JOURNAL OF COMBINATORIAL THEORY, Series A **61**, 142–146 (1992)

# Note

# The Equivalence of Two Problems on the Cube

C. GOTSMAN* AND N. LINIAL[†]

*Department of Computer Science, The Hebrew University, Jerusalem 91904, Israel*

*Communicated by the Managing Editors*

Received November 5, 1990

Denote by $Q_n$ the graph of the hypercube $C^n = \{+1, -1\}^n$. The following two seemingly unrelated questions are equivalent: 1. Let $G$ be an induced subgraph of $Q_n$ such that $|V(G)| \neq 2^{n-1}$. Denote $\Delta(G) = \max_{x \in V(G)} \deg_G(x)$ and $\Gamma(G) = \max(\Delta(G), \Delta(Q_n - G))$. Can $\Gamma(G)$ be bounded from below by a function of $n$?; 2. Let $f: C^n \to \{+1, -1\}$ be a boolean function. The *sensitivity* of $f$ at $x$, denoted $s(f, x)$, is the number of neighbors $y$ of $x$ in $Q_n$ such that $f(x) \neq f(y)$. The sensitivity of $f$ is $s(f) = \max_{x \in C^n} s(f, x)$. Denote by $d(f)$ the degree of the unique representation of $f$ as a real multilinear polynomial on $C^n$. Can $d(f)$ be bounded from above by a function of $s(f)$? 

## 1. PRELIMINARIES

Denote by $Q_n$ the graph on the $n$-dimensional cube $C^n = \{+1, -1\}^n$, where any two vertices are adjacent iff they differ in exactly one component. For an induced subgraph $G$ of $Q_n$, denote the *maximal degree* of $G$ by $\Delta(G)$, i.e.,

$$\Delta(G) = \max_{x \in V(G)} \deg_G(x).$$

In [1], it was shown that if $G$ contains more than $2^{n-1}$ vertices, then

$$\Delta(G) > \tfrac{1}{2}(\log n - \log\log n + 1)$$

* Supported in part by an Eshkol doctoral fellowship, administered by the National Council for R&D, Israel Ministry of Science.

† Research supported in part by US–Israel Binational Science Foundation Grant #0378114 and a grant from the Israel Academy of Sciences.



0097-3165/92 $5.00

and there exists a $G$ such that

$$\Delta(G) < \sqrt{n} + 1.$$

The value $\sqrt{n}$ is also conjectured to be the correct order of magnitude for a lower bound on $\Delta(G)$. Denote $\Gamma(G) = \max(\Delta(G), \Delta(Q_n - G))$.

Let $f\colon C^n \to \{+1, -1\}$ be a boolean function. The *sensitivity* of $f$ at $x$, denoted by $s(f, x)$, is the number of neighbors $y$ of $x$ for which $f(x) \neq f(y)$. The sensitivity of $f$ is

$$s(f) = \max_{x \in C^n} s(f, x)$$

The sensitivity of $f$ is sometimes called the *critical complexity* of $f$.

In theoretical computer science, much effort has been expended in the definition of various measures of complexity of boolean functions. Some are derived from an underlying computational model, such as *decision tree depth.* Here the function is computed by repeatedly reading input bits, until the function can be determined from the bits accessed. The *cost* of an algorithm is the number of bits read on the worst case input, and the complexity of a function is the cost of the best algorithm for this function. A similar measure is the *certificate complexity.* A 1-certificate (0-certificate) for $f$ is an assignment to some subset of the variables that forces the value of $f$ to 1 (0). The certificate complexity of $f$ on $x$, denoted $C(f, x)$, is the size of the smallest certificate that agrees with $x$. The certificate complexity of $f$ is

$$C(f) = \max_{x \in C^n} C(f, x).$$

Other measures of complexity are of a combinatorial nature, e.g., sensitivity. A related measure is *block-sensitivity*, defined: Denote $[n] = \{1, ..., n\}$ and let $R \subset [n]$. If $x$ is the vector $(x_1, ..., x_n)$, then $x^{(R)}$ is defined as the vector with coordinates:

$$x_i^{(R)} = \begin{cases} x_i, & i \notin R \\ -x_i, & i \in R. \end{cases}$$

The block sensitivity of $f$ at $x$, denoted $bs(f, x)$, is the largest number $t$ such that there exist $t$ *disjoint* sets $R_1, ..., R_t$ such that for all $1 \leqslant i \leqslant t$, $R_i \subset [n]$, and $f(x) \neq f(x^{(R_i)})$. The block-sensitivity of $f$ is

$$bs(f) = \max_{x \in C^n} bs(f, x).$$

A central activity in this field is determining the relation between various measures. The measures of complexity $s_1$ and $s_2$ are *equivalent* if they are polynomially related; i.e., there exist polynomials $p_1(x)$ and $p_2(x)$ such that

$$\forall f, \qquad s_1(f) \leqslant p_2(s_2(f)), \qquad s_2(f) \leqslant p_1(s_1(f)).$$

Nisan [3] showed that decision tree depth, certificate complexity, and block-sensitivity are equivalent. Nisan first considered the more natural measure of sensitivity (which is block-sensitivity restricted to singletons), but was unable to prove equivalence to decision tree depth and certificate complexity. However, only after introducing block-sensitivity was equivalence obtained.

Yet another complexity measure is obtained from the unique representation of the boolean function $f$ as a real multilinear polynomial over the cube:

$$f(x) = \sum_{I \subset [n]} \left[ \alpha_I \prod_{i \in I} x_i \right].$$

The coefficient $\alpha_I$ (which satisfies $-1 \leqslant \alpha_I \leqslant 1$ for all $I \subset [n]$) is also called $\hat{f}(I)$, the *Fourier transform* of $f$ at $I$. Denote by $d(f)$ the *degree* of this polynomial, i.e.,

$$d(f) = \max_{I \subset [n]} \{ |I| : \alpha_I \neq 0 \}.$$

Nisan and Szegedy [4] show that $d(f)$ is also equivalent to the three complexity measures mentioned above. As for the relation between sensitivity and degree, Szegedy [6] showed that

$$d(f) \geqslant \sqrt{s(f)}.$$

This can easily be shown to be tight. Whether $s(f)$ is also equivalent to all of the above is still unknown. In particular, an upper bound on $d(f)$ in terms of $s(f)$ is sought and is conjectured to be $s^2(f)$. Such a bound would mean that sensitivity is equivalent to all the previously mentioned quantities. In the next section we show that this upper bound is equivalent to a lower bound on $\Gamma(\ )$.

## 2. The Equivalence Theorem

THEOREM 2.1. *The following are equivalent for any function $h: N \to R$:*

1. *For any induced subgraph $G$ of $Q_n$ such that $|V(G)| \neq 2^{n-1}$, $\Gamma(G) \geqslant h(n)$.*
2. *For any boolean function $f$, $d(f) < h^{-1}(s(f))$.*

*Proof.* We first transform 1 into a statement concerning boolean functions: Associate with the subgraph $G$ a boolean function $g$ such that $g(x)=1$ iff $x \in V(G)$. Note that $\deg_G(x)=n-s(g,x)$ for $x \in V(G)$ and the same holds in $Q_n - G$ for $x \notin V(G)$. Denote by $\mathbf{E}(g)$ the average value of $g$ on $C^n$. Now 1 and 2 are clearly equivalent to the following:

1′. For any boolean function $g$, $\mathbf{E}(g) \neq 0$ implies $\exists x\colon s(g,x) \leqslant n-h(n)$.

2′. For any boolean function $f$, $s(f) < h(n)$ implies $d(f) < n$.

To see the equivalence of 1′ and 2′, define

$$g(x)=f(x)\,p(x),$$

where $p(x)$ is the parity function of $x$: $p(x)=\prod_{i=1}^{n} x_i$. Note that for all $x \in C^n$, $s(g,x)=n-s(f,x)$ and for all $I \subset [n]$, $\hat{g}(I)=\hat{f}([n]-I)$, therefore $\mathbf{E}(g)=\hat{g}(\varnothing)=\hat{f}([n])$, where $\hat{f}([n])$ is the Fourier transform of $f$ at $[n]$, i.e., the highest order coefficient in the representation of $f$ as a polynomial.

1′ → 2′. Assume that $d(f)=n$, i.e., $\hat{f}([n]) \neq 0$. This is equivalent to $\mathbf{E}(g) \neq 0$. By 1′, $\exists x\colon s(g,x) \leqslant n-h(n)$; therefore $\exists x\colon s(f,x) \geqslant h(n)$, contradicting the premise.

2′ → 1′. Assume that $\forall x$, $s(g,x) > n-h(n)$. This implies that $s(f) < h(n)$. By 2′, $d(f) < n$, which is equivalent to $\hat{f}([n])=\hat{g}(\varnothing)=\mathbf{E}(g)=0$, contradicting the premise. ∎

## 3. Conclusion

Substituting $h(x)=\sqrt{x}$ in Theorem 2.1 shows that the two bounds $\Delta(G) \geqslant \sqrt{n}$ and $d(f) < s^2(f)$ are equivalent. The example from [1] which shows that there exists $G$ such that $\Delta(G) \leqslant \sqrt{n}+1$ can be used to show that the upper bound on $d(f)$ would be tight if it were true. All of this means that a proof of $\Delta(G) \geqslant \sqrt{n}$ would imply that boolean function sensitivity is equivalent to all other complexity measures mentioned in [3].

The sensitivity complexity measure $s(f)$ is especially important, since it also lower bounds $T(f)$—the time needed by a parallel RAM to compute $f$ (a parallel RAM is a collection of synchronous parallel processors sharing a global memory with no write-conflicts allowed). Cook and Dwork [2] have shown that $T(f) \geqslant \log s(f)$. In fact, Nisan [3] later improved this to $T(f) \geqslant \log bs(f)$ (this is a stronger inequality, since for any $f$, $bs(f) \geqslant s(f)$). Simon [5] has also shown that a $n$-variable boolean function which depends on all its variables must have sensitivity at least $\Omega(\log n)$.

## REFERENCES


1. F. R. K. CHUNG, Z. FÜREDI, R. L. GRAHAM, AND P. SEYMOUR, On induced subgraphs of the cube, *J. Combin. Theory Ser. A* **49** (1988), 180–187.
2. S. COOK AND C. DWORK, Bounds on the time for parallel RAM's to compute simple functions, *in* "Proceedings, 14th Symposium on the Theory of Computing," pp. 231–233, Assoc. Comput. Mach., New York, 1982.
3. N. NISAN, CREW PRAMS and decision trees, *SIAM J. Comput.* **20** (6), (1991), 999–1007.
4. N. NISAN AND M. SZEGEDY, On the degree of boolean functions as real polynomials, *in* "Proceedings, 24th Symposium on the Theory of Computing," Assoc. Comp. Mach., New York, 1992.
5. H. U. SIMON, A tight $O(\log \log n)$ bound on the time for parallel RAM's to compute non-degenerate functions, *in* "Foundations of Computing Theory," pp. 439–444, Lecture Notes in Computer Science, Vol. 158, Springer-Verlag, New York/Berlin, 1983.
6. M. SZEGEDY, Ph.D. thesis, University of Chicago, 1989.