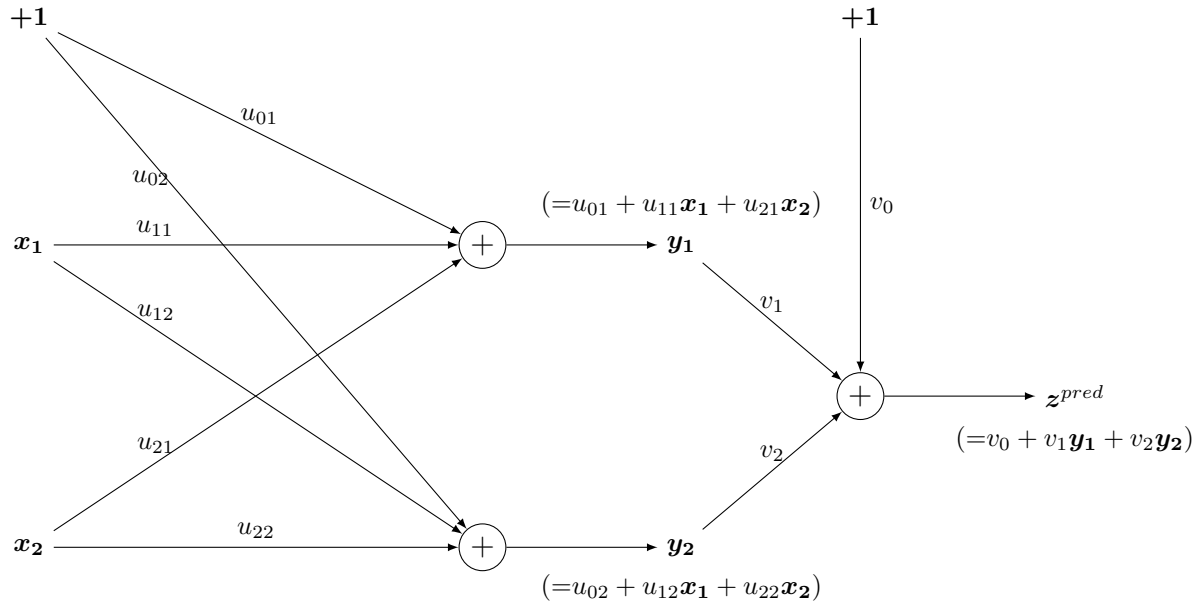
1 Model outline

Let us suppose we want to use the following linear neural network architecture, with no activation function:



Let us define the following matrices, which model the above network:

$$\mathbf{X}_0 = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \\ x_{41} & x_{42} \end{bmatrix} \quad \mathbf{X} = \begin{bmatrix} 1 & \mathbf{X}_0 \end{bmatrix} = \begin{bmatrix} 1 & x_{11} & x_{12} \\ 1 & x_{21} & x_{22} \\ 1 & x_{31} & x_{32} \\ 1 & x_{41} & x_{42} \end{bmatrix} \quad \mathbf{u} = \begin{bmatrix} u_{01} & u_{02} \\ u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$$

$$\mathbf{Y}_0 = \mathbf{X}\mathbf{u} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \\ y_{31} & y_{32} \\ y_{41} & y_{42} \end{bmatrix} \qquad \mathbf{Y} = \begin{bmatrix} 1 & \mathbf{Y}_0 \end{bmatrix} = \begin{bmatrix} 1 & y_{11} & y_{12} \\ 1 & y_{21} & y_{22} \\ 1 & y_{31} & y_{32} \\ 1 & y_{41} & y_{42} \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} v_0 \\ v_1 \\ v_2 \end{bmatrix}$$

$$\mathbf{z}^{pred} = \mathbf{Y}\mathbf{v} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}$$

The final output of the network can also be written as:

$$\begin{aligned} z_i^{pred} &= v_0 + v_1 y_{i1} + v_2 y_{i2} && (1) \\ &= u_0 + v_1(u_{01} + u_{11}x_{i1} + u_{21}x_{i2}) + v_2(u_{02} + u_{12}x_{i1} + u_{22}x_{i2}) && (2) \\ &= v_0 + u_{01}v_1 + u_{02}v_2 + u_{11}v_1x_{i1} + u_{21}v_1x_{i2} + u_{12}v_2x_{i1} + u_{22}v_2x_{i2} && (3) \end{aligned}$$

2 Training with gradient descent

Let us use the following cost function, which we want to minimize:

$$C = \frac{1}{2}\sum_{i=1}^{n}(z_i^{reality} - z_i^{pred})^2$$

The derivative of C with respect to any coefficient w is:

$$\frac{\partial C}{\partial w} = -\sum_{i=1}^{n}\frac{\partial z_i^{pred}}{\partial w}(z_i^{reality} - z_i^{pred})$$

Using (3) we can calculate the derivative of C with respect to each weight u_{ij} and v_i of the network and write them in vector form:

$$\frac{\partial z_i^{pred}}{\partial \mathbf{u}} = \begin{bmatrix} \frac{\partial z_i^{pred}}{\partial u_{01}} & \frac{\partial z_i^{pred}}{\partial u_{02}} \\ \frac{\partial z_i^{pred}}{\partial u_{11}} & \frac{\partial z_i^{pred}}{\partial u_{12}} \\ \frac{\partial z_i^{pred}}{\partial u_{21}} & \frac{\partial z_i^{pred}}{\partial u_{22}} \end{bmatrix} = \begin{bmatrix} v_1 & v_2 \\ v_1 x_{i1} & v_2 x_{i1} \\ v_1 x_{i2} & v_2 x_{i2} \end{bmatrix}$$

$$\frac{\partial z_i^{pred}}{\partial \mathbf{v}} = \begin{bmatrix} \frac{\partial z_i^{pred}}{\partial v_0} \\ \frac{\partial z_i^{pred}}{\partial v_1} \\ \frac{\partial z_i^{pred}}{\partial v_2} \end{bmatrix} = \begin{bmatrix} 1 \\ u_{01} + u_{11}x_{i1} + u_{21}x_{i2} \\ u_{02} + u_{12}x_{i1} + u_{22}x_{i2} \end{bmatrix}$$

Let us define d_i such that

$$d_i = -(z_i^{reality} - z_i^{pred})$$

By plugging in the corresponding derivatives found above in to (dC), we obtain:

$$\frac{\partial C}{\partial u_{01}} = v_1\sum_{i=1}^{n} d_i \qquad \frac{\partial C}{\partial u_{02}} = v_2\sum_{i=1}^{n} d_i$$

$$\frac{\partial C}{\partial u_{11}} = v_1\sum_{i=1}^{n} x_{i1}d_i \qquad \frac{\partial C}{\partial u_{12}} = v_2\sum_{i=1}^{n} x_{i1}d_i$$

$$\frac{\partial C}{\partial u_{21}} = v_1\sum_{i=1}^{n} x_{i2}d_i \qquad \frac{\partial C}{\partial u_{22}} = v_2\sum_{i=1}^{n} x_{i2}d_i$$

$$\frac{\partial C}{\partial v_0} = \sum_{i=1}^{n} d_i$$
$$\frac{\partial C}{\partial v_1} = \sum_{i=1}^{n} (u_{01} + u_{11} x_{i1} + u_{21} x_{i2}) d_i$$
$$\frac{\partial C}{\partial v_2} = \sum_{i=1}^{n} (u_{02} + u_{12} x_{i1} + u_{22} x_{i2}) d_i$$

Which can be cast in matrix form as:

$$\frac{\partial C}{\partial \mathbf{u}} = \mathbf{X}^T \mathbf{d} \begin{bmatrix} & v_1 & v_2 \end{bmatrix}$$
$$\frac{\partial C}{\partial \mathbf{v}} = \mathbf{Y}^T \mathbf{d} \qquad \text{with } \mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix}$$

3 Proof that this model is equivalent to a one layer model

The coefficients v_1 and v_2 are constant, so equations (1) to (6) reduce to

$$\sum_{i=1}^{n} d_i = 0 \tag{4}$$
$$\sum_{i=1}^{n} x_{i1} d_i = 0 \tag{5}$$
$$\sum_{i=1}^{n} x_{i2} d_i = 0 \tag{6}$$

Moreover, we can rewrite equations (8) as

$$u_{01} \sum_{i=1}^{n} d_i + u_{11} \sum_{i=1}^{n} x_{i1} d_i + u_{21} \sum_{i=1}^{n} x_{i2} d_i \tag{7}$$

which is true if equations (10) to (12) are satisfied, and similarly for equation (9). So only equations (10), (11) and (12) need to be solved.

Noting that

$$d_i = (z_i^{reality} - z_i^{pred}) \tag{8}$$

$$z = v_0 + u_{01}v_1 + u_{02}v_2 + u_{11}v_1x_{i1} + u_{21}v_1x_{i2} + u_{12}v_2x_{i1} + u_{22}v_2x_{i2} \tag{9}$$

$$z = w_0 + w_1x_{i1} + w_2x_{i2} \tag{10}$$

We can rewrite (10)-(12) as

$$\sum_{i=1}^{n} z_i^r - w_0 - w_1x_{i1} - w_2x_{i2} = 0 \tag{11}$$

$$\sum_{i=1}^{n} x_{i1}(z_i^r - w_0 - w_1x_{i1} - w_2x_{i2}) = 0 \tag{12}$$

$$\sum_{i=1}^{n} x_{i2}(z_i^r - w_0 - w_1x_{i1} - w_2x_{i2}) = 0 \tag{13}$$

These are the normal equations of the least squares regression, so the coefficients w need to satisfy the normal equations. There are possibly multiple solutions for u_{ij} and v_j, but they are constrained the normal equations.