An algebraic reduction of Hedetniemi's conjecture

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Abstract

For a graph G, let $\chi(G)$ denote the chromatic number. In graph theory, the following famous conjecture posed by Hedetniemi has been studied: For two graphs G and H, $\chi(G \times H) = \min\{\chi(G), \chi(H)\}$, where $G \times H$ is the tensor product of G and H. In this paper, we give a reduction of Hedetniemi's conjecture to an inclusion relation problem on ideals of polynomial rings, and we demonstrate computational experiments for partial solutions of Hedetniemi's conjecture along such a strategy using Gröbner basis.

Key words and phrases. Hedetniemi's conjecture; tensor product of graphs; Gröbner basis.

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1 Introduction

In this paper, we consider only finite undirected simple graphs. Let G be a graph. Let $V(G)$ and $E(G)$ denote the *vertex set* and the *edge set* of G, respectively. For $u \in V(G)$, let $N_G(u)$ and $d_G(u)$ denote the *neighborhood* and the *degree* of u, respectively; thus $N_G(u) = \{v \in V(G) : uv \in E(G)\}$ and $d_G(u) = |N_G(u)|$. Let $\delta(G)$ denote the *minimum degree* of G. For $u \in V(G)$ and $X \subseteq V(G)$, let $\mathrm{dist}_G(u, X)$ denote the length of a shortest path of G joining u and some vertex in X. For $X \subseteq V(G)$, let $G[X]$ denote the subgraph of G induced by X. A k-subset $X \subseteq V(G)$ is called a *k-clique* (or just a *clique*) of G if $G[X]$ is a complete graph. Let $\omega(G)$ denote the largest positive integer k such that G contains a k-clique. Let K_n and C_n denote the *complete graph* and the *cycle* of order n, respectively. For a positive integer k, a mapping $c : V(G) \to [k]$ is a *proper k-coloring* of G if $c(u) \neq c(v)$ for all adjacent vertices u and v of G, where $[k] = \{1, 2, \dots, k\}$. The smallest positive integer k such that G has a proper k-coloring is called the *chromatic number* of G, and it is denoted by $\chi(G)$. For terms and symbols not defined here, we refer the reader to [3].

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The product operations of graphs have been widely studied because they can produce important illustrations for many graph properties. The readers might find many interesting results in, for example, [11]. In the deep studies for products, some primitive (but essential) problems and conjectures were posed. In this paper, we focus on a classical conjecture concerning the chromatic number of a product of graphs. Let G and H be two graphs. The *tensor product* $G \times H$ of G and H is the graph on $V(G) \times V(H)$ such that two vertices (u, v) and (u', v') are adjacent in $G \times H$ if and only if $uu' \in E(G)$ and $vv' \in E(H)$. For a proper k-coloring c of G, the mapping $c_0 : V(G) \times V(H) \to [k]$ with $c_0(u, v) = c(u)$ $(u \in V(G),\ v \in V(H))$ is clearly a proper k-coloring of $G \times H$. By the symmetry of G and H, this leads to

$$\chi(G \times H) \leq \min\{\chi(G), \chi(H)\}. \tag{1}$$

Hedetniemi [10] conjectured that the equality in (1) always holds for all graphs G and H as follows.

Conjecture 1 (Hedetniemi [10]) *Let G and H be graphs. Then $\chi(G \times H) = \min\{\chi(G), \chi(H)\}$.*

Conjecture 1 has been studied for more than 50 years, and some approaches for the conjecture and its analogies were also studied (see surveys [16, 18]). Hedetniemi [10] verified that Conjecture 1 is true for the case where $\min\{\chi(G), \chi(H)\} \leq 3$, and El-Zahar and Sauer [8] proved that Conjecture 1 is true if $\min\{\chi(G), \chi(H)\} = 4$. On the other hand, Shitov [14] recently constructed counterexamples for Conjecture 1, and he proved that if an integer k is sufficiently large, then there exist infinitely many pairs (G, H) of graphs such that $\min\{\chi(G), \chi(H)\} > k$ and $\chi(G \times H) = k$. We remark that Conjecture 1 is still open for the case where $\min\{\chi(G), \chi(H)\}$ is small. Our aim of this paper is to propose a new effective approach to solve Conjecture 1 for small k's; a reduction of the conjecture to an inclusion relation problem on ideals of polynomial rings. Indeed, we demonstrate computational experiments for partial solutions of the conjecture using Gröbner basis.

We are inspired from the results given by Margulies and Hicks [13] concerning Vizing's conjecture, that is a conjecture on the domination number of Cartesian product of graphs. They also reduced Vizing's conjecture to an inclusion relation problem of ideals. However, the chromatic number and the domination number have major difference for the criticality. When we consider a reduction of graph-theoritical problems to an inclusion relation of ideal, the criticality concerning edge-deletion or vertex-deletion is a useful tool. For a given graph G, although a subgraph of G might have larger domination number than G, deleting a vertex or an edge cannot increase the chromatic number, that is, every graph contains a subgraph with a criticality for the chromatic number. This property gives a strong advantage if we adopt the reduction strategy to Hedetniemi's conjecture.

This paper is organized as follows: In Section 2, we list some known results concerning Conjecture 1. In order to clear the standpoint of the cases treated in our computational experiments, we indicate the cases which force Conjecture 1 to be true from known results in Section 3. Some results proved in Section 3 might been known, but to keep the paper self-contained we give their proofs. Thus readers not interested in its detail are advised to skip the proof. The main results are in Section 4. In Subsection 4.1, we reduce Conjecture 1 to a problem concerning graphs with the criticality for chromatic numbers. Using the reduction, we further reduce the conjecture to an inclusion relation problem on ideals of polynomial rings in Subsection 4.2. In Subsection 4.3, more feasible reductions for computer analysis are considered. In Section 5, we give computational experiments along the strategy developed in Section 4.

2 Preliminary results

In this section, we list some useful results for our argument.

Theorem A (Burr, Erdős and Lovász [4]) *Let $k \geq 2$ be an integer. Let G and H be graphs with $\chi(G) = \chi(H) = k$, and suppose that each vertex of G belongs to a $(k-1)$-clique of G. Then $\chi(G \times H) = k$.*

Theorem B (Duffus, Sands and Woodrow [7]; Welzl [17]) *Let $k \geq 2$ be an integer. Let G and H be graphs with $\chi(G) = \chi(H) = k$, and suppose that both G and H contain $(k-1)$-cliques. Then $\chi(G \times H) = k$.*

As we mentioned in Section 1, the criticality for chromatic number plays a crucial role in this paper. Thus we next focus on such a concept and related results.

A graph G is said to be *k-critical* if $\chi(G) = k$ and $\chi(G') \leq k-1$ for all subgraphs G' of G with $G' \neq G$. In many papers, edge-critical graphs (i.e., graphs G with $\chi(G-e) \leq \chi(G)-1$ for all $e \in E(G)$) and vertex-critical graphs (i.e., graphs G with $\chi(G-u) \leq \chi(G)-1$ for all $u \in V(G)$) are individually considered. Note that the concept of critical graphs defined above contains such two criticality concepts. It is clear that K_k is the unique k-critical graph if $k \in \{1, 2\}$. Furthermore, a graph is 3-critical if and only if the graph is an odd cycle. On the other hand, nobody knows an explicit characterization of 4-critical graphs, and 4-critical graphs have been studied.

For two vertex-disjoint graphs G_1 and G_2, the *join* of G_1 and G_2, denoted by G_1+G_2, is obtained from G_1 and G_2 by joining each vertex of G_1 to all vertices of G_2. A graph G is *decomposable* if the complement $\overline{G}$ of G is disconnected. A non-decomposable graph is said to be *indecomposable*. We can easily verify that a k-critical graph G is decomposable if and only if G is the join of a k_1-critical graph G_1 and a k_2-critical graph G_2 with $k_1 + k_2 = k$. On the other hand, indecomposable critical graphs have many vertices as follows (here the second statement was proved by Gallai [9]):

Theorem C (Stehlík [15]) *Let $k \geq 3$ be an integer, and let G be an indecomposable k-critical graph. Then for any $u \in V(G)$, $G-u$ has a proper $(k-1)$-coloring such that every color class contains at least two vertices. In particular, $|V(G)| \geq 2k-1$.*

Furthermore, the following result closely related to the k-criticality is well-known and we can find it in many textbooks of graph theory (for example, in [3, Theorem 14.7]).

Theorem D *For a positive integer k, every k-critical graph G satisfies $\delta(G) \geq k-1$.*

3 Hedetniemi's conjecture for small graphs

In this section, we focus on small graphs G and H satisfying Conjecture 1 and finally prove the following theorem.

Theorem 3.1 *Let $k \geq 5$ be an integer. Let G and H be graphs with $\min\{\chi(G), \chi(H)\} = k$, and suppose that*

(i) $\min\{|V(G)|, |V(H)|\} \leq k+2$;

(ii) $|V(G)| = |V(H)| = k+3$; *or*

(iii) $k = 5$ *and* $(|V(G)|, |V(H)|) \in \{(8,8), (8,9), (8,10), (9,8), (10,8)\}$.

Then $\chi(G \times H) = k$.

By Theorem 3.1, the first nontrivial cases for Conjecture 1 are

- $|V(G)| = 8$ and $|V(H)| = 11$ if $k = 5$; and
- $|V(G)| = k+3$ and $|V(H)| = k+4$ if $k \geq 6$.

We can refine the latter case as follows. (Here, for a graph H, we regard $K_0 + H$ as H.)

Theorem 3.2 *Let $k \geq 6$ be an integer. Then all graphs G and H with $|V(G)| \leq k+3$, $|V(H)| \leq k+4$ and $\min\{\chi(G), \chi(H)\} = k$ satisfy $\chi(G \times H) = k$ if and only if*

$$\chi((K_{k-4} + H_0) \times (K_{k-6} + C_5 + C_5)) = k,$$

where H_0 denotes the graph depicted in Figure 1.

The following theorem is a useful tool in the proof of our argument.

Theorem E (Chvátal [5]; Jensen and Royle [12]) *For $k \in \{4,5\}$, if a K_{k-1}-free graph G satisfies $\chi(G) = k$, then $|V(G)| \geq 11$.*

We first prove that K_k is the unique k-critical graph of order at most $k+1$.

Lemma 3.3 *Let $k \geq 1$ be an integer, and let G be a k-critical graph of order at most $k+1$. Then $G = K_k$.*

Proof. It is clear that if $|V(G)| \leq k$, then $G = K_k$. Thus it suffices to show that $|V(G)| \neq k+1$. By way of contradiction, suppose that $|V(G)| = k+1$. Let c be a proper k-coloring of G. We may assume that $|c^{-1}(i)| = 1$ for every i $(1 \leq i \leq k-1)$ (and so $|c^{-1}(k)| = 2$). Note that $G[\bigcup_{i=1}^{k-1} c^{-1}(i)]$ is a complete graph. Write $c^{-1}(i) = \{u_i\}$ for each i $(1 \leq i \leq k-1)$ and $c^{-1}(k) = \{v_1, v_2\}$. If for each $j \in \{1,2\}$, there exists a vertex $w_j \in \{u_i : 1 \leq i \leq k-1\}$ with $v_j w_j \notin E(G)$, then the mapping $c' : V(G) \to [k-1]$ with

$$c'(a) = \begin{cases} c(a) & (a \notin \{v_1, v_2\}) \\ c(w_j) & (a = v_j) \end{cases}$$

is a proper $(k-1)$-coloring of G, which contradicts the fact that $\chi(G) = k$. Thus, without loss of generality, we may assume that $N_G(v_1) = V(G) \setminus \{v_1, v_2\}$. Then $G - v_2$ is a complete graph of order k, and so $\chi(G - v_2) = k$, which contradicts the fact that G is k-critical. $\square$

Let $\mathcal{A}_4$ be the family of 4-critical graphs of order 7. Then every k-critical graph with at most $k+3$ vertices can be characterized as follows.

Lemma 3.4 *For an integer $k \geq 3$, a graph G of order at most $k+3$ is k-critical if and only if*

(i) $G = K_k$;

(ii) $G = K_{k-3} + C_5$; *or*

(iii) $k \geq 4$ *and* $G = K_{k-4} + A$ *for a graph* $A \in \mathcal{A}_4$.

Proof. The "if" part is trivial. Thus we show the "only if" part. Since a graph is 3-critical if and only if it is an odd cycle, the lemma holds for $k = 3$. Thus we may assume that $k \geq 4$.

Claim 3.1 *Let $l \geq 4$ be an integer, and let H be an l-critical graph of order at most $l+3$. Then either $(l, |V(H)|) = (4,7)$ or $H = K_{l-l_0} + H'$ for an l_0-critical graph H' with $1 \leq l_0 \leq l-1$.*

Proof. Note that $|V(H)| \leq l+3$ and $l+3 \leq 2l-1$ (i.e. $l \geq 4$) holds by our assumption. Hence, we have $|V(H)| \leq l+3 \leq 2l-1$ and all the equalities hold if and only if $(l, |V(H)|) = (4,7)$. Thus, to prove the claim, we may assume $|V(H)| < 2l-1$. Then by Theorem C, H is decomposable, and hence $H = H_1 + H_2$ for an l_1-critical graph H_1 and l_2-critical graph H_2 with $l_1 + l_2 = l$. If both H_1 and H_2 are non-complete, then it follows from Lemma 3.3 that $|V(H_i)| \geq l_i + 2$ $(i \in \{1,2\})$, and so $|V(H)| = |V(H_1)| + |V(H_2)| \geq (l_1+2) + (l_2+2) > l+3$, which is a contradiction. Thus we may assume that H_1 is complete. Then $H_1 = K_{l_1} = K_{l-l_2}$, as desired. $\square$

By Lemma 3.3, we may assume that $|V(G)| \in \{k+2, k+3\}$, and so G is non-complete. If $(k, |V(G)|) = (4,7)$, then (iii) holds. Thus we may assume that $(k, |V(G)|) \neq (4,7)$. Then by Claim 3.1, $G = K_{k-k_0} + G'$ for a k_0-critical graph G' with $1 \leq k_0 \leq k-1$. Note that G' is non-complete. Choose k_0 and G' so that $|V(G')|$ is as small as possible.

In the case $k_0 \leq 3$, since G' is non-complete, $k_0 = 3$ and G' is an odd cycle of order at least 5. Since $k-3 = k-k_0 = |V(G) \setminus V(G')| \in \{k+2-|V(G')|, k+3-|V(G')|\}$, it follows that $|V(G)| = k+2$ and $|V(G')| = 5$, i.e., $G = K_{k-3} + C_5$, which implies (ii).

Let $k_0 \geq 4$. Note that $k \geq 5$. Since $|V(G')| = |V(G)| - (k-k_0) \leq k+3-(k-k_0) = k_0+3$, it follows from Claim 3.1 that either $(k_0, |V(G')|) = (4,7)$ or $G' = K_{k_0-k_1} + G''$ for a k_1-critical graph G'' with $1 \leq k_1 \leq k_0 - 1$. If the latter holds, then G is the join of a complete graph of order $k - k_1$ and G'', which contradicts the choice of k_0 and G'. Thus $(k_0, |V(G')|) = (4,7)$, i.e., G is the join of K_{k-k_0} $(= K_{k-4})$ and G' belonging to $\mathcal{A}_4$, which implies (iii). $\square$

We will use Lemma 3.4 to prove Theorem 3.1. We can verify that $\mathcal{A}_4$ consists of graphs $H_0, H_1, \ldots, H_6$ depicted in Figure 1, and so Lemma 3.4 gives a complete characterization of small k-critical graphs. However, the characterization of $\mathcal{A}_4$ might be proved by tedious argument (or computer search), and so we omit the detail. Indeed, in order to prove Theorem 3.1, it suffices to prove a more restricted characterization as follows.

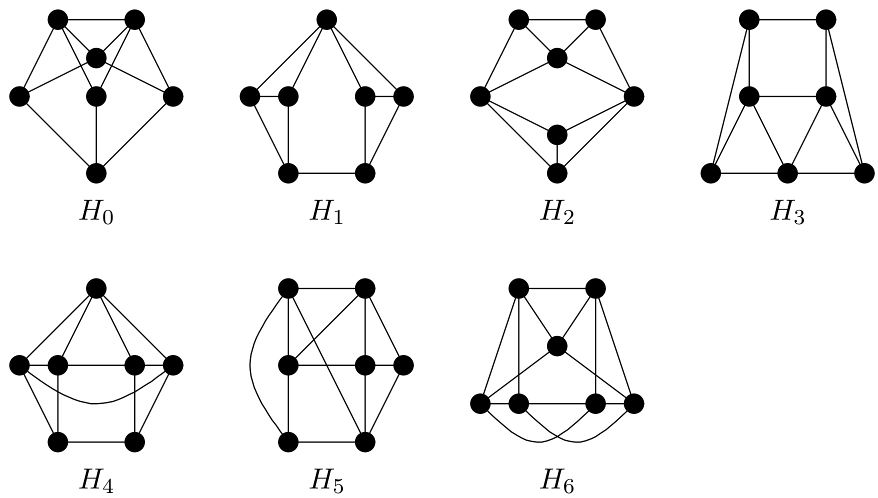

Figure 1: The 4-critical graphs of order 7.

Lemma 3.5 *If a graph $G \in \mathcal{A}_4$ has a vertex u belonging to no triangle, then $G = H_0$.*

Proof. By Theorem D, we have

$$\delta(G) \geq 3. \tag{2}$$

Suppose that G contains no triangle. Since G has no proper 2-coloring, G contains an odd cycle of order at least 5. This together with (2) implies that G contains an induced odd cycle C of order 5 and $N_G(v) \setminus V(C) \neq \emptyset$ for all $v \in V(C)$. Since $|V(G) \setminus V(C)| = 2$, a vertex in $V(G) \setminus V(C)$ is adjacent to two consecutive vertices on C, and so G contains a triangle, which contradicts the assumption that G contains no triangle.

Thus, G contains a triangle $T = v_1v_2v_3v_1$. Choose T so that $\mathrm{dist}_G(u, V(T))$ is as large as possible. By the definition of u, we have $\mathrm{dist}_G(u, V(T)) \geq 1$.

Suppose that $\mathrm{dist}_G(u, V(T)) = 1$. Note that $|N_G(u) \cap V(T)| = 1$. Without loss of generality, we may assume that $N_G(u) \cap V(T) = \{v_1\}$. If $V(G) \setminus \{u, v_2, v_3\} \subseteq N_G(u)$, then by (2) and the definition of u, G is a graph depicted in Figure 2, and so G has a proper 3-coloring, which is a

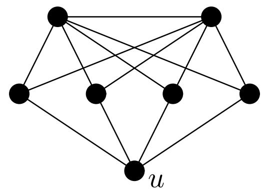

Figure 2: A graph appearing in the proof of Lemma 3.5.

contradiction. Thus $V(G) \setminus \{u, v_2, v_3\} \not\subseteq N_G(u)$. This together with (2) implies that $d_G(u) = 3$ and $V(G) \setminus (\{u, v_1, v_2\} \cup N_G(u))$ contains exactly one vertex, say x. If $\{v_2, v_3\} \subseteq N_G(x)$, then $\mathrm{dist}_G(u, xv_2v_3x\}) \geq 2$, which contradicts the choice of T. Thus, without loss of generality, we may assume that $xv_2 \notin E(G)$. Now we consider the mapping $c : V(G) \to [3]$ with

$$c(a) = \begin{cases} 1 & (a \in \{u, v_2, x\}) \\ 2 & (a = v_3) \\ 3 & (a \in N_G(u)). \end{cases}$$

Then c is a proper 3-coloring of G because $\{u, v_2, x\}$ and $N_G(u)$ are independent sets of G, which is a contradiction. Thus we may assume that $\mathrm{dist}_G(u, V(T)) \geq 2$.

Note that $V(G) = \{u\} \cup N_G(u) \cup V(T)$ and $d_G(u) = 3$. Write $N_G(u) = \{w_1, w_2, w_3\}$. For $i \in \{1, 2, 3\}$, if $V(T) \subseteq N_G(w_i)$, then the subgraph of G induced by $V(T) \cup \{w_i\}$ is a complete graph of order 4, and so $\chi(G - u) \geq 4$, which contradicts the 4-criticality of G. Since $\delta(G) \geq 3$, this implies that $|V(T) \cap N_G(w_i)| = 2$ for each $i \in \{1, 2, 3\}$. Suppose that $V(T) \cap N_G(w_i) = V(T) \cap N_G(w_{i'})$ for $1 \leq i < i' \leq 3$. By the symmetry, we may assume that $V(T) \cap N_G(w_1) = V(T) \cap N_G(w_2) = \{v_1, v_2\}$ and $V(T) \setminus N_G(w_3) = \{v_j\}$ for $j \in \{2, 3\}$. Now we consider the mapping $c' : V(G) \to [3]$ with

$$c'(a) = \begin{cases} 1 & (a \in \{u, v_1\}) \\ 2 & (a = v_2) \\ 3 & (a \in \{v_3, w_1, w_2\}) \\ j & (a = w_3). \end{cases}$$

Then we can easily verify that c' is a proper 3-coloring of G, which is a contradiction. Thus $V(T) \cap N_G(w_i) \neq V(T) \cap N_G(w_{i'})$ for all $1 \leq i < i' \leq 3$. This implies that $G = H_0$. $\square$

Lemma 3.6 *Let $k \geq 5$ be an integer, and let n be a positive integer. Then all graphs G and H with $|V(G)| \leq k+3$, $|V(H)| \leq n$ and $\min\{\chi(G), \chi(H)\} = k$ satisfy $\chi(G \times H) = k$ if and only if all K_{k-1}-free k-critical graphs H with $k+4 \leq |V(H)| \leq n$ satisfy $\chi((K_{k-4} + H_0) \times H) = k$.*

Proof. The "only if" part is trivial. Thus we show the "if" part. We suppose that

$$\text{all } K_{k-1}\text{-free } k\text{-critical graphs } H \text{ with } k+4 \leq |V(H)| \leq n \text{ satisfy } \chi((K_{k-4} + H_0) \times H) = k. \quad (3)$$

Let G' and H' be k-critical subgraphs of G and H, respectively. Since $G' \times H'$ is a subgraph of $G \times H$, we have $\chi(G \times H) \geq \chi(G' \times H')$. Considering (1), it suffices to show that $\chi(G' \times H') = k$.

We first assume that $\min\{|V(G')|, |V(H')|\} \leq k+2$. Without loss of generality, we may assume that $|V(G')| \leq k+2$. Then by Lemma 3.4, G' is either K_k or $K_{k-3} + C_5$. In particular, each vertex of G' belongs to a $(k-1)$-clique of G'. Hence by Theorem A, $\chi(G' \times H') = k$, as desired. Thus we may assume that $|V(G')| = k+3$ and $|V(H')| \geq k+3$.

By Lemma 3.4, $G' = K_{k-4} + A$ for some $A \in \mathcal{A}_4$. Suppose that $A \neq H_0$. Then by Lemma 3.5, each vertex of A belongs to a triangle. Since $G' = K_{k-4} + A$, this implies that each vertex of G' belongs to a $(k-1)$-clique of G'. This together with Theorem A implies that $\chi(G' \times H') = k$. Thus we may assume that $A = H_0$.

If H' contains $(k-1)$-clique, then both G' and H' contain $(k-1)$-cliques, and hence by Theorem B, $\chi(G' \times H') = k$, as desired. Thus we may assume that H' is K_{k-1}-free. If $|V(H')| = k+3$, then by similar argument in the previous paragraph, we have $H' = K_{k-4} + H_0$, which contradicts the K_{k-1}-freeness of H'. Thus $k+4 \leq |V(H')| \leq |V(H)| \leq n$. Then by (3), $\chi(G' \times H') = k$. □

Now we prove Theorem 3.1.

Proof of Theorem 3.1. Applying Lemma 3.6 with $n = k+3$, we obtain that if one of the assumptions (i) and (ii) of the theorem holds, then $\chi(G \times H) = k$. Thus we may assume that $k = 5$ and $(|V(G)|, |V(H)|) \in \{(8,8), (8,9), (8,10), (9,8), (10,8)\}$. We may assume that $|V(G)| = 8$. Then by Lemma 3.6 with $n = 10$, it suffices to show that all K_4-free 5-critical graphs H' with $9 \leq |V(H)| \leq 10$ satisfy $\chi((K_1 + H_0) \times H) = 5$. However, it follows from Theorem E that every K_4-free 5-critical graph has at least 11 vertices, and so there is no target graph.

This completes the proof of Theorem 3.1. □

To prove Theorem 3.2, we prepare the following lemma.

Lemma 3.7 *For an integer $k \geq 6$, a K_{k-1}-free graph G of order $k+4$ is k-critical if and only if $G = K_{k-6} + C_5 + C_5$.*

Proof. The "if" part is trivial. Thus we show the "only if" part by induction on k. Note that $|V(G)| = k+4$ and $k+4 < 2k-1$ (i.e. $k \geq 6$) holds by our assumption. Hence, we have $|V(G)| = k+4 < 2k-1$. Then G is decomposable by Theorem C, and hence $G = G_1 + G_2$ for a

k_1-critical graph G_1 and a k_2-critical graph G_2 with $k_1 + k_2 = k$ and $k_1 \geq k_2$. Choose G_1 and G_2 so that k_2 is as small as possible.

For the moment, we suppose that G_1 and G_2 are non-complete. Then by Lemma 3.3, $|V(G_i)| \geq k_i + 2$ for each $i \in \{1, 2\}$. Since

$$k + 4 = |V(G)| = |V(G_1)| + |V(G_2)| \geq (k_1 + 2) + (k_2 + 2) = k + 4,$$

we have $|V(G_i)| = k_i + 2$. It follows from Lemma 3.4 that $G_1 = K_{k_1-3} + C_5$ and $G_2 = K_{k_2-3} + C_5$. Hence, $G = G_1 + G_2 = K_{k_1+k_2-6} + C_5 + C_5$, as desired.

If $k = 6$, then G_1 is a K_4-free 5-critical graph of order $|V(G) \setminus V(G_2)|$ $(= 9)$, which contradicts Theorem E. Thus $k \geq 7$ (and the first step of the induction is completed). Since G_1 is a K_{k-2}-free $(k-1)$-critical graph of order $k+3$, we have $G_1 = K_{(k-1)-6} + C_5 + C_5$ by the induction hypothesis. Consequently, $G = G_1 + K_1 = K_{k-6} + C_5 + C_5$, as desired. $\square$

Combining Lemmas 3.6 and 3.7, we obtain Theorem 3.2.

4 Algebraic reduction of Conjecture 1

4.1 Equivalence conjecture for Conjecture 1 via the criticality

In this subsection, we focus on the following conditions for given graphs G and H:

(X1) $\chi(G \times H) \leq k - 1$;

(W1) $\min\{\chi(G), \chi(H)\} \leq k - 1$;

(V1) $\delta(G) \geq 1$ and there exists a proper $(k-1)$-coloring c of $G - uv$ with $c(u) = c(v) = 1$ for all $uv \in E(G)$;

(V2) $\delta(H) \geq 1$ and there exists a proper $(k-1)$-coloring c' of $H - u'v'$ with $c'(u') = c'(v') = 1$ for all $u'v' \in E(H)$;

(V3) there exists a vertex of G belonging to no $(k-1)$-clique of G;

(V4) there exists a vertex of H belonging to no $(k-1)$-clique of H;

(V5) $\max\{\omega(G), \omega(H)\} \leq k - 1$ and $\min\{\omega(G), \omega(H)\} \leq k - 2$;

(V6) $\delta(G) \geq k - 1$; and

(V7) $\delta(H) \geq k - 1$.

Note that the conditions (V1)–(V7) derive from the definitions or the previous results in Section 2 as follows:

- (V1) and (V2) derive from the definition of the criticallity.
- If (V3) or (V4) is not satisfied for graphs G and H with $\chi(G) = \chi(H) = k$, then Conjecture 1 is automatically true by Theorem A.
- Assume G and H are k-critical graphs. On (V5), if $\max\{\omega(G), \omega(H)\} \geq k$, say, $\omega(G) \geq k$, then we see that G should be a complete graph of order k by the criticality of G, so Conjecture 1 is automatically true by Theorem A. If $\min\{\omega(G), \omega(H)\} \geq k-1$, then Conjecture 1 is automatically true by Theorem B.

- (V6) and (V7) always hold for k-critical graphs G and H, respectively, by Theorem D.

For an integer $n \geq 1$, let $\mathcal{G}_n$ be the set of graphs of order at most n. We define two sets as follows:

$$W_{k,n,n'} = \{(G,H) \in \mathcal{G}_n \times \mathcal{G}_{n'} : (G,H) \text{ satisfies (X1) and (W1)}\}; \text{ and}$$
$$V_{k,n,n'} = \{(G,H) \in \mathcal{G}_n \times \mathcal{G}_{n'} : (G,H) \text{ satisfies (X1) and (V1)–(V7)}\}.$$

The following is the key proposition for our argument.

Proposition 4.1 *For integers $k \geq 2$, $n \geq 1$ and $n' \geq 1$, the following are equivalent:*
(H1) $V_{k,n,n'} \subseteq W_{k,n,n'}$;
(H2) *if $G \in \mathcal{G}_n$ and $H \in \mathcal{G}_{n'}$ are k-critical, then $\chi(G \times H) = k$; and*
(H3) *if $G \in \mathcal{G}_n$ and $H \in \mathcal{G}_{n'}$ satisfy $\min\{\chi(G), \chi(H)\} = k$, then $\chi(G \times H) = k$.*

Proof. We first prove "(H1) $\Rightarrow$ (H2)". Suppose that (H1) holds and there exist k-critical graphs $G \in \mathcal{G}_n$ and $H \in \mathcal{G}_{n'}$ such that $\chi(G \times H) \leq k-1$ (i.e., (X1) holds). By the k-criticality of G and H, (V1) and (V2) clearly hold. If each vertex of G belongs to a $(k-1)$-clique of G, then by Theorem A, $\chi(G \times H) = k$, a contradiction. Thus both (V3) and (V4) hold. If one of G and H contains a k-clique, then by its k-criticality, it is a complete graph of order k, which contradicts (V3) or (V4). Thus $\max\{\omega(G), \omega(H)\} \leq k-1$. If $\min\{\omega(G), \omega(H)\} \geq k-1$, then by Theorem B, $\chi(G \times H) = k$, a contradiction. Therefore (V5) holds. Furthermore, it follows from Theorem D that (V6) and (V7) hold. Consequently, we have $(G,H) \in V_{k,n,n'}$. By our assumption, we have $(G,H) \in W_{k,n,n'}$. In particular, $\min\{\chi(G), \chi(H)\} \leq k-1$, which contradicts the assumption that G and H are k-critical.

We next prove "(H2) $\Rightarrow$ (H1)". Suppose that (H2) holds. Let $(G,H) \in V_{k,n,n'}$. Then G and H satisfy (X1) and (V1)–(V7). If $\chi(G) \geq k$ and $\chi(H) \geq k$, then the conditions (V1) and (V2) force both G and H to be k-critical, and hence $\chi(G \times H) = k$ by (H2), which contradicts (X1). Thus $\chi(G) \leq k-1$ or $\chi(H) \leq k-1$. In particular, G and H satisfy (W1), and so $(G,H) \in W_{k,n,n'}$.

Finally, we prove "(H2) $\Leftrightarrow$ (H3)". Since "(H3) $\Rightarrow$ (H2)" trivially holds, it suffices to show that "(H2) $\Rightarrow$ (H3)" holds. Suppose that (H2) holds. Let $G \in \mathcal{G}_n$ and $H \in \mathcal{G}_{n'}$ be graphs with $\min\{\chi(G), \chi(H)\} = k$. Then G contains a k-critical subgraph G' and H contains a k-critical subgraph H'. By (H2), we have $\chi(G' \times H') = k$, and hence $\chi(G \times H) \geq \chi(G' \times H') = k$ because $G' \times H'$ is a subgraph of $G \times H$. This together with (1) implies that $\chi(G \times H) = k$.

This completes the proof of the proposition. $\square$

By Theorem 3.1 and Proposition 4.1, we can translate Conjecture 1 into an inclusion relation problem concerning $W_{k,n,n'}$ and $V_{k,n,n'}$ as follows.

Corollary 4.2 *Let $k \geq 3$, $n \geq 1$ and $n' \geq 1$ be integers. Then the following are equivalent:*

(i) *Conjecture 1 is true for the case where $\min\{\chi(G), \chi(H)\} = k$;*

(ii) $V_{k,n,n'} \subseteq W_{k,n,n'}$ *for any integers $n \geq k+3$ and $n' \geq k+3$ with $(n,n') \neq (k+3,k+3)$.*

4.2 Equivalence conjecture for Conjecture 1 via ideals of polynomial rings

Throughout this section, we fix integers $k \geq 3$, $n \geq 1$ and $n' \geq 1$. We start with an easy algebraic proposition.

Proposition 4.3 *Let e, x_1 and x_2 be three variables satisfying $e(e-1) = 0$ and $x_1^k - 1 = x_2^k - 1 = 0$. Then $e(x_1^{k-1} + x_1^{k-2}x_2 + \cdots + x_2^{k-1}) = 0$ if and only if $e = 0$ or $x_1 \neq x_2$.*

Proof. Note that $x_i \neq 0$ for each $i \in \{1, 2\}$. If $x_1 \neq x_2$, then $x_1^{k-1} + x_1^{k-2}x_2 + \cdots + x_2^{k-1} = 0$ because

$$0 = (x_1^k - 1) - (x_2^k - 1) = (x_1 - x_2)(x_1^{k-1} + x_1^{k-2}x_2 + \cdots + x_2^{k-1}).$$

Conversely, if $x_1 = x_2$, then $x_1^{k-1} + x_1^{k-2}x_2 + \cdots + x_2^{k-1} = kx_1^{k-1} \neq 0$. Hence $x_1 \neq x_2$ if and only if $x_1^{k-1} + x_1^{k-2}x_2 + \cdots + x_2^{k-1} = 0$, which proves the proposition. $\square$

We prepare the variables

$$x_1, \ldots, x_n, e_{ij}\ (1 \leq i < j \leq n), \quad y_1, \ldots, y_{n'}, f_{i'j'}\ (1 \leq i' < j' \leq n'), \quad z_{ii'}\ (1 \leq i \leq n,\ 1 \leq i' \leq n').$$

Then, considering Proposition 4.3, we obtain the following: The solutions of system of equations

$$\begin{cases} e_{ij}(e_{ij} - 1) = 0 \quad (1 \leq i < j \leq n) \\ x_i^k - 1 = 0 \quad (1 \leq i \leq n) \\ e_{ij}(x_i^{k-1} + x_i^{k-2}x_j + \cdots + x_j^{k-1}) = 0 \quad (1 \leq i < j \leq n) \end{cases}$$

one-to-one correspond to the pairs of a labeled graph on $[n]$ and its proper k-coloring; the solutions of system of equations

$$\begin{cases} f_{i'j'}(f_{i'j'} - 1) = 0 \quad (1 \leq i' < j' \leq n') \\ y_{i'}^k - 1 = 0 \quad (1 \leq i' \leq n') \\ f_{i'j'}(y_{i'}^{k-1} + y_{i'}^{k-2}y_{j'} + \cdots + y_{j'}^{k-1}) = 0 \quad (1 \leq i' < j' \leq n') \end{cases}$$

one-to-one correspond to the pairs of a labeled graph on $[n']$ and its proper k-coloring; and the solutions of system of equations

$$\begin{cases} e_{ij}(e_{ij} - 1) = 0 \quad (1 \leq i < j \leq n) \\ f_{i'j'}(f_{i'j'} - 1) = 0 \quad (1 \leq i' < j' \leq n') \\ z_{ii'}^k - 1 = 0 \quad (1 \leq i \leq n,\ 1 \leq i' \leq n') \\ e_{ij}f_{i'j'}(z_{ii'}^{k-1} + z_{ii'}^{k-2}z_{jj'} + \cdots + z_{jj'}^{k-1}) = 0 \quad (1 \leq i < j \leq n,\ 1 \leq i' < j' \leq n') \end{cases}$$

one-to-one correspond to the pairs of the tensor product of labeled graphs on $[n]$ and $[n']$ and its proper k-coloring. We explain an outline of, for example, the first fact. Consider a graph on $[n]$. We regard a solution of $e_{ij}(e_{ij} - 1) = 0$ $(1 \leq i < j \leq n)$ as its adjacency matrix, and a solution of $x_i^k - 1 = 0$ as a color assigned to the vertex i (here x_i can take exactly k solutions because x_i is a k-th root of unity). Then $e_{ij}(x_i^{k-1} + x_i^{k-2}x_j + \cdots + x_j^{k-1}) = 0$ $(1 \le i < j \le n)$ implies by Proposition 4.3 that if two vertices i and j are adjacent, then the color assigned to i differs from the color assigned to j.

Based on the above facts, we associate solutions of some systems of equations with the members in $W_{k,n,n'}$ and $V_{k,n,n'}$ appearing in Subsection 4.1.

Description of $W_{k,n,n'}$

All the ideals below (i.e., the ideals $E_{n,n'}$, $X_{k,n,n'}$, $Z_{k,n,n'}$, $I_{k,n}$, $I'_{k,n'}$ and $J_{k,n,n'}$) are regarded as the ones of the polynomial ring

$$\mathbb{C}[e_{ij}, f_{i'j'}, x_s, y_{s'}, z_{ss'} : 1 \le i < j \le n, 1 \le i' < j' \le n', 1 \le s \le n, 1 \le s' \le n']$$

of $(\binom{n}{2} + \binom{n'}{2} + n + n' + nn')$ variables.

We define several ideals as follows:

$$\begin{aligned}
E_{n,n'} &= (e_{ij}(e_{ij}-1) : 1 \le i < j \le n) + (f_{i'j'}(f_{i'j'}-1) : 1 \le i' < j' \le n'), && (E(G) \text{ and } E(H))\\
X_{k,n,n'} &= (x_i^{k-1} - 1 : 1 \le i \le n) + (y_{i'}^{k-1} - 1 : 1 \le i' \le n'), && ((k-1)\text{-colorings of } G \text{ and } H)\\
Z_{k,n,n'} &= (z_{ii'}^{k-1} - 1 : 1 \le i \le n, 1 \le i' \le n'), && ((k-1)\text{-coloring of } G \times H)\\
I_{k,n} &= (e_{ij}(x_i^{k-2} + x_i^{k-3}x_j + \cdots + x_j^{k-2}) : 1 \le i < j \le n), && (\mathrm{W1}(G))\\
I'_{k,n'} &= (f_{i'j'}(y_{i'}^{k-2} + y_{i'}^{k-3}y_{j'} + \cdots + y_{j'}^{k-2}) : 1 \le i' < j' \le n'). && (\mathrm{W1}(H))
\end{aligned}$$

Note that the solutions of $E_{n,n'} + X_{k,n,n'} + I_{k,n} \cdot I'_{k,n'}$ one-to-one correspond to the pairs of graphs $(G, H) \in \mathcal{G}_n \times \mathcal{G}_{n'}$ satisfying (W1) with their proper $(k-1)$-colorings. Furthermore, let

$$J_{k,n,n'} = (e_{ij}f_{i'j'}(z_{ii'}^{k-2} + z_{ii'}^{k-3}z_{jj'} + \cdots + z_{jj'}^{k-2}) : 1 \le i < j \le n, 1 \le i' < j' \le n'). \qquad (\mathrm{X1})$$

Let

$$\mathfrak{J}_{k,n,n'} = E_{n,n'} + X_{k,n,n'} + Z_{k,n,n'} + I_{k,n} \cdot I'_{k,n'} + J_{k,n,n'}$$

be the ideal, and set

$$\tilde{\mathfrak{J}}_{k,n,n'} = \mathfrak{J}_{k,n,n'} \cap \mathbb{C}[e_{ij}, f_{i'j'}]. \qquad (W_{k,n,n'})$$

Then we can verify that the solutions of $\tilde{\mathfrak{J}}_{k,n,n'}$ one-to-one correspond to the members of $W_{k,n,n'}$.

Description of $V_{k,n,n'}$

All the ideals below (i.e., the ideals appearing in $\mathfrak{I}_{k,n,n'}$) are regarded as the ideals of the polynomial ring

$$\begin{gathered}\mathbb{C}[e_{ij}, f_{i'j'}, x_{pq\ell}, y_{p'q'\ell'}, z_{ss'} : 1 \le i < j \le n, 1 \le i' < j' \le n', 1 \le p < q \le n, 1 \le p' < q' \le n',\\ 1 \le \ell \le n, 1 \le \ell' \le n', 1 \le s \le n, 1 \le s' \le n']\end{gathered}$$

of $(\binom{n}{2}(n+1)+\binom{n'}{2}(n'+1)+nn')$ variables.

We define several ideals as follows:

$$\begin{aligned}P_{k,n} = (&(\prod_{1\le j<i}(e_{ji}-1))\cdot(\prod_{i<j\le n}(e_{ij}-1)) : 1\le i\le n)\\&+(x_{pqi}^{k-1}-1 : 1\le p<q\le n,\ 1\le i\le n)+(x_{pqi}-1 : 1\le p<q\le n,\ i\in\{p,q\})\\&+(e_{pq}e_{ij}(x_{pqi}^{k-2}+x_{pqi}^{k-3}x_{pqj}+\cdots+x_{pqj}^{k-2}) : 1\le p<q\le n,\ 1\le i<j\le n,\ (p,q)\ne(i,j)),\end{aligned}\tag{V1}$$

$$\begin{aligned}P'_{k,n'} = (&(\prod_{1\le j'<i'}(f_{j'i'}-1))\cdot(\prod_{i'<j'\le n'}(f_{i'j'}-1)) : 1\le i'\le n')\\&+(y_{p'q'i'}^{k-1}-1 : 1\le p'<q'\le n',\ 1\le i'\le n')+(y_{p'q'i'}-1 : 1\le p'<q'\le n',\ i'\in\{p',q'\})\\&+(f_{p'q'}f_{i'j'}(y_{p'q'i'}^{k-2}+y_{p'q'i'}^{k-3}y_{p'q'j'}+\cdots+y_{p'q'j'}^{k-2}) : 1\le p'<q'\le n',\ 1\le i'<j'\le n',\ (p',q')\ne(i',j')),\end{aligned}\tag{V2}$$

$$Q_{k,n} = ((\prod_{i\in X}e_{1i})\cdot(\prod_{\substack{i,j\in X\\ i<j}}e_{ij}) : X\subseteq[n]\setminus\{1\}\text{ with }|X|=k-2),\tag{V3}$$

$$Q'_{k,n'} = ((\prod_{i'\in X'}f_{1i'})\cdot(\prod_{\substack{i',j'\in X'\\ i'<j'}}f_{i'j'}) : X'\subseteq[n']\setminus\{1\}\text{ with }|X'|=k-2),\tag{V4}$$

$$R_{k,n} = (\prod_{\substack{i,j\in X\\ i<j}}e_{ij} : X\subseteq[n]\text{ with }|X|=k),\tag{$\omega(G)\le k-1$}$$

$$R'_{k,n'} = (\prod_{\substack{i',j'\in X'\\ i'<j'}}f_{i'j'} : X'\subseteq[n']\text{ with }|X'|=k).\tag{$\omega(H)\le k-1$}$$

Note that the condition that $\omega(G)\le k-1$ *and* $\omega(H)\le k-1$ hold is equivalent to $\max\{\omega(G),\omega(H)\}\le k-1$, while the condition that $\omega(G)\le k-2$ *or* $\omega(H)\le k-2$ holds is equivalent to $\min\{\omega(G),\omega(H)\}\le k-2$. Let

$$S_{k,n} = ((\prod_{\substack{i\in X\\ i<\ell}}(e_{i\ell}-1))\cdot(\prod_{\substack{i\in X\\ \ell<i}}(e_{\ell i}-1)) : \ell\in[n],\ X\subseteq[n]\setminus\{\ell\}\text{ with }|X|=n-k+1),\tag{V6}$$

$$S'_{k,n'} = ((\prod_{\substack{i'\in X'\\ i'<\ell'}}(f_{i'\ell'}-1))\cdot(\prod_{\substack{i'\in X'\\ \ell'<i'}}(f_{\ell'i'}-1)) : \ell'\in[n'],\ X'\subseteq[n']\setminus\{\ell'\}\text{ with }|X'|=n'-k+1).\tag{V7}$$

Furthermore, let

$$\begin{aligned}\mathcal{I}_{k,n,n'} = {}&E_{n,n'}+Z_{k,n,n'}+J_{k,n,n'}+P_{k,n}+P'_{k,n'}+Q_{k,n}+Q'_{k,n'}\\&+\underbrace{R_{k,n}+R'_{k,n'}+R_{k-1,n}\cdot R'_{k-1,n'}}_{\text{(V5)}}+S_{k,n}+S'_{k,n'}\end{aligned}$$

and set

$$\tilde{\mathfrak{I}}_{k,n,n'} = \mathfrak{I}_{k,n,n'} \cap \mathbb{C}[e_{ij}, f_{i'j'}]. \tag{$V_{k,n,n'}$}$$

Then we can verify that the solutions of $\tilde{\mathfrak{I}}_{k,n,n'}$ one-to-one correspond to the members of $V_{k,n,n'}$.

Consequently, it follows from Proposition 4.1 that the following theorem holds.

Theorem 4.4 *Let $k \geq 3$, $n \geq 1$ and $n' \geq 1$ be integers. Then the following are equivalent:*

(i) *Conjecture 1 is true for the case where $G \in \mathcal{G}_n$, $H \in \mathcal{G}_{n'}$ and $\min\{\chi(G), \chi(H)\} = k$;*

(ii) $\tilde{\mathfrak{J}}_{k,n,n'} \subseteq \tilde{\mathfrak{I}}_{k,n,n'}$.

Remark 4.5 For the computations of $\tilde{\mathfrak{J}}_{k,n,n'}$ and $\tilde{\mathfrak{I}}_{k,n,n'}$, we have to *eliminate* the variables. For example, $\tilde{\mathfrak{J}}_{k,n,n'}$ is defined by $\mathfrak{J}_{k,n,n'} \cap \mathbb{C}[e_{ij}, f_{i'j'}]$, where $\mathfrak{J}_{k,n,n'}$ is the ideal of the polynomial ring $\mathbb{C}[e_{ij}, f_{i'j'}, x_s, y_{s'}, z_{ss'}]$. Such ideal, i.e., the ideal obtained by eliminating some variables, can be computed by using the theory of *Gröbner basis*. For the detail, we refer the reader to [6, Section 3].

4.3 Refinement of Conjecture 1 using a characterization of critical graphs

Now we consider an additional condition that

(V8) G and H are k-critical graphs.

Let

$$V'_{k,n,n'} = \{(G,H) \in V_{k,n,n'} : (G,H) \text{ satisfies (V8)}\}.$$

Then the following holds.

Proposition 4.6 *For integers $k \geq 2$, $n \geq 1$ and $n' \geq 1$, the conditions (H1)–(H3) in Proposition 4.1 are equivalent to*

(H′1) $V'_{k,n,n'} \subseteq W_{k,n,n'}$.

Proof. By Proposition 4.1, it suffices to show that "(H′1) ⇔ (H1)". "(H1) ⇒ (H′1)" clearly holds. Thus we suppose that (H′1) holds and show that (H1) holds.

Let $(G,H) \in V_{k,n,n'}$. Then G and H satisfy (X1) and (V1)–(V7). If G and H are k-critical, then $(G,H) \in V'_{k,n,n'}$, and so $(G,H) \in W_{k,n,n'}$ because (H′1) holds. Thus, without loss of generality, we may assume that G is not k-critical. Since G satisfies (V1), this implies that $\chi(G) \leq k-1$. In particular, $\min\{\chi(G), \chi(H)\} \leq k-1$. Hence $(G,H) \in W_{k,n,n'}$, which proves that (H1) holds. □

Therefore, if k-critical graphs of order at most n can be characterized, then the information for such critical graphs directly effect the system of equation corresponding to $V'_{k,n,n'}$. By Theorem 3.1, the smallest nontrivial case for Conjecture 1 is $\chi(G) = \chi(H) = 5$, $|V(G)| = 8$ and $|V(H)| = 11$.

Jensen and Royle [12] also claimed that there exist 56 K_4-free graphs with chromatic number 5. Although they are not always 5-critical, we expect that there are a lot of 5-critical ones among

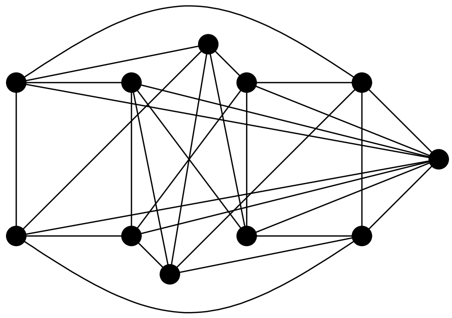

Figure 3: A K_4-free 5-critical graph H^* of order 11.

them. Now, we focus on the graph H^* depicted in Figure 3, which is one of K_4 free 5-critical graph of order 11, and demonstrate a partial solution of the problem that $V'_{5,8,11} \subseteq W_{5,8,11}$.

We translate the situations into ideal type formulas. All the ideals below are in the polynomial ring

$$\mathbb{C}[e_{ij}, f_{i'j'}, z_{ss'} : 1 \leq i < j \leq 8, 1 \leq i' < j' \leq 11, 1 \leq s \leq 8, 1 \leq s' \leq 11]$$

of $28 + 55 + 88 = 171$ variables. Let

$$\begin{aligned} E = {} & (e_{12} - 1, e_{13} - 1, e_{14} - 1, e_{25} - 1, e_{26} - 1, e_{35} - 1, e_{37} - 1, e_{46} - 1, e_{47} - 1, e_{56} - 1, e_{57} - 1, e_{67} - 1) \\ & + (e_{i8} - 1 : 1 \leq i \leq 7) + (e_{ij} : \text{otherwise}) \end{aligned} \tag{W1($G = K_1 + H_0, k = 5$)}$$

and

$$\begin{aligned} E' = {} & (f_{12} - 1, f_{13} - 1, f_{17} - 1, f_{24} - 1, f_{28} - 1, f_{34} - 1, f_{36} - 1, f_{45} - 1, f_{56} - 1, f_{57} - 1, f_{68} - 1, f_{78} - 1) \\ & + (f_{19} - 1, f_{29} - 1, f_{59} - 1, f_{69} - 1, f_{3,10} - 1, f_{4,10} - 1, f_{7,10} - 1, f_{8,10} - 1, f_{9,10} - 1) \\ & + (f_{i,11} - 1 : 1 \leq i \leq 8) + (f_{ij} : \text{otherwise}). \end{aligned} \tag{W1($H = H^*, k = 5$)}$$

Let

$$\mathcal{L} = E + E' + Z_{5,8,11} + J_{5,8,11}.$$

To show the equality

$$\mathcal{L} = \mathbb{C}[e_{ij}, f_{i'j'}, z_{ii'}], \tag{4}$$

which implies that $\mathcal{L} \cap \mathbb{C}[e_{ij}, f_{i'j'}] = \mathbb{C}[e_{ij}, f_{i'j'}]$, gives a partial solution of the problem that $V'_{5,8,11} \subseteq W_{5,8,11}$. We will focus on (4) and related problems in Section 5. Remark that to get a complete solution of $V'_{5,8,11} \subseteq W_{5,8,11}$, it suffices to prove a similar inclusion problem for all K_4-free 5-critical graphs H except for H^*.

5 Computational experiment

For confirming that Conjecture 1 is true or finding a counterexample in the case the graphs are small, we implement Theorem 4.4 and other related functions by an open source general computer algebra system Risa/Asir [1]. All source codes of our programming are put at the webpage [2]. More precisely, we implement the computations whether the following inclusion or the equality are true or not:

(a) $\tilde{\mathfrak{J}}_{k,n,n'} \subseteq \tilde{\mathfrak{I}}_{k,n,n'}$ for given n, n' and k;
(b) $\mathcal{L} = \mathbb{C}[e_{ij}, f_{i'j'}, z_{ii'}]$;

Note that [2] contains many other functions related to our problem, some of which will be explained below. Theorem 4.4 says that the inclusion (a) is equivalent to that Conjecture 1 is true. On the other hand, the discussions developed in Section 4.3 say that the confirmations of the equality (b) implies the search of the smallest non-trivial unknown case of Conjecture 1.

For example, we can perform the computations in Mac or Linux OS as follows:

Example 5.1 Let $k = 3, n = 5, n' = 5$. On Risa/Asir running with terminal, we will check the inclusion (a), i.e., we will check the condition Theorem 4.4 (ii) as follows:

```
[1895] load("**certan path**/Hedetniemi.rr")$
[1949] K = 3$
[1950] N = 5$
[1951] N' = 5$
[1952] hedetniemi.theorem_4_4(N, N', K);
### Theorem 4.4 (ii): k = 3, n = 5, n' = 5
(omitted)
### True: k = 3, n = 5, n' = 5
```

Similarly, we can check the equality (b) by `hedetniemi.section_4_3()`.

We performed the above computational experiments (a). All computations have been performed in Ubuntu OS equipped with 64 GB memory, Intel Xeon(R) W-2135, CPU 3.7 GHz. The following table shows the times took for each experiment. As the tables show, it took a huge time for checking even trivial cases of Conjecture 1.

	Time	Result
$k = 3, n = 4, n' = 4$	2 seconds	True
$k = 3, n = 4, n' = 5$	58 seconds	True
$k = 3, n = 5, n' = 5$	4409 seconds $\risingdotseq$ 73 minutes	True
$k = 3, n = 5, n' = 6$	241116 seconds $\risingdotseq$ 67 hours	True
$k = 3, n = 6, n' = 6$	more than two weeks	Still running
$k = 4, n = 4, n' = 4$	167 seconds	True
$k = 4, n = 4, n' = 5$	480818 seconds $\risingdotseq$ 133 hours	True
$k = 4, n = 5, n' = 5$	more than one month	Still running

Unfortunately, the computation (b) did not stop even after one month. We have to upgrade the machine performance or devise the algorithm or the theoretical part in order to push the boundary of the computable cases (more concretely, to complete the coputation of `hedetniemi.section_4_3()`).

Instead, we implemented the following experimental computations:

(c-1) we replace E (resp. E') with the ideal corresponding to C_{2m+1} (resp. $C_{2m'+1}$), and $\mathcal{L} = E + E' + Z_{3,2m+1,2m'+1} + J_{3,2m+1,2m'+1}$;

(c-2) we replace both E and E' with the ideal corresponding to the graph H_0, and $\mathcal{L} = E + E' + Z_{4,7,7} + J_{4,7,7}$;

(c-3) we replace E (resp. E') with the ideal corresponding to the graph H_0 (resp. the graph depicted in Figure 4), and $\mathcal{L} = E + E' + Z_{4,7,11} + J_{4,7,11}$;

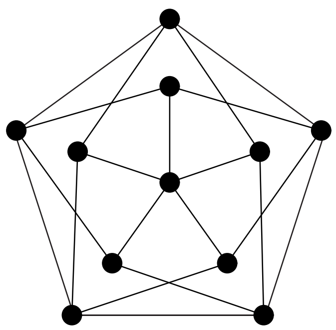

Figure 4: A triangle-free 4-critical graph of order 11.

We note that C_{2m+1} is 3-critical, H_0 is 4-critical of order 11. We also note that we already know theoritcally that the equality (4) holds for any case (c-1)–(c-3). Those computations stopped within some minutes or hours as shown below:

E	E'	Time	Result
C_{13}	C_{13}	1270 seconds $\risingdotseq$ 21 minutes	True
C_{13}	C_{15}	3855 seconds $\risingdotseq$ 64 minutes	True
C_{15}	C_{15}	9890 seconds $\risingdotseq$ 164 minutes	True
C_{15}	C_{17}	27743 seconds $\risingdotseq$ 7.7 hours	True
C_{17}	C_{17}	63158 seconds $\risingdotseq$ 17.5 hours	True

E	E'	Time	Result
H_0	H_0	1 second	True
H_0	Figure 4	18 seconds	True

6 Concluding remark

In this paper, we presented a reduction of Conjecture 1 using the inclusion of the ideals of a polynomial ring (Theorem 4.4). Since our reduction strongly depends on the structure of critical graphs as we verified in Subsection 4.3, the advance of the research of the criticality directly gives favorable effects on Conjecture 1.

We remark that Shitov [14] used the existence of graphs with large fractional chromatic number and large girth, and so his counterexamples implicitly depend on so-called probabilistic method. In particular, it seems to be difficult to give their specific constructions. Since our main result (Theorem 4.4) gives a reduction for each case, we expect that it offers not only a new approach for Conjecture 1 (in small chromatic number case) but the smallest specific counterexample of Conjecture 1.

Furthermore, every Shitov's counterexample contains a large clique. Hence the following weaker conjecture than Conjecture 1 is naturally posed.

Conjecture 2 *Let G and H be triangle-free graphs. Then $\chi(G \times H) = \min\{\chi(G), \chi(H)\}$.*

Conjecture 2 is still interesting because the chromatic number of graphs with large girth has deeply studied in graph theory. Note that the triangle-freeness of a labeled graph on $[n]$ is corresponding the following condition:

$$e_{ij}e_{jl}e_{il} = 0 \quad (1 \leq i < j < l \leq n), \tag{5}$$

where e_{ij}, e_{jl} and e_{il} are in Subsection 4.2. Since every subgraph of a triangle-free graph is also triangle-free, the criticality argument in Subsection 4.3 can work if we consider the triangle-free graphs. Consequently, our reduction can be applied to Conjecture 2.

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