Master's Thesis - High Level Description

John AOUSSOU
Graduate Research Assistant
MIT 2013-2016

Imagine you put a gas inside a recipient. Let us assume the gas is heavier than air and will not escape. You then place a cover that perfectly fits the inside of that recipient, and you push on the cover. Assuming it is perfectly tight and the gas can't escape, if you push hard enough, the cover will go down, the gas will contract, its volume will get smaller, but you will feel more pressure on your hand. This is because a gas is a compressible fluid.

Contrast this with water: even if you push the cover hard, it won't move, water will not contract or shrink in volume. Water is an incompressible fluid. Its movement is described by the *incompressible* Navier-Stokes equation:

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} = \nu \nabla^2 \mathbf{u} - \mathbf{u} \cdot \nabla \mathbf{u} - \nabla p + \mathbf{f} & (1) \\ \nabla \cdot \mathbf{u} = 0 & (2) \end{cases}$$

Equation (1) describes how the fluid moves. Equation (2) is a constraint that tells us that the water may not be compressed.

Imagine we are observing water in a channel.

- $\mathbf{u}$ is the velocity of the water in a given location, at a given time (that's what we want to solve for)
- $\frac{\partial \mathbf{u}}{\partial t}$ describes how velocity changes with time (i.e. acceleration)
- $\nu \nabla^2 \mathbf{u}$ imagine that the water is perfectly still. This explains how water will eventually settle if it is disturbed at some location (like you if stir it), because of viscosity (internal friction).
- $\mathbf{u} \cdot \nabla \mathbf{u}$ explains how the velocity changes depending on where you are in the water (like how the flow in the channel gets faster when its width gets narrower)
- ∇p describes how pressure is different at different locations
- $\mathbf{f}$ means that external forces might be applied to the water, like gravity, or the wind blowing etc...

A common way of solving this equation in time is to use the projection method, where you calculate a intermediate value for the velocity, here $\bar{\mathbf{u}}$

$$\begin{cases} \dfrac{\bar{\mathbf{u}} - \mathbf{u}^n}{\Delta t} = \nu \nabla^2 \bar{\mathbf{u}} - \bar{\mathbf{u}} \cdot \nabla \bar{\mathbf{u}} - \nabla p^n + \mathbf{f}^{n+1} & (3a) \\ \nabla^2 \phi^{n+1} = \dfrac{1}{\Delta t} \nabla \cdot \bar{\mathbf{u}} & (3b) \\ p^{n+1} = p^n + \phi^{n+1} & (3c) \\ \mathbf{u}^{n+1} = \bar{\mathbf{u}} - \Delta t \nabla \phi^{n+1} & (3d) \end{cases}$$

To improve the accuracy of the result, one would typically reduce the time step. The main point of my thesis is to prove that it is computationally cheaper to instead plug back in the intermediate value that we calculated to get a better results. In other words, it is more efficient to iterate over k in the following algorithm rather than making Δt smaller.

$$
\begin{cases}
p_0^{n+1} = p^n & \text{(4a)} \\
\dfrac{\bar{\mathbf{u}}_{k+1} - \mathbf{u}^n}{\Delta t} = \nu \nabla^2 \bar{\mathbf{u}}_{k+1} - \bar{\mathbf{u}}_{k+1} \cdot \nabla \bar{\mathbf{u}}_{k+1} - \nabla p_k^{n+1} + \mathbf{f}^{n+1} & \text{(4b)} \\
\nabla^2 \phi_{k+1}^{n+1} = \dfrac{\nabla \cdot \bar{\mathbf{u}}_{k+1}}{\Delta t} & \text{(4c)} \\
p_{k+1}^{n+1} = p^n + \phi_{k+1}^{n+1} & \text{(4d)} \\
\mathbf{u}_{k+1}^{n+1} = \bar{\mathbf{u}}_{k+1} - \Delta t \nabla \phi_{k+1}^{n+1} & \text{(4e)}
\end{cases}
$$

In practice, the algorithm is different (for example, advection is typically solved in time using an explicit scheme), however this document is only meant to offer an intuitive, high level explanation.