Pigeon Hole Principles

1. Let $n > 1$ and let $P(x)$ be a polynomial with integer coefficients and degree at most n. Suppose that $|P(x)| < n$ for all $|x| < n^2$. Show that P is constant.
2. Let A be any set of 20 distinct integers chosen from the arithmetic progression 1,4,7,...,100. Prove that there must be two distinct integers in A whose sum is 104.
3. Let there be given nine lattice points (points with integral coordinates) in three dimensional Euclidean space. Show that there is a lattice point on the interior of one of the line segments joining two of these points.) [To test your understanding, how many lattice points does one need in four dimensions to reach the same conclusion?]
4. Given any five points on a sphere, show that some four of them must lie on a closed hemisphere.
5. Show that at any party there are two people who have the same number of friends at the party (assume that all friendships are mutual).
6. Let S be a set of n integers. Show that there is a subset of S, the sum of whose elements is a multiple of n.
7. Show that if 101 integers are chosen from the set $\{1,2,3,\cdots,200\}$ then one of the chosen integers divides another.
8. Show that for some integer $k > 1$, 3k ends with 0001 (in its decimal representation).
9. Let n be a positive integer. Show that there is a positive multiple of n whose digits (in the base 10 representation) are all 0's and 1's.
10. Show that some pair of any 5 points in the unit square will be at most $\sqrt{2}/2$ units apart, and that some pair of any 8 points in the unit square will be at most $\sqrt{5}/4$ units apart.
11. There are 50 baskets of apples. Each basket contains no more than 24 apples. Show that there are at least 3 baskets containing the same number of apples
12. Show that among any 4 numbers one can find 2 numbers so that their difference is divisible by 3.
13. Show that among any n+1 numbers one can find 2 numbers so that their difference is divisible by n.
14. Show that for any natural number n there is a number composed of digits 5 and 0 only and divisible by n.
15. Given 12 different 2-digit numbers, show that one can choose two of them so that their difference is a two-digit number with identical first and second digit.
16. There are five points inside an equilateral triangle of side length 2. Show that at least two of the points are within 1 unit distance from each other.
17. There are 10 (possibly overlapping) small line segments marked on a bigger line segment of length 1. If we add up the lengths of the marked segments, we get 1.1. Show that at least two of the marked segments have a common point

18. There are 13 squares of side 1 positioned inside a circle of radius 2. Show that at least 2 of the squares have a common point
19. In the following fraction every letter represents a different digit. Blueberry/icecream Knowing that the value of the fraction is a real number, find its value. Justify your answer!
20. Every point on the plane is colored either red or blue. Prove that no matter how the coloring is done, there must exist two points, exactly a mile apart, that are the same color.
21. Let S be a region in the plane (not necessarily convex) with area greater than the positive integer n. Show that it is possible to translate S(i.e., slide without turning or distorting) so that S covers at least n+1 lattice points.
22. Let n be a positive integer. Show that if you have n integers, then either one of them is a multiple of n or a sum of several of them is a multiple of n.
23. Prove that from a set of ten distinct two-digit numbers (in the decimal system), it is possible to select two disjoint subsets whose members have the same sum.
24. Color the plane in three colors. Prove that there are two points of the same color one unit apart.
25. Color the plane in two colors. Prove that one of these colors contains pairs of points at every mutual distance.

Useful references:

Coloring: http://www.cut-the-knot.org/proofs/two_color.shtml

Pigeon Hole principles:
https://www.math.uwaterloo.ca/~snew/Contests/ProblemSessions/Problems2015/Lesson2soln.pdf

Arts and craft of problem solving by Paul Zeit : https://kheavan.files.wordpress.com/2010/06/paul-zeitz-author-the-art-and-craft-of-problem-solving-2edwiley20060471789011.pdf

Some small practice problems:
http://euclid.ucc.ie/MATHENR/Exercises/PigeonholePrinciple2Solutions.pdf