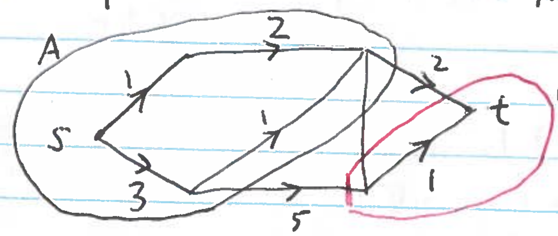
Recall:

Cut: A partition of vertices into A, B s.t $s \in A, t \in B$



$Cap(A,B) = 5+2=7$

$$Cap(A,B) = \sum_{\substack{uv \in E \\ u \in A \\ v \in B}} C_{uv}$$

Recall: For a flow $f: E \to \mathbb{R}^+$, $val(f) = \sum_{su \in E} f(su)$

Claim: For any s-t-cut (A,B); $val(f) = f^{out}(A) - f^{in}(A)$

$$f^{out}(A) = \sum_{\substack{uv \in E \\ u \in A \\ v \in B}} f(u,v) \qquad f^{in}(A) = \sum_{\substack{uv \in E \\ u \in B \\ v \in A}} f(u,v)$$

Pf:

Note: $Val(f) = \sum_{su \in E} f(su) = f^{out}(s)$

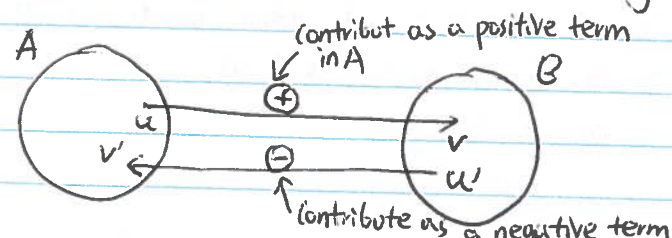
$$Val(f) = \sum_{u \in A} \underbrace{f^{out}(u) - f^{in}(u)}_{\text{equals to 0 except for } u = s}$$

$$\sum_{v \in A} f^{out}(u) - f^{in}(u)$$

$$= \sum_{v \in A} \left(\sum_{uw \in E} f(uw) - \sum_{vu \in E} f(vu) \right) = f^{out}(A) - f^{in}(A)$$

Now, if e is edge with endpoints in B, f(e) doesn't appear in the sum, if e is an edge with endpoints in A then it will be canceled out. If e is an edge with one end in A the other in B, then they appear once

eg.



Q: why $val(f) = f^{in}(t)$?

Take the cut $A = E \setminus \{t\}$, $B = \{t\}$

Then since $val(f) = f^{out}(A) - f^{in}(A)$

$$= \sum_{ut \in E} f(ut) = f^{in}(t)$$

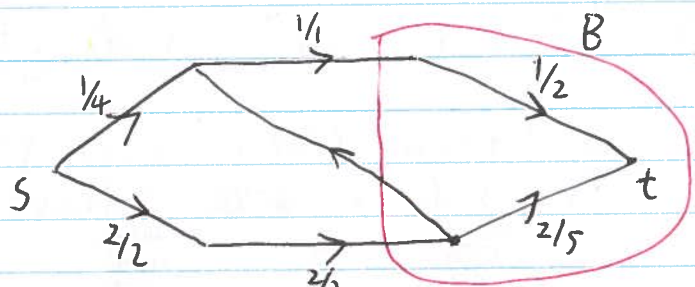
Cor: Let (A, B) be a cut, f be a flow, then

$val(f) \leq Cap(A, B)$

$Val(f) = f^{out}(A) - f^{in}(A)$

$$\leq f^{out}(A) = \sum_{\substack{u \in A \\ v \in B \\ uv \in E}} f(u,v) \leq \sum_{\substack{u \in A \\ v \in B}} Cap(u,v) = Cap(A, B)$$

Ex:



$Cap(A,B) = 3$

$\Rightarrow$

max flow ≤ 3

and since flow is 3 $\Rightarrow$ it is maximal.

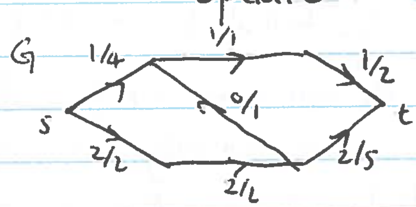
Proof of Flow returned by F-Fulkerson is maximal

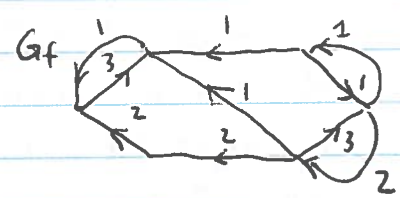
F-F: Start with $f \equiv$

while s-t parth P in G_f

augment (f, P)

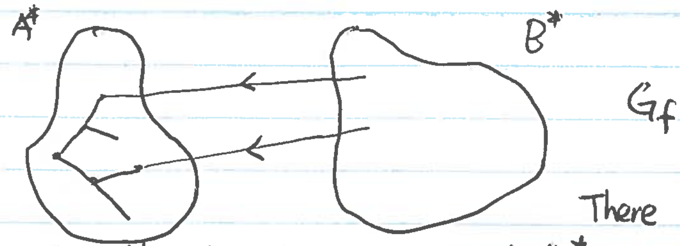
update





Consider the point where F-F terminates. Let A^* be the set of the vertices that can be reached from S in the residual graph.

Note $t \notin A^*$.



There are no edges from A^* to B^* as otherwise we can extend A^*

Thus: If uv is an edge in the original network

with $u \in A^*$, $v \in B^*$ $\quad f(u,v) = C_{uv}$

If $u \in B^*$, $v \in A^*$ $\quad f(u,v) = 0$

$f^{in}(A^*) = 0$

$f^{out}(A^*) = Cap(A^*, B^*) = val(f)$ as $f^{in}(A^*) = 0$

Problem: Given a network, can we find a min-cut?

Run F-F.

As $val(f) \leq$ max-flow $\leq$ min-cut $\leq$ Any cut, Say $Cap(A^*, B^*)$

and after ff, we found a flow f s.t

$val(f) = Cap(A^*, B^*) \Rightarrow$ max-flow = min-cut.

Thm: For any flow network, maxflow = min-cut