Brownian Motion

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Abstract

Brownian motion is a natural phenomenon first formally discovered by botanist Robert Brown in 1827. Robert Brown was studying the particles in grains of pollen when he noticed the particles were in constant motion. Over the course of the next century the phenomenon was studied further by mathematicians and physicists, who developed ideas based on this constant random motion. In this thesis I will go over the history of this phenomenon and its importance in the subject of mathematics.

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1 History

1.1 Jan Ingenhousz

In 1784, biologist, Jan Ingenhousz observed the irregular movement of coal dust on the surface of alcohol. Ingenhousz interpreted this motion to be the currents of the liquid creating movement in inert objects. This interpretation was incorrect, however, it is believed to be the first observation of Brownian motion over 40 years before Robert Brown.

1.2 Robert Brown

In 1827 Robert Brown observed Brownian motion while studying the pollen particles of the *Clarkia* flower. Brown originally believed that the motion of the particles was due to the pollen being alive. Brown subsequently conducted the same experiment on dead samples of plants as well as inorganic matter such as glass particles and dust that came from the Sphinx. Seeing the same movement in the other experiments, Brown concluded that the random motion was from a physical action and not biological.

1.3 Louis Bachelier

Inspired by Brownian motion, Bachelier introduced random motion to model stock prices. In his Ph.D Thesis(1900) "*Théorie de la Spéculation*" the idea of a "random-walk" was used to model the price of stocks, which is now called a barrier option. This "random-walk" is what developed into the mathematical "Wiener process" thus Bachelier was credited as the first person to derive Brownian motion and also the inventor of *Mathematical Finance*.

1.4 Albert Einstein

Independent of Bachelier's work, in 1905 Einstein published his own ideas on Brownian motion. Using the Molecular-Kinetic Theory of Heat, Einstein worked out a quantitative description of Brownian motion or Einstein's predicates. Einstein was then able to use Brownian motion with the diffusion equation in order to estimate Avogadro's number which helped prove the existence of atoms.

1.5 Norbert Wiener

In 1923 Norbert Wiener proved that a stochastic process existed that satisfied Einstein's predicates, and provided the first rigorous construction of Brownian motion. This stochastic process is dubbed as "Brownian motion" or the "Wiener process". In mathematics, the term Brownian motion is referring to this stochastic process and not the natural phenomenon.

2 Properties

2.1 Bachelier

Bachelier presented a "random-walk" in his Ph.D thesis which developed from his idea of an efficient market hypothesis, or rather the idea that the influences on the market are so numerous that past, current and anticipated events have no obvious connection with its changes. This idea helps describe two fundamental facts about Brownian motion:

1. Brownian motion has a **Markovian Character**. Given position at time t, you do not need the prior positions to predict the future behavior.

2. Brownian motion has the **Reflection property**. If $W(s)$ denotes the position of the Brownian motion at time s, then the maximal displacement by time t has the same distribution as the absolute displacement at time t.

The second has a simple distribution which leads to Bachelier's calculation:

$$P\{\max_{0 \leq s \leq t} W(s) \leq \lambda\} = \sqrt{\frac{2}{\pi t}} \int_0^\lambda e^{\frac{-x^2}{2t}} dx \tag{2.1}$$

2.2 Einstein's Predicates

Einstein predicted that the one-dimensional Brownian motion is a random function of time written as $W(t)$ for time $t \geq 0$ such that:

1. At time $t = 0$, the random movement starts at the origin, i.e. $W(0) = 0$.

2. At any given time $t > 0$, the position $W(t)$ of the particle has the normal distribution with mean 0 and variance t.

3. If $t > s > 0$, then the displacement from time s to time t is independent of the past until time s. i.e. $W(t) - W(s)$ is independent of $W(r)$; $r < s$.

4. The displacement is time-homogeneous; i.e., the distribution of $W(t) - W(s)$ is the same as the distribution of $W(t - s)$ which is in turn normal with mean 0 and variance $t - s$.

5. The random function W is continuous.

Einstein never formally constructed a stochastic process that satisfied these predicates. It was left until mathematician Norbert Wiener provided a rigorous mathematical construction of Brownian motion, which is why Brownian motion is often also called as Wiener process.These predicates became the basis for the Wiener process. Next we will define the properties of Brownian motion or Wiener process.

2.3 Properties of Brownian Motion

Definition 2.1: A d-valued stochastic process W_t, $t \geq 0$, is called a d-dimensional **Brownian motion** (also called a Wiener process) starting at $x \in \mathbb{R}$ if:

1. ***Origin*** - At time $t = 0$, $W_0 = x$ where x is the origin.
2. ***Independence*** - For all $0 \leq s \leq t$, the process $[W_t]_{t \geq 0}$ has stationary, independent increments, $W_t - W_s$ which are independent random variables.
3. ***Normality*** - For all $t \geq 0$ and $s \geq 0$ the increments $W_{t+s} - W_s$ are distributed normally.
4. ***Continuity*** - With probability 1, W_t is continuous.

The first property is what defines the origin of the stochastic process. The second property describes the continually random nature of the particle's collisions within a fluid. The displacement of the particle should be proportional to the time and symmetrically distributed about the origin thus the need for the third property. Lastly the final property is clear since physical motion is continuous.

The construction of Brownian motion is beyond the scope of this paper, however, we should note that rather than the construction, it is the properties of Brownian motion that define it. There are numerous different constructions, thus taking the time to depict a specific construction is more confusing and rigorous than necessary.

2.4 Nondifferentiability

Theorem 2.1: *Almost surely, Brownian motion is nowhere differentiable.*

Rather than by proof, this theorem can be understood by the definition of Brownian motion. If a function W_t were a Brownian motion and was differentiable at some point s, we would be able to predict where it would go in the future, but by the independent property of W_t we should know that this isn't the case. Thus we can be confident that W_t is not differentiable anywhere.

This gives us some insight into the nature of the random motion and depicts how erratically the Brownian motion jumps. This also relates back to one of Bachelier's facts that Brownian motion has a **Markovian Character**. Since the Brownian motion is differentiable nowhere, then it follows that the previous positions cannot help predict the future behavior.

Theorem 2.2: (Marcov Property) *Let W_t be a Brownian motion and fix $s \geq 0$. $B_{t+s} - B_s$ is a standard Brownian motion independent of B_t.*

proof. Clearly B_{t+s} is a Brownian motion. If we subtract a constant from B_{t+s} we only change the initial point, subtracting by B_s makes this a standard Brownian motion. The independence of B_t before time s follows from the independence of increments of Brownian motion. □

Brownian motion is an important stochastic process, it is a Gaussian process, a diffusion process, a Lévy process, a Markov process, and a self-similar process. Each of these properties became the starting point to a new theory of stochastic processes. we will briefly cover what these properties mean.

2.5 More Properties

We will briefly go over some of these properties of Brownian motion, primarily, let us define what a stochastic process is.

Definition 2.2: Let N be a subset of $[0,\infty)$. A group of random variables $\{X_n\}_{n\in N}$, is called a **stochastic process**. When $N=\mathbb{N}$, $\{X_n\}_{n\in N}$ is a *discrete-time process*, and when $N=[0,\infty)$, it is called a *continuous-time process*.

When N is a singleton, the process is just a single random variable. If N is finite, we get a random vector. Therefore a stochastic process is just a generalization of random vectors. Thus we can say that Brownian motion is a continuous-time stochastic process.

Definition 2.3: A stochastic process $X_t, t\geq 0$ is a **Gaussian process** if $X_{t_1},\ldots,X_{t_n}$ has a normal distribution for all $t_1,\ldots,t_n$.

Brownian motion is a Gaussian process since we know that Brownian motion has normal distributions, and from Einstein's predicates that state for a standard Brownian motion; mean $=0$ and variance $=t-s$, or:

$$E[W_t]=0, \quad E[W_tW_s]=min(t,s) \tag{2.2}$$

Thus we can get the following proposition.

Proposition2.1: If $X_t, t\geq 0$ is a standard Brownian motion, then $Z_t = X_t - tX_1, 0\leq t\leq 1$. It follows that Z_t is a *Gaussian process*.

Definition 2.4: A stochastic process $X(t), t\geq 0$ is a **Lévy process** if:

1. For any sequence $0\leq t_1<t_2<...<t_n$ the random variables $X(t_0), X(t_1)-X(t_0), ..., X(t_n)-X(t_{n-1})$ are independent.
2. For any positive times $s\leq t$ the random variable $X(t)$ is homogeneous, meaning $X(t-s)$ and $X(t)-X(s)$ are the same.
3. The process is almost surely continuous.

In general, a stochastic process is a Lévy process if it has stationary, independent increments. Thus from Einstein's predicates and the principles of Brownian motion we can see that a Brownian motion is surely a Lévy process.

Definition 2.5: A stochastic process $X_t, t\geq 0$ is said to be self-similar if there exists a real number $H>0$ such that for any $c>0$ the processes X_{ct} and c^HX_t have the same finite dimensional distributions.

Showing that Brownian motion is a self-similar process goes beyond the scope of this paper, however, we can show that Brownian motion is self-similar by showing that there exists a self-similar process that satisfies the properties of Brownian motion. We can show that Brownian motion is a diffusion process, observe in the next chapter the application of Brownian motion in diffusion.

3 Application

3.1 Diffusion

Einstein's work with the diffusion equation and Brownian motion allowed him to estimate Avogadro's number and thus the diameter of the hydrogen atom. This contributed a strong argument in the debate for the existence of atoms, which was a controversial topic at the time. Einstein noticed that there were random qualities of the gas particles, along with qualities that were measurable. Einstein noticed that one such quality was the density ρ of the gas. Observe the equation:

$$\frac{\partial \rho}{\partial t} = D\frac{\partial^2 \rho}{\partial x^2} \tag{3.1}$$

Einstein showed that the diffusion coefficient D could be calculated from the diffusion equation satisfied by the density. Thus we can solve and predict something from a random model. From this we can see that the diffusion process is related to Brownian motion.

We can show this connection by creating numerous random-walks. Using Matlab, we create a program that plots M random-walks of length N where each walk changes direction randomly at each iteration. The code generates the following figure:

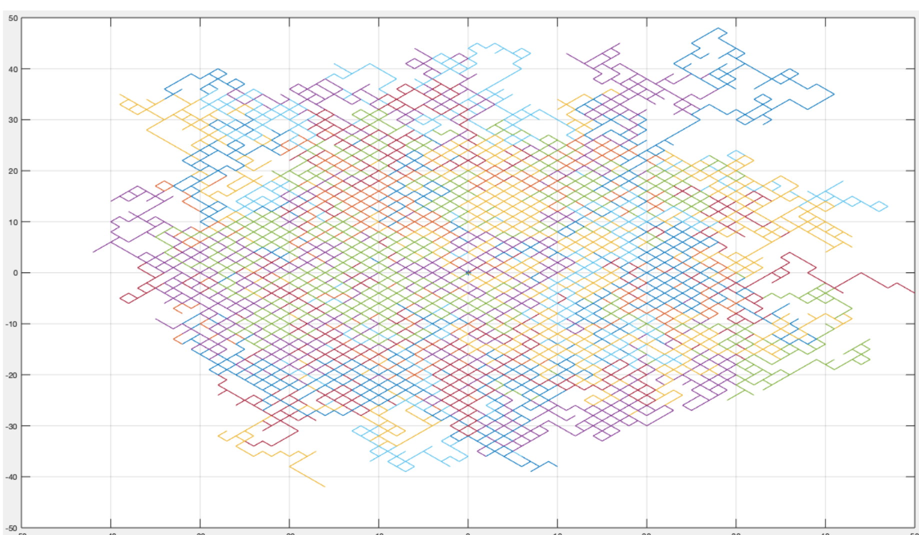


Figure 3.1: Diffusion demonstrated by random-walks

Figure 3.1 shows the random-walks spreading throughout the plane. These random-walks demonstrate how particles diffuse in a fluid, for example a drop of ink falling into a cup of water the ink can be seen spreading throughout the cup randomly. We know that this is a Brownian motion as the ink particles collide with the particles of water. Next we will show how the random-walk is involved in probability theory.

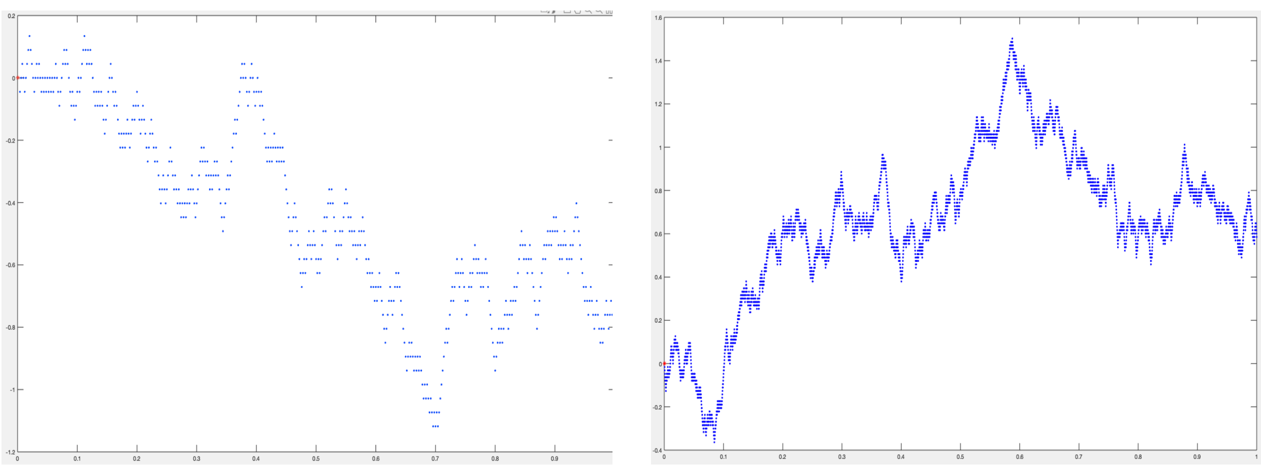
3.2 Brownian Motion as a Limit of Random-Walks

Brownian motion's significance to probability theory is due to it being, in some way, a limit of re-scaled simple random walks. Let $\xi_1, \xi_2, \ldots$ be independent, normally distributed random variables with mean 0 and variance 1. For $n \geq 1$ we define a continuous-time stochastic process $\{W_n(t)\}_{t \geq 0}$ by:

$$W_n(t) = \frac{1}{\sqrt{n}} \sum_{1 \leq j \leq \lfloor nt \rfloor} \xi_j \tag{3.2}$$

This function has random jumps of size $\frac{\pm 1}{\sqrt{n}}$ at each time $\frac{k}{n}$, where k is a positive integer. We know that the random variables ξ_j are independent, and thus we know that the increments of $W_n(t)$ are also independent. Moreover, we can assume that for a large enough n the distribution of $W_n(t+s) - W_n(s)$ is similar to a normal distribution[1]. Thus we can assume that as $n \to \infty$, the random-walk function $W_n(t)$ approaches a standard Brownian motion.

Using this, we can make the random-walks look like a Brownian motion over a large enough n. Again we use Matlab to illustrate this process with two figures, first with $n = 500$ and then with $n = 4000$. Observe how increasing the n creates a better visualisation of a Brownian motion.



(a) Random-walks with $n = 500$ (b) Random-walks with $n = 4000$

Figure 3.2: Brownian motion as a limit of random-walks

It is clear that the figure with $n = 4000$ is a more accurate depiction of a Brownian motion, however, even with this quantity of n there are still visible gaps between the points. These gaps show that even 4000 is not a high enough n to completely represent a Brownian motion. This is because the number of collisions occurring on a specific particle in a gas chamber is at a rate of 10^{14} collisions per second. Thus we can see that with an n at only 4000, there still aren't nearly enough n to fully represent a Brownian motion.

[1] By Central Limit theorem

3.3 Modeling a Standard Wiener Process

We can construct a standard Wiener Process in Matlab by setting up an algorithm that increments with normal distribution and with zero mean and unit variance. Observe the equation:

$$W(t) - W(s) \sim \sqrt{t-s}N(0,1) \tag{3.3}$$

Where $W(t) - W(s)$ is the increment at each step. We can discretize the previous equation with time step dt, and we get:

$$dW \sim \sqrt{dt}N(0,1) \tag{3.4}$$

Where dt is one over the number of iterations, and $N(0,1)$ is obtained with the 'rand' function, which generates arrays of random numbers whose elements are uniformly distributed in the interval (0,1). Inputting this into Matlab gives us the following figure:

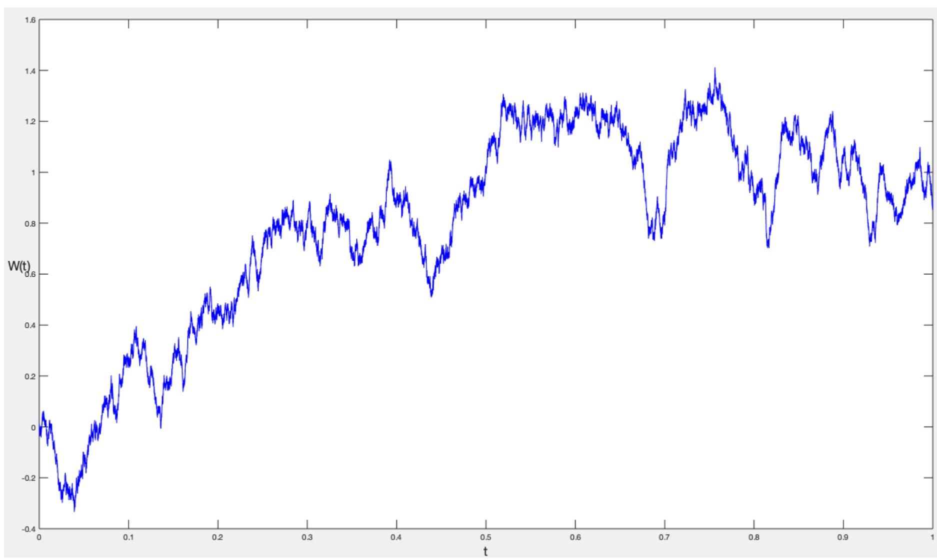


Figure 3.3: Wiener Process

This model of the Wiener process is much more clear in its shape. The figure is clearly continuous on the bound $[0,1]$, and without the gaps that appeared in the random-walk figure. This is due to having 10^6 increments with normal distribution over the bounds. This model also helps illustrate the self-similarity property of Brownian motion. In the figure the bound was $[0,1]$, however, if the bound was [0,0.1] or [0,100] the shape of the figure would be generally the same.

4 Conclusion

Brownian motion has impacted the fields of mathematics, physics, chemistry, economics and many more. The natural phenomenon discovered over a hundred years ago caused the growth of numerous different fields. Einstein's use of Brownian motion helped settle the debate over the existence of atoms. Louis Bachelier's thesis was the first to use this random motion to model markets. Bachelier's use of advanced mathematics in the market was the beginning of what is mathematical finance. Those ideas in mathematical finance continued to develop with new mathematicians. Brownian motion continues to be used to uncover more ideas about the connection with mathematics and natural behavior.

5 Code Used

Code written in Matlab for Figures 3.1, 3.2, and 3.3.

```matlab
close all;clear;clc;clearvars;
N = 300; % Length of the x-axis, also known as the length of the random walks.
%M = 1;
M = 200; % The amount of random walks.
x_t(1) = 0;
y_t(1) = 0;
plot(x_t(1), y_t(1), '*')
hold on
for m=1:M
  for n = 1:N % Looping all values of N into x_t(n).
    A = sign(randn); % Generates either +1/-1 depending on the SIGN of RAND.
    x_t(n+1) = x_t(n) + A;
    A = sign(randn); % Generates either +1/-1 depending on the SIGN of RAND.
    y_t(n+1) = y_t(n) + A;
  end
  plot(x_t, y_t);
  hold on
end
grid on;
% Enlarge figure to full screen.
set(gcf, 'Units', 'Normalized', 'Outerposition', [0, 0.05, 1, 0.95]);
%axis square;
```

Figure 5.1: Diffusion with Random-walks

```matlab
close all; clear; clc; clearvars;
%n = 50;
%n = 500;
n = 4000;
x = 0;

for i = 1:n
    W(i) = x;
end

plot(0,W(1),'r*')
hold on
ones = [-1, 1];

for i = 2:n
    m = randi([1,2],1);
    Z = ones(m);%chooses positive or negative one
    W(i) = Z/sqrt(n) + W(i-1);% takes a random step depending on Z
    plot(i/n,W(i),'b.')
end
```

Figure 5.2: Limit of Random-walks

```matlab
close all; clear; clc; clearvars;
randn('state',100)              % set the state of randn
T = 1;
%N = 50;                        % different size N for presentation
%N = 500;
%N = 2500;
N = 10.0e6;
dt = T/N;
dW = zeros(1,N);                % preallocate arrays ...
W = zeros(1,N);                 % for efficiency

dW(1) = sqrt(dt)*randn;         % first approximation outside the loop ...
W(1) = dW(1);                   % since W(0) = 0 is not allowed
for j = 2:N
    dW(j) = sqrt(dt)*randn;     % general increment
    W(j) = W(j-1) + dW(j);
end

plot([0:dt:T],[0,W],'b-')       % plot W against t
xlabel('t','FontSize',16)
ylabel('W(t)','FontSize',16,'Rotation',0)
set(gcf, 'Units', 'Normalized', 'Outerposition', [0, 0.05, 1, 0.95]);
```

Figure 5.3: Wiener Process

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