Lecture 26 Acousto-Optics

Acousto-optic effect describes changes of optical properties (i.e. permittivity tensor) imposed by the strain. Why the changes occur can be seen from Fig.26.1 (a) When the stress is applied to the material the ions move away from their equilibrium positions. The projections of motion along three axes are

$$u(x,y,z),\ v(x,y,z),\ w(x,y,z) \tag{26.1}$$

The change of the positions of ions surrounding electron cloud changes the forces acting on the electron cloud which formally can be represented by the changes in "spring coefficients" K_{xyz} and can be seen in Fig.26.1(b) as changes in the effective potential that binds electron.

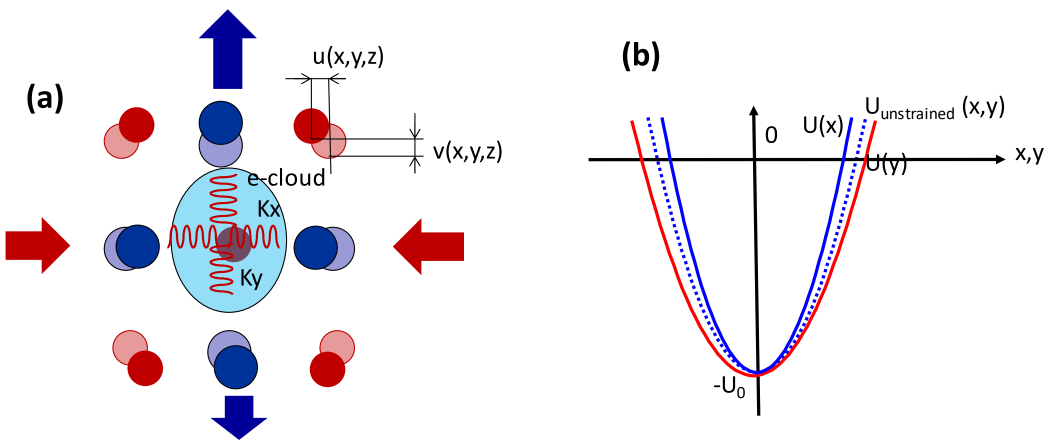


Figure 26.1 (a) Strain (b) its effect on the potential seen by the electrons

Introduce the strain tensor (dimensionless) as

$$S = \begin{pmatrix} S_{xx} & S_{xy} & S_{xz} \\ S_{yx} & S_{yy} & S_{yz} \\ S_{zx} & S_{zy} & S_{zz} \end{pmatrix} \tag{26.2}$$

Where the diagonal components

$$S_{xx} = \frac{\partial u}{\partial x};\ S_{yy} = \frac{\partial v}{\partial y}; S_{zz} = \frac{\partial w}{\partial x}; \tag{26.3}$$

describe tension (if positive) or compression (if negative), while the off-diagonal components

$$S_{xy} = S_{yx} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x};\ S_{xz} = S_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x};\ S_{yz} = S_{zy} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \tag{26.4}$$

describe the shear strain. To make things easier to put on paper we can represent the strain tensor as a1x6 vector

$$S = \begin{pmatrix} S_1 = S_{xx} \\ S_2 = S_{yy} \\ S_3 = S_{yy} \\ S_4 = S_{yz} \\ S_5 = S_{zx} \\ S_6 = S_{xy} \end{pmatrix} \tag{26.5}$$

Next we introduce 6x6 strain-optical or photo-elstensor P that causes changes in the impermeability tensor

$$\Delta\varepsilon_r^{-1} = \Delta\left(\frac{1}{n^2}\right)_i = \sum_{j=1}^{6} P_{ij} S_j \tag{26.6}$$

where the original impermeability tensor is

$$\varepsilon_r^{-1} = \begin{pmatrix} 1/n_1^2 & 0 & 0 \\ 0 & 1/n_2^2 & 0 \\ 0 & 0 & 1/n_3^2 \end{pmatrix} \tag{26.7}$$

As a result original index ellipsoid shown in Fig.26.2(a) and described by the equation

$$\frac{x^2}{n_1^2} + \frac{y^2}{n_2^2} + \frac{z^2}{n_3^2} = 1 \tag{26.8}$$

now becomes

$$\begin{gathered} \left[\left(\frac{1}{n^2}\right)_1 + \sum_{j=1}^{6} P_{1j} S_j\right] x^2 + \left[\left(\frac{1}{n^2}\right)_2 + \sum_{j=1}^{6} P_{2j} S_j\right] y^2 + \left[\left(\frac{1}{n^2}\right)_3 + \sum_{j=1}^{6} P_{3j} S_j\right] z^2 + \\ +2\left[\left(\frac{1}{n^2}\right)_4 + \sum_{j=1}^{6} P_{4j} S_j\right] yz + 2\left[\left(\frac{1}{n^2}\right)_5 + \sum_{j=1}^{6} P_{5j} S_j\right] xz + 2\left[\left(\frac{1}{n^2}\right)_6 + \sum_{j=1}^{6} P_{6j} S_j\right] xy = 1 \end{gathered} \tag{26.9}$$

as shown in Fig. 26.2(b)

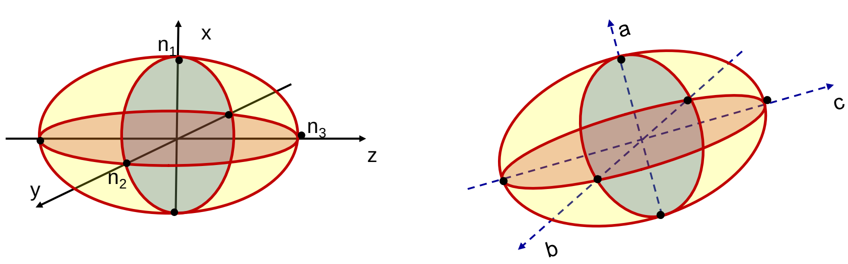


Figure 26.2 index ellipsoid (a) without and (b) with strain

Consider the examples. In the isotropic medium photoblastic tensor has just two independent components

$$P=\begin{pmatrix} P_{11} & P_{12} & P_{12} & 0 & 0 & 0 \\ P_{12} & P_{11} & P_{12} & 0 & 0 & 0 \\ P_{12} & P_{12} & P_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}\left(P_{11}-P_{12}\right) & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}\left(P_{11}-P_{12}\right) & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\left(P_{11}-P_{12}\right) \end{pmatrix} \tag{26.10}$$

In Fused Silica $p_{11}=0.121\, p_{12}=0.270$ and in water $p_{11}=0.131\, p_{12}=0.31$.

Consider longitudinal acoustic wave propagating along z direction in isotropic medium

$$w(z)\equiv w_0\cos\left(Kz-\Omega t\right) \tag{26.11}$$

The strain is

$$S_{zz}=w_0K\left(Kz-\Omega t\right)=S_3\sin\left(Kz-\Omega t\right) \tag{26.12}$$

Then impermeability tensor becomes

$$\varepsilon_r^{-1}=\begin{pmatrix} 1/n^2 \\ 1/n^2 \\ 1/n^2 \\ 0 \\ 0 \\ 0 \end{pmatrix}+\begin{pmatrix} P_{11} & P_{12} & P_{12} & 0 & 0 & 0 \\ P_{12} & P_{11} & P_{12} & 0 & 0 & 0 \\ P_{12} & P_{12} & P_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}\left(P_{11}-P_{12}\right) & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}\left(P_{11}-P_{12}\right) & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\left(P_{11}-P_{12}\right) \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ S_3 \\ 0 \\ 0 \\ 0 \end{pmatrix}=\begin{pmatrix} 1/n^2+P_{12}S_3 \\ 1/n^2+P_{12}S_3 \\ 1/n^2+P_{11}S_3 \\ 0 \\ 0 \\ 0 \end{pmatrix}=\begin{pmatrix} 1/n_0^2 \\ 1/n_0^2 \\ 1/n_e^2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

(26.13)

i.e. the medium becomes birefringent and uniaxial with ordinary and extraordinary indices

$$n_o\approx n-\frac{1}{2}n^3P_{12}S_3 \;\; n_e\approx n-\frac{1}{2}n^3P_{11}S_3 \tag{26.14}$$

Consider some other materials such as $LiNbO_3$ which is normally a uniaxial material. The photoelastic tensor has 8 independent components

$$P = \begin{pmatrix} P_{11} & P_{12} & P_{13} & P_{14} & 0 & 0 \\ P_{12} & P_{11} & P_{13} & -P_{14} & 0 & 0 \\ P_{31} & P_{31} & P_{33} & 0 & 0 & 0 \\ P_{41} & -P_{41} & 0 & P_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & P_{44} & P_{41} \\ 0 & 0 & 0 & 0 & P_{14} & \frac{1}{2}\left(P_{11}-P_{12}\right) \end{pmatrix} \tag{26.15}$$

Consider transverse (shear) acoustic wave propagating along z direction and polarized along y direction in $LiNbO_3$

$$v = v_0 \cos\left(Kz - \Omega t\right) \tag{26.16}$$

Strain is

$$S_{zy} \equiv S_{yz} = v_0 K \cos\left(Kz - \Omega t\right) = S_4 \sin\left(Kz - \Omega t\right) \tag{26.17}$$

$$\varepsilon_r^{-1} = \begin{pmatrix} 1/n_o^2 \\ 1/n_o^2 \\ 1/n_e^2 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} P_{11} & P_{12} & P_{13} & P_{14} & 0 & 0 \\ P_{12} & P_{11} & P_{13} & -P_{14} & 0 & 0 \\ P_{31} & P_{31} & P_{33} & 0 & 0 & 0 \\ P_{41} & -P_{41} & 0 & P_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & P_{44} & P_{41} \\ 0 & 0 & 0 & 0 & P_{14} & \frac{1}{2}\left(P_{11}-P_{12}\right) \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ S_4 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/n_o^2 + P_{14}S_3 \\ 1/n_o^2 - P_{14}S_4 \\ 1/n_e^2 \\ P_{44}S_4 \\ 0 \\ 0 \end{pmatrix} \tag{26.18}$$

The index ellipsoid becomes

$$\left(\frac{1}{n_o^2} + P_{14}S_4\right)x^2 + \left(\frac{1}{n_o^2} - P_{14}S_4\right)y^2 + \frac{z^2}{n_e^2} + 2P_{44}S_4 yz = 0 \tag{26.19}$$

And the medium is now biaxial with principal axes rotated around x axis.

Photoelastic properties of important materials are shown below

Material	Photoelastic coefficients		Figure of merit $M_2(\times 10^{-15} s^3/kg)$
Fused silica	$p_{11} = +0.121$,	$p_{12} = +0.270$	1.51
Water	$p_{11} = 0.31$,	$p_{12} = 0.31$	160
Dense flint SF-4	$p_{11} = +0.232$,	$p_{12} = +0.256$	4.53
Lithium niobate ($LiNbO_3$)	$p_{11} = -0.02$,	$p_{12} = +0.08$	13.6
	$p_{13} = +0.13$,	$p_{14} = -0.08$	
	$p_{31} = +0.17$,	$p_{33} = +0.07$	
	$p_{41} = -0.15$,	$p_{44} = +0.12$	
Lithium tantalate ($LiTaO_3$)	$p_{11} = 0.08$,	$p_{12} = 0.08$	1.37
	$p_{13} = 0.09$,	$p_{14} = 0.03$	
	$p_{31} = 0.09$,	$p_{33} = 0.15$	
	$p_{41} = 0.02$,	$p_{44} = 0.02$	
α-Quartz (SiO_2)	$p_{11} = +0.16$,	$p_{12} = +0.27$	
	$p_{13} = +0.27$,	$p_{14} = -0.03$	
	$p_{31} = +0.29$,	$p_{33} = +0.10$	
	$p_{41} = -0.047$,	$p_{44} = -0.079$	
Tellurium dioxide (TeO_2)	$p_{11} = 0.0074$,	$p_{12} = +0.187$	793
	$p_{13} = +0.340$,	$p_{31} = +0.090$	
	$p_{33} = +0.240$,	$p_{44} = -0.17$	
	$p_{66} = -0.046$		
KDP	$p_{11} = +0.251$,	$p_{12} = +0.249$	3.8
	$p_{13} = +0.246$,	$p_{31} = +0.225$	
	$p_{33} = +0.221$,	$p_{44} = ?$	
	$p_{66} = 0.058$		

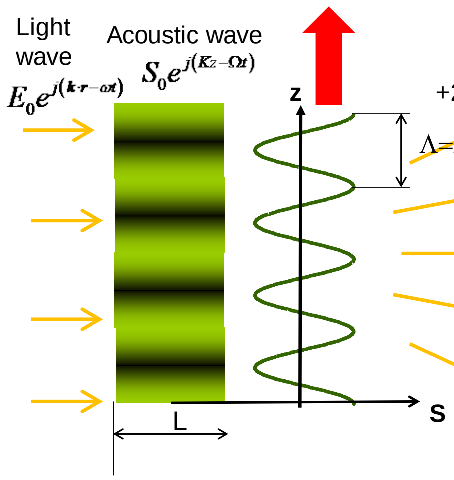
Raman Nath diffraction

Raman Nath diffraction on sound waves occurs when the interaction length is relatively short

$$L << \frac{k}{K^2} = \frac{\Lambda^2}{2\pi\lambda} \tag{26.20}$$

and it is characterized by multiple diffraction orders. The order of magnitude of critical length can be found for the example of $F = 10MHz$ acoustic wave propagating in the glass with acoustic wave velocity of $v_s = 3000m/s$ Then acoustic wavelength Λ is $\Lambda = v_s / F = 300\mu m$ - for the visible light of $\lambda_0 = 600nm$ and refractive index $n = 1.5$ we obtain $L << 3.5cm$. In the opposite limit the Acousto-optic interactions will occur in Bragg regime.

Let us now consider Raman-Nath diffraction as in Fig.26.3. Acoustic wave produces **moving** refractive index grating.



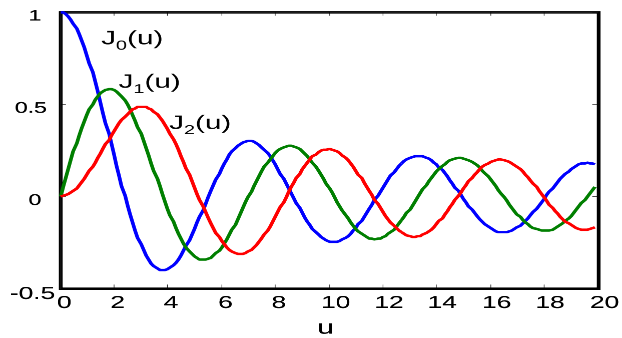


Figure 26.3 (a) Raman Nath Diffraction (b) Bessel functions

$$n(z,t) = n_0 + \Delta n \sin(\Omega t - Kz) \tag{26.21}$$

For **thin** plate (small L) phase delay is

$$\varphi = \frac{2\pi}{\lambda_0} n_0 L + \frac{2\pi}{\lambda_0} \Delta n L \sin(\Omega t - Kz) = \varphi_0 + \Delta\varphi \sin(\Omega t - Kz) \tag{26.22}$$

Therefore, if the incident field is the transmitted filed is $E_i = E_0 e^{-j(\omega t - kx)}$

$$E_T = E_i e^{-j\omega t} e^{j\Delta\varphi \sin(\Omega t - Kz)} \tag{26.23}$$

where we have dropped the constant phase term φ_0. Expand (26.23)into the Fourier series using properties of Bessel functions,

$$e^{ju\sin\theta} = J_0(u) + 2\sum_{n=1}^{\infty} J_{2n}(u)\cos(2n\theta) - 2j\sum_{n=1}^{\infty} J_{2n-1}(u)\sin\left[(2n-1)\sin\theta\right] \tag{26.24}$$

Thus

$$E_T = E_i e^{-j\omega t} e^{j\Delta\varphi \sin(\Omega t - Kz)} = E_i J_0\left(\Delta\varphi\right) e^{-j\omega t} +$$
$$-E_i J_1\left(\Delta\varphi\right) e^{-j\omega t} \times 2j\sin(\Omega t - Kz) + E_i J_2\left(\Delta\varphi\right) e^{-j\omega t} \times 2\cos(2\Omega t - 2Kz) =$$
$$= E_0 J_i\left(\Delta\varphi\right) e^{-j\omega t} + E_i J_1\left(\Delta\varphi\right)\left[e^{-j\left[(\omega+\Omega)t - Kz\right]} - e^{-j\left[(\omega-\Omega)t + Kz\right]}\right] + E_i J_2\left(\Delta\varphi\right)\left[e^{-j\left[(\omega+2\Omega)t - 2Kz\right]} - e^{-j\left[(\omega-2\Omega)t + 2Kz\right]}\right] + \ldots\ldots$$

(26.25)

With the amplitudes of the Fourier components shown in Fig.26.2.(b).

The first, zeroth order term is an undiffracted wave.

$$E_{T,0}(x) = E_0 J_0\left(\Delta\varphi\right) e^{-j(\omega t - kx)} \tag{26.26}$$

The +1 order term at frequency $\omega_1 = \omega + \Omega$ is

$$E_{T,1}(x,z) = -E_0 J_1\left(\Delta\varphi\right) e^{-j(\omega t - kx)} e^{-j[\Omega t - Kz]} = -E_0 J_1\left(\Delta\varphi\right) e^{-j\left[(\omega+\Omega)t - k^+ x - Kz\right]} \tag{26.27}$$

Similarly, the (-1)-order term at frequency $\omega_{-1} = \omega - \Omega$ is

$$E_{T,-1}(x,z) = -E_0 J_1\left(\Delta\varphi\right) e^{-j\left[(\omega-\Omega)t - k^- x + Kz\right]} \tag{26.28}$$

Since the transverse component is added to the wavevector, its projection on x axis changes to assure momentum conservation

$$\begin{aligned} \left(k^{\pm}\right)^2 + K^2 &= \frac{\left(\omega \pm \Omega\right)^2}{c^2} \\ k^{\pm} &\approx (k_0^2 - K^2)^{1/2} \end{aligned} \tag{26.29}$$

The direction of propagation as shown in Fig. 26.4(a) is

$$\sin\theta_{\pm 1} = \pm\frac{K}{k_0} = \pm\frac{\lambda_0}{\Lambda} \tag{26.30}$$

For m-th order the diffraction angle is

$$\sin\theta_{\pm m} = \pm m\frac{K}{k_0} = \pm m\frac{\lambda_0}{\Lambda} \tag{26.31}$$

Essentially the acoustic wavelength plays the role of the period of ”normal”, i.e. stationary diffraction grating a. The magnitude of m-th maximum is determined by $J_m^2\left(\Delta\varphi\right)$ shown in Fig.26.4(b). If Δϕ=2.405 all the light is diffracted and if Δϕ =1.85- intensity of the 1st order reaches maximum 34%. Now, for small $\Delta\varphi << \pi/2$ one can approximate Bessel function as

$$J_m\left(\Delta\varphi\right) \approx \frac{1}{m!}\left(\frac{\Delta\varphi}{2}\right)^n \tag{26.32}$$

And most of the diffraction is in the first order , i.e.

$$E_T = E_0 J_0\left(\Delta\varphi\right) e^{-j\omega t} - \tfrac{1}{2} E_0 \Delta\varphi e^{-j\left[(\omega+\Omega)t - Kz\right]} + \tfrac{1}{2} E_0 \Delta\varphi e^{-j\left[(\omega-\Omega)t + Kz\right]} \tag{26.33}$$

With the diffraction efficiency into the first order equal to

$$\eta = \frac{1}{4}\left(\Delta\varphi\right)^2 = \frac{\pi^2 (\Delta n)^2 L^2}{\lambda^2} = \frac{\pi^2 L^2}{\lambda^2}\left(\frac{1}{2} n^3 PS\right)^2 \tag{26.34}$$

The energy density of the acoustic wave is

$$U_{ac} = \frac{1}{2}\delta S^2 \tag{26.35}$$

Where δ is elastic modulus (ratio of strain to stress causing it). The velocity of sound is $v_s = \sqrt{\delta / \rho}$ where ρ is the density. Therefore, the intensity (power density) of the acoustic wave is

$$I_{ac} = U_{ac} v_s = \frac{1}{2}\rho v_s^3 S^2 \tag{26.36}$$

The acoustic power is $\mathrm{P}_{ac} = LHI_{ac}$ where $L \times H$ is the cross-section of acoustic wave – therefore

$$S^2 = \frac{2}{\rho v_s^3}\frac{1}{LH}\mathrm{P}_{ac} \tag{26.37}$$

and substituting it into (26.34)

$$\eta = \frac{\pi^2}{2\lambda^2}\frac{n^6}{\rho v_s^3}P^2\frac{L}{H}\mathrm{P}_{ac} = \frac{\pi^2}{2\lambda^2}M_2\frac{L}{H}\mathrm{P}_{ac} \tag{26.38}$$

Where the *acousto-optic figure of merit* is

$$M_2 = \frac{n^6}{\rho v_s^3}P^2 \tag{26.39}$$

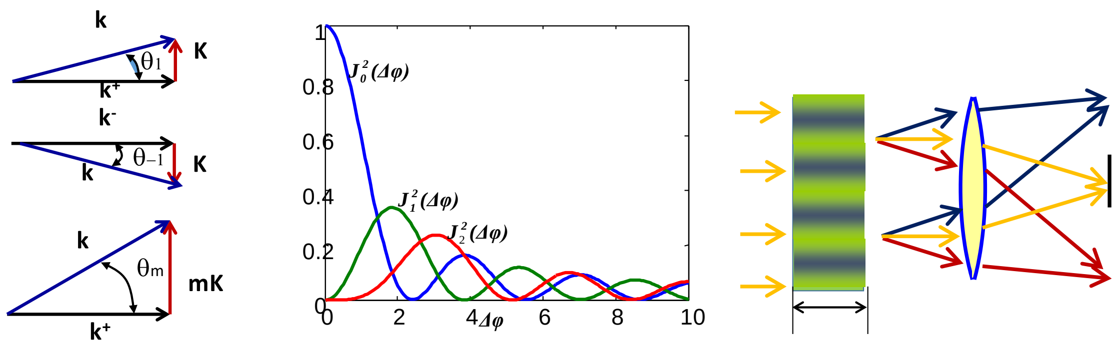


Figure 26.4 (a) momentum conservation in Raman Nath diffraction (b) Intensities of diffracted light (c) Raman Nath Acousto-optic modulator.

In Figure 26.4 (c) one can see the RN Acousto-optic modulator where in the absence of acoustic wave all the light is topped but in presence of acoustic wave it goes around the stop. For high efficiency one needs not only high figure of merit but also a large aspect ratio, i.e. long interaction length. However, to operate in Raman Nath regime according to (26.20)

$$L << \frac{\Lambda^2}{2\pi\lambda} = \frac{v_s^2}{2\pi\lambda F^2} \tag{26.40}$$

Hence efficiency decreases quadratic ally with frequency. For wide bandwidth modulation one has to resort to Bragg modulators.

Bragg diffraction

Consider the light propagating at the oblique angle to the wave front of the acoustic wave as in Fig. 26.5 (a). In the absence of the acoustic wave the wave equation is

$$\nabla^2 E - \frac{n_0^2}{c^2}\frac{\partial^2}{\partial t^2}E = 0 \tag{26.41}$$

with a ubiquitous plane wave solution

$$E = E_0 e^{j(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} \tag{26.42}$$

where

$$k(\omega) = n_0\omega / c \tag{26.43}$$

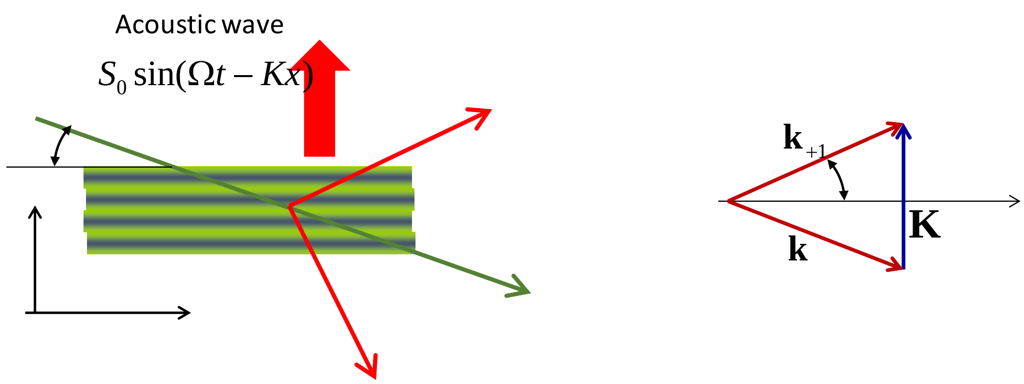


Figure 26.5 (a) Bragg diffraction (b) Bragg condition

When the acoustic wave $S_0 \sin(\Omega t - Kx)$ with frequency $\Omega << \omega$ propagates the dielectric constant is modulated as

$$\varepsilon_r(x,t) = n^2 = n_0^2 + \Delta\varepsilon_r \sin(\Omega t - Kx) \tag{26.44}$$

and the wave equation becomes

$$\nabla^2 E - \frac{n_0^2}{c^2}\frac{\partial^2}{\partial t^2}E = \frac{\Delta\varepsilon_r}{2jc^2}\frac{\partial^2}{\partial t^2}E\left(e^{j(\Omega t - Kx)} - e^{-j(\Omega t - Kx)}\right) \tag{26.45}$$

Obviously terms with frequencies $\omega \pm \Omega, \omega \pm 2\Omega, ...$ are generated as light propagates and therefore we shall look for a solution

$$E(x,z,t)=\sum_{m=0,\pm1,\ldots} A_m(z)e^{-j(\omega+m\Omega)t}e^{j\boldsymbol{k}_m\boldsymbol{r}} \tag{26.46}$$

where

$$k_m^2=\frac{n_0^2}{c^2}\left(\omega+m\Omega\right)^2 \tag{26.47}$$

and $A_m(z)$ are slowly (relative to wavelength) variable amplitudes for which

$$\frac{d^2A_m(z)}{dz^2} << k_m\frac{dA_m(z)}{dz} \tag{26.48}$$

Then

$$\begin{aligned}\nabla^2\left[A_m(z)e^{-j(\omega+m\Omega)t}e^{j\boldsymbol{k}_m\boldsymbol{r}}\right]&=e^{-j(\omega+m\Omega)t}e^{j\boldsymbol{k}_m\boldsymbol{r}}\left[\frac{\partial^2A_m}{\partial z^2}+2jk_{mz}\frac{\partial A_m}{\partial z}-k_m^2A_m\right]\approx\\ &e^{-j(\omega+m\Omega)t}e^{j\boldsymbol{k}_m\boldsymbol{r}}\left[+2jk_{mz}\frac{\partial A_m}{\partial z}-\frac{n_0^2(\omega+m\Omega)^2}{c^2}A_m\right]\end{aligned} \tag{26.49}$$

and

$$\frac{n_0^2}{c^2}\frac{\partial^2}{\partial t^2}A_m(z)e^{-j(\omega+m\Omega)t}e^{j\boldsymbol{k}_m\boldsymbol{r}}=-\frac{n_0^2(\omega+m\Omega)^2}{c^2}e^{-j(\omega+m\Omega)t}e^{j\boldsymbol{k}_m\boldsymbol{r}} \tag{26.50}$$

when (26.49) and (26.50) are substituted into (26.45) the last term from (26.49) cancels the only term from (26.50) . At the same time for the term on the r.h.s. of (26.45)

$$\begin{aligned}&\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left[A_m(z)e^{-j(\omega+m\Omega)t}e^{j\boldsymbol{k}_m\boldsymbol{r}}\left(e^{j(\Omega t-Kx)}-e^{-j(\Omega t-Kx)}\right)\right]=\\ &-\frac{A_m\left[\omega+(m-1)\Omega\right]^2}{c^2}e^{-j\left[\omega+(m-1)\Omega\right]t}e^{j(\boldsymbol{k}_m\boldsymbol{r}-Kx)}+\frac{A_m\left[\omega+(m+1)\Omega\right]^2}{c^2}e^{-j\left[\omega+(m+1)\Omega\right]t}e^{j(\boldsymbol{k}_m\boldsymbol{r}+Kx)}\end{aligned} \tag{26.51}$$

And so we obtain

$$\begin{aligned}&\sum_{m=0,\pm1,}2jk_{mz}\frac{\partial A_m}{\partial z}e^{-j(\omega+m\Omega)t}e^{j\boldsymbol{k}_m\boldsymbol{r}}=\\ &=\frac{\Delta\varepsilon_r}{2j}\sum_{m=0,\pm1,}\left[-\frac{A_m\left[\omega+(m-1)\Omega\right]^2}{c^2}e^{-j\left[\omega+(m-1)\Omega\right]t}e^{j(\boldsymbol{k}_m\boldsymbol{r}-Kx)}+\frac{A_m\left[\omega+(m+1)\Omega\right]^2}{c^2}e^{-j\left[\omega+(m+1)\Omega\right]t}e^{j(\boldsymbol{k}_m\boldsymbol{r}+Kx)}\right]\end{aligned}$$

(26.52)

Now equalize the terms with the same time dependence. The terms with the same time dependence $e^{-j(\omega+m\Omega)t}$ are the terms with A_m on the l.h.s. and the terms with A_{m+1} and A_{m-1} on the r.h.s.

$$2\,jk_{mz}\frac{\partial A_m}{\partial z}e^{j\boldsymbol{k}_m\cdot\boldsymbol{r}}=\frac{\Delta\varepsilon_r}{2j}\frac{A_{m+1}\omega_m^2}{c^2}e^{j(\boldsymbol{k}_{m+1}\cdot\boldsymbol{r}-Kx)}-\frac{\Delta\varepsilon_r}{2j}\frac{A_{m-1}\omega_m^2}{c^2}e^{j(\boldsymbol{k}_{m-1}\cdot\boldsymbol{r}+Kx)} \tag{26.53}$$

where $\omega_m=\omega+m\Omega$, or

$$\frac{\partial A_m}{\partial z}=-\frac{\Delta\varepsilon_r}{4}\frac{\omega_m^2}{c^2k_{mz}}A_{m+1}e^{j(\boldsymbol{k}_{m+1}\cdot\boldsymbol{r}-Kx-\boldsymbol{k}_m\cdot\boldsymbol{r})}+\frac{\Delta\varepsilon_r}{4}\frac{\omega_m^2}{c^2k_{mz}}A_{m-1}e^{j(\boldsymbol{k}_{m-1}\cdot\boldsymbol{r}+Kx-\boldsymbol{k}_m\cdot\boldsymbol{r})} \tag{26.54}$$

Introduce now coupling strength

$$\kappa_m=\frac{\Delta\varepsilon_r}{4}\frac{\omega_m^2}{c^2k_{mz}} \tag{26.55}$$

and the wave vector (momentum) mismatch

$$\begin{aligned}\Delta\mathbf{k}_{m,m+1}&=\left(\mathbf{k}_{m+1}-\mathbf{k}_m\right)-\mathbf{K}\\ \Delta\mathbf{k}_{m,m-1}&=\left(\mathbf{k}_{m-1}-\mathbf{k}_m\right)-\mathbf{K}\end{aligned} \tag{26.56}$$

Then we finally arrive at

$$\frac{\partial A_m}{\partial z}=-\kappa_mA_{m+1}e^{j\Delta\mathbf{k}_{m,m+1}\cdot\mathbf{r}}+\kappa_mA_{m-1}e^{j\Delta\mathbf{k}_{m,m-1}\cdot\mathbf{r}} \tag{26.57}$$

Obviously solution grows linearly only when Δ**k**~0 otherwise it oscillates as a *sinc* function. Clearly no more than a single wave vector mismatch can be equal to zero simultaneously. Let us say the very first one, between 0 and +1 order,

$$\Delta\mathbf{k}_{0,1}=\mathbf{k}_{+1}-\mathbf{k}_m-\mathbf{K}=0 \tag{26.58}$$

or

$$\mathbf{k}_{+1}=\mathbf{k}_m+\mathbf{K} \tag{26.59}$$

as shown in Fig. 26.5(b). Since absolute values of wavevectors are nearly equal $k\approx k_{+1}$ we obtain *Bragg condition*

$$K=2k\sin\theta_B;\;\;\sin\theta_B=\frac{\lambda}{2\Lambda} \tag{26.60}$$

which can also be written as

$$k_x=\frac{K}{2};\;k_x\Lambda=\pi \tag{26.61}$$

Indicating that acoustic wave creates a moving multilayer reflector similar to the ones studied in Chapter 16.

Now we can write coupled wave equations for two coupled by Bragg diffraction waves (Fig.26.6a)

$$
\begin{aligned}
\frac{\partial A_0}{\partial z} &= -\kappa A_{+1} e^{j\Delta k_z z} \\
\frac{\partial A_{+1}}{\partial z} &= +\kappa A_0 e^{-j\Delta k_z z}
\end{aligned}
\tag{26.62}
$$

where according to (26.55)

$$
\kappa = \frac{\Delta\varepsilon_r}{4}\frac{\omega^2}{c^2 k_{0z}}
\tag{26.63}
$$

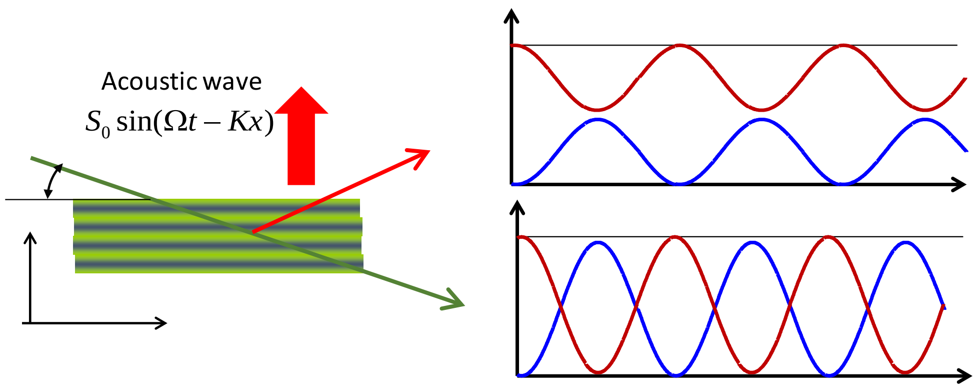


Figure 26.6. (a) Bragg diffraction –coupled waves (b) Power transfer with wavevector mismatch (c) Power transfer with perfect matching

Introduce new variables

$$
\begin{aligned}
A_0(z) &= \tilde{A}_0(z) e^{j\Delta k_z/2} \\
A_{+1}(z) &= \tilde{A}_{+1}(z) e^{-j\Delta k_z/2}
\end{aligned}
\tag{26.64}
$$

and substitute it into (26.62) to obtain

$$
\begin{aligned}
\frac{\partial \tilde{A}_0}{\partial z} + j\frac{\Delta k_z}{2}\tilde{A}_0 &= -\kappa \tilde{A}_{+1} \\
\frac{\partial \tilde{A}_{+1}}{\partial z} - j\frac{\Delta k_z}{2}\tilde{A}_{+1} &= \kappa \tilde{A}_0
\end{aligned}
\tag{26.65}
$$

Look for the usual exponential solution for the system of two first order differential equations

$$
\begin{aligned}
\tilde{A}_0(z) &= \tilde{A}_0 e^{j\gamma z} \\
A_{+1}(z) &= \tilde{A}_{+1} e^{j\gamma z}
\end{aligned}
\tag{26.66}
$$

and substitute it into (26.65) to obtain

$$\begin{pmatrix} \gamma + j\dfrac{\Delta k_z}{2} & \kappa \\ -\kappa & \gamma - j\dfrac{\Delta k_z}{2} \end{pmatrix} \begin{pmatrix} \tilde{A}_0 \\ \tilde{A}_{+1} \end{pmatrix} = 0 \tag{26.67}$$

To have nontrivial solution the determinant of this equation must be equal to zero and we obtain characteristic equation

$$\begin{vmatrix} j\left(\gamma + \dfrac{\Delta k_z}{2}\right) & \kappa \\ -\kappa & j\left(\gamma - \dfrac{\Delta k_z}{2}\right) \end{vmatrix} = 0 \tag{26.68}$$

with the solution

$$\begin{aligned} &\gamma^2 - \frac{\Delta k_z^2}{4} - \kappa^2 = 0 \\ &\gamma_{1,2} = \pm\sqrt{\kappa^2 + \frac{\Delta k_z^2}{4}} \end{aligned} \tag{26.69}$$

Therefore the solution of (26.65) is

$$\begin{aligned} &\tilde{A}_0(z) = B\cos\gamma z + C\sin\gamma z \\ &\tilde{A}_{+1}(z) = \frac{\gamma}{\kappa}\left(B\sin\gamma z - C\cos\gamma z\right) - j\frac{\Delta k_z}{2\kappa}\left(B\cos\gamma z + C\sin\gamma z\right) \end{aligned} \tag{26.70}$$

The boundary conditions are

$$\begin{aligned} &\tilde{A}_0(0) = 1 \\ &\tilde{A}_+(0) = 0 \end{aligned} \tag{26.71}$$

immediately yield

$$\begin{aligned} &B = 1 \\ &C = -j\frac{\Delta k_z}{2\gamma} \end{aligned} \tag{26.72}$$

Resulting in

$$\begin{aligned} &\tilde{A}_0(z) = \cos\gamma z - j\frac{\Delta k_z}{2\gamma}\sin\gamma z \\ &\tilde{A}_{+1}(z) = \left(\frac{\gamma}{\kappa} - \frac{\Delta k_z^2}{4\kappa\gamma}\right)\sin\gamma z = \frac{\gamma^2 - \Delta k_z^2/4}{\kappa\gamma}\sin\gamma z = \frac{\kappa}{\gamma}\sin\gamma z \end{aligned} \tag{26.73}$$

Taking absolute value squared we obtain the equations of power transfer (here power is dimensionless relative to the input power)

$$
\begin{aligned}
P_0(z) &= \cos^2 \gamma z + \left(\frac{\Delta k_z}{2\gamma}\right)^2 \sin^2 \gamma z \\
P_+(z) &= \left(\frac{\kappa}{\gamma}\right)^2 \sin^2 \gamma z
\end{aligned}
\tag{26.74}
$$

as shown in Fig.26.6(b). Adding two equation in (26.74) and using (26.69) we get

$$
P_0(z) + P_+(z) = \cos^2 \gamma z + \frac{\frac{\Delta k_z^2}{4} + \kappa^2}{\gamma^2} \sin^2 \gamma z = 1 \tag{26.75}
$$

i..e the total power is conserved as it should be. One can achieve complete power transfer only for the perfectly matched wavevectors, i.e. at Bragg angle (26.60) Then

$$
\begin{aligned}
P_0(z) &= \cos^2 \kappa z \\
P_+(z) &= \sin^2 \kappa z
\end{aligned}
\tag{26.76}
$$

As shown in Fig.26.6©.

Let us calculate the efficiency of Bragg modulator at Bragg incidence angle. The projection of wavevector on z axis is $k_{0z} = k\cos\theta_B$, therefore, according to (26.63) coupling coefficient is

$$
\kappa = \frac{\Delta\varepsilon_r}{4} \frac{\omega^2}{c^2 k \cos\theta_B} \tag{26.77}
$$

Since $\Delta\varepsilon_r = 2n\Delta n = n^4 PS$ and $k = n\omega / c$ we have

$$
\kappa = \frac{n^3}{2} \frac{\pi}{\lambda\cos\theta_B} PS = \frac{n^3}{2} \frac{\pi}{\lambda\cos\theta_B} P \left(\frac{2}{\rho v_s^3} \frac{1}{LH} \mathrm{P}_{ac} \right)^{1/2} = \frac{\pi}{\sqrt{2}\lambda\cos\theta_B} \left(\frac{M_2 \mathrm{P}_{ac}}{LH} \right)^{1/2} \tag{26.78}
$$

where we have used (26.37) and (26.39).

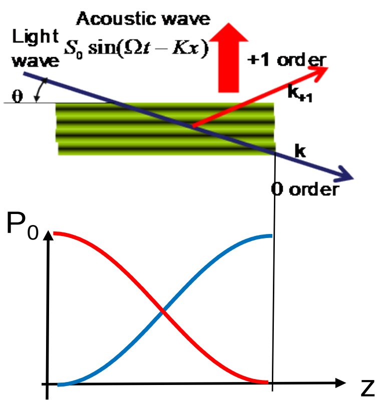
According to (26.76) full power transfer takes place when $\kappa L = \pi / 2$ i.e.

$$
\frac{\pi}{\sqrt{2}\lambda\cos\theta_B} \left(\frac{L M_2 \mathrm{P}_{ac}}{H} \right)^{1/2} = \frac{\pi}{2} \tag{26.79}
$$

And the acoustic power required for the complete power transfer is

$$
\mathrm{P}_{ac} = \frac{\lambda^2 \cos^2\theta_B}{2M_2} \frac{H}{L} \tag{26.80}
$$

Completer power transfer (100% modulation) is shown in Fig.26.7



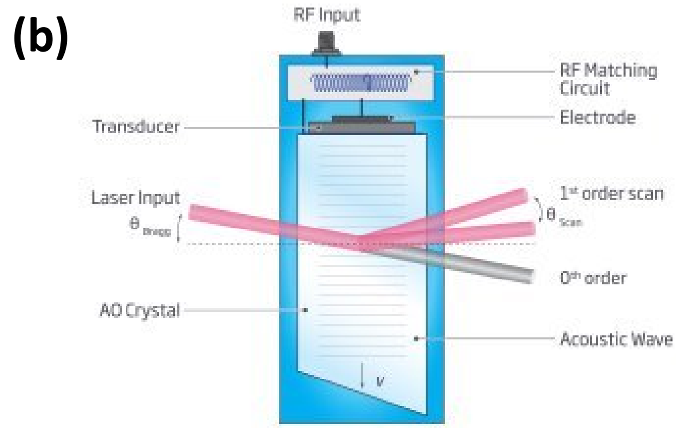


Figure 26.7 (a) 100% Acousto optic modulation. (b) AO defector

Let us consider an example. Use TeO_2 with $M_2 = 34.5\times10^{-15} s^3 / kg$, $H = 2mm$, $L = 50mm$, $n = 2.2$, $v_s = 4,250 m/s$ acoustic frequency $F = 50MHz$ and $\lambda = 600nm$. The sound wavelength $\Lambda = v_s / F = 85\mu m$. The Bragg angle is according to (26.60) $\theta_B = \sin^{-1}(600nm/2\cdot 85\mu m) \approx 0.2^\circ$ i.e $\cos\theta_B \approx 1$. Substituting in (26.80) we obtain

$$\mathrm{P}_{ac} = \frac{0.6^2 \times 10^{-12}}{2\cdot 34.5\times10^{-5}} \frac{2}{50} \approx 0.2W \tag{26.81}$$

Note that for small acoustic powers (or short length) (26.76) becomes

$$\eta = P_+(L) \approx \kappa^2 L^2 = \frac{\pi^2}{2\lambda^2 \cos\theta_B} M_2 \mathrm{P}_{ac} \frac{L}{H} \tag{26.82}$$

Which is precisely the same expression as for Raman-Nath diffraction (26.38)

The modulation speed of AO modulator is limited by the sound transit time across the beam waist, and is equal to the inverse of that time $\Delta F_{\max} = v_s \cos\theta_B / w_0$. For the waist size of $w_0 = 200\mu m$ one gets $\Delta F_{\max} \approx 21MHz$.

Obviously one can use the Bragg modulator as an AO deflector (Fig.26.7 b) where by changing acoustic frequency one can change the deflection angle. Since deflection angle is twice Bragg angle ,

$$\theta_d = 2\theta_B \approx \lambda / n\Lambda = (\lambda / nv_s) F \tag{26.83}$$

the deflection angel can be changed as

$$\Delta\theta_d = (\lambda / nv_s)\Delta F \tag{26.84}$$

Of course there is a limit to deflection angel because eventually one cannot get good momentum matching.

Acousto-Optic Tunable Filter (AOTF)

Consider collinear Bragg diffraction between extraordinary (z-polarized) and ordinary (y-polarized) waves as shown in Fig.26.8(a). Both waves and acoustic wave as well propagate in the same direction z. Acoustic wave is polarized in x direction, hence the only strain is the sheer strain S_6.

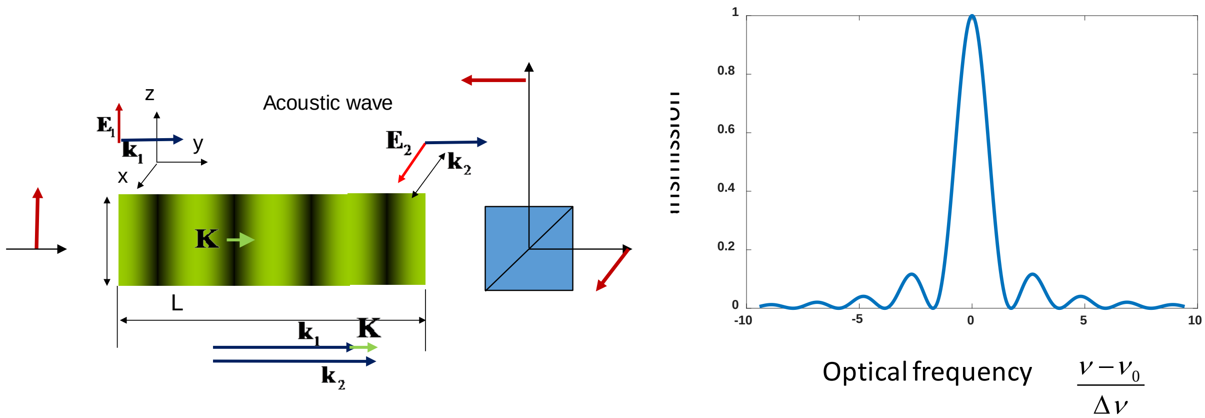


Figure 26.8 (a) AOTF based on collinear propagation in $LiNbO_3$ (b) Transmission spectrum.

The impermeability tensor becomes

$$\varepsilon_r^{-1} = \begin{pmatrix} 1/n_o^2 \\ 1/n_o^2 \\ 1/n_e^2 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} P_{11} & P_{12} & P_{13} & P_{14} & 0 & 0 \\ P_{12} & P_{11} & P_{12} & -P_{14} & 0 & 0 \\ P_{31} & P_{31} & P_{33} & 0 & 0 & 0 \\ P_{41} & -P_{41} & 0 & P_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & P_{44} & P_{41} \\ 0 & 0 & 0 & 0 & P_{41} & \frac{1}{2}\left(P_{11}-P_{12}\right) \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ S_6 \end{pmatrix} = \begin{pmatrix} 1/n_o^2 \\ 1/n_o^2 \\ 1/n_e^2 \\ 0 \\ P_{41}S_6 \\ \frac{1}{2}\left(P_{11}-P_{12}\right)S_6 \end{pmatrix} \quad (26.85)$$

The off-diagonal element $\Delta\varepsilon_{r,zx}^{-1} = P_4 S_6$ and accordingly $\Delta\varepsilon_{r,zx} = -n_o^2 n_e^2 P_4 S_6$ means that the wave $\mathbf{E}_2$ polarized in x direction is coupled with the wave $\mathbf{E}_1$ polarized in z direction, i.e. there is coupling between ordinary and extraordinary waves. The mismatch of wavevectors is

$$\Delta\mathbf{k}_{0,1} = \mathbf{k}_o - \mathbf{k}_e \text{ - } \mathbf{K} \quad (26.86)$$

and for the collinear propagation

$$\Delta k = \frac{2\pi\nu}{c} n_o - \frac{2\pi(\nu+F)}{c} n_e - \frac{2\pi F}{v_a} \approx \frac{2\pi\nu}{c}(n_o - n_e) - \frac{2\pi F}{v_a} \quad (26.87)$$

Optical frequency at which the wave vectors are matched is determined from

$$\Delta k(\nu_0) = \frac{2\pi\nu_0}{c}(n_o - n_e) - \frac{2\pi F}{v_a} = 0 \quad (26.88)$$

and is equal to

$$\nu_0 = \frac{c\Delta n}{v_a} F \tag{26.89}$$

Where $\Delta n = (n_o - n_e)$. Then at frequency $\nu_0 + \delta\nu$ the wave vector mismatch is

$$\Delta k(\delta\nu) = 2\pi\delta\nu\Delta n / c \tag{26.90}$$

The power transfer equation is according to (26.74) is

$$P_2(L) = \frac{\kappa^2}{\kappa^2 + \frac{\Delta k^2}{4}} \sin^2\left(\sqrt{\kappa^2 + \frac{\Delta k^2}{4}} L\right) \tag{26.91}$$

Where coupling constant

$$\kappa = \frac{\pi}{\sqrt{2}\lambda\cos\theta_B}\left(\frac{M_2 \mathrm{P}_{ac}}{HW}\right)^{1/2} \tag{26.92}$$

Length and power are adjusted for complete power transfer (polarization rotation) for light of frequency ν_0,

$$\kappa L = \pi / 2 \tag{26.93}$$

Then

$$P_{out}(L) = \frac{\kappa^2}{\kappa^2 + \Delta k^2/4}\sin^2\left(\frac{\pi\sqrt{\kappa^2 + \Delta k^2/4}}{2\kappa}\right) = \frac{1}{1 + \Delta k^2/4\kappa^2}\sin^2\left(\frac{\pi}{2}\sqrt{1 + \Delta k^2/4\kappa^2}\right) = \frac{\sin^2\left(\frac{\pi}{2}\sqrt{1 + \left(\delta\nu/\Delta\nu\right)^2}\right)}{1 + \left(\delta\nu/\Delta\nu\right)^2}$$

(26.94)

where

$$\Delta\nu = \frac{c\kappa}{\pi\Delta n} = \frac{c}{2\Delta n L} \tag{26.95}$$

If one now places a polarizing beamsplitter at the output its transmission will be a rather narrow passband around ν_0 as shown in Fig. 26.8b. Of course, one can build a spectrometer based on the AOTF.

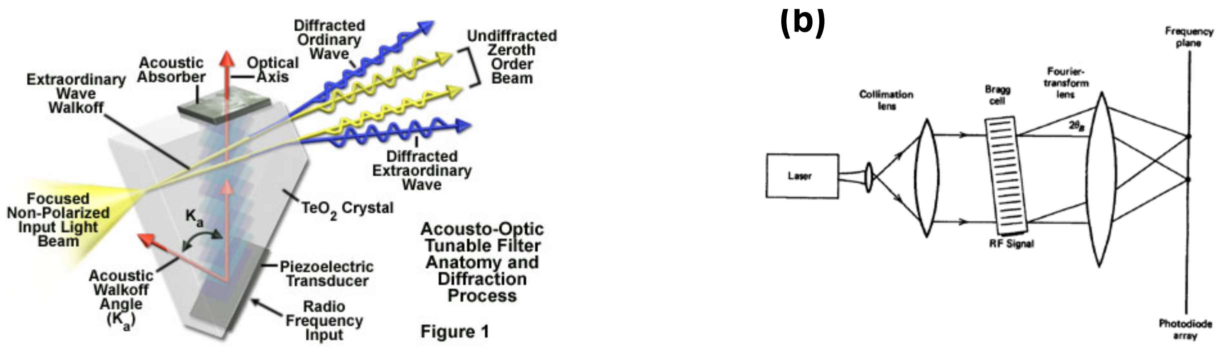


Figure 26.9 (a) Non-collinear AOTF (b) Acousto-optic spectrum analyzer

AOTF does not gave to be collinear as shown in Fig.26.9a. Furthermore, one can use Bragg diffraction to measure the frequency spectrum of the radio-signal F as shown in Fig.26,9b