = 3.1415 \ .$$

1.4.6 Exercises

Exercise 1.4.3
Calculate the area of the polygon to the right.



Solution:
The values of the indicated subareas are calculated separately.

- F_1 is a triangle: $F_1 = \frac{15 \cdot 22}{2} = 165$.
- F_2 is a trapezoid that can be divided into two triangles with the altitude 25: $F_2 = \frac{22 \cdot 25}{2} + \frac{26 \cdot 25}{2} = 275 + 325 = 600$.
- F_3 is a triangle: $F_3 = \frac{14 \cdot 26}{2} = 182$.
- The surface F_4 is also a triangle: $F_4 = \frac{(15+25+14) \cdot 18}{2} = 486$.

Finally, we obtain the area of the entire polygon by summing up all these subareas: $F_1 + F_2 + F_3 + F_4 = 165 + 600 + 182 + 486 = 1433$.

1.5 Simple Geometric Solids

1.5.1 Introduction

The shapes of common objects such as a notepad, a mobile as well as of technical structures as tunnels can be described by simple basic solids, apart form the "rounded vertices". Why is that?

If a broom is moved straight across a plane floor covered with dust, a rectangular section of the clean floor becomes visible. Geometrically idealised, a rectangle is formed by shifting a line segment (the broom). If the broom is rotated, a circle can be created. In this way, from simple objects more complex objects are constructed that can nevertheless be described easily.

1.5.2 Simple Geometric Solids

Points are the simplest basic geometric objects. Translations of points result in line segments, and transformations such as translation or rotation of line segments result in simple geometric figures. For example, polygons and circles are obtained in a way described above.

If figures are shifted or rotated out of their plane, then new objects are created that are denoted as solids. In the following, some simple solids will be described whose shapes can be identified easily in many everyday objects and technical constructions.

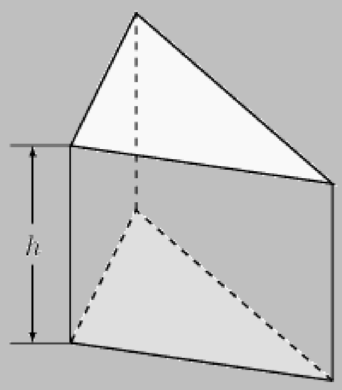
Example 1.5.1

Let us consider a rectangle and shift it perpendicular to the drawing plane. In this way, a rectangular cuboid (or informally a rectangular box) is constructed. Its surface consists of the given rectangle and a copy of that (two faces). Moreover, by the four sides of the given rectangle four further rectangles (faces) are formed.

Taking any polygon and shifting it perpendicular to the drawing plane, results in a solid that is called a prism. The term also denotes a transparent optical element of this shape used to refract light waves. Because of the fact that the refractive index depends on the wavelength (i.e. the colour of the light), the different wave lengths of seemingly white light are refracted differently. In this way, the different colours of white light become visible.

Prism 1.5.2

Let a polygon G be given. A **prism** is a solid resulting from a perpendicular translation of a polygon G by a line segment of length h. The two faces, i.e. the given polygon and the shifted copy of this polygon, are then called base faces. They are parallel to each other. All other faces together form the lateral surface M.



The figure above shows a prism with a triangle as its base. The other faces adjacent to the base face are rectangles.

The volume V of the prism is the product of the area of the polygon G and the height h: We have $V = G \cdot h$.

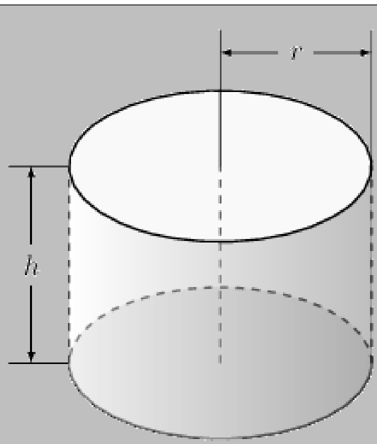
The area O of the surface is the sum of twice the area of the base face G and the area of the lateral surface M. If U is the circumference of the given polygon, we have $O = 2 \cdot G + M = 2 \cdot G + U \cdot h$.

In the introductory example a rectangular cuboid was described. Using the definition above, it can be considered as a special case of a prism, namely a prism with a rectangle as its base face. If all faces are squares, the prism is called a cube.

The construction principle can be varied in different ways. For example, the polygon can be replaced by a disk that is shifted. By a perpendicular translation of the disk a solid is created that is especially symmetric, namely a cylinder. A tunnel drilling machine creates – considered in a simplified manner – a cylindrical tube.

Cylinder 1.5.3

Let a disk G be given. A **cylinder** is a solid created by a perpendicular translation of a disk G by a line segment h. The two faces, i.e. the given disk and its copy, are then called the base faces of the cylinder. They are parallel to each other. The curved part of the surface between the two disks forms the lateral face M of the cylinder.



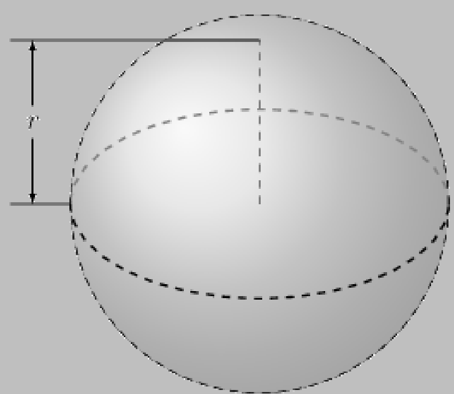
The volume V of the cylinder is the product of the area of the disk G with radius r and the height h of the cylinder: We have $V = G \cdot h = \pi \cdot r^2 \cdot h$.

The area O of the surface is the sum of twice the area of the disk G and the area of the lateral surface M. With the circumference $U = 2 \cdot \pi \cdot r$ of the disk we have $O = 2 \cdot G + M = 2 \cdot \pi \cdot r^2 + 2 \cdot \pi \cdot r \cdot h = 2 \cdot \pi \cdot r \cdot (r + h)$.

If the disk is not translated but rotated, where the axis of rotation passes trough the centre of the disk and one of its boundary points, then the resulting solid is a sphere.

Sphere 1.5.4

Let a disk with centre M and radius r be given. If the disk M is rotated around an axes trough M and a boundary point of the disk, the resulting solid is a sphere with radius r.



The volume V of the sphere is $V = \frac{4}{3} \cdot \pi \cdot r^3$.

The area O of the surface is given by $O = 4 \cdot \pi \cdot r^2$.

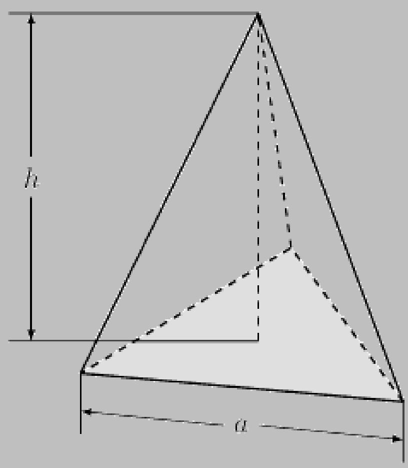
A sphere can also be described as a solid consisting of all points that have a distance from M less or equal to r (see also Chapter **??** on page ??).

In this approach, a prism is a solid consisting of all points that lie on a connecting line between the base face and its copy.

In the following, two variations of this approach will be considered. We start again with a polygon as base face. Moreover, instead of a copy of the base face now only a point is given.

Pyramid 1.5.5

Let a polygon G and a point S with distance $h > 0$ from G be given. A pyramid with the base G and the apex S is a solid consisting of all points lying on a line segment between S and a point of the base face G.



The figure above shows a pyramid with a triangular base face.

The volume V of the pyramid is proportional to the area of the base face G and the height h. We have $V = \frac{1}{3} \cdot G \cdot h$.

The area O of the surface is the sum of the area of the base face G and the area of the lateral surface M, where the area of the lateral surfaces is the sum of the areas of its triangular faces D_k $(1 \leq k \leq n)$. Thus, we have $O = G + M = G + D_1 + \ldots + D_n$.

In special situations one obtains simple formulas that can be used to calculate the volume and the surface area of the solid. One example is the pyramid shown above. There, the base face is an equilateral triangle. The following exercise illustrates how to derive a formula for the surface area of a special case of a pyramid from the properties of equilateral triangles.

Exercise 1.5.1
Calculate the surface area O of a pyramid whose faces are all equilateral triangles with sides of length a.

Answer: O =

Solution:
A pyramid whose faces are all equilateral triangles has in total four faces: a triangular base face and three further adjacent faces. Since all faces are equal, the surface area of this pyramid is given by $O = 4 \cdot F$, where F is the area if a single equilateral triangle. The height (altitude) ℓ of an equilateral triangle with sides of length a is, according to Pythagoras' theorem,

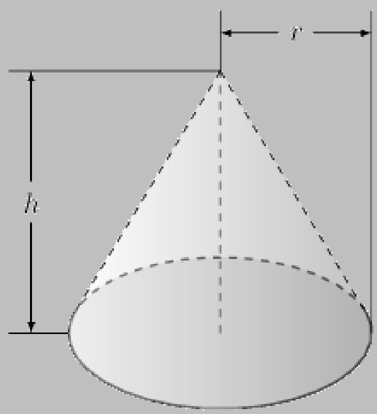
$$a^2 = \ell^2 + \left(\frac{a}{2}\right)^2 ,$$

equal to $\ell = \sqrt{a^2 - \frac{1}{4} \cdot a^2} = \frac{1}{2} \cdot a \cdot \sqrt{3}$. Hence, $F = \frac{1}{2} \cdot a \cdot \ell = \frac{1}{4} \cdot a^2 \cdot \sqrt{3}$, such that $O = 4 \cdot F = a^2 \cdot \sqrt{3}$.

The considerations above, that a prism and a cylinder share the same constructing principle for different base faces, can be applied to the new situation of a pyramid as well. One obtains another solid if instead of a polygon (as for the case of a pyramid) now a disk is used as base face.

Cone 1.5.6

Let a disk G with radius r and a point S with distance $h > 0$ from G be given. A cone with the base face G and the apex S is the solid consisting of all points lying on a line segment between S and a point of the base face G.



The volume V of the cone is proportional to the area of the disk G and its height h. We have $V = \frac{1}{3} \cdot G \cdot h = \frac{1}{3} \cdot \pi \cdot r^2 \cdot h$.

A cone whose apex is perpendicularly above the centre of the disk is called **right circular cone**.

The area of the surface of a right circular cone is the sum of the area of the disk G and the area of the lateral surface M. If ℓ is the distance of the apex from the boundary of the disk, then with the circumference of a circle $U = 2\pi r$ we have $O = G + M = \pi \cdot r^2 + \pi \cdot r \cdot \ell = \pi \cdot r \cdot (r + \ell)$.

In the figure above the height h of the cone is given. However, the formulas given above contain the distance between the apex and the boundary of the disk. How these quantities are related?

Exercise 1.5.2
Let's imagine that the right circular cone is divided into two equal pieces by a plane trough the apex perpendicular to the base face. Then, one face of the two pieces is a triangle whose sides are defined by the height h, the distance ℓ, and the radius r or the diameter of the circle, respectively. Here, r and h are given.

a. Describe ℓ as a function of h and r:

   $\ell = \boxed{}$

   Solution:
   The length of ℓ is the hypotenuse of the triangle whose legs are the height h of the circular cone and the radius r of the disk. Form Pythagoras' theorem we have $\ell = \sqrt{r^2 + h^2}$.

b. Describe the surface area O of the cone as a function of h and r:

   $O = \boxed{}$

   Solution:
   The result for $\ell = \sqrt{r^2 + h^2}$ obtained in the first part of the exercise is inserted in the formula for the surface area O given above. Thus, we have

$$O = \pi \cdot r \cdot (r + \ell) = \pi \cdot r \cdot \left(r + \sqrt{r^2 + h^2}\right) = \pi \cdot r^2 \cdot \left(1 + \sqrt{1 + \left(\frac{h}{r}\right)}\right).$$

   The last formula shows how the surface area of the cone compared to the area of the disk $G = \pi \cdot r^2$ depends on the ratio from the height h to the radius r.

1.5.3 Exercises

Exercise 1.5.3
Calculate the volume of a prism of height $h = 8\,\mathrm{cm}$ with a triangle as its base. Two sides of this triangle are of length $5\,\mathrm{cm}$, and one side is of length $6\,\mathrm{cm}$.

Answer: $\boxed{}$ cm^3

Exercise 1.5.4
The surface area of a cylinder of height $h = 6\,\mathrm{cm}$ is to be covered with a coloured sheet. The surface area shall be $O = 200\,\mathrm{cm}^2$. Calculate the diameter d of the disk and the volume of the cylinder. Use for π the approximate value 3.1415 and round off your result to millimetre.

Answers:

a. $d = \boxed{}$ cm

b. $V = \boxed{}$ cm^3

Exercise 1.5.5
Let a piece of wood with the shape of a rectangular cuboid with the volume V be given. The height of the cuboid is $h = 120\,\mathrm{cm}$, and the base face is a square with sides of length $s = 40\,\mathrm{cm}$. From the piece of wood, a cylindrical hole of height g with a diameter $d = 20\,\mathrm{cm}$ is drilled "centrically" (i.e. the intersection point of the diagonals of the quadratic base face is the centre of the base disk of the cylinder). Use for π the approximate value 3.1415 and round off your result to integers. Calculate

a. the volume V_Z of the drilled hole:

   $V_Z = \boxed{}$ cm^3
   Solution:
   The volume of the cylinder is

$$V_Z = \pi \cdot \left(\frac{d}{2}\right)^2 \cdot h = 3.1415 \cdot \left(\frac{20\,\mathrm{cm}}{2}\right)^2 \cdot 120\,\mathrm{cm} = 3.1415 \cdot 12000\,\mathrm{cm}^3 = 37698\,\mathrm{cm}^3$$

b. the percentage of the volume V_1 of the new piece of wood remaining after drilling of the volume V_0:

   Answer: $\boxed{}$ %
   Solution:
   The volume V of the piece of wood is

$$V = s^2 \cdot h = (40\,\mathrm{cm})^2 \cdot 120\,\mathrm{cm} = 1600 \cdot 120\,\mathrm{cm}^3 = 16 \cdot 12000\,\mathrm{cm}^3$$

   The percentage p_Z of the drilled cylinder is

$$p_Z = \frac{V_Z}{V} = \frac{\pi \cdot 12000\,\mathrm{cm}^3}{16 \cdot 12000\,\mathrm{cm}^3} \approx \frac{3.1415}{16} \approx 19\%$$

   and thus, $p = (100 - 19)\% = 81\%$ is the percentage of the new piece of wood compared to the original wooden rectangular cuboid.

1.6 Trigonometric Functions: Sine and et cetera

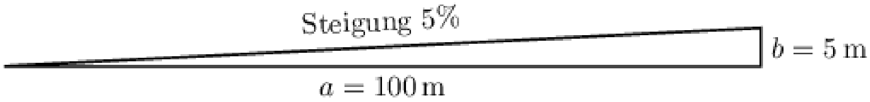
1.6.1 Introduction

At mountain roads signs are put up warning if the road goes steeply downhill. The percentage describes how steep the terrain slopes compared to a horizontal movement. Questions for the conditions of movements on an inclined plane in physics have been investigated by Galileo Galilee. The results are also relevant for technical constructions.

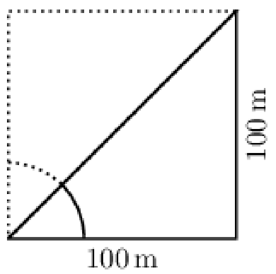
Trigonometric functions serve as a mathematical tool. They describe a geometric situation by means of a mathematical expression. This section describes how the relation between the percentage of the slope and the corresponding angle can be expressed. A first investigation of the properties of the trigonometric functions gives an idea of the various possible applications that are far beyond geometry and will be revisited repeatedly in the later sections.

1.6.2 Trigonometry in Triangles

If one drives downhill on a road with a slope of five percent, then the height falls five metres for every 100 metres. Here, the difference in height is considered in comparison to the horizontal line.



Accordingly, the slope is 100% if the difference in height between two positions with a horizontal distance of 100 m is 100 m. Geometrically, the connecting line segment between the two points is a diagonal of a square. Hence, the angle between the horizontal line and the diagonal, i.e. the road on which ones moves, has a degree measure of 45°.

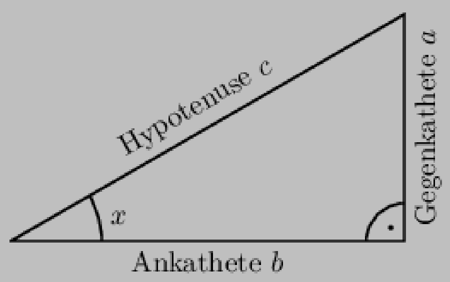


In other words: An angle of 45° corresponds to a slope of $\frac{100\,\mathrm{m}}{100\,\mathrm{m}} = 1$, i.e. the ratio of the vertical line segment to the vertical line segment is 1. According to the intercept theorem, this ratio does not depend on the lengths of the individual segments. It only depends on the position of the two rays with respect to each other, i.e. the measure of the angle they enclose. If this assignment of a ratio of the line segments to an angle is also known for other angles, many constructive problems can be solved. For example, for a given angle the height can be determined.

Even the question which ratio corresponds to an angle of 30° shows, however, that in general it is not that simple to determine the assignment of a ratio of line segments to an angle. Therefore, the time-consumingly determined values that we considered initially were listed in mathematical tables such that they could be looked up later again easily. Meanwhile, these values are available practically everywhere provided by calculators and computers. The most common assignments of an angle to a ratio of line segments are presented in the following. They are called circular functions or trigonometric functions, the branch of mathematics dealing with their properties is called **trigonometry**.

Trigonometric Functions in the Right Triangle 1.6.1

Here, the most common **circular functions** are described as assignments of ratios of the sides in a right triangle to an angle. The circular functions are also called **trigonometric functions**. Here, x denotes an angle in a right triangle that is not a right angle. The **opposite (side)** is the side opposite to the angle x, and the other leg is called **adjacent (side)**.



- The assignment of the ratio of the opposite side a to the adjacent side b to an angle is called tangent function:
$$\tan(x) := \frac{\text{opposite side}}{\text{adjacent side}} = \frac{a}{b}$$
- The assignment of the ratio of the adjacent side b to the hypotenuse c to an angle is called cosine function:
$$\cos(x) := \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{b}{c}$$
- The assignment of the ratio of the opposite side a to the hypotenuse c to an angle is called sine function:
$$\sin(x) := \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{a}{c}$$

Accordingly, the tangent function describes the assignment of the ratio of height to width to the angle of inclination, i.e. the slope. In Chapter ?? on page ?? this is also relevant in the context to the geometrical interpretation of the derivative.

According to the definition, the tangent function of the angle α is

$$\tan(\alpha) = \frac{a}{b} = \frac{a}{b} \cdot \frac{c}{c} = \frac{a}{c} \cdot \frac{c}{b} = \frac{\sin(\alpha)}{\cos(\alpha)} .$$

Thus, it suffices to know the values of sine and cosine to be able to calculate the tangent function.

Example 1.6.2

Let a triangle with a right angle $\gamma = \frac{\pi}{2} = 90°$ be given. The side c is of length 5 cm, and the side a is of length 2.5 cm. Calculate the sine, cosine and tangent function of the angle α.

The sine can be calculated immediately from the given values:

$$\sin(\alpha) = \frac{a}{c} = \frac{2.5\,\text{cm}}{5\,\text{cm}} = 0.5\ .$$

To calculate the cosine the length of the side b is required obtained by means of Pythagoras' theorem:

$$b^2 = c^2 - a^2$$

Hence,

$$\cos(\alpha) = \frac{b}{c} = \frac{\sqrt{c^2 - a^2}}{c} = \frac{\sqrt{(5\,\text{cm})^2 - (2.5\,\text{cm})^2}}{5\,\text{cm}} = 0.866\ .$$

Thus, the tangent of the angle α is

$$\tan(\alpha) = \frac{\sin(\alpha)}{\cos(\alpha)} = \frac{0.5}{0.866} = 0.5773\ .$$

Exercise 1.6.1
Determine some approximate values of the trigonometric functions sine, cosine and tangent graphically. Let a right triangle with the hypotenuse $c = 5$ be given. Use Thales' circle to draw right triangles for the angles

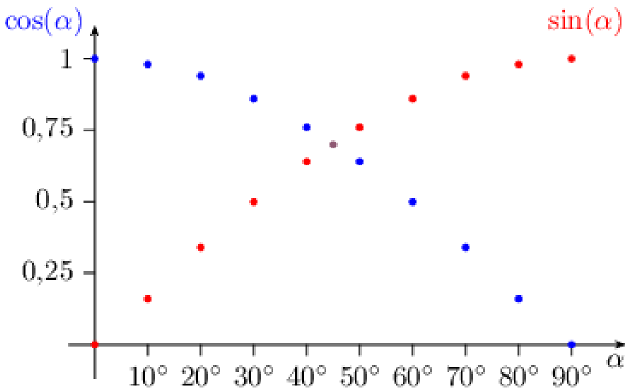
$$\alpha \in \{10°; 20°; 30°; 40°; 45°; 50°; 60°; 70°; 80°\}\ .$$

Use a drawing scale of 1 unit length $\hat{=}$ 2 cm, and fill in the measured values for the sides a and b in a table. From the measured values calculate to each angle the sine, cosine, and tangent and decide subsequently for which functions also values for $\alpha = 0°$ and $\alpha = 90°$ do exist. After that, plot the calculated values of sine and cosine against the angle α.

Solution:
In the process of measurement errors do always occur! Therefore, the values in your table will be slightly different from the ones given in the table below. The table could look as follows:

α	a	b	$\sin(\alpha)$	$\cos(\alpha)$	$\tan(\alpha)$
0	0.0	5.0	0.0	1.0	0.0
10°	0.8	4.9	0.160	0.98	0.1633
20°	1.7	4.7	0.34	0.94	0.3617
30°	2.5	4.3	0.5	0.86	0.5814
40°	3.2	3.8	0.64	0.76	0.8421
45°	3.5	3.5	0.7	0.7	1.0
50°	3.8	3.27	0.76	0.64	1.1875
60°	4.3	2.5	0.86	0.5	1.7200
70°	4.7	1.7	0.94	0.34	2.7647
80°	4.9	0.8	0.98	0.160	6.1250
90°	5.0	0.0	1.0	0.0	–

Then, the corresponding diagram looks as follows:



If we once again look closer at the results obtained in the last exercise, we can find different ways how to interpret them, and then identify some relations.

- With increasing angle α the opposite side a increases and the adjacent side b decreases.

  Likewise, $\sin(\alpha) \sim a$ and $\cos(\alpha) \sim b$.

- With increasing angle α the opposite side a increases to the same extent as the adjacent side b decreases with the angle α decreasing from 90°. In the Thales circle, the two triangles with the opposite values of a and b are two solutions for the construction of a right triangle with a given hypotenuse and a given altitude (see also Example 1.3.6 on page 20).

- In the right triangle the adjacent side of the angle $\beta = 90° - \alpha$ is the same side as the opposite side of the angle α (and vice versa). Thus,

$$\sin(\alpha) = \cos(90° - \alpha) = \cos\left(\frac{\pi}{2} - \alpha\right)$$

  and

$$\cos(\alpha) = \sin(90° - \alpha) = \sin\left(\frac{\pi}{2} - \alpha\right) .$$

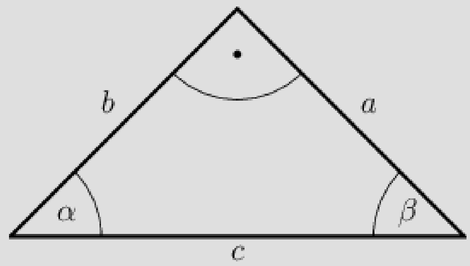
- For $\alpha = 45°$ opposite side and adjacent side are equal, and thus sine and cosine are equal as well. Conversely, this observation was used at the beginning of this section for the determination of the slope.
- The tangent function, i.e. the ratio of a to b, increases with increasing angle α from zero to "infinity".

In the following example we will continue our considerations from the beginning of this section leading to a triangle with an angle of 45° to calculate the value of the corresponding sine value exactly.

Example 1.6.3

Calculate the sine of the angle $\alpha = 45°$ now exactly, i.e. unlike as in Exercise 1.6.1 on page 49, where the sine was calculated from measured (and hence error-prone) values.

If in a right triangle with $\gamma = 90°$ the angle α is equal to 45°, then, because of the formula for the sum of interior angles in a right triangle, $\alpha + \beta + \gamma = \pi = 180°$, the angle β also needs to be equal to $45° = \pi/4$, and the two legs a and b are of equal length. A triangle with two sides of equal length is called **isosceles**.



We have:

$$\sin(\alpha) = \sin(45°) = \frac{a}{c}\,.$$

Moreover:

$$a^2 + b^2 = 2a^2 = c^2 \quad \Rightarrow \quad c = \sqrt{2} \cdot a$$

$$\Rightarrow \quad \sin(45°) = \sin(\pi/4) = \frac{a}{\sqrt{2} \cdot a} = \frac{1}{2} \cdot \sqrt{2}\,.$$

In Exercise 1.6.1 on page 49 the value of the sine of 45° was approximated by a value of 0.7 which is quite close to the actual value of $\frac{1}{2} \cdot \sqrt{2}$.

In the next example we will calculate the sine of the angle $\alpha = 60°$. For this purpose, we will first do not consider a right triangle but a equilateral triangle. By a clever decomposition of the triangle and by using another "auxiliary quantity" we will obtain the required result.

Example 1.6.4

Consider a **equilateral** triangle to calculate $\sin(60°)$. As the term implies the sides of this triangle are all of equal length, and the angles are also all of the same magnitude, namely $\alpha = \beta = \gamma = \frac{180°}{3} = 60° = \frac{\pi}{3}$. According to the theorem for congruent triangles "sss", the triangle is defined uniquely by the specification of a side a. This triangle is constructed by drawing the side a and then drawing a circle with radius r around both endpoints of the side. Now, the intersection point of the two circles is the third vertex.

This triangle is not right-angled. If an altitude h is drawn on one of the sides a, the triangle can be divided into two congruent right triangles.
We have:

$$\sin(\alpha) = \sin(60°) = \frac{h}{a} .$$

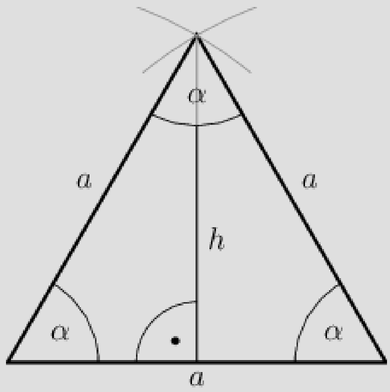
According to Pythagoras' theorem we have

$$\left(\frac{a}{2}\right)^2 + h^2 = a^2 .$$

Therefore,

$$h^2 = \frac{3}{4}a^2 \quad \text{and hence} \quad h = \frac{1}{2}\sqrt{3} \cdot a .$$

As a result we obtain the required value



$$\sin(60°) = \sin\left(\frac{\pi}{3}\right) = \frac{h}{a} = \frac{1}{2} \cdot \sqrt{3} .$$

From this triangle also the sine of another angle can be calculated: The altitude h bisects the above angle such that in the two congruent smaller triangles the above angle is $30° = \frac{\pi}{6}$. Now we have

$$\sin(30°) = \sin\left(\frac{\pi}{6}\right) = \frac{a/2}{a} = \frac{1}{2} .$$

Exercise 1.6.2
Calculate the exact value of the cosine of the angles $\alpha_1 = 30°$, $\alpha_2 = 45°$, and $\alpha_3 = 60°$. To do this, use the results obtained in the example above and in Exercise 1.6.1 on page 49.

Solution:
From Exercise 1.6.1 on page 49 it is known that $\cos(\alpha) = \sin(90° - \alpha)$. With the results obtained in the example above it follows

$$\begin{aligned}
\cos(30°) &= \sin(90° - 30°) = \sin(60°) = \frac{1}{2} \cdot \sqrt{3} , \\
\cos(45°) &= \sin(90° - 45°) = \sin(45°) = \frac{1}{2} \cdot \sqrt{2} , \\
\cos(60°) &= \sin(90° - 60°) = \sin(30°) = \frac{1}{2} .
\end{aligned}$$

The following small table lists the values for frequently used angles: In the first row denoted by x the angle is given in degree measure, and in the last row denoted by α the angle is given in radian measure.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
sin	$0 = \frac{1}{2} \cdot \sqrt{0}$	$\frac{1}{2} = \frac{1}{2} \cdot \sqrt{1}$	$\frac{1}{2} \cdot \sqrt{2}$	$\frac{1}{2} \cdot \sqrt{3}$	$\frac{1}{2} \cdot \sqrt{4} = 1$
cos	$1 = \frac{1}{2} \cdot \sqrt{4}$	$\frac{1}{2} \cdot \sqrt{3}$	$\frac{1}{2} \cdot \sqrt{2}$	$\frac{1}{2} \cdot \sqrt{1} = \frac{1}{2}$	$\frac{1}{2} \cdot \sqrt{0} = 0$
tan	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	–
α	$0°$	$30°$	$45°$	$60°$	$90°$

These values you should know by heart. The values of the trigonometric functions for other angles are listed in tables or saved in the calculator.

Hence, a height can be calculated very easily from an angle and a distance. Namely, if s is the distance of a building with a flat roof, which is observed at an angle of x, then from $\tan(x) = \frac{h}{s}$ we have $h = s \cdot \tan(x)$. Likewise, sine and cosine can be used to calculate lengths. This relation between angles and lengths is often used.

For example, in this way an area can be calculated even if the required length us not given directly. In the following example, the altitude of a triangle is to be calculated. Since h emanating from a vertex C is perpendicular to the line of the opposite side $c = \overline{AB}$, the vertices of h and A or B, respectively, form a right triangle. If an angle and the adjacent side is given, then the altitude can be calculated from $\sin(\alpha) = \frac{h}{b}$ or from $\sin(\beta) = \frac{h}{a}$, where the standard notation was used.

Exercise 1.6.3
Calculate the area F of a triangle with the sides $c = 7$, $b = 3$, and the angle $\alpha = 30°$ between the two sides c and b.

Result: F =

Solution:
The area F can be calculated from $F = \frac{1}{2} \cdot c \cdot h_c$, where we still need to determine h_c. From $\sin(\alpha) = \frac{h_c}{b}$ we have

$$h_c = b \cdot \sin(\alpha) = 3 \cdot \sin(30°) = 3 \cdot \frac{1}{2} \,.$$

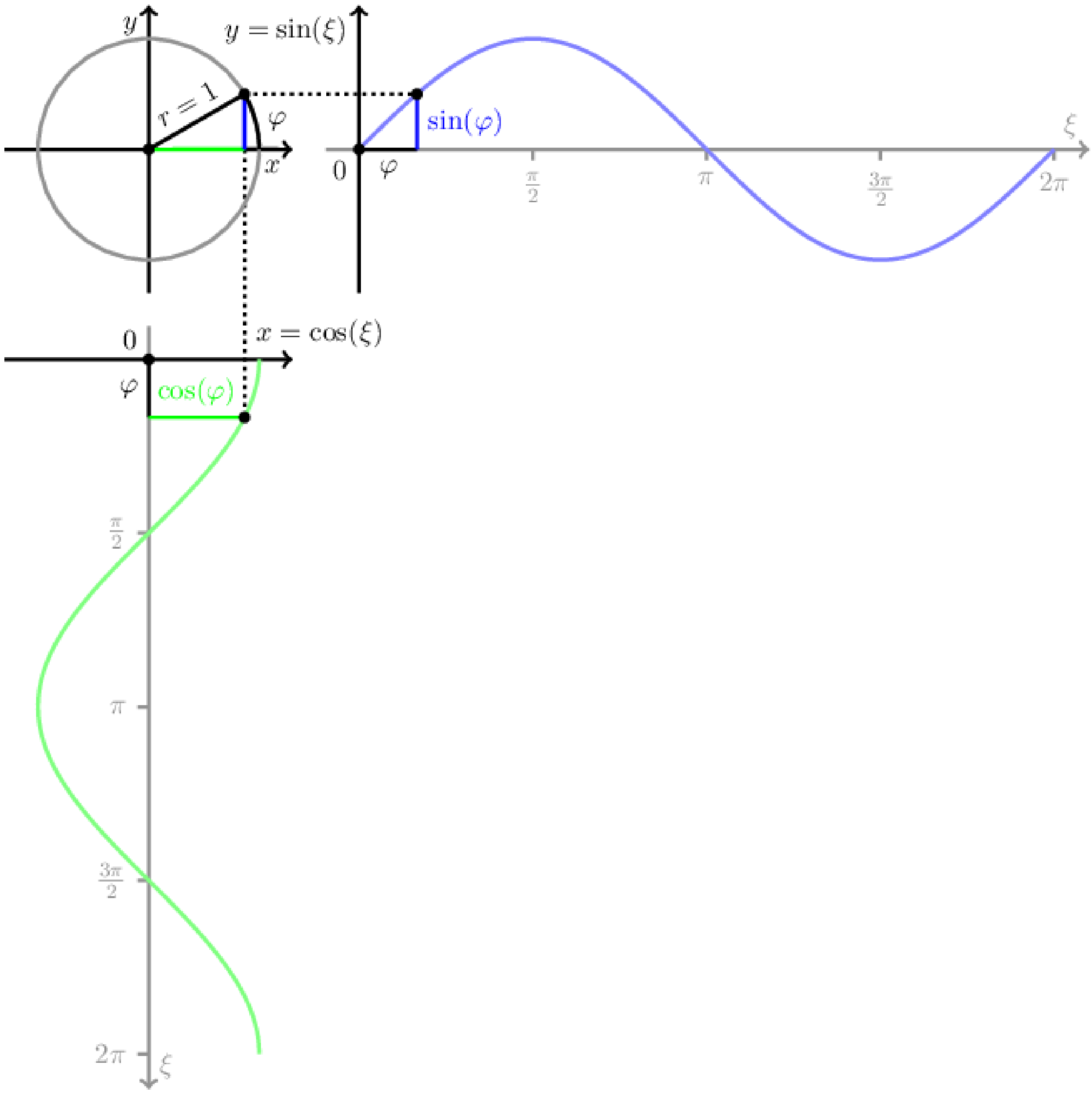
Hence,

$$F = \frac{1}{2} \cdot c \cdot b \cdot \sin(\alpha) = \frac{1}{2} \cdot 7 \cdot 3 \cdot \frac{1}{2} = \frac{21}{4} \,.$$

1.6.3 Trigonometry in the Unit Circle

In the previous section the trigonometric functions were introduced by means of a right triangle. Hence, the properties described above are valid for an angle ranging from 0° to 90° or from 0 to $\frac{\pi}{2}$, respectively.

To extend the acquired insights to angles greater than $\pi/2$, it is particularly useful to investigate the so called unit circle.



The unit circle is a circle with a radius of 1. Its centre is positioned at the origin in the Cartesian coordinate system. Consider a line segment of length 1 emanating from the centre. From its horizontal initial position on the positive x-axis, this segment is now rotated counterclockwise, i.e. in mathematical positive direction, around its centre. In this process, its rotating end point is sweeping the unit circle enclosing the angle φ with the positive x-axis. During rotation, the angle φ increases from 0 to 2π or $360°$, respectively. Thus, to any angle φ corresponds a point with the coordinates x_φ and y_φ on the unit circle.

For φ from 0 to $\frac{\pi}{2}$, the line segment, the corresponding segment on the x-axis, and the the corresponding segment on the y-axis can be regarded as a right triangle. The hypotenuse is the line segment of length 1, the x-intercept is the adjacent side, and the y-intercept is the opposite side. This matches the situation described in the previous section.

Hence, the sine of the angle φ is

$$\sin(\varphi) = \frac{y_\varphi}{1} = y_\varphi$$

and the cosine is

$$\cos(\varphi) = \frac{x_\varphi}{1} = x_\varphi .$$

Based on the description above, these definitions now are also valid for angles $\varphi > \pi/2$. Here, the values of x_φ and y_φ can be negative as well, hence also sine and cosine can be negative. If the y-values are plotted against the angle φ, one obtains for the sine function the blue curve. Plotting y-values against the angle φ one obtains for the cosine function the green curve. If the line segment is rotated in the opposite direction, values for negative angles can be defined accordingly.

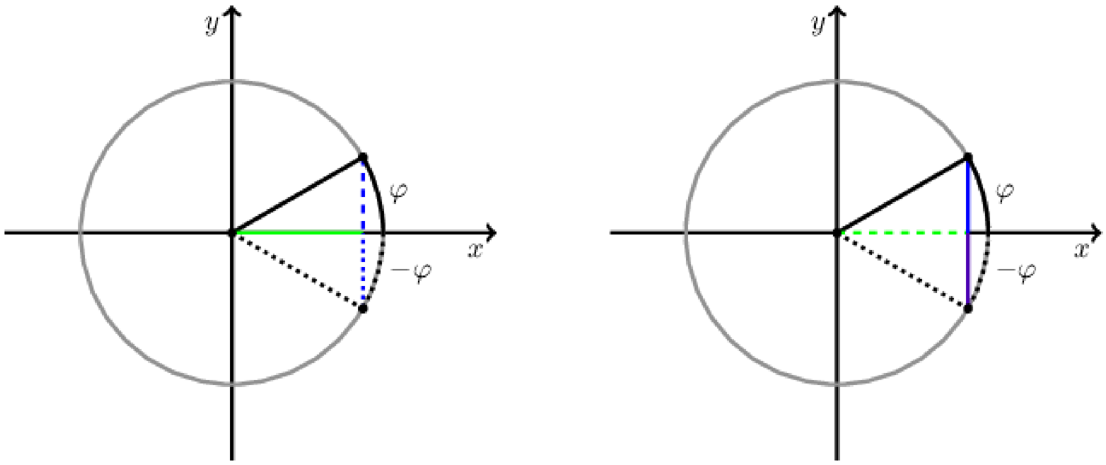
Furthermore, using Pythagoras' theorem, we have

$$x_\varphi^2 + y_\varphi^2 = 1 \ .$$

Replacing x_φ and y_φ by the corresponding relations to the trigonometric functions results for any φ in the important relation

$$\sin^2(\varphi) + \cos^2(\varphi) = 1 \ .$$

Additionally, from the description of the sine and cosine function, it can be seen that the values of the cosine function do not change if the line segment is reflected across the x-axis. Hence, the cosine value of the angle φ is equal to the cosine value of the angle $-\varphi$ (indicated in the figure below by the green line). For the sine function, a reflection across the x-axis results in a change of sign of the sinus value (indicated in the figure below by the blue line and the violet line, respectively)



Expressed in formulas, this is

$$\cos(-\varphi) = \cos(\varphi) \qquad \text{and} \qquad \sin(-\varphi) = -\sin(\varphi)$$

for every angle φ. These symmetry properties are useful for many calculations. An elementary example is the calculation of the angle between the x-axis and the connecting line from the origin to a point in the Cartesian coordinate system (see also Exercise 1.6.4 on the following page).

Example 1.6.5

Find the values of the sine, cosine, and tangent function of the angle $\alpha = 315°$.

For $\alpha = 315°$, the point P_α lies in the fourth quadrant. On the unit circle it es also described by the negative angle $\varphi = 315° - 360° = -45°$. Therefore, we have $\sin(315°) = \sin(-45°) = -\sin(45°) = \frac{1}{2}\sqrt{2}$ and $\cos(315°) = \cos(-45°) = \cos(45°) = \frac{1}{2}\sqrt{2}$ as well as $\tan(315°) = \tan(-45°) = -1$.

1.6.4 Exercises

Exercise 1.6.4
What is the degree measure of the angle φ between the x-axis and the connecting line from the origin in the Cartesian coordinate system to the point $P_\varphi = (-0.643; -0.766)$ on the unit circle? Use a calculator, but do not trust it blindly!

Result: φ = [ ] °

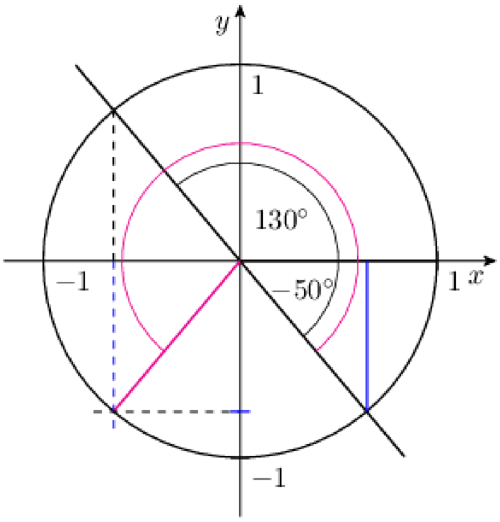
Solution:
From the coordinates of the point P_φ we have

$$\cos(\alpha) = -0.643 \quad \text{and} \quad \sin(\alpha) = -0.766\ .$$

If you enter

- `invers(cos(-0.643))` or $\cos^{-1}$(-0.643) in the calculator, you obtain approximately 130°
- `invers(sin(-0.766))` or $\sin^{-1}$(-0.766) in the calculator, you obtain approximately $-50°$.

Moreover, you know that the point lies in the third quadrant. Thus, the angle must be in the range from 180° to 270°.



The figure to the left shows that the negative cosine value corresponds to the angle $-130°$ and to the angle $\varphi = -130° = -130° + 360° = 230°$.
Likewise, the negative sine value can correspond to the angle $-50°$ and to the angle $\varphi = -(-50°) + 180° = 230°$.
Since this last value lies in the range stated above, the required value of the angle is $\varphi = 230°$ indicated in the figure by a pink line.

Exercise 1.6.5 1. Let a right triangle with the right angle at the vertex C and the sides $b = 2.53\,\text{cm}$ and $c = 3.88\,\text{cm}$ be given. Calculate the values of $\sin(\alpha)$, $\sin(\beta)$, and a.

Results:

- $\sin(\alpha)$ = [ ]
- $\sin(\beta)$ = [ ]
- a = [ ] cm

Solution:

We have

$$a = \sqrt{c^2 - b^2} = \sqrt{(3.88\,\mathrm{cm})^2 - (2.53\,\mathrm{cm})^2} = \sqrt{15.0544\,\mathrm{cm}^2 - 6.4009\,\mathrm{cm}^2} = \sqrt{8.6535}\,\mathrm{cm}\ ,$$

and

$$\sin(\alpha) = \frac{a}{c} = \frac{\sqrt{8.6535}\,\mathrm{cm}}{3.88\,\mathrm{cm}} = \frac{\sqrt{86535}}{388} \qquad \text{and} \qquad \sin(\beta) = \frac{b}{c} = \frac{2.53\,\mathrm{cm}}{3.88\,\mathrm{cm}} = \frac{253}{388}\ .$$

Numerically, we obtain $a \approx 2.9417\,\mathrm{cm}$, $\sin(\alpha) \approx 0.7587$, and $\sin(\beta) \approx 0.65201$.

2. Calculate the area F of a triangle with the sides $a = 4\,\mathrm{m}$, $c = 60\,\mathrm{cm}$, and the angle $\beta = \measuredangle(a, c) = \frac{11\pi}{36}$.

   Result: F = ☐ m^2

   Solution:

$$\frac{(a \cdot \sin(\beta)) \cdot c}{2} = \sin\left(\tfrac{11\pi}{36}\right) \cdot 1.2\,\mathrm{m}^2 \approx 0.98298\,\mathrm{m}^2\ .$$

1.7 Final Test

1.7.1 Final Test Modul 7

Exercise 1.7.1
Identify the figures below as precise as possible by specifying the name of the type (preceded by an adjective if necessary) describing as many properties of the figure as possible.

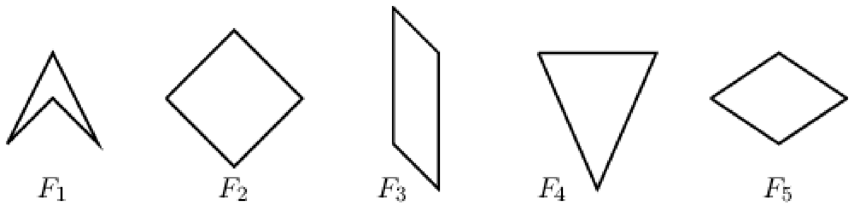


Figure:	Description of the Type:
F_1	
F_2	
F_3	
F_4	
F_5	

Exercise 1.7.2
Are the following results and statements right or wrong?

right	wrong	
☐	☐	Every rectangle is a rhombus.
☐	☐	Every square is a parallelogram.
☐	☐	It exists exactly one square with a diagonal of 5 cm.
☐	☐	A triangle with the angles 36° and 54° is right-angled.
☐	☐	In a rectangle the sum of all (interior) angles in radian measure is equal to 4π.

Exercise 1.7.3
Let a triangle ABC with side lengths $a = 5\,\text{cm}$, $b = 6\,\text{cm}$, and $c = 9\,\text{cm}$ be given. On the side c a point P and on the side b a point Q are chosen such that PQ is parallel to the side a and $[\overline{PQ}] = 0.50\,\text{cm}$. Calculate the lengths of the line segments $[\overline{PB}]$ and $[\overline{QC}]$ specified in centimetre.

a. $[\overline{PB}] =$ ☐ cm

b. $[\overline{QC}] =$ ☐ cm

Exercise 1.7.4
Let a square with sides of length a be given. Find the formulas for the area and the circumference for the largest circle inscribed to the square as well as for the smallest circle containing the square completely:

a. Circumference of the circle within the square as a function of the side length a: ☐

b. Area of the circle within the square as a function of the side length a:

c. Circumference of the circle around the square as a function of the side length a:

d. Area of the circle around the square as a function of the side length a:

Do not enter any brackets or radical terms. Enter, for example, $2^{0.5}$ instead of $\sqrt{2}$ to avoid the radical.

Index

n-gon, 32

acute (triangle), 17
adjacent (side), 48
alternate angle, 10
altitude of a triangle, 18, 36
angle, 8
angle (acute), 13
angle (obtuse), 13
angle (reflex), 13
angle (triangle), 17
arc, 12
arms, 9

cathetus, 17
circle, 4
circular cone, right, 44
circular functions, 48
complete angle (angle), 10
congruent (triangle), 22
corresponding angle, 10
cylinder, 41

diagonal (polygon), 32
diagonal (quadrilateral), 27
diamond, 28

equilateral quadrilateral, 28
exterior angle, 17

half-line, 3
hypotenuse, 17

interior angle, 17
intersection point, 4

line, 3
line length, 3
line segment, 3

obtuse (triangle), 17
opposite (side), 48

parallel, 4
parallelogram, 28
perpendicular (line), 10
perpendicular foot, 18
polygon, 32
prism, 40
Pythagoras (theorem), 18

quadrilaterals, 27

radian measure, 12
radius, 4
ray, 3
rectangle, 28
regelar n-gon, 32
regular polygon, 32
rhombus, 28
right angle (angle), 10
right-angled (triangle), 17

side (polygon), 32
side (triangle), 17
similarity (triangles), 24
square, 28
supplementary, 9

Thales (theorem), 19
theorem (congruent triangles), 22
trapezoid, 28
triangle, 16
trigonometric function, 48
trigonometry, 48

unit square, 28

vertex (angle), 9
vertex (polygon), 32
vertex (triangle), 17
vertical angle, 9