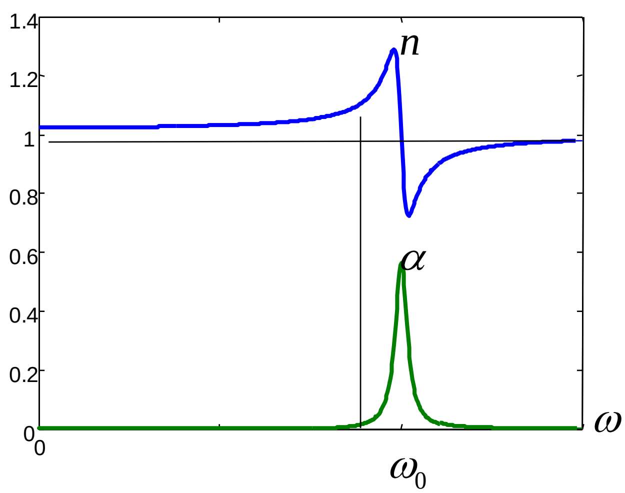
Lecture 21 Dispersive medium: group velocity, energy density, energy velocity

What speed of light?

Earlier on, we have shown that fro plane monochromatic waves the speed of propagation is the speed of propagation of the phase front, or phase velocity,

$$v_p(\omega) = \omega / k(\omega) = c / n(\omega) \tag{21.1}$$

In other words, each frequency has its own refractive index and there therefore its own phase velocity. That immediately creates the ground for concern. Consider the refractive index dispersion in Fig.21.1(a) and the dispersion curve $\omega(k)$ in Fig.21.1.(b) for the case of a simple resonance. It is easy to see that at frequencies above ω_0 refractive index $n(\omega) < 1$ which immediately leads to $v_p(\omega) > c$ which immediately creates conflict with special relativity theory according to which no signal (information) can propagates with velocity exceeding speed of light in vacuum. The conflict can be resolved by noting that by definition plane monochromatic wave that starts at infinity and ends at infinity can carry no information. One cannot detect phase – only the energy! The information can be carried by the modulated waves. And it is this definition that we shall use to define the group velocity of light.



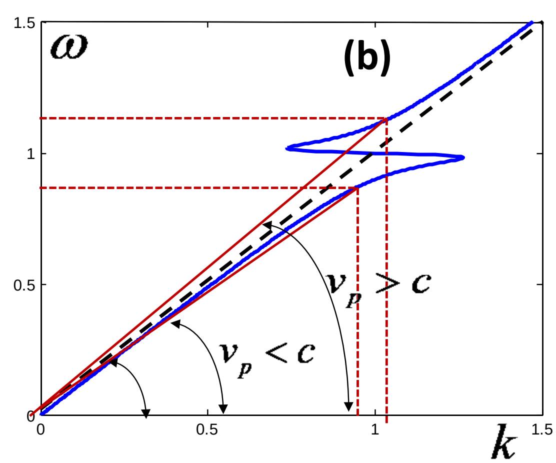


Figure 21.1 (a) Dispersion of the refractive index and (b) dispersion curve $\omega(k)$ indicating the region where phase velocity exceeds speed of light.

In Fig. 21.2 (a) we show a monochromatic wave $E = E_0 \cos(k_0 z - \omega_0 t)$ and in Fig. 21.2(b) its spectrum $E(\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$ (we only show positive frequencies) To introduce the group velocity we consider a modulated wave obtained by superimposing two monochromatic waves with closely spaced frequencies ω_1 and ω_2 as shown in Fig.21.2(c) and (d).

$$
\begin{aligned}
E(z,t) &= \frac{1}{2}E_0\cos(k_1 z-\omega_1 t)+\frac{1}{2}E_0\cos(k_2 z-\omega_2 t)= \\
&= E_0\cos\left(\frac{k_1+k_2}{2}z-\frac{\omega_1+\omega_2}{2}t\right)\cos\left(\frac{k_1-k_2}{2}z-\frac{\omega_1-\omega_2}{2}t\right)= \\
&= E_0\cos(\delta k\cdot z-\delta\omega\cdot t)\cos(\bar{k}z-\bar{\omega}t)
\end{aligned}
\tag{21.2}
$$

where we have introduced mean, *or carrier frequency and wavevecto*r as

$$
\begin{aligned}
\bar{\omega} &= (\omega_1+\omega_2)/2 \\
\bar{k} &= (k_1+k_2)/2
\end{aligned}
\tag{21.3}
$$

and the difference, or *modulation frequency and wavevector*

$$
\begin{aligned}
\delta k &= (k_2-k_1)/2 \\
\delta\omega &= (\omega_2-\omega_1)/2
\end{aligned}
\tag{21.4}
$$

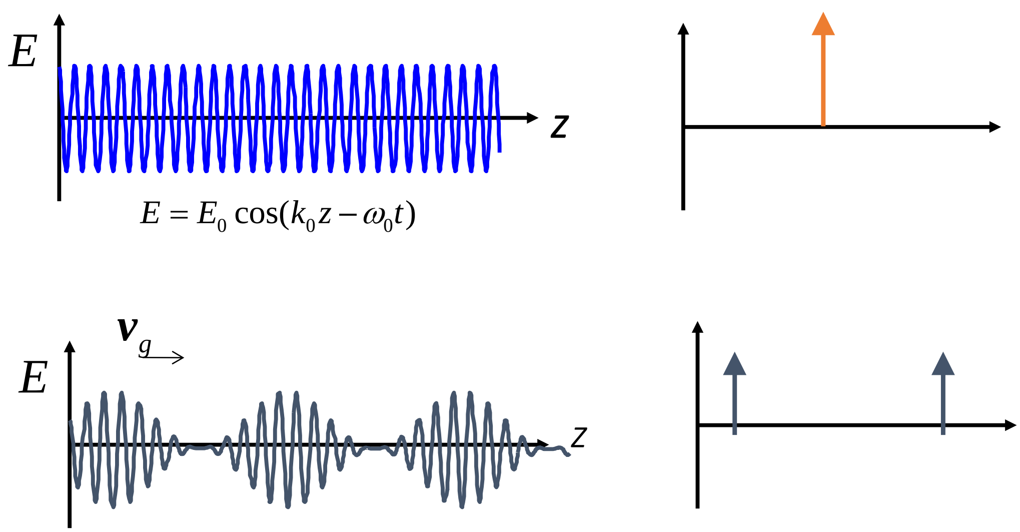


Figure 21.2 (a) Monochromatic wave and (b) its spectrum (c) Modulated wave and (d) its spectrum

So, the carrier wave is modulated by a “slow” envelope $E_{env}(z,t)=E_0\cos(\delta k\cdot z-\delta\omega\cdot t)$ that carries the information –if we want to see the speed with which the peak of envelope propagates, obviously it is

$$
v_g = \frac{\delta\omega}{\delta k} \tag{21.5}
$$

Which in the limit of small $\delta\omega$ becomes

$$
v_g = \frac{\partial\omega}{\partial k} \tag{21.6}
$$

i.e. group velocity is the slope of the dispersion curve $\omega(k)$

Wave packets

In general, the information is being carried by the "wave packets", i.e. superposition of many plane waves,

$$E(z,t)=\sum_i A_i e^{j(k_i z-\omega_i t)} \tag{21.7}$$

If the frequencies are close to each other one can change summation to integration

$$E(z,t)=\int A(\omega)e^{j(k(\omega)z-\omega t)}d\omega \tag{21.8}$$

where $A(\omega)$ is obviously a (scaled) Fourier transform of the wave packet as shown in Fig. 21.3

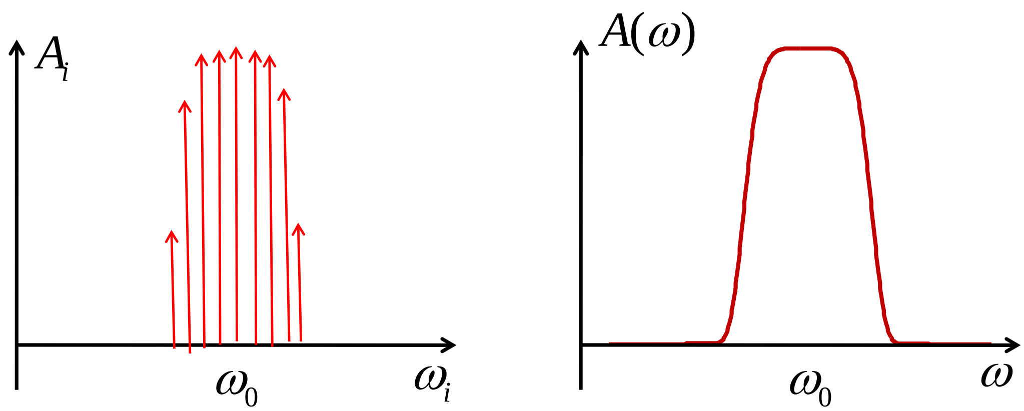


Figure 21.3 Wave packet spectra, (a) discrete and (b) continuous

Consider now a Gaussian packet of plane waves whose spectrum is shown in Fig.21.4(a),

$$A(\omega)=\frac{A_0}{\sqrt{2\pi}\Delta\omega}e^{-\frac{(\omega-\omega_0)^2}{2(\Delta\omega)^2}} \tag{21.9}$$

Let us see how the packet looks in time domain. Substituting (21.9) into (21.8) we obtain

$$E(z,t)=\frac{A_0}{\sqrt{2\pi}\Delta\omega}\int e^{-\frac{(\omega-\omega_0)^2}{2(\Delta\omega)^2}}e^{j(k(\omega)z-\omega t)}d\omega=\frac{A_0}{\sqrt{2\pi}\Delta\omega}\int e^{-\frac{(\omega-\omega_0)^2}{2(\Delta\omega)^2}+j[(k-k_0)z-(\omega-\omega_0)t]}d\omega e^{j(k_0z-\omega_0t)}\equiv E_{env}(z,t)e^{j(k_0z-\omega_0t)}$$
(21.10)

where $E_{env}(z,t)$ is the envelope modulating the carrier wave $e^{j(k_0z-\omega_0t)}$. Let us take a closer look at the terms in the exponent of envelope in (21.10)

$$B=-\frac{(\omega-\omega_0)^2}{2(\Delta\omega)^2}+j\left[v_g^{-1}(\omega-\omega_0)z-(\omega-\omega_0)t\right]=-\frac{(\omega-\omega_0)^2}{2(\Delta\omega)^2}+j\left(v_g^{-1}z-t\right)(\omega-\omega_0) \tag{21.11}$$

We can add and subtract $\frac{(\Delta\omega)^2}{2}\left(v_g^{-1}z-t\right)^2$ to (21.11) and obtain

$$B=-\left[\frac{\omega-\omega_0}{\sqrt{2}\Delta\omega}-j\frac{\Delta\omega}{\sqrt{2}}(z/v_g-t)\right]^2+\frac{(\Delta\omega)^2}{2}\left(t-z/v_g\right)^2 \tag{21.12}$$

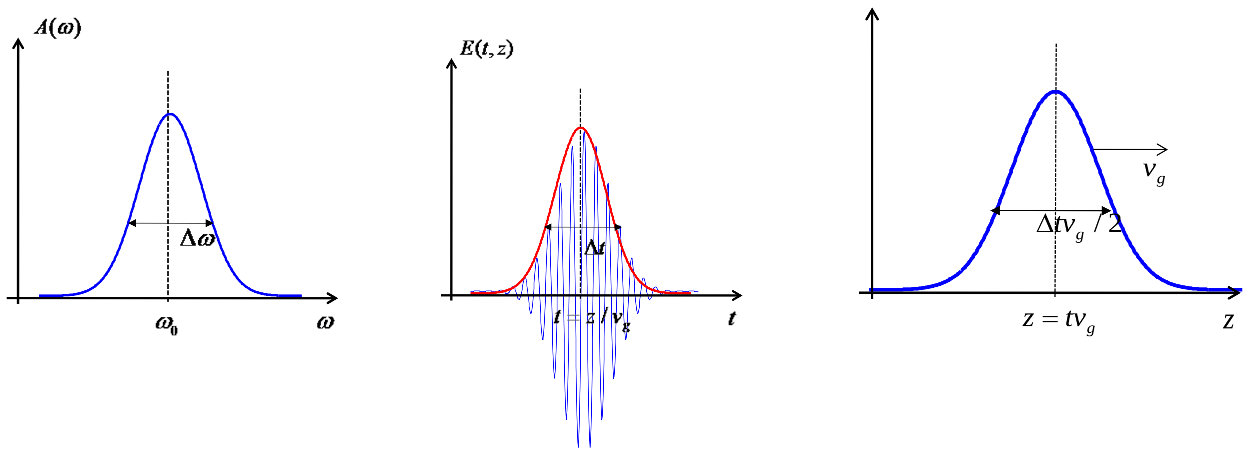


Figure 21.4 (a) spectrum of Gaussian wave packet (b) Temporal pulse shape corresponding to this wave packet (c) Power density of the wave packet in space.

We can now take and integral

$$\frac{A_0}{\sqrt{2\pi}\Delta\omega}\int e^{-\left[\frac{\omega-\omega_0}{\sqrt{2}\Delta\omega}-j\frac{\Delta\omega}{\sqrt{2}}(z/v_g-t)\right]^2}d\omega=\frac{A_0}{\sqrt{2\pi}\Delta\omega}\sqrt{2}\Delta\omega\int_{-\infty}^{\infty}e^{-x^2}dx=A_0 \tag{21.13}$$

and finally obtain the envelope

$$E_{env}(z,t)=A_0e^{-\frac{\left(t-z/v_g\right)^2}{2(\Delta t)^2}} \tag{21.14}$$

Where temporal spread of the pulse is $\Delta t=1/\Delta\omega$ and the electric field is

$$E(z,t)=A_0e^{-\frac{\left(t-z/v_g\right)^2}{2(\Delta t)^2}}e^{j(k_0z-\omega_0t)} \tag{21.15}$$

as shown in Fig.21.4(b). The envelope of the Gaussian packet (bit) preserves its shape and moves with the group velocity, hence group velocity describes the envelope's propagation, and, obviously, it also describes propagation of energy since the time averaged Poynting vector is

$$\langle S\rangle_t(z,t)=\frac{\left|E_{env}(z,t)\right|^2}{2\eta}=S_0e^{-\frac{\left(z-tv_g\right)^2}{\left(\Delta tv_g\right)^2}}=S_0e^{-\frac{\left(z-tv_g\right)^2}{(\Delta z)^2}} \tag{21.16}$$

where $\Delta z = \Delta t \cdot v_g$ is the spatial extent of the pulse as shown in Fig. 21.4(c). Note that since $\Delta t \Delta\omega = 1$ and $\Delta\omega = \Delta k \cdot v_g$ there exists an uncertainty relation

$$\Delta z \cdot \Delta k = 1 \tag{21.17}$$

Or

$$\Delta z \sim \lambda \frac{\lambda}{2\pi\Delta\lambda} \tag{21.18}$$

i.e. the minimum spread of the pulse in space depends on the spectral width.

Note that for pulses that are not Gaussian $\Delta t \Delta\omega > 1$ and $\Delta z \cdot \Delta k > 1$

Group velocity dispersion

One can use stream of pulses to transmit information, as shown in Fig.21.5 , but the question arises what will happen with these pulses if we consider the fact that group velocity itself is not the same for all the frequencies in the pulse spectrum as one can see from Fig.21.5(b) where the slope of the dispersion curve is not constant.

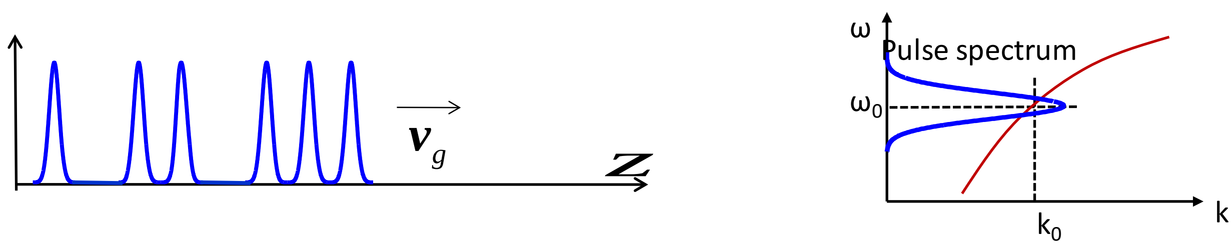


Figure 21.5 (a) Return-to-zero optical signal (b) Group velocity dispersion

Let us approximate the dispersion curve of Fig.21.5(b) up to the second order

$$k(\omega) \approx k(\omega_0) + \frac{dk}{d\omega}(\omega-\omega_0) + \frac{1}{2}\frac{d^2k}{d\omega^2}(\omega-\omega_0)^2 = k(\omega_0) + v_g^{-1}(\omega-\omega_0) + \frac{1}{2}\beta_2(\omega-\omega_0)^2 \tag{21.19}$$

Where group velocity dispersion (GVD) is $\beta_2 = d^2k/d\omega^2$ (typically measured in units of ps^2/km because of applications to fiber optics) . The envelope propagation is now described as in (21.10), but this time keeping the second derivative,

$$\begin{aligned} E_{env}(z,t) &= \frac{A_0}{\sqrt{2\pi}\Delta\omega}\int e^{-\frac{(\omega-\omega_0)^2}{2(\Delta\omega)^2}+j\left[(k-k_0)z-(\omega-\omega_0)t\right]}d\omega = \\ &= \frac{A_0}{\sqrt{2\pi}\Delta\omega}\int e^{-\frac{\omega_1^2}{2(\Delta\omega)^2}+j\omega_1\left[v_g^{-1}z-t\right]+j\frac{1}{2}\beta_2\omega_1^2 z}d\omega_1 = \frac{A_0}{\sqrt{2\pi}\Delta\omega}\int e^{-\frac{\omega_1^2}{2(\Delta\omega)^2}+j\frac{1}{2}\beta_2\omega_1^2 z-j\omega_1 t_1}d\omega_1 \end{aligned} \tag{21.20}$$

where we have introduced relative frequency $\omega_1 = \omega - \omega_0$ and the delayed time $t_1 = t - z / v_g$. Let us consider the exponent in (21.20)

$$-\frac{\omega_1^2}{2(\Delta\omega)^2} + j\frac{1}{2}\beta_2\omega_1^2 z - j\omega_1 t_1 = -\frac{\omega_1^2}{2}(\Delta t_0^2 + j\beta_2 z) - j\omega_1 t_1 = -\frac{\Delta t_0^2 + j\beta_2 z}{2}\left(\omega_1^2 + 2j\frac{\omega_1 t_1}{\Delta t_0^2 + j\beta_2 z}\right) =$$
$$= -\frac{\Delta t_0^2(1 + jz / z_0)}{2}\left(\omega_1^2 + 2j\frac{\omega_1 t_1}{\Delta t_0^2(1 + jz / z_0)}\right)$$

(21.21)

Where we have introduced the original (z=0) pulse width $\Delta t_0 = 1 / \Delta\omega$, and then an important parameter of *dispersion length*

$$z_0 = \Delta t_0^2 / \beta_2 \tag{21.22}$$

As before, we now complete the term in parenthesis of (21.21) to a full square

$$\omega_1^2 + 2j\frac{\omega_1 t_1}{\Delta t_0^2(1 + z / z_0)} = \left(\omega_1 + \frac{jt_1}{\Delta t_0^2(1 + jz / z_0)}\right)^2 + \frac{t_1^2}{\Delta t_0^2(1 + jz / z_0)^2} \tag{21.23}$$

And upon substituting it first into (21.21) and then the result into (21.20)

$$E_{env}(z, t_1) = \frac{A_0}{\sqrt{2\pi}\Delta\omega}\int e^{-\frac{1 + jz/z_0}{2\Delta\omega^2}\left(\omega_1 + \frac{jt_1}{2\Delta t_0^2(1+z/z_0)}\right)^2} d\omega_1 \times e^{-\frac{t_1^2}{2\Delta t_0^2(1 + jz/z_0)}} = \frac{A_0}{\sqrt{1 + jz / z_0}} e^{-\frac{t_1^2}{2\Delta t_0^2 1 + jz/z_0)}}$$

(21.24)

The pulse is still Gaussian, but it has a "complex width". If we now separate the phase and amplitude in (21.24) we obtain

$$E_{env}(z, t_1) = \frac{A_0}{\sqrt[4]{1 + (z / z_0)^2}} e^{-\frac{t_1^2(1 - jz/z_0)}{2\Delta t_0^2[1 + (z/z_0)^2]}} e^{-j\phi/2} \tag{21.25}$$

where $\phi = \tan^{-1}(z / z_0)$. Now we introduce the time-dependent width of the pulse,

$$\Delta t(z) = \Delta t_0\sqrt{1 + (z / z_0)^2} = \Delta t_0\sqrt{1 + (\beta_2 z / \Delta t_0)^2} \tag{21.26}$$

Then

$$E_{env}(z, t_1) = A_0\sqrt{\frac{\Delta t_0}{\Delta t}} e^{-\frac{t_1^2}{2\Delta t^2}} e^{j\frac{z}{z_0}\frac{t_1^2}{2\Delta t^2} - j\phi/2} \tag{21.27}$$

So, the amplitude of the pulse (and hence its power) remains Gaussian, but the pulse envelope expand according to (21.26), which is a complete analogy to the Gaussian beam propagating and diffracting in space studies in Lecture 19. At one dispersion length it increases by a factor of $2^{1/2}$, and, realistically, if one uses Gaussian pulses for communication, the symbols (bits) in Fig.21.5(a) will start interfering with each other and communicating links will start giving large bit error rate. Therefore, dispersion must be somehow compensated which will be studied later in the course. But now let us look at the phase. The pulse temporal shape is

$$E(z,t_1) = A_0\sqrt{\frac{\Delta t_0}{\Delta t}}e^{-\frac{t_1^2}{2\Delta t^2}}e^{j\Phi(z,t_1)} \tag{21.28}$$

Where the phase is

$$\Phi(z,t_1) = -\frac{zt_1^2}{z_0\Delta t^2} - \frac{\phi}{2} + k_0 z - \omega_0 t \tag{21.29}$$

Introduce the instant frequency as

$$\omega_{inst} = -\frac{d\Phi}{dt} = \omega_0 + \frac{z}{z_0\Delta t^2}t_1 = \omega_0 + Ct_1 \tag{21.30}$$

Where C is the linear chirp parameter. As one can see the instant frequency of the pulse increases linearly with time as illustrated in Fig. 21.6 . And this is easily understandable – as one can see from Fig. 21.5(b) the slope of the $\omega(k)$ dispersion curve decreases with frequency, and since slope is a group velocity, the spectral component with high frequencies ("blue") get delayed relative to the lower frequencies, or "red" components, so the instant frequency increases with time – so-called positive chirp. Of course one can engineer medium with negative GVD that will produce negative chirp.

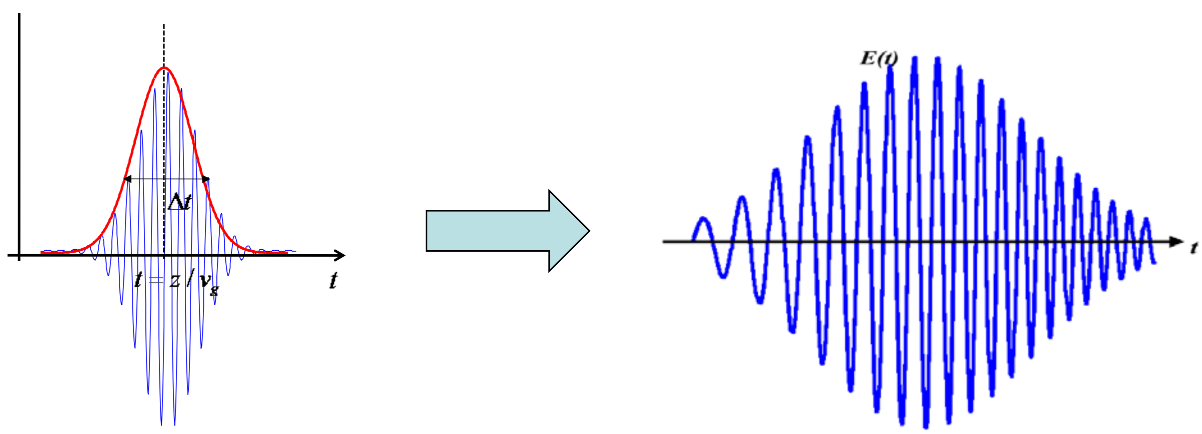


Figure 21.6 Chirped pulse formation due to GVD.

Relation between group and phase velocities

In Fig. 21.7 one can see that the slope of the dispersion curve, i.e. the group velocity is always less than phase velocity in the region of normal dispersion, i.e. when $dn/d\omega > 0$. Indeed, we can write

$$v_g^{-1}=\frac{dk}{d\omega}=\frac{d\left(n\frac{\omega}{c}\right)}{d\omega}=\frac{\omega}{c}\frac{dn}{d\omega}+\frac{n}{c}=v_p^{-1}+\frac{\omega}{c}\frac{dn}{d\omega}>v_p^{-1} \tag{21.31}$$

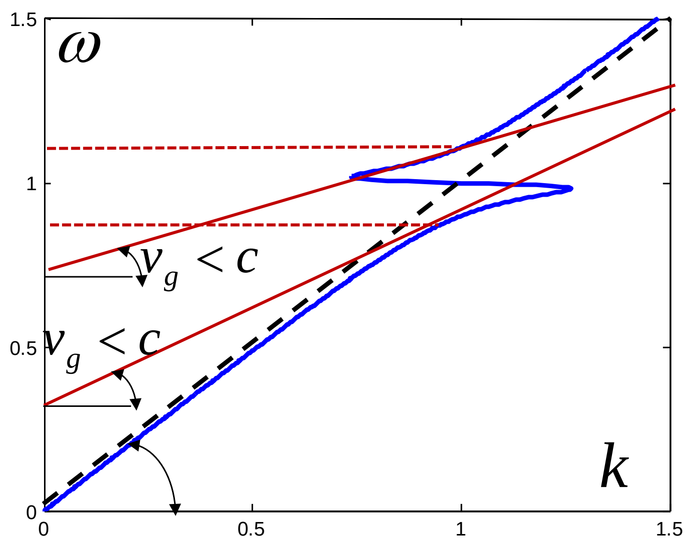


Figure 21.7 group velocity in the region of normal dispersion

We can also introduce a *group index* as

$$n_g = n+\omega\frac{dn}{d\omega}>n \tag{21.32}$$

Then we can express group velocity a simply

$$v_g = \frac{c}{n_g} \tag{21.33}$$

Energy density in the dispersive medium.

The energy density of the medium has been earlier (lecture 1) taken as in Eq. 1.62

$$\begin{aligned} U_E &= \frac{1}{2}\mathbf{E}\cdot\mathbf{D}=\frac{1}{2}\varepsilon\mathbf{E}\cdot\mathbf{E} \\ U_M &= \frac{1}{2}\mathbf{H}\cdot\mathbf{B}=\frac{1}{2}\mu\mathbf{H}\cdot\mathbf{H} \end{aligned} \tag{21.34}$$

For the medium with dielectric constant $\varepsilon_r(\omega)$ the time-averaged electrical energy density is then

$$\langle U_E\rangle_t = \frac{1}{4}\varepsilon_0\varepsilon_r E^2 \tag{21.35}$$

But what if the dielectric constant (consider just the real part for now) (Eq. 3.15) shown in Fig.21.8

$$\varepsilon_r(\omega) = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - j\omega\gamma} \tag{21.36}$$

is negative, as is also the case for the range frequencies above the resonant frequency ω_0 as shown in Fig. 21.8 including the case of the metal ?

$$\varepsilon_r(\omega) = 1 - \frac{\omega_p^2}{\omega^2 + j\omega\gamma} \tag{21.37}$$

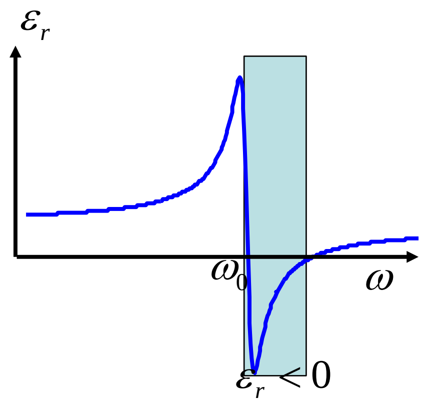


Figure 21.8 Dielectric constant dispersion showing the region of negative ε_r

Evan though $\varepsilon_r < 0$ we know that electromagnetic field can exist inside the material as an evanescent wave. But according to (21.35) the energy density is negative!. That is clearly unphysical, so we need to find out where we went wrong. Let is derive the expression for the change of energy density.

The rate of power transfer from electro-magnetic field to the medium (amount of work per unit volume performed by the field) is

$$W = \mathbf{E} \cdot \mathbf{J} \tag{21.38}$$

Substitute current density $\mathbf{J}$ from the Maxwell's equation

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{21.39}$$

Into (21.38) to get

$$W = \mathbf{E} \cdot (\nabla \times \mathbf{H}) - \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} \tag{21.40}$$

Use vector identity

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{H} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{H}) \tag{21.41}$$

and obtain

$$W = \mathbf{H}\cdot(\nabla\times\mathbf{E}) - \nabla\cdot(\mathbf{E}\times\mathbf{H}) - \mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t} \tag{21.42}$$

Now $\mathbf{E}\times\mathbf{H} = \mathbf{S}$ (Poynting vector) and $\nabla\times\mathbf{E} = -\partial\mathbf{B}/\partial t$ so from (21.42) we obtain

$$W = -\mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t} - \mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t} - \nabla\cdot\mathbf{S} \tag{21.43}$$

Clearly, the low of conservation of energy dictates that the rate of change of energy density is

$$\frac{dU}{dt} = \mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t} + \mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t} = -\nabla\cdot\mathbf{S} - W \tag{21.44}$$

Hence the rate of change of magnetic energy is

$$\frac{dU_M}{dt} = \mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t} \tag{21.45}$$

and the rate of change of electric energy is

$$\frac{dU_E}{dt} = \mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t} \tag{21.46}$$

In general, the energy is only defined relative to some “zero” level hence it is only the change of energy that matters so we may write for the magnetic field

$$\frac{dU_M}{dt} = \mathbf{H}\cdot\frac{\partial\mu_0\mathbf{H}}{\partial t} = \frac{\partial}{\partial t}\frac{1}{2}\mu_0 H^2 \tag{21.47}$$

From which the second equation in (21.34) follows. But for the electric field the situation is different since in general we have the relation between the **D** and **E** as a convolution (Eq. 1.23)

$$\mathbf{D}(t) = \varepsilon_0\int_{-\infty}^{0}\varepsilon_r(\tau)\mathbf{E}(t-\tau)d\tau \tag{21.48}$$

Therefore

$$\frac{\partial}{\partial t}\mathbf{D}(t) = \varepsilon_0\int_{-\infty}^{0}\varepsilon_r(\tau)\dot{\mathbf{E}}(t-\tau)d\tau \neq \varepsilon_0\varepsilon_r\mathbf{E} \tag{21.49}$$

To estimate the temporal derivative of $\mathbf{D}$ we shall go to the frequency domain using the relations 1.26

$$\begin{aligned}\varepsilon_r(\omega) &= \int_{-\infty}^{\infty}\varepsilon_r(t)e^{j\omega t}dt \\ \varepsilon_r(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\varepsilon_r(\omega)e^{-j\omega t}d\omega\end{aligned} \tag{21.50}$$

The spectrum of $\varepsilon_r(\omega)$ is shown in Fig. 21.9 in the narrow region near the carrier frequency ω_0 where we also plot the spectrum of the signal – a wave packet

$$E(t) = E_{env}(t)\cos(\omega_0 t) = \frac{1}{2}E_{env}(t)e^{-j\omega_0 t} + c.c.. \tag{21.51}$$

where the envelope is

$$E_{env}(t) = \int E(\alpha)e^{-j\alpha t}d\alpha \tag{21.52}$$

and therefore

$$E(t) = \frac{1}{2}\int E(\alpha)e^{-j(\alpha+\omega_0)t}d\alpha + c.c. \tag{21.53}$$

(note that $\alpha = \omega - \omega_0$ is the same as ω_1 in (21.20). Also noet that we have assumed a\n isotropic medium so we have us escalars for the field and displacement.

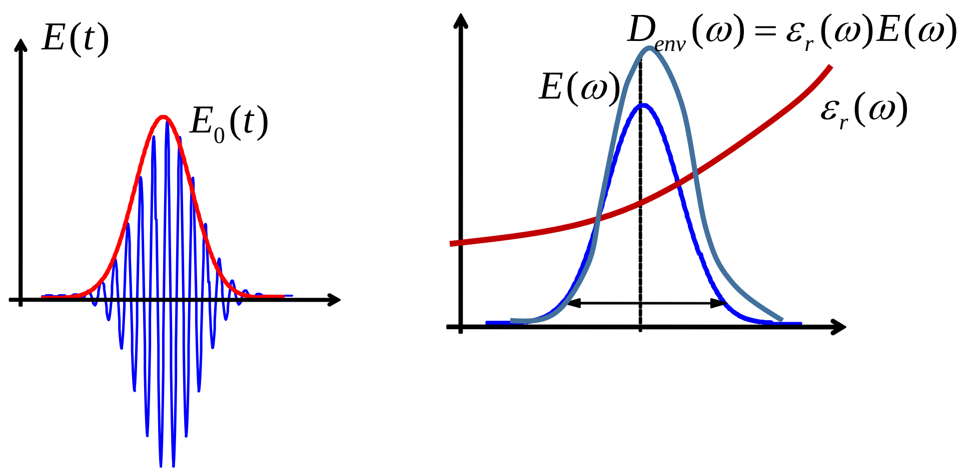


Figure 21.9 (a) The signal centered at carrier frequency ω_0, (b) its spectrum and the dielectric constant dispersion in the vicinity of ω_0

Now our task is to find the derivative of the displacement

$$\frac{\partial D}{\partial t} = \frac{\partial}{\partial t}\frac{1}{2}\int D(\alpha)e^{-j(\alpha+\omega_0)t}d\alpha + c.c = \frac{1}{2}\varepsilon_0\frac{\partial}{\partial t}\int \varepsilon_r(\alpha+\omega_0)E(\alpha)e^{-j(\alpha+\omega_0)t}d\alpha + c.c. \tag{21.54}$$

Next, for a reasonably narrow spectral width $\delta\alpha$ we can linearize the dispersion of the dielectric constant around carrier frequency

$$\varepsilon_r(\alpha+\omega_0) \approx \varepsilon_r(\omega_0) + \alpha\frac{d\varepsilon_r}{d\omega} \tag{21.55}$$

and substitute it into (21.54)

$$\frac{\partial D}{\partial t} = \frac{\varepsilon_0}{2}\frac{\partial}{\partial t}\int E(\alpha)\left[\varepsilon_r(\omega_0) + \alpha\frac{d\varepsilon_r}{d\omega}\right]e^{-j(\alpha+\omega_0)t}d\alpha + c.c. \tag{21.56}$$

Since the only time-dependent term in (21.56) is the exponential one

$$\begin{aligned}\frac{\partial D}{\partial t} &= -j\frac{\varepsilon_0}{2}\int E(\alpha)(\alpha+\omega_0)\left[\varepsilon_r(\omega_0)+\alpha\frac{d\varepsilon_r}{d\omega}\right]e^{-j(\alpha+\omega_0)t}d\alpha + c.c. = \\ &= -j\frac{\varepsilon_0}{2}\int E(\alpha)\left[\alpha\varepsilon_r(\omega_0)+\omega_0\varepsilon_r(\omega_0)+\alpha\omega_0\frac{d\varepsilon_r}{d\omega}+\alpha^2\frac{d\varepsilon_r}{d\omega}\right]e^{-j(\alpha+\omega_0)t}d\alpha + c.c.\end{aligned} \tag{21.57}$$

Let us look at the four terms inside e the square brackets. The last term is proportional to α^2 and can be neglected when compared to the third term since $\alpha << \omega_0$. The first and the third terms can be combined since

$$\left.\frac{d(\omega\varepsilon_r)}{d\omega}\right|_{\omega=\omega_0} = \varepsilon_r(\omega_0)+\omega_0\left.\frac{d\varepsilon_r(\omega)}{d\omega}\right|_{\omega=\omega_0} \tag{21.58}$$

and the second term stays as is. Hence

$$\frac{\partial D}{\partial t} = -j\frac{\varepsilon_0}{2}\int E(\alpha)\left[\omega_0\varepsilon_r(\omega_0)+\alpha\left.\frac{d(\omega\varepsilon_r)}{d\omega}\right|_{\omega=\omega_0}\right]e^{-j(\alpha+\omega_0)t}d\alpha + c.c. \tag{21.59}$$

First let us take all the terms that do not contain α outside the integral

$$\frac{\partial D}{\partial t} = -j\frac{\varepsilon_0}{2}\omega_0\varepsilon_r(\omega_0)e^{-j\omega_0 t}\int E(\alpha)e^{-j\alpha t}d\alpha - j\frac{\varepsilon_0}{2}e^{-j\omega_0 t}\frac{d(\omega\varepsilon_r)}{d\omega}\int \alpha E(\alpha)e^{-j\alpha t}d\alpha + c.c. \tag{21.60}$$

Now use (21.52) for the first term and also recognize that multiplication by $-j\alpha$ of the Fourier transform corresponds to the time derivative if the signal for the second term, hence

$$\begin{aligned}\frac{\partial D}{\partial t} &= -\frac{1}{2}j\omega_0\varepsilon_0\varepsilon_r(\omega_0)E_{env}(t)e^{-j\omega_0 t}+\frac{1}{2}\varepsilon_0\frac{d}{dt}\left[\frac{d(\omega\varepsilon_r)}{d\omega}E_{env}(t)\right]e^{-j\omega_0 t}+c.c. = \\ &= \omega_0\varepsilon_0\varepsilon_r(\omega_0)E_{env}(t)\sin(\omega_0 t)+\varepsilon_0\frac{d}{dt}\left[\frac{d(\omega\varepsilon_r)}{d\omega}E_{env}(t)\right]\cos(\omega_0 t)\end{aligned} \tag{21.61}$$

Using (21.51) we now obtain

$$\frac{dU_E}{dt} = E(t)\frac{\partial D(t)}{\partial t} = \omega_0\varepsilon_0\varepsilon_r(\omega_0)E_{env}^2(t)\sin(\omega_0 t)\cos(\omega_0 t)+\varepsilon_0\frac{d}{dt}\left[\frac{d(\omega\varepsilon_r)}{d\omega}E_{env}(t)\right]E_{env}(t)\cos(\omega_0 t)\cos(\omega_0 t)$$

(21.62)

All that is left is to perform time-averaging over the time interval that is long compared to optical period but short compared to the temporal extent of the signal. Obviously the first term in (21.62) average sto zero while the second one leave us with

$$\left\langle \frac{dU_E}{dt} \right\rangle_{\delta t} = \frac{1}{2}\varepsilon_0 \frac{d}{dt}\left[\frac{d\left(\omega\varepsilon_r\right)}{d\omega} E_{env}(t) \right] E_{env}(t) = \frac{1}{4}\varepsilon_0 \frac{d}{dt}\left[\frac{d\left(\omega\varepsilon_r\right)}{d\omega} E_{env}^2(t) \right] \quad (21.63)$$

Note that even we have a very long time duration and the wave is nearly monochromatic it all stands and hence we can simply use $E_0(t)$-amplitude in place of $E_{env}(t)$ and the density of electric energy is

$$\left\langle U_E \right\rangle_t = \frac{1}{4}\varepsilon_0 \frac{d\left(\omega\varepsilon_r\right)}{d\omega} E_0^2 = \frac{1}{4}\varepsilon_0\varepsilon_r E_0^2 + \frac{1}{4}\varepsilon_0\omega \frac{d\varepsilon_r}{d\omega} E_0^2 > 0 \quad (21.64)$$

There are two contributions here – the first one present in any medium but the second one owes its existence to the presence of dispersion.

To see if all we did make sense, let us consider the case of metal with dielectric constant

$$\varepsilon_r(\omega) = 1 - \frac{\omega_p^2}{\omega^2} < 0 \quad (21.65)$$

Find

$$\varepsilon_r(\omega) + \omega \frac{d\varepsilon_r(\omega)}{d\omega} = 1 - \frac{\omega_p^2}{\omega^2} + \omega \times 2\frac{\omega_p^2}{\omega^3} = 1 + \frac{\omega_p^2}{\omega^2} > 0 \quad (21.66)$$

Therefore

$$\left\langle U_E \right\rangle_t = \frac{1}{4}\varepsilon_0 E_0^2 + \frac{1}{4}\varepsilon_0 \frac{\omega_p^2}{\omega^2} E_0^2 > 0 \quad (21.67)$$

So the energy density is positive, but what is the physical reason for this result? Let us expand plasma frequency

$$\frac{1}{4}\varepsilon_0 \frac{\omega_p^2}{\omega^2} E_0^2 = \frac{1}{4}\varepsilon_0 \frac{N_e e^2}{\varepsilon_0 m_0 \omega^2} E_0^2 \quad (21.68)$$

where N_e is the density of electrons. According to Lecture when the electric field $E_0 \cos\omega t$ is applied the velocity of electrons is $v(t) = (-eE_0 / m_0)\sin(\omega t)$ and therefore

$$\frac{1}{4}\varepsilon_0 \frac{\omega_p^2}{\omega^2} E_0^2 = \frac{1}{2} N_e \left\langle m_0 \left(\frac{e}{m_0\omega} E_0 \sin(\omega t) \right)^2 \right\rangle_t = N_e \left\langle \frac{1}{2} m_0 v(t)^2 \right\rangle_t = N_e \left\langle \mathrm{E}_{kin}(t) \right\rangle_t \quad (21.69)$$

where $\mathrm{E}_{kin}(t)$ is kinetic energy of electron. Thus the second term in (21.67) is the density of kinetic energy of the carriers added to the first term which is energy of the electric field in the vacuum. Therefore, we can now access the energy density of dispersive medium as the sum of potential and kinetic energies.

Let us first neglect the dispersion. The time averaged energy density is the sum of electric and magnetic energies

$$\langle U \rangle_t = \langle U_E \rangle_t + \langle U_M \rangle_t = \frac{1}{4}\varepsilon_0\varepsilon_r E_0^2 + \frac{1}{4}\mu_0 H_0^2 \tag{21.70}$$

And since

$$H_0 = \frac{E_0 n}{\eta_0} = \frac{E_0 \varepsilon_0^{1/2}\varepsilon_r^{1/2}}{\mu_0^{1/2}} \tag{21.71}$$

We have

$$\langle U_M \rangle_t = \langle U_E \rangle_t = \frac{1}{4}\varepsilon_0\varepsilon_r E_0^2 \tag{21.72}$$

And

$$\langle U \rangle_t = \frac{1}{2}\varepsilon_0\varepsilon_r E_0^2 \tag{21.73}$$

On the microscopic level, we assume the Lorentz oscillator model as shown in Fig.21.10 at two different moments, t=0 when $\mathbf{r} = \mathbf{r}_{\max}$ and $\mathbf{v} = 0$ and t=T/4 (quarter of optical period) when $\mathbf{r} = 0$ and $\mathbf{v} = \mathbf{v}_{\max}$ a

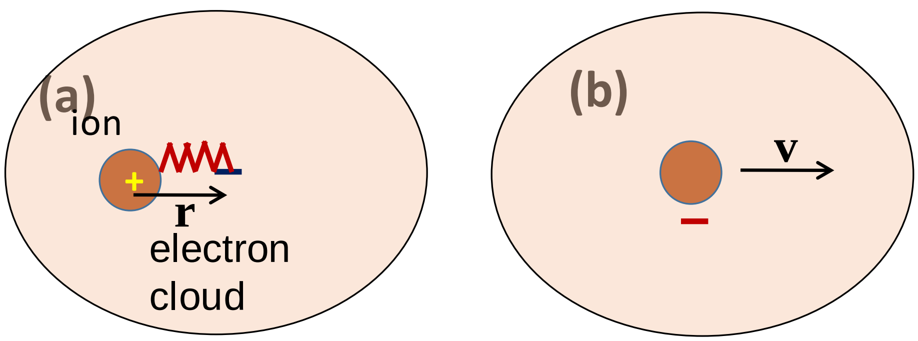


Figure 21. 10. Oscillating Lorentz dipole at (a) t=0 and (b) t=T/4

The dielectric function is (21.36) and therefore

$$\langle U_E \rangle_t = \frac{1}{4}\varepsilon_0 E_0^2 + N\frac{1}{4}\frac{e^2/m}{\omega_0^2 - \omega^2}E_0^2 \tag{21.74}$$

The first term is the “vacuum contribution” and the second term is the energy of the dipoles

$$\langle \mathrm{E}_d \rangle_t = \frac{1}{4}\frac{e^2/m}{\omega_0^2 - \omega^2}E_0^2 \tag{21.75}$$

But each electron in the dipole has alternating kinetic (Fig.20.10(a)) at times t=0,T/2… and kinetic energies (Fig.20.10(b)) at times t=T/4, 3T/4,…energies. Are they properly taken into account?

Dipole potential energy

Using Lorentz model the equation of motion (neglecting damping) is

$$m\frac{d^2\boldsymbol{r}}{dt^2} = -K\boldsymbol{r} - e\boldsymbol{E}_0\cos(\omega t) \tag{21.76}$$

where $K = \omega_0^2 m$ and the electron displacement ("extension of the spring") is

$$r(t) = -\frac{eE_0 / m}{\omega_0^2 - \omega^2}\cos(\omega t) \tag{21.77}$$

Therefore, time averaged potential energy of the dipole is

$$\left\langle \mathrm{E}_{pot} \right\rangle_t = \left\langle \frac{1}{2}Kr^2 \right\rangle_t = \frac{1}{2}K\frac{e^2/m^2}{\left(\omega_0^2 - \omega^2\right)^2}E_0^2\left\langle \cos^2(\omega t) \right\rangle_t = \frac{1}{4}\frac{\omega_0^2 e^2/m}{\left(\omega_0^2 - \omega^2\right)^2}E_0^2 \tag{21.78}$$

Clearly, far from the resonance $\omega << \omega_0$; $\omega_0^2 / (\omega_0^2 - \omega^2) \approx 1$ and

$$\left\langle \mathrm{E}_{pot} \right\rangle_t \approx \frac{1}{4}\frac{e^2/m}{\omega_0^2 - \omega^2}E_0^2 = \left\langle \mathrm{E}_d \right\rangle_t \tag{21.79}$$

Now everything almost makes sense, but what happens close to the resonance? Let us find out the difference

$$\left\langle \mathrm{E}_{pot} \right\rangle_t - \left\langle \mathrm{E}_d \right\rangle_t = \frac{1}{4}\frac{e^2/m}{\omega_0^2 - \omega^2}E_0^2\left[\frac{\omega_0^2}{\omega_0^2 - \omega^2} - 1\right] = \frac{1}{4}\frac{\omega^2 e^2/m}{\left(\omega_0^2 - \omega^2\right)^2}E_0^2 \tag{21.80}$$

Therefore, the potential energy can be written as a sum of two parts, static and dynamic which becomes prominent only near resonance

$$\mathrm{E}_{pot} = \frac{1}{2}\frac{e^2/m}{\omega_0^2 - \omega^2}E_0^2\cos^2(\omega t) + \frac{1}{2}\frac{\omega^2 e^2/m}{\left(\omega_0^2 - \omega^2\right)^2}E_0^2\cos^2(\omega t) \tag{21.81}$$

Dipole Kinetic energy

Velocity of the electron can be found by differentiating (21.77),

$$v(t) = \omega\frac{eE_0/m}{\omega_0^2 - \omega^2}\sin(\omega t) \tag{21.82}$$

Hence the kinetic energy is

$$\mathrm{E}_{kin} = \frac{1}{2}mv^2 = \frac{1}{2}\frac{\omega^2 e^2/m}{\left(\omega_0^2 - \omega^2\right)^2}E_0^2\sin^2(\omega t) \tag{21.83}$$

Our dipole is not only a capacitor, but also an inductor (that is why it has resonant frequency). Kinetic energy becomes significant only near the resonance, i.e. in the dispersive region. In fact, kinetic energy is exactly equal to the “dynamic” part of the potential energy in (21.81).

Total energy of the oscillating dipole

Adding (21.81) and (21.83) we obtain for the total energy of the dipole

$$\mathrm{E}_{pot}+\mathrm{E}_{kin}=\frac{1}{2}\frac{e^2/m}{\omega_0^2-\omega^2}E_0^2\cos^2(\omega t)+\frac{1}{2}\frac{\omega^2e^2/m}{\left(\omega_0^2-\omega^2\right)^2}E_0^2\cos^2(\omega t)+\frac{1}{2}\frac{\omega^2e^2/m}{\left(\omega_0^2-\omega^2\right)^2}E_0^2\sin^2(\omega t)=$$
$$=\frac{1}{2}\frac{e^2/m}{\omega_0^2-\omega^2}E_0^2\cos^2(\omega t)+\frac{1}{2}\frac{\omega^2e^2/m}{\left(\omega_0^2-\omega^2\right)^2}E_0^2$$

(21.84)

The time-averaged electrical energy density with dynamic contribution is then

$$\left\langle U_E\right\rangle_t=\frac{1}{4}\varepsilon_0E_0^2+\frac{1}{4}\frac{Ne^2/m}{\omega_0^2-\omega^2}E_0^2+\frac{1}{2}\frac{\omega^2Ne^2/m}{\left(\omega_0^2-\omega^2\right)^2}E_0^2 \qquad (21.85)$$

i.e. it has three parts- energy of electric field in vacuum, static energy of the dipoles and dynamic (resonant) contribution of the dipoles. The first two terms obviously add up to $\frac{1}{4}\varepsilon_0\varepsilon_rE_0^2$-static result. To find out what is the last term amounts to, take the derivative of $\varepsilon_r(\omega)$ in (21.36)

$$\frac{\partial\varepsilon_r(\omega)}{\partial\omega}=\frac{\partial}{\partial\omega}\left(1+\frac{Ne^2/\varepsilon_0m}{\omega_0^2-\omega^2}\right)=\varepsilon\frac{2\omega Ne^2/m\varepsilon_0}{\left(\omega_0^2-\omega^2\right)^2} \qquad (21.86)$$

and therefore the dynamic term is

$$\frac{\omega^2Ne^2/m}{\left(\omega_0^2-\omega^2\right)^2}=\frac{1}{2}\varepsilon_0\omega\frac{\partial\varepsilon_r(\omega)}{\partial\omega} \qquad (21.87)$$

Thus

$$\left\langle U_E\right\rangle_t=\frac{1}{4}\varepsilon_0\varepsilon_rE_0^2+\frac{1}{4}\varepsilon_0\omega\frac{\partial\varepsilon_r}{\partial\omega}E_0^2=\frac{1}{4}\frac{\partial\left(\omega\varepsilon\right)}{\partial\omega}E_0^2 \qquad (21.88)$$

Which is precisely the result (21.64), but now obtained using microscopic picture.

Energy velocity

The total energy density is

$$\left\langle U\right\rangle_t=\left\langle U_E\right\rangle_t+\left\langle U_M\right\rangle_t=\frac{1}{4}\frac{\partial\left(\omega\varepsilon\right)}{\partial\omega}E^2+\frac{1}{4}\mu_0H^2 \qquad (21.89)$$

Using (21.71) we obtain

$$\langle U\rangle_t = \frac{1}{2}\varepsilon_0\varepsilon_r E_0^2 + \frac{1}{4}\varepsilon_0\omega\frac{\partial n^2}{\partial\omega}E_0^2 = \frac{1}{2}\varepsilon_0 n^2 E_0^2 + \frac{1}{2}\varepsilon_0 n\omega\frac{\partial n}{\partial\omega}E_0^2 = \frac{1}{2}\varepsilon_0 n E_0^2\left[n+\omega\frac{\partial n}{\partial\omega}\right] = \frac{1}{2}\varepsilon_0 n n_g E_0^2$$

(21.90)

The power flow is

$$\langle S\rangle_t = \frac{nE_0^2}{2\eta_0} = v_E\langle U\rangle_t \tag{21.91}$$

Where we have introduced the energy velocity

$$v_E = \frac{\langle S\rangle_t}{\langle U\rangle_t} = \frac{1}{\eta_0\varepsilon_0 n_g} = \frac{c}{n_g} = v_g \tag{21.92}$$

The energy is moving with the group velocity.