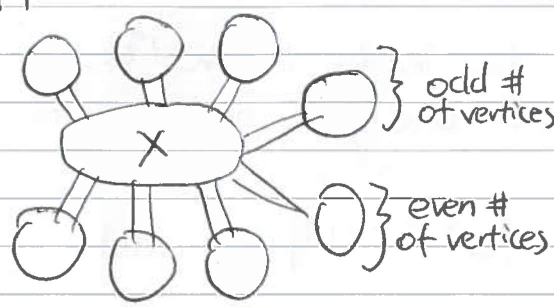
Thm: Every bridgeless cubic graph has a perfect matching (Peterson's th

Thm (Tutte's theorem): G has a perfect matching $\Leftrightarrow$ $\forall S \subseteq V(G)$, the graph induced by $V(G) \setminus S$ has a number of components with an odd # of vertices less than or equal to the number of vertices in S.

Pf:



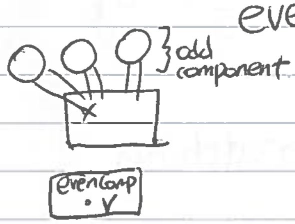
If G has a perfect matching, then $|\text{odd Components of } |G \setminus X|| \leq |X|$ $\forall x \in V(G)$

Now we show that if $|\text{oddcomp}(G \setminus X)| \leq X$ $\forall x \subseteq V(G)$, then G has a perfect matching. We apply induction on $|V(G)|$

Claim 1: $|V(G)|$ is even, as $|V(G)| = |X| + \text{odd}(G \setminus X)$, take $|X| = 0$ gives $|V(G)| = 0 \pmod 2$

We say X is critical if $\text{odd}(G \setminus X) = |X|$, By Claim 1, if $\text{odd}(G \setminus X) \geq |X| - 1$, then X is critical.

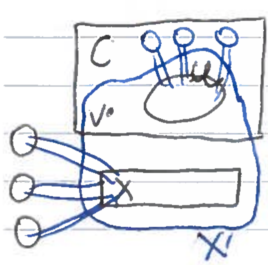
Claim : Since ϕ is a critical set, G has at least 1 critical set. Let X be the maximal critical set, $G \setminus X$ has no even components. Otherwise, Let $v \in V(C)$ and $X' = X \cup \{v\}$



$\text{odd}(G | X') \geq \text{odd}(G \setminus X) = |X| = |X'| - 1$.
So X' is critical. $\lightning$

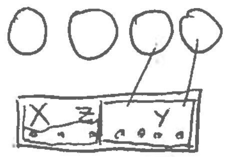
Claim : If C is an odd component of $G | X$ and $v \in V(C)$, then $C | v$ has a perfect matching.

We can show $\forall$ subset U of $C | v$, $\text{odd}(C | v | U) \leq |U|$ for all $U \subseteq V(C | v)$. then apply induction.



Let $X' = X \cup \{v\} \cup U$

$$\begin{aligned} \text{odd}(G \setminus X') &= \text{odd}(G | X) - 1 + \text{odd}(C | v | U) \\ &= |X| - 1 + \text{odd}(C \setminus v | U) \\ &\leq |X'| - 2 \quad \text{as } X' \text{ is not critical} \\ &= (|X| + 1 + |U|) - 2 = |X| + |U| - 1 \Rightarrow \text{odd}(C | v | U) \leq |U| \end{aligned}$$



} odd components

We want a matching covering X and hitting every odd component of $G \setminus X$ exactly once.

Suppose it doesn't exist, by Hall's theorem, there exists $Y \subseteq X$ such that $\#$ odd components adjacent to $Y) < |Y|$.
Let $Z = X - Y$, then $odd(G \setminus Z) > |Z|$ ↯ So ☒

Pf (Peterson's theorem):
We will prove that $\forall\, U \subseteq V(G)$, $odd(G-U) \leq |U|$, then the theorem will follow from tutte's theorem.
Let G_i be an odd component of $G \setminus U$. Let V_i be the vertices in G_i and m_i be the set of edges with one end in U and the other in V_i,
then $\sum_{v \in V_i} deg(v) = 2|E_i| + m_i$, $\sum_{v \in V_i} deg(v) = 3|V_i|$, is odd
hence m_i is odd.
As G is bridgeless, $m_i \geq 3$.
Let m be the number of edges with one end in U and the other in $G \setminus U$. then All odd component contribute to at least 3 vertices in m. Hence, the number of component is at most $m/3$
In worse case, all edges with one end in U contribute to m
So $m \leq 3|U| \Rightarrow$ $odd(U) \leq m/3 \leq \frac{3|U|}{3} = |U|$
So by tutte thm, G has a perfect matching ☒

Def: $m(G)$ is used to denote the number of perfect matching in a graph G.

Theorem: For any $\alpha > 0$ there exists a constant $\beta > 0$ such that every n-vertex cubic bridgeless graph has at least $\alpha n - \beta$ perfect matchings.