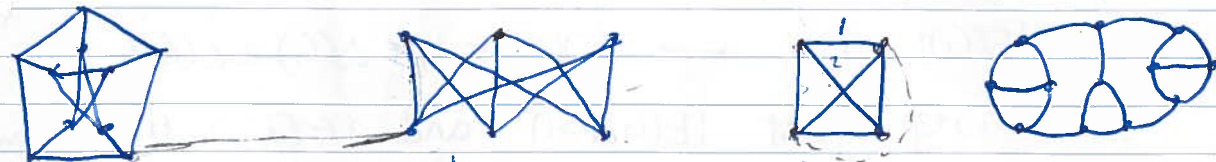
Def: A cubic graph is a graph in which all vertices have degree three. Other names includes 3-regular, trivalent.

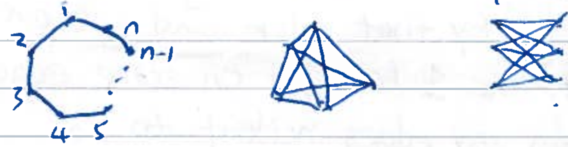
Ex:



peterson $K_{3,3}$ K_4

Def: a bridge, isthmus, cut-edge, cut-arc is an edge whose deletion increases its number of connected component. a bridgeless or isthmus free graph is a graph with no bridges.

Ex: C_n K_n $K_{n,m}$



Def: a cycle double cover is a collection of cycles in an undirected graph that include each edge of the graph exactly twice.

Ex:

Cycle double cover conjecture: Whether every bridgeless graph has a cycle double cover.

Vizing's theorem: Let G be a simple undirected graph, edge-chromatic number is atmost $\Delta+1$. We use $\chi'(G)$ to denote edge-chromatic #

Note that edge-chromatic number is at least Δ. We can therefore partition the graphs into two classes:

Class 1: edge-chromatic number is Δ, **Class 2:** edge chromatic number is $\Delta+1$.

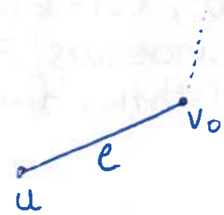
Ex: $\Delta=1$, all graph G of $\Delta(G)=1$. is disconnected edges,
hence $\chi'(G)=1$

$\Delta=2$ G is disjoint Union of paths and cycles.
G is of Class one iff it is bipartite.

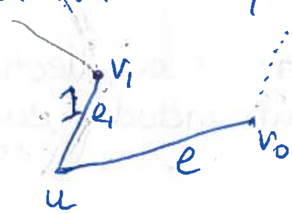
Pf (Vising): We proof by induction on # of edges.

Base: $|E(G)|=1$ —— $\chi'(G)=1=\Delta(G)\le\Delta(G)$ ✓

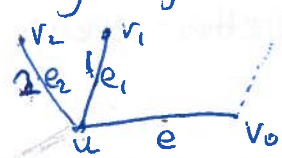
Induction Step: let $|E(G)|=n$ and $e\in E(G)$, then $\Delta(G|e)\le\Delta(G)$. Let's color $G|e$ using at most $\Delta(G)+1$ colors. and Let $e=uv_0$,



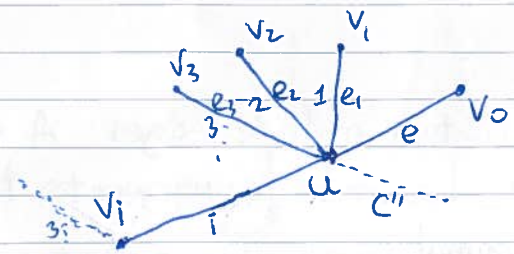
If there is a color 1 that is not used on any edges incident to u and v_0 then color e by that color and we are done. So assume not, then Let color 1 be used on some edge incident to u but not on any edges incident to v_0



If there is a color 2 that is not used on any edges incident to v_1 or u, we color e_1 with 2 and e with 1 and we are done. otherwise, the is color 2 used on some edge e_2 incident to u but not to any edges incident to v_1.

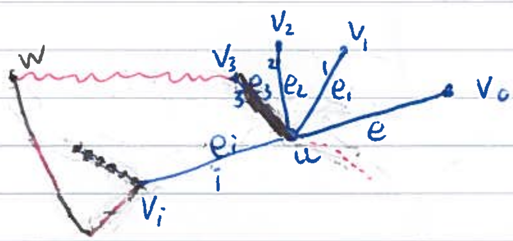


Again as before, if $\exists$ color 3 that is not used on any edges incident to u and v_2, then color e_2, 3 then e_1 2 then e, 1. and we are done. Else continue above process until stuck. Say we stop after steps.



So $\exists$ a color, WLOG, 3 on a edge incident to u, namely e_3, that does not arise on any edges incident to v_i. By degree consideration, Color C'' doesn't arise on any edges incident to u. then C'' most have arise on some edge incident to each of v_i, $i \geq 1$

Consider the subgraph with edges colored only 3 and C''. it is a graph of degree at most 2, it must be eigher path or cycles



Look at path extended from v_i, It might end up in w.
If $w = v$, then it must pass through v_3. then change the red to blue and blue to red in the path in the above picture.
Then there is no more red edge incident to v_i and v. so Color e_i red, e_{i-1}, i ..... e, e_1 and we are done.
If $w \neq v_i$ $\forall i$ and $w \neq v$. then change color of the path as before and note that red is not used on any ~~next~~ edges incident to v_i and v then we are again done by apply same recolor as befor on each v_i's.
Finally if $w = v_i$ for some i then alternate red and black and change v_i. $\boxtimes$

Def: A cactus is a connected graph in which any two simple cycle have at most one vertex in common. Equivalently, it is a connected graph in which every edge belongs to at most one simple cycle, or (for non-trivial cactus) in which every block is an edge or a cycle.