Lecture 29 Planar and Channel waveguides

Planar waveguide

Consider a symmetric planar (slab) waveguide consisting of core with index n_1 of thickness a and a cladding with lower index $n_2 < n_1$ as shown in Fig.29.1.a.Then light propagating at angles exceeding critical is expected to stay confined inside the core (waveguiding) layer. Let us now approach this using wave theory, i.e. Maxwell's equations.

$$\begin{aligned}\nabla \times \mathbf{H} &= \varepsilon(x)\frac{\partial \mathbf{E}}{\partial t} \\ \nabla \times \boldsymbol{E} &= -\mu\frac{\partial \mathbf{H}}{\partial t}\end{aligned} \tag{29.1}$$

We assume that the waves are harmonic with frequency ω, i.e. $E,H \sim e^{-j\omega t}$, and that the waveguide is infinite along the direction y, i.e.

$$\frac{\partial \boldsymbol{H}}{\partial y} = \frac{\partial \boldsymbol{E}}{\partial y} = 0 \tag{29.2}$$

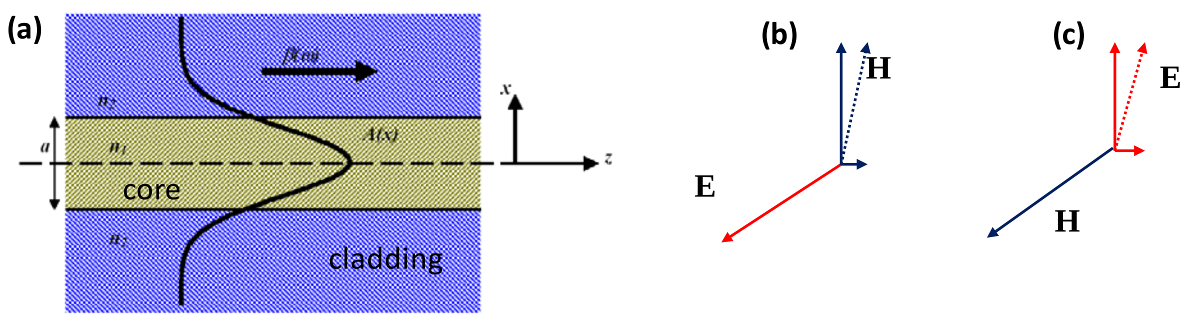

Figure 29.1 (a) Geometry of a planar waveguide (b) Fields in TE mode (c) Fields in TM mode

Let us write out the Maxwell's equations for reach projection of fields,

$$\begin{aligned}
&\frac{\partial H_y}{\partial z} = j\omega\varepsilon(x)E_x && \frac{\partial E_y}{\partial z} = -j\omega\mu H_x \\
&\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = -j\omega\varepsilon(x)E_y && \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = j\omega\mu H_y \\
&\frac{\partial H_y}{\partial x} = -j\omega\varepsilon(x)E_z && \frac{\partial E_y}{\partial x} = j\omega\mu H_z
\end{aligned} \tag{29.3}$$

We can re-group the equations as

$$
\begin{aligned}
&\frac{\partial E_y}{\partial z} = -j\omega\mu H_x && \frac{\partial H_y}{\partial z} = j\omega\varepsilon(x)E_x \\
&\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = -j\omega\varepsilon(x)E_y && \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = j\omega\mu H_y \\
&\frac{\partial E_y}{\partial x} = j\omega\mu H_z && \frac{\partial H_y}{\partial x} = -j\omega\varepsilon(x)E_z
\end{aligned}
\tag{29.4}
$$

Clearly the two systems of three equations each are independent of each other. The first one contains no projection of the electric field on the direction of propagation z and is called transverse-electric (TE) wave having the following three components

$$
\begin{aligned}
&\mathbf{E} = E_y\hat{\mathbf{y}} \\
&H_x = \frac{j}{\omega\mu}\frac{\partial E_y}{\partial z} \\
&H_z = -\frac{j}{\omega\mu}\frac{\partial E_y}{\partial x}
\end{aligned}
\tag{29.5}
$$

And represented in Fig.29.1b. The second family has no projection of the magnetic field on the direction of propagation z and is called transverse-magnetic (TM) shown in Fig.29.1c

$$
\begin{aligned}
&\mathbf{H} = H_y\hat{\mathbf{y}} \\
&E_x = -\frac{j}{\omega\mu\varepsilon(x)}\frac{\partial H_y}{\partial z} \\
&E_z = \frac{j}{\omega\varepsilon(x)}\frac{\partial H_y}{\partial x}
\end{aligned}
\tag{29.6}
$$

TE waves in planar waveguide

Since the electric field is polarized along y and we consider isotropic medium (or if y is an optical axis of uniaxial medium) we shall use scalar wave equation

$$\nabla^2 E_y(x,z,t) - \frac{n^2(x)}{c^2}\frac{\partial^2}{\partial t^2}E_y(x,z,t) = 0; \tag{29.7}$$

And look for a solution

$$E_y(x,z,t) = \frac{1}{2}A(x)e^{j(\beta z - \omega t)} + c.c. \tag{29.8}$$

where β is a propagation constant. Substituting (29.8) into (29.7) we obtain

$$\frac{\partial^2 A(x)}{\partial x^2} - \left[\beta^2 - \frac{\omega^2}{c^2}n_{1,2}^2\right]A(x) = 0; \tag{29.9}$$

or

$$\frac{\partial^2 A(x)}{\partial x^2} + k_x^2 A(x) = 0; \tag{29.10}$$

where the transverse wavevector is

$$k_{x,1(2)} = \sqrt{\frac{\omega^2}{c^2} n_{1(2)}^2 - \beta^2} \tag{29.11}$$

Solution of (29.10) is obviously

$$A(x) = A_{1,2} e^{\pm jk_{x,1(2)} x} \tag{29.12}$$

And it is confined within core region 1 hence k_{x1} is real,

$$k_{x,1} = q = \sqrt{\frac{\omega^2}{c^2} n_1^2 - \beta^2}, \ \ n_2 \omega / c < \beta < n_1 \omega / c \tag{29.13}$$

Due to symmetry the light power density distribution is also symmetric $A^2(x) = A^2(-x)$ or $A(x) = \pm A(-x)$. So, there are two possible solutions inside the core, odd and even,

$$A(|x| < a/2) = \begin{cases} A_1 \cos(qx) \\ A_1 \sin(qx) \end{cases}; \tag{29.14}$$

In the cladding on the other hand the field decays exponentially, i..e transverse wavevector is imaginary

$$k_{x,2} = jp; \ p = \sqrt{\beta^2 - \frac{\omega^2}{c^2} n_2^2} \tag{29.15}$$

and

$$A(|x| > a/2) = A_2 e^{-p(|x|-a/2)} \tag{29.16}$$

Let us now find the in-plane magnetic field applying the third equation in (29.5) to (29.14) and (29.15):

$$\begin{gathered} H_z(x \le |a|) = \begin{cases} \dfrac{jq}{\omega\mu} A_1 \sin(qx) \\ -\dfrac{jq}{\omega\mu} A_1 \cos(qx) \end{cases} \\ H_z(x > |a|) = \frac{jp}{\omega\mu} A_2 e^{-p(|x|-a/2)} \end{gathered} \tag{29.17}$$

The boundary conditions consist of continuity for the tangential fields E_y and H_z, i.e.

$$A_1 \cos(qa/2) = A_2$$
$$A_1 q \sin(qa/2) = pA_2 \tag{29.18}$$

for even modes and

$$A_1 \sin(qa/2) = A_2$$
$$-A_1 q \cos(qa/2) = pA_2 \tag{29.19}$$

Dividing the second equation in (29.18),(29.19) by the first one yields

$$p = q \tan(qa/2)$$
$$p = -q \cot(qa/2) \tag{29.20}$$

for the even and odd modes respectively. Introduce dimensionless variables

$$u = qa/2,\ v = pa/2 \tag{29.21}$$

and obtain from (29.20)

$$v = u \tan(u) \text{ for even modes}$$
$$v = -u \cot(u) = u \tan(u + \pi/2) \text{ for odd modes} \tag{29.22}$$

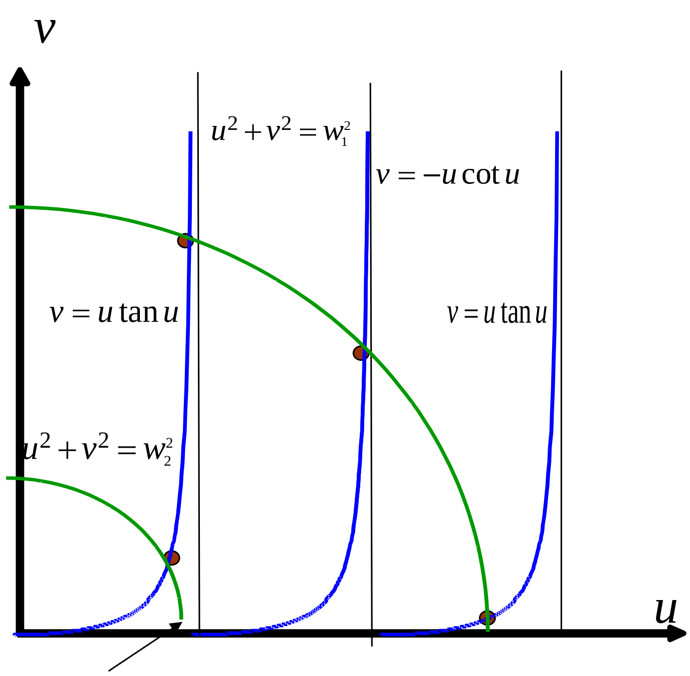

Figure 29.2 Graphic method o finding the solutions for the TE modes

This result is plotted in Fig. 29.2 as an almost periodic series of curves. Next from (29.13) and (29.15) we obtain

$$p^2 + q^2 = \beta^2 - \frac{\omega^2}{c^2} n_2^2 + \frac{\omega^2}{c^2} n_1^2 - \beta^2 = \frac{\omega^2}{c^2}(n_1^2 - n_2^2) \qquad (29.23)$$

Next, multiply both sides of (29.23) by $a/2$ introduce normalized thickness

$$w = \frac{\omega}{c}\frac{a}{2}\sqrt{n_1^2 - n_2^2} \approx \frac{\pi n a}{\lambda}\sqrt{2\Delta n / n} \qquad (29.24)$$

and finally we obtain

$$u^2 + v^2 = w^2 \qquad (29.25)$$

which is the equation of a circle with radius w as shown in Fig.29.2 One can see that as w increases so does the total number of solutions. Note, however that there is always at least one solution, no matter how thin is the core region. This is only true for symmetric planar waveguide. The graphic solution yield values of u and v which then can be used to find p, q, and finally β from (29.13) or (29.15).

The fields of the first two TE0 and TE1 modes in a planar waveguide are shown in Fig.29.3 a-d. The index of the mode means the number of nodes of electric field for aSi core with SiO_2 cladding for the wavelength of 1550nm for two different waveguide thicknesses, 600nm and 800nm As one can see, the mode with a higher index penetrates deeper into the cladding.

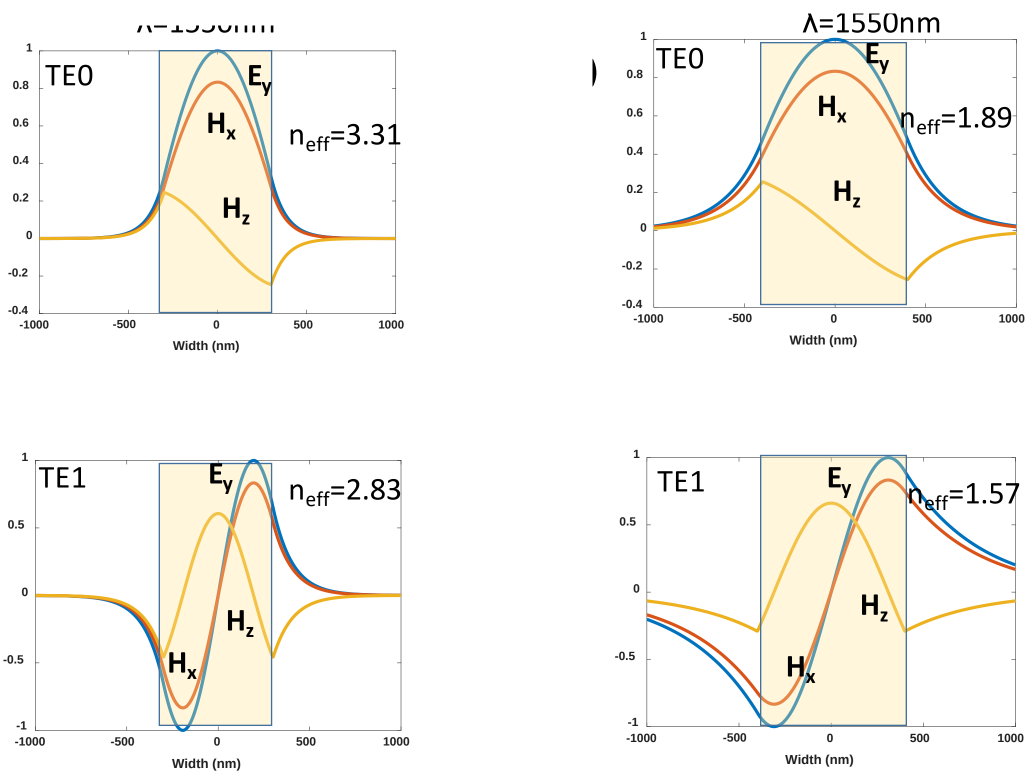

Figure 29.3 TE modes in Si/SiO_2 symmetric planar waveguides (a) TE0, a=600nm, (b) TE0 a=800nm, (c0 TE1, a=600nm, (d) TE1, a=800nm

TM waves in planar waveguide

Now it is the transverse magnetic field that we are defining as

$$H_y(x,z,t) = B(x)e^{j(\beta z-\omega t)} \tag{29.26}$$

Inside the core we have

$$B(|x| < a/2) = \begin{cases} B_1\cos(qx) \\ B_1\sin(qx) \end{cases}; \tag{29.27}$$

Aad in the cladding

$$B(|x| > a/2) = B_2 e^{-p(|x|-a/2)} \tag{29.28}$$

where just as for TE waves

$$\begin{aligned} q &= \sqrt{\frac{\omega^2}{c^2}n_2^2 - \beta^2} \\ p &= \sqrt{\beta^2 - \frac{\omega^2}{c^2}n_2^2} \\ p^2 + q^2 &= \frac{\omega^2}{c^2}(n_2^2 - n_1^2) \end{aligned} \tag{29.29}$$

In-plane electric field is according to (29.6)

$$\begin{aligned} E_z(x \le |a|) &= \begin{cases} -\dfrac{jq}{\omega\varepsilon_0 n_1^2}B_1\sin(qx) \\ \dfrac{jq}{\omega\varepsilon_0 n_1^2}B_1\cos(qx) \end{cases} \\ E_z(x > |a|) &= -\frac{jp}{\omega\varepsilon_0 n_2^2}B_2 e^{-p(|x|-a/2)} \end{aligned} \tag{29.30}$$

Applying the boundary conditions for continuous in-plane field we obtain

$$\begin{aligned} B_1\cos(qa/2) &= B_2 \\ B_1 q\sin(qa/2)\frac{n_2^2}{n_1^2} &= pB_2 \end{aligned} \tag{29.31}$$

for even modes and

$$\begin{aligned} B_1\sin(qa/2) &= B_2 \\ B_1 q\cos(qa/2)\frac{n_2^2}{n_1^2} &= -pB_2 \end{aligned} \tag{29.32}$$

for odd modes. Dividing the second equation by the first in (29.31) and (29.32) we obtain

$$
\begin{aligned}
p &= \frac{n_2^2}{n_1^2} q \tan(qa/2) \text{ for even modes} \\
p &= -\frac{n_2^2}{n_1^2} q \cot(qa/2) \text{ for odd modes}
\end{aligned}
\tag{29.33}
$$

So, finally, using (29.21) we obtain

$$
v = \begin{cases} \dfrac{n_2^2}{n_1^2} u \tan(u) & \text{even modes} \\ \dfrac{n_2^2}{n_1^2} u \tan(u+\pi/2) & \text{odd modes} \end{cases}
\tag{29.34}
$$

$$
u^2 + v^2 = w^2
$$

which only differs from (29.22) for TE modes by the presence of a multiplier n_2^2/n_1^2. Hence one obtains graphic solution just like shown in Fig.29.3 2 TM modes are shown for the first two TM modes in Fig.29.4.

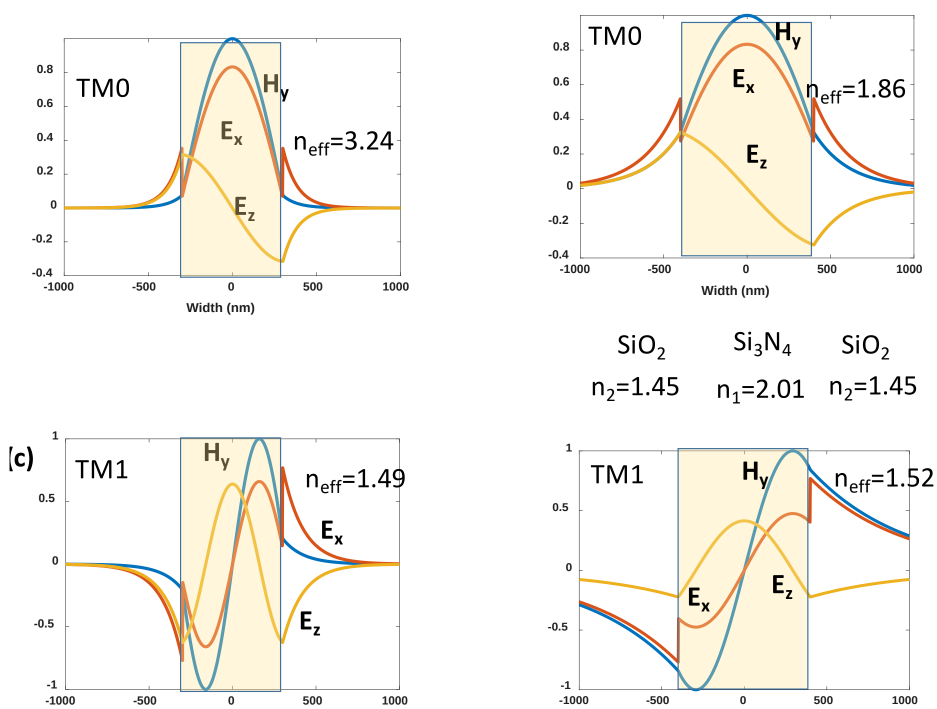

Figure 29.4 TM modes in Si/SiO2 symmetric planar waveguides (a) TE0, a=600nm, (b) TE0 a=800nm, (c0 TE1, a=600nm, (d) TE1, a=800nm

The most interesting feature that distinguish TM from TE mode is that normal E-field (29.6) is

$$E_x = -\frac{j}{\omega\mu\varepsilon(x)}\frac{\partial H_y}{\partial z} = \frac{\beta}{\omega\varepsilon_0 n_{1,2}^2} H_y \tag{29.35}$$

Experiences discontinuity at the interface – which is expected as normal displacement $D_{z,1,2} = \varepsilon_0 n_{1,2}^2 E_{z,1,2}$, must remain continuous at the interface. The field is just stronger in the evanescent region which can be used for various sensors. Basically it is the same principle that drives total internal reflection sensors that we have studies earlier on in Chapter 5.

Effective and group indices

Let us introduce the effective index as the ratio of the phase velocity of light in vacuum c and in the waveguide $v_p = \omega / \beta$

$$n_{eff} = \beta / k_0 = \beta c / \omega \tag{29.36}$$

The effective index is shown as a function of wavelength or frequency in Fig.29.5a,b for the 600nm Si waveguide of Figs 29.3,4. As one can see, all the modes, except TE0 and TM0 experience cut-off at long wavelength, and, as expected, the effective index value is always betwwen the indices of core and cladding,

$$n_2 < n_{eff} < n_1 \tag{29.37}$$

With the effective index defined by (29.37), the mode (29.8) can be described as

$$\mathbf{E}(x,z,t) = \frac{1}{2}\mathbf{A}(x)e^{j(n_{eff}\omega z/c - \omega t)} + c.c. = \frac{1}{2}\mathbf{A}(x)e^{-j\omega(t - n_{eff} z/c)} + c.c. \tag{29.38}$$

Also shown in Fig.29.5c is the dispersion of propagation constant. The allowed region is squeezed between two curves, $\beta_{high}(\omega) = n_1(\omega)\omega / c$ and $\beta_{low}(\omega) = n_2(\omega)\omega / c$ corresponding to free waves propagating in core and cladding respectively.

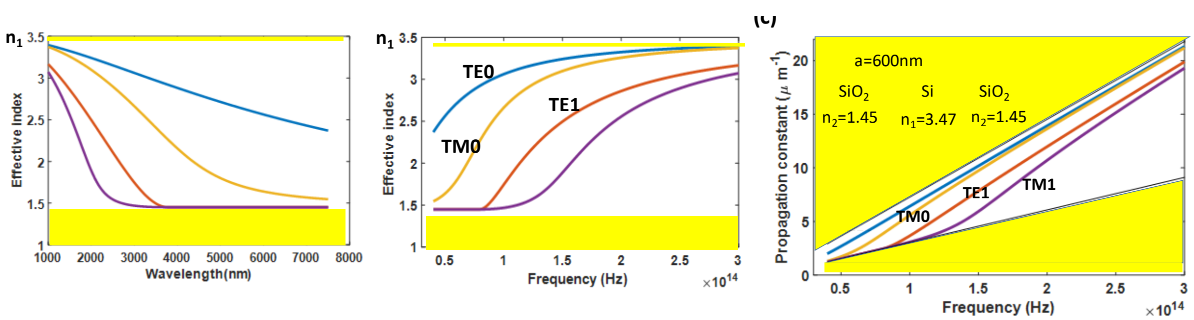

Figure 29.5 Dispersion of effective index of 600nm Si waveguide vs (a) wavelength and (b) frequency, and (c) dispersion of the propagation constant.

We can also estimate the group velocity as

$$v_g = \frac{d\omega}{d\beta} \tag{29.39}$$

and group index

$$n_g = \frac{c}{v_g} = c\frac{d\beta}{d\omega} \tag{29.40}$$

The dispersion of group index is shown in Fig,29.6 as a function of wavelength (a) or frequency (b). The dependence of group index is more interesting than of effective index- there exists a maximum with $n_g > n_1$ before the group index settle asymptotically at $n_g = n_1$.

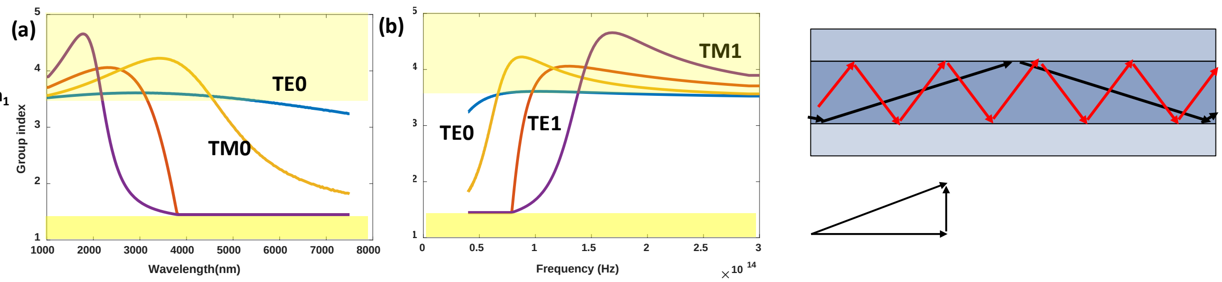

Figure 29.6 Dispersion of group index of 600nm Si waveguide vs (a) wavelength and (b) frequency, and (c) interpretation of slow group velocity using geometrical optics.

This, of course, can be easily understood from the geometrical optics interpretation of the propagation in the waveguide (Fig.29.6.c), where the mode propagates inside the core at an angle

$$\theta = \sin^{-1}\frac{q}{n_1 k_0} \tag{29.41}$$

and experiences total internal reflections at the interface. Since the light travels along longer path, for a well-confined mode

$$v_g \approx \frac{c}{n_1}\cos\theta \tag{29.42}$$

Higher order modes, according to Figs. 29.2, have larger transverse wavevector q and angle θ so the group index in them peaks at larger values.

Power flow and effective thickness

Let us consider TE wave (29.8)

$$E_y(x,z,t) = \frac{1}{2}A(x)e^{j(\beta z-\omega t)} + c.c.$$
$$H_x = \frac{j}{\omega\mu}\frac{\partial E_y}{\partial z} = -\frac{\beta}{2\omega\mu}A(x)e^{j(\beta z-\omega t)} + c.c. \tag{29.43}$$
$$H_z = -\frac{j}{\omega\mu}\frac{\partial E_y}{\partial x} = -\frac{j}{\omega\mu}\frac{\partial A(x)}{\partial x}e^{j(\beta z-\omega t)} + c.c.$$

Note that

$$\frac{\beta}{\omega\mu} = n_{eff}\frac{\omega}{c\omega\mu} = \frac{n_{eff}}{\eta_0} \tag{29.44}$$

Therefore, we can find the time-averaged power density propagating in the z direction as

$$\overline{S}_z(x) = -\left\langle E_y(t)H_x(t)\right\rangle_t = \frac{n_{eff}}{2\eta_0}\left|A(x)\right|^2 \tag{29.45}$$

As far as the time averaged power propagating in x direction, H_z is 90 degrees out of phase with E_y, hence $\overline{S}_x(x) = 0$.

Let us find the power density peer unit width,

$$P' = \int \overline{S}_z(x)dx = \frac{n_{eff}}{2\eta_0}\int |A(x)|^2\, dx \tag{29.46}$$

We can now introduce the effective thickness as

$$t_{eff} = \int |A(x)|^2\, dx / A_{\max}^2 \tag{29.47}$$

where $A_{\max}$ is the maximum amplitude as shown in Fig.29.7a. Then

$$P' = \frac{n_{eff}}{2\eta_0}A_{\max}^2 t_{eff} \tag{29.48}$$

In other words, we approximate the real mode by a “square wave mode” filling the equivalent waveguide with thickness t_{eff}. In Fig.29.7b we plot effective thickness of Si/SiO_2 waveguide as function of its real thickness a for two modes at wavelength $\lambda = 1550nm$ As one can see, for a very thin waveguide the effective thickness sis very large because the mod espread out into the cladding. As thickness increases mode gets more confined and effective thickness decreases, but when strong confinement is achieved, effective thickness starts growing with a, asymptotically reaching $t_{eff} = a/2$.

We can also introduce the confinement factor – fraction of power contained inside the core as

$$\Gamma = \frac{\int_{-a/2}^{a/2} \bar{S}_z(x)dx}{\int_{-\infty}^{\infty} \bar{S}_z(x)dx} = \frac{\int_{-a/2}^{a/2} |A(x)|^2 \, dx}{\int_{-\infty}^{\infty} |A(x)|^2 \, dx} = \frac{\int_{-a/2}^{a/2} |A(x)|^2 \, dx}{A_{\max}^2 t_{eff}} \tag{29.49}$$

The confinement factor is plotted in Fig.29.7c and approaches unity for thicker waveguides.

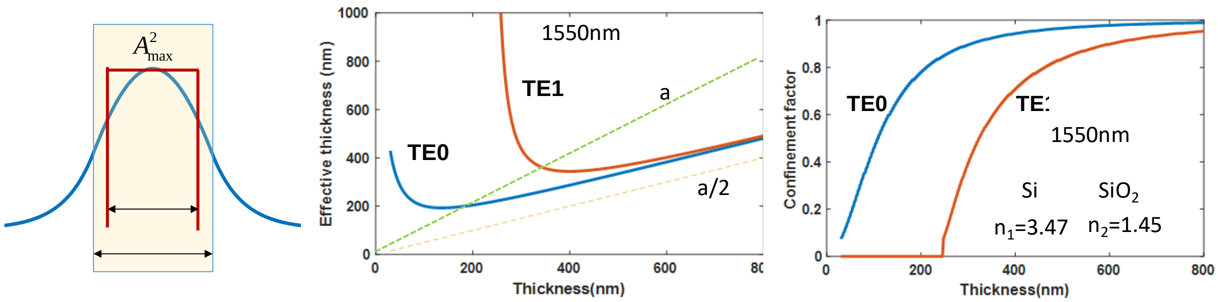

Figure 29.7 (a) definition of effective thickness (b) Effective thickness of Si/SiO₂ waveguide versus core thickness a .(c0 Confinement factor Γ versus core thickness.

Slab waveguide that is not symmetric

Consider now a slab waveguide of Fig.29.8a consisting of three layers – substrate, waveguide and a cover (often it would be air) such that $n_1 > n_2 > n_3$. One can proceed to solve the Maxwell's equation for TE and TM modes – only the boundary conditions are slightly different. As a result, one can identify three types of modes all shown in Fig 29.9.

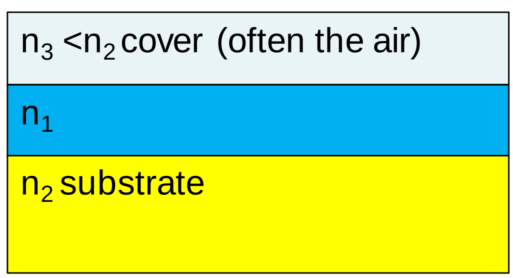

Figure 29.9 Asymmetric slab waveguide

First there are the guided modes (Fig.29.10.a), similar to the guided modes in symmetric waveguide. From the ray optics point of view, the ray experiences total internal reflection on both interfaces and the field is confined inside the guiding (core) layer and exponentially decays in both substrate and cover. The effective index is $n_2 < n_{eff} < n_1$. Next, there is a substrate mode (Fig.29.10b) – the ray gets totally reflected at the cover interface but, but leaks into the substrate. This mode is therefore quite lossy. For this mode $n_3 < n_{eff} < n_2$. Finally, when the rays do not get totally reflected at either interface (Fig.29.10.c) one has a radiating mode situation, for which the filed escapes into both substrate allowe dand cover and $n_{eff} < n_3$

. The substrate and radiating modes, unlike the guided ones, do not have discrete values of propagation constant – β's are continuous.

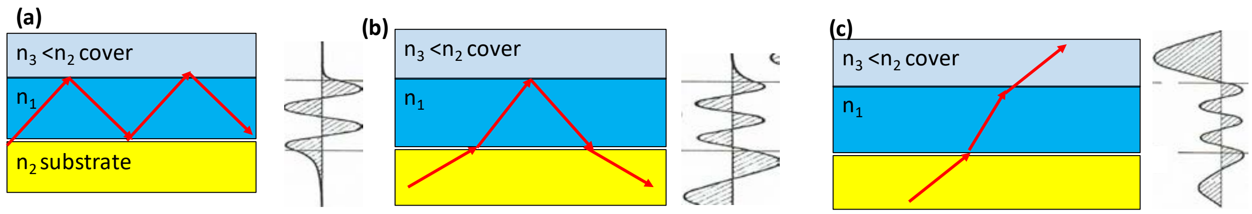

Figure 29.10 Modes in asymmetric planar waveguide, (a) Guided mode (b) Substarte mode (c0 Radiation mode

The dispersion diagram of the modes in asymmetric planar waveguide is shown in Fig.29.11 – not ethat even the fundamental mode TE0 has a cut-off.

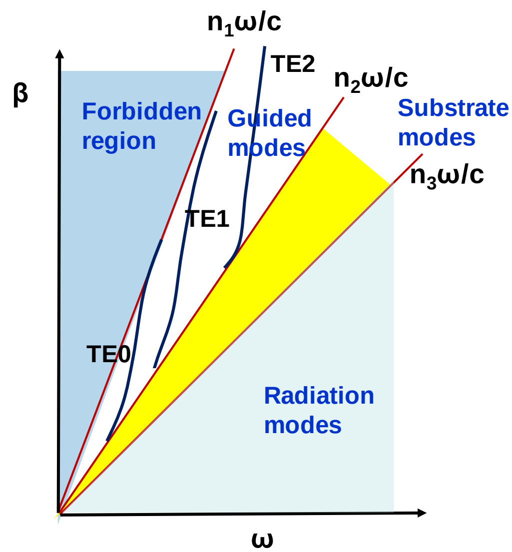

Figure 29.11 Dispersion of modes in the asymmetric planar waveguide.

Channel waveguides

In order to provide confinement in the lateral direction the slab waveguide need sto be modified. There is more than one way to achieve it, depending on the fabrication method as shown in Fig.29.12. Ridge and rib waveguides are the most common, but buried and diffused ones are also used quite often. ARROW stands for anti-resonant reflecting optical waveguide

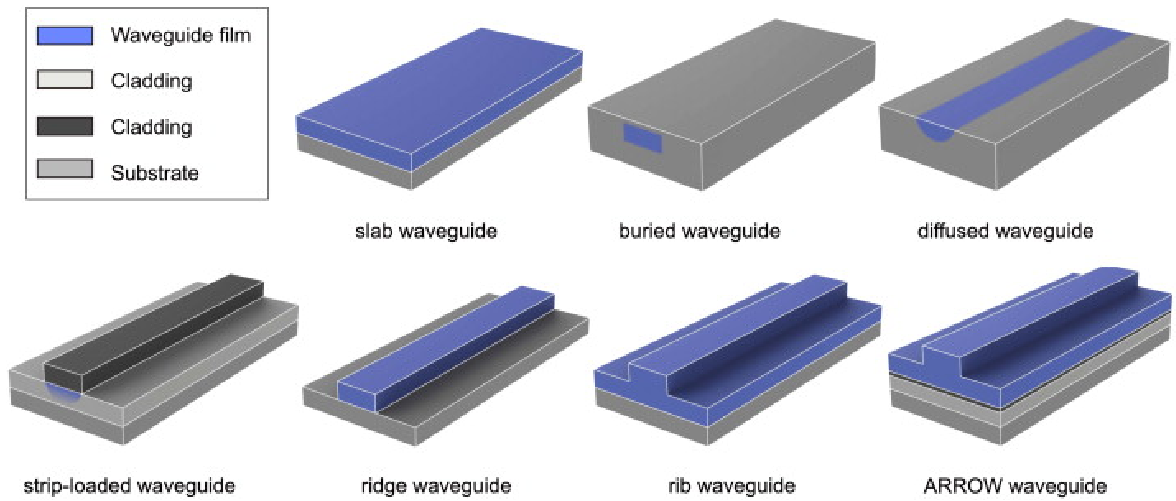

Figure 29.12 Channel waveguides – different types

In Channel waveguides you cannot separate pure “TE” and “TM” modes – there is always a longitudinal component of the field, so “TE” and “TM” simply means “almost horizontal” or “almost vertical” polarization. Also, depending on the number of nodes, the modes have two indices – first one is number of nodes in vertical direction and the second – in horizontal direction. In Fig. 29.13 and Fig. 29.14 the distributions of the absolute value of electric field and Power density are shown for TE and TM modes respectively for the Si on SiO_2 800nm x 500nm ridge waveguide (λ=1.55µm)

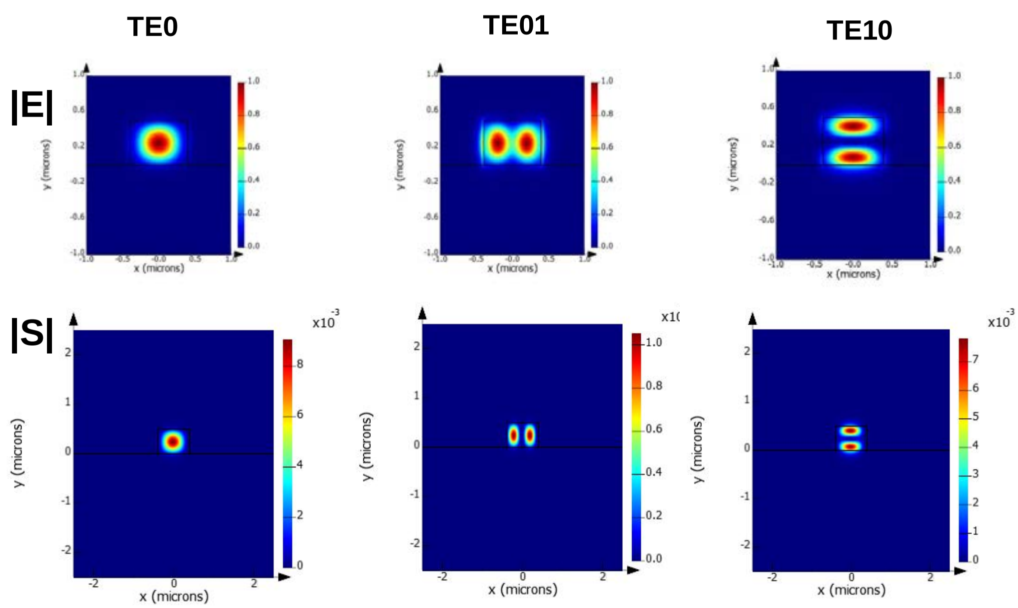

Figure 29.13 TE modes in a Si on SiO_2 800nm x 500nm ridge waveguide (λ=1.55µm). Top row- electric field bottom row-power density

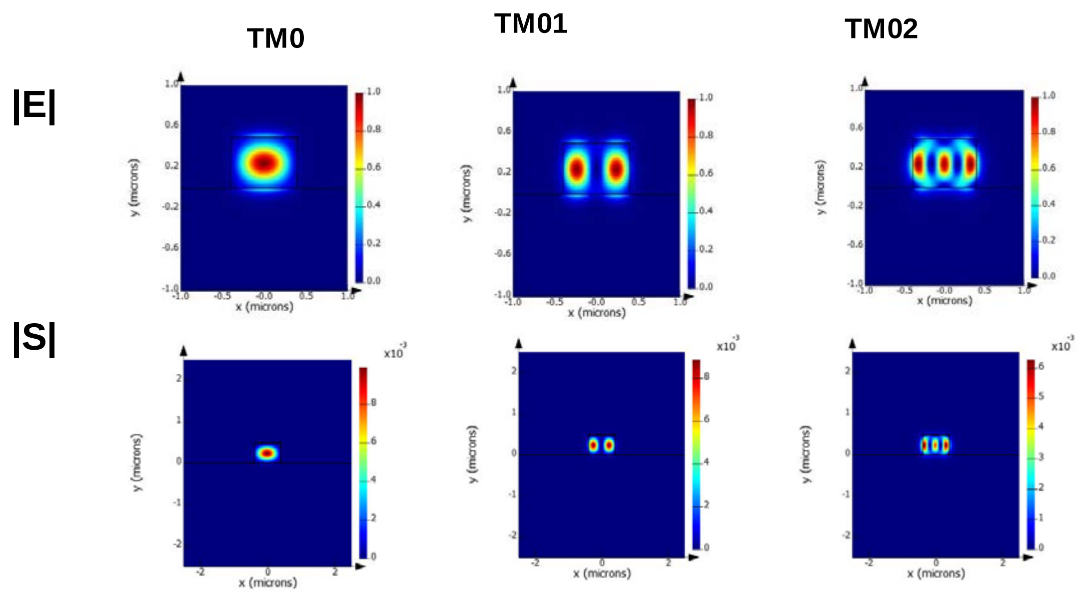

Figure 29.14 TE modes in a Si on SiO_2 800nm x 500nm ridge waveguide (λ=1.55μm). Top row- electric field bottom row-power density

The effective index and group velocity dispersion $\beta_2 = d^2\beta / d\omega^2$ for this waveguide are plotted in Fig. 29.15. Also plotted is the imaginary part of effective index since Si waveguides can be quite lossy.

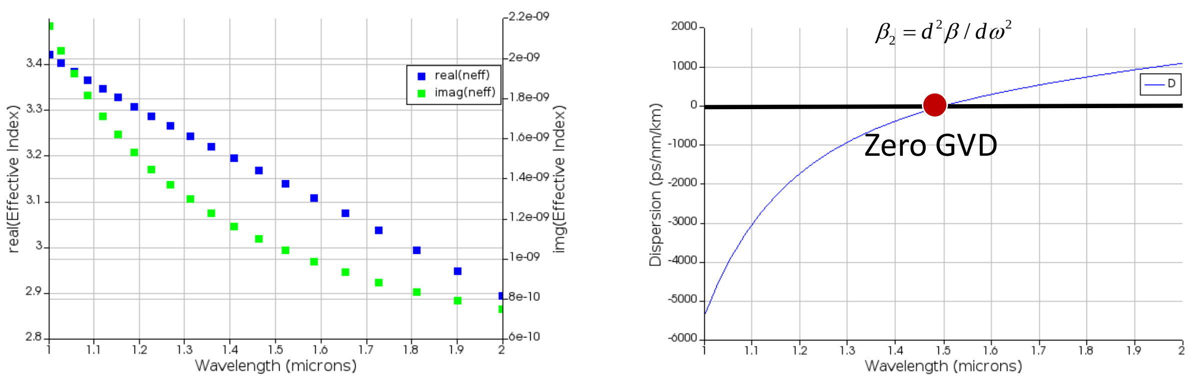

Figure 29.15 (a) effective index (real and imaginary part) and (b) GVD in a Si on SiO_2 800nm x 500nm ridge waveguide (λ=1.55μm).

Note that the waveguide has zero GVD near 1500nm – that has important implications for nonlinear optics –as it allows very good phasematching for the four wave mixing.

To avoid excessive loss, it is more common to use silicon nitride Si_3N_4 on SiO_2 ridge waveguides. The index contrast is lower hence the waveguide must be larger to provide good confinement. In Fig. 29.16and Fig. 29.17 the distributions of the absolute value of electric field and Power density are shown for TE and TM modes respectively for the 2000nm x 1000nm Si_3N_4 on SiO_2 ridge waveguide (λ=1.55μm).

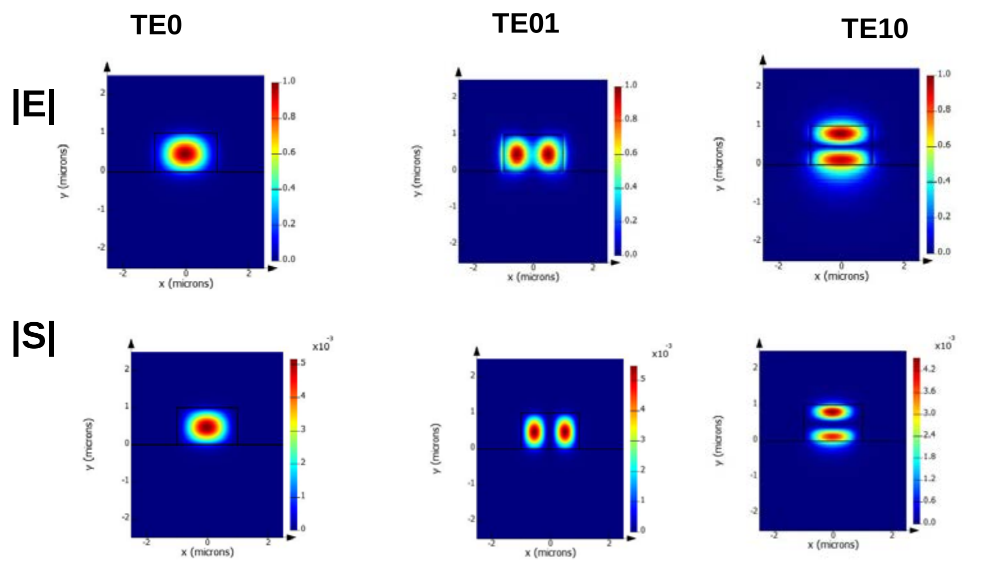

Figure 29.16 TE modes in a 2000nm x 1000nm Si_3N_4 on SiO_2 ridge waveguide (λ=1.55μm). Top row- electric field bottom row-power density

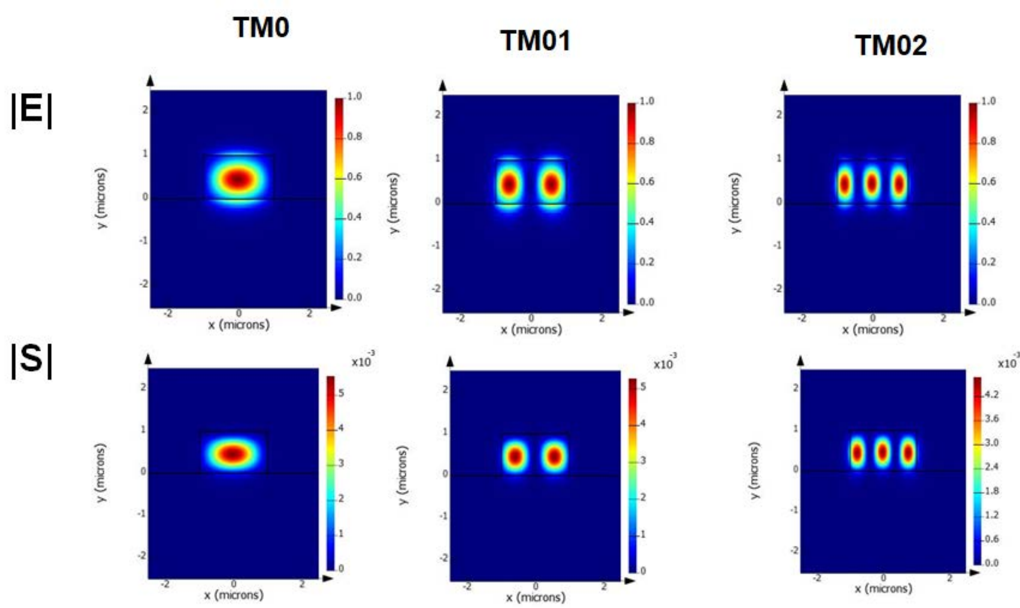

Figure 29.17 TE modes in a 2000nm x 1000nm Si_3N_4 on SiO_2 $_r$ridge waveguide (λ=1.55μm). Top row-electric field bottom row-power density

In Fig.29.18 the dispersion of effective index and GVD for the Si nitride waveguide are plotted. GVD remain positive through the entire range. Note that imaginary part of effective index is essentially zero – this is a very low loss platform

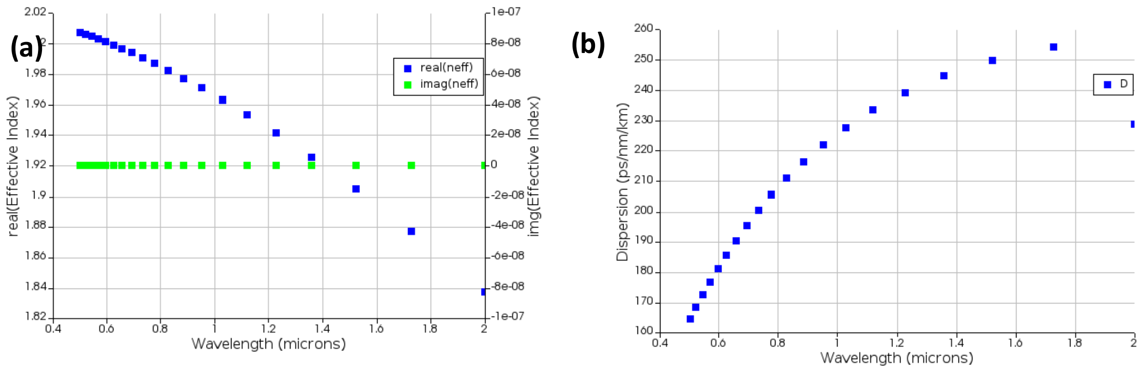

Figure 29.18(a) effective index (real and imaginary part) and (b) GVD in a 2000nm x 1000nm Si_3N_4 on SiO_2 $_r$ridge waveguide (λ=1.55μm).

Reciprocity and orthogonality

Let us start with Maxwell's equations (29.1) and assume that electro-magnetic field can be written frequency, , $\mathbf{E}_{1(2)}(\mathbf{r})e^{-j\omega t}+c.c.$ where index 1(2) simply represents two different field distributions Then we can write for each field (assuming no free charges and currents)

$$\begin{aligned}\nabla\times\mathbf{E}_1 &= j\omega\mu_0\mathbf{H}_1\\ \nabla\times\mathbf{H}_2 &= -j\omega\varepsilon_0\varepsilon_r\mathbf{E}_2\end{aligned}\tag{29.50}$$

Take a complex conjugate of the second equation and dot multiply it by $\mathbf{E}_1$, while multiplying the first equation by $\mathbf{H}_2^*$. We obtain

$$\begin{aligned}\mathbf{H}_2^*\cdot\nabla\times\mathbf{E}_1 &= j\omega\mu_0\mathbf{H}_1\cdot\mathbf{H}_2^*\\ \mathbf{E}_1\cdot\nabla\times\mathbf{H}_2^* &= j\omega\varepsilon_0\varepsilon_r\mathbf{E}_2^*\cdot\mathbf{E}_1\end{aligned}\tag{29.51}$$

Subtract the second equation in (29.51) from the first one

$$\mathbf{H}_2^*\cdot\nabla\times\mathbf{E}_1-\mathbf{E}_1\cdot\nabla\times\mathbf{H}_2^*=\nabla\cdot(\mathbf{E}_1\times\mathbf{H}_2^*)=j\omega\left(\mu_0\mathbf{H}_1\cdot\mathbf{H}_2^*-\varepsilon_0\varepsilon_r\mathbf{E}_2^*\cdot\mathbf{E}_1\right)\tag{29.52}$$

(where we have used vector identity $\nabla\cdot(\mathbf{A}\times\mathbf{B})=\mathbf{B}\cdot\nabla\times\mathbf{A}-\mathbf{A}\cdot\nabla\times\mathbf{B}$. We can now conjugate and simultaneously switch indices 1 and 2 in (29.52) to obtain

$$\nabla\cdot(\mathbf{E}_2^*\times\mathbf{H}_1)=-j\omega\left(\mu_0\mathbf{H}_1\cdot\mathbf{H}_2^*-\varepsilon_0\varepsilon_r\mathbf{E}_2^*\cdot\mathbf{E}_1\right)\tag{29.53}$$

Adding up (29.52) and (29.53) we obtain the reciprocity condition,

$$\nabla\cdot(\mathbf{E}_1\times\mathbf{H}_2^*+\mathbf{E}_2^*\times\mathbf{H}_1)=0\tag{29.54}$$

Now consider two modes of the waveguide

$$\begin{aligned}
\mathbf{E}_1 &= \mathbf{E}_1(x,y)e^{j\beta_1 z} \\
\mathbf{E}_2 &= \mathbf{E}_2(x,y)e^{j\beta_2 z} \\
\mathbf{H}_1 &= \mathbf{H}_1(x,y)e^{j\beta_1 z} \\
\mathbf{H}_2 &= \mathbf{H}_2(x,y)e^{j\beta_2 z}
\end{aligned} \tag{29.55}$$

Then l.h.s. of (29.55) can be written as

$$\nabla\cdot(\mathbf{E}_1\times\mathbf{H}_2^*+\mathbf{E}_2^*\times\mathbf{H}_1)=\nabla_\perp\cdot(\mathbf{E}_1\times\mathbf{H}_2^*+\mathbf{E}_2^*\times\mathbf{H}_1)_\perp+\frac{\partial}{\partial z}(\mathbf{E}_1\times\mathbf{H}_2^*+\mathbf{E}_2^*\times\mathbf{H}_1)_z=0 \tag{29.56}$$

where $\perp$ is normal to propagation direction plane (xy) and

$$\nabla_\perp=\hat{\mathbf{x}}\frac{\partial}{\partial x}+\hat{\mathbf{y}}\frac{\partial}{\partial y} \tag{29.57}$$

Next, considering the fact that z-projection of the vector product contains is a product of their normal (x,y)component s, we obtain

$$\begin{aligned}
&\frac{\partial}{\partial z}(\mathbf{E}_1\times\mathbf{H}_2^*+\mathbf{E}_2^*\times\mathbf{H}_1)_z=\left(\frac{\partial}{\partial z}\mathbf{E}_{1\perp}\right)\times\mathbf{H}_{2\perp}^*+\mathbf{E}_1\times\left(\frac{\partial}{\partial z}\mathbf{H}_{2\perp}^*\right)+\left(\frac{\partial}{\partial z}\mathbf{E}_{2\perp}^*\right)\times\mathbf{H}_{1\perp}+\mathbf{E}_{2\perp}^*\times\left(\frac{\partial}{\partial z}\mathbf{H}_{1\perp}\right)= \\
&=j\beta_1\mathbf{E}_{1\perp}\times\mathbf{H}_{2\perp}^*-j\beta_2\mathbf{E}_{1\perp}\times\mathbf{H}_{2\perp}^*-j\beta_1\mathbf{E}_{2\perp}^*\times\mathbf{H}_{1\perp}+j\beta_2\mathbf{E}_{2\perp}^*\times\mathbf{H}_{1\perp}=j\left(\beta_1-\beta_2\right)(\mathbf{E}_{1\perp}\times\mathbf{H}_{2\perp}^*+\mathbf{E}_{2\perp}^*\times\mathbf{H}_{1\perp})
\end{aligned} \tag{29.58}$$

Now putting (29.58) where it belongs, i.e. (29.56) we obtain

$$\nabla_\perp\cdot(\mathbf{E}_1\times\mathbf{H}_2^*+\mathbf{E}_2^*\times\mathbf{H}_1)_\perp+j\left(\beta_1-\beta_2\right)(\mathbf{E}_1\times\mathbf{H}_2^*+\mathbf{E}_2^*\times\mathbf{H}_1)_z=0 \tag{29.59}$$

Now we integrate (29.59) in plane, using Stokes theorem we obtain

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\nabla_\perp\cdot(\mathbf{E}_1\times\mathbf{H}_2^*+\mathbf{E}_2^*\times\mathbf{H}_1)_\perp dxdy=\oint_\infty(\mathbf{E}_1\times\mathbf{H}_2^*+\mathbf{E}_2^*\times\mathbf{H}_1)_\perp dl=0 \tag{29.60}$$

(since at infinity the fields vanish) – hence we arrive at the orthogonality condition for two modes

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\mathbf{E}_1\times\mathbf{H}_2^*+\mathbf{E}_2^*\times\mathbf{H}_1)_z\,dxdy=0 \tag{29.61}$$

Normalization

Let us now expand fields into the modes with amplitudes A_m

$$\mathbf{E} = \sum_m \frac{1}{2} A_m e^{j\beta_m z} \mathbf{e}_m(x,y) + c.c.$$
$$\mathbf{H} = \sum_m \frac{1}{2} A_m e^{j\beta_m z} \mathbf{h}_m(x,y) + c.c. \qquad (29.62)$$

where $\mathbf{e}_m, \mathbf{h}_m$ are normalized mode shapes.

If we designate $\mathbf{E}_1 = \frac{1}{2} A_m e^{j\beta_m z} \mathbf{e}_m(x,y) \quad \mathbf{E}_2 = \frac{1}{2} A_n e^{j\beta_n z} \mathbf{e}_n(x,y)$ and substitute it into (29.61) we obtain the mode orthogonality condition

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\mathbf{e}_m \times \mathbf{h}_n^* + \mathbf{e}_n^* \times \mathbf{h}_m)_z \, dxdy = 0, \quad m \neq n \qquad (29.63)$$

But what about $m = n$? Power along the direction of propagation in each mode is

$$P_m = \iint \left(\mathbf{E}_m \times \mathbf{H}_m^* + \mathbf{E}_m^* \times \mathbf{H}_m \right)_z dxdy = \frac{|A_m|^2}{4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\mathbf{e}_m \times \mathbf{h}_m^* + \mathbf{e}_m^* \times \mathbf{h}_m)_z \, dxdy \qquad (29.64)$$

If we now define the normalization condition as

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\mathbf{e}_m \times \mathbf{h}_m^* + \mathbf{e}_m^* \times \mathbf{h}_m)_z \, dxdy = 4 \qquad (29.65)$$

Then we rather conveniently get

$$P_m = |A_m|^2 \qquad (29.66)$$

We can combine (29.63) and (29.65) into one ortho-normality condition

$$\frac{1}{4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\mathbf{e}_m \times \mathbf{h}_n^* + \mathbf{e}_n^* \times \mathbf{h}_m)_z \, dxdy = \delta_{mn} \qquad (29.67)$$

This condition can be greatly simplified for the “quasi” TE and TM waves. For TE wave electric field is horizontal and magnetic field is vertical,

$$E_y \approx \frac{1}{2} A_m e_m(x,y)$$
$$H_x = \frac{1}{2} A_m h_m(x,y) \approx -\frac{n_{eff,m}}{2\eta_0} A_m e_m(x,y) \qquad (29.68)$$

(where we have used (29.44)). Therefore, substituting

$$h_m = -n_{eff,m} e_m / 2\eta_0 \qquad (29.69)$$

Into (29.67) we obtain

$$\frac{n_{eff,m}+n_{eff,n}}{4\eta_0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e_m e_n \; dxdy=\delta_{mn} \tag{29.70}$$

For TM wave we can write the relation between electric and magnetic fields as

$$E_x=\frac{\beta}{\omega\varepsilon_0 n^2(x,y)}H_y=\frac{\eta_0 n_{eff}}{n^2(x,y)}A_m h_m(x,y) \tag{29.71}$$

Since $E_x \approx \frac{1}{2}A_m e_m(x,y)$,

$$h_m(x,y)\approx\frac{n^2(x,y)}{\eta_0 n_{eff}}e_m(x,y) \tag{29.72}$$

Substituting (29.72) into (29.67) we obtain the orthogonality condition for TM modes

$$\frac{n_{eff,m}^{-1}+n_{eff,n}^{-1}}{4\eta_0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} n^2(x,y)e_m e_n dxdy=\delta_{mn} \tag{29.73}$$

Obviously, for weak waveguides all the indices are roughly the same and

$$\frac{\overline{n}}{2\eta_0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e_m e_n \; dxdy\approx\delta_{mn} \tag{29.74}$$

where $\overline{n}$ is some mean index. Note that according to (29.74) dimensionality of e is $\Omega^{1/2}\cdot m^{=1}$, dimensionality of h is $m^{-1}\Omega^{-1/2}$ and dimensionality of amplitude A is $W^{1/2}$

More on reciprocity

Let us consider the situation when there are current sources. Maxwell equations are

$$\begin{aligned}\nabla\times\mathbf{E}&=-\mu\frac{\partial\mathbf{H}}{\partial t}\\ \nabla\times\mathbf{H}&=\varepsilon\frac{\partial\mathbf{E}}{\partial t}+\mathbf{J}\end{aligned} \tag{29.75}$$

and for two different cases 1 and 2 we have

$$\begin{aligned}\nabla\times\mathbf{E}_1&=j\omega\mu_0\mathbf{H}_1\\ \nabla\times\mathbf{H}_2&=-j\omega\varepsilon_0\varepsilon_r\mathbf{E}_2+\mathbf{J}_2\end{aligned} \tag{29.76}$$

Multiply first equation by $\mathbf{H}_2$ and the second one by $\mathbf{E}_1$ to obtain

$$\mathbf{H}_2 \cdot \nabla \times \mathbf{E}_1 = j\omega\mu_0 \mathbf{H}_1 \cdot \mathbf{H}_2$$
$$\mathbf{E}_1 \cdot \nabla \times \mathbf{H}_2 = -j\omega\varepsilon_0\varepsilon_r \mathbf{E}_2 \cdot \mathbf{E}_1 + \mathbf{J}_2 \cdot \mathbf{E}_1 \tag{29.77}$$

Add up these two equations in (29.77)

$$\mathbf{H}_2 \cdot \nabla \times \mathbf{E}_1 - \mathbf{E}_1 \cdot \nabla \times \mathbf{H}_2 = \nabla \cdot (\mathbf{E}_1 \times \mathbf{H}_2) = j\omega\left(\mu_0 \mathbf{H}_1 \cdot \mathbf{H}_2 - \varepsilon_0\varepsilon_r \mathbf{E}_2 \cdot \mathbf{E}_1\right) + \mathbf{J}_2 \cdot \mathbf{E}_1 \tag{29.78}$$

Now switch indices 1 and 2

$$\nabla \cdot (\mathbf{E}_2 \times \mathbf{H}_1) = j\omega\left(\mu_0 \mathbf{H}_1 \cdot \mathbf{H}_2 - \varepsilon_0\varepsilon_r \mathbf{E}_2 \cdot \mathbf{E}_1\right) + \mathbf{J}_1 \cdot \mathbf{E}_2 \tag{29.79}$$

And subtract (29.79) from (29.78) to obtain

$$\nabla \cdot (\mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1) = \mathbf{J}_2 \cdot \mathbf{E}_1 - \mathbf{J}_1 \cdot \mathbf{E}_2 \tag{29.80}$$

Integrate over the volume and use Gauss theorem,

$$\iiint (\mathbf{J}_2 \cdot \mathbf{E}_1 - \mathbf{J}_1 \cdot \mathbf{E}_2) dV = \iint (\mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1) \cdot \hat{\mathbf{n}} dS = 0 \tag{29.81}$$

The surface integral is zero because fields vanish at infinity, so we have a mor efamiliar form of reciprocity relation

$$\iiint (\mathbf{J}_2 \cdot \mathbf{E}_1 - \mathbf{J}_1 \cdot \mathbf{E}_2) dV = 0 \tag{29.82}$$

Time reversal

If we change simultaneously sign of time $t' = -t$ and magnetic field $\mathbf{H}' = -\mathbf{H}$, the Maxwell's equations (29.1) will still hold

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}'}{\partial t'}$$
$$\nabla \times \mathbf{H}' = \varepsilon \frac{\partial \mathbf{E}}{\partial t'} \tag{29.83}$$

If we use complex form of Maxwell's equations

$$\nabla \times \mathbf{E} = j\omega\mu_0 \mathbf{H}$$
$$\nabla \times \mathbf{H} = -j\omega\varepsilon_0\varepsilon_r \mathbf{E}, \tag{29.84}$$

take complex conjugate of it

$$\nabla \times \mathbf{E}^* = -j\omega\mu_0 \mathbf{H}^*$$
$$\nabla \times \mathbf{H}^* = j\omega\varepsilon_0\varepsilon_r \mathbf{E}^* \tag{29.85}$$

and re-write it as

$$\begin{aligned}\nabla \times \mathbf{E}^* &= j\omega\mu_0(-\mathbf{H}^*) \\ \nabla \times (-\mathbf{H}^*) &= -j\omega\varepsilon_0\varepsilon_r\mathbf{E}^*\end{aligned} \tag{29.86}$$

We can see that change $\mathbf{E} \Rightarrow \mathbf{E}^*,\ \mathbf{H} \Rightarrow -\mathbf{H}^*$ does not change the equations, hence reciprocity condition (29.80) (without currents) can be written as $\nabla \cdot (\mathbf{E}_1^* \times \mathbf{H}_2 + \mathbf{E}_2 \times \mathbf{H}_1^*) = 0$, which is, of course, Eq. (29.54)

More on orthogonality

The power along the direction of propagation is given by

$$P = \iint \left(\mathbf{E} \times \mathbf{H}^* + \mathbf{E}^* \times \mathbf{H}\right)_z dxdy \tag{29.87}$$

Substituting (29.62) we obtain

$$P = \frac{1}{4}\sum_{mn} A_m A_n^* \iint \left(\mathbf{e}_m \times \mathbf{h}_n^* + \mathbf{e}_n^* \times \mathbf{h}_m\right)_z dxdy \tag{29.88}$$

But the power does not change as the wave propagates along the waveguide (in the absence of loss) , hence

$$\frac{dP}{dz} = \frac{j}{4}\sum_{mn} (\beta_m - \beta_n) A_m A_n^* \iint \left(\boldsymbol{e}_m \times \mathbf{h}_n^* + \boldsymbol{e}_n^* \times \mathbf{h}_m\right)_z dxdy = 0 \tag{29.89}$$

This expression should be true for any combination of A’s, hence

$$\iint \left(\mathbf{e}_m \times \mathbf{h}_n^* + \mathbf{e}_n^* \times \mathbf{h}_m\right)_z dxdy = 4\delta_{mn} \tag{29.90}$$

We can actually further simplify this expression – assume that one of the waves, m, propagate backward, i.e.

$$\iint \left(\mathbf{e}_{-m} \times \mathbf{h}_n^* + \mathbf{e}_n^* \times \mathbf{h}_{-m}\right)_z dxdy = 0 \tag{29.91}$$

But from time reversal we know that $\mathbf{e}_{-m} = \mathbf{e}_m \quad \mathbf{h}_{-m} = -\,\mathbf{h}_m$, hence

$$\iint \left(\mathbf{e}_m \times \mathbf{h}_n^* - \mathbf{e}_n^* \times \mathbf{h}_m\right)_z dxdy = 0 \tag{29.92}$$

Adding up (29.91) and (29.92) we obtain yet another expression for the orthogonality

$$\iint \left(\mathbf{e}_m \times \mathbf{h}_n^*\right)_z dxdy = 2\delta_{mn} \tag{29.93}$$

Propagation in a waveguide with perturbation

Consider a waveguide characterized by the modes $\mathbf{e}_m(x,y)$ with propagation constants β_m in which a small perturbation of permittivity $\delta\varepsilon(x,y)$ is introduced as shown in Fig.28.19. We expect that the propagation constant will change its value to $\beta_m + \delta\beta_m$.

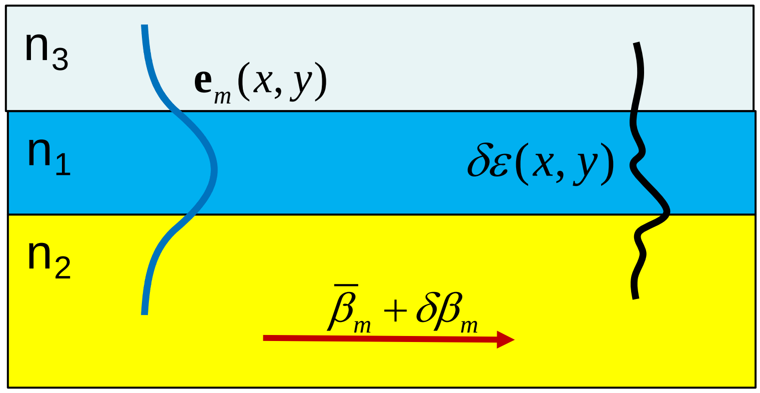

Figure 29.19 Waveguide with a perturbation of permittivity (and refractive index)

To find the change of propagation constant (and effective index) we shall use Maxwell's equation and avoid using slow variable approach. The fields in mode m are

$$\mathbf{E} = \frac{1}{2} A e^{j\beta_m z} \mathbf{e}_m(x,y)$$
$$\mathbf{H} = \frac{1}{2} A e^{j\beta_m z} \mathbf{h}_m(x,y) \qquad (29.94)$$

And substituting them into Maxwell equations (29.1) yields

$$\nabla_\perp \times \mathbf{e}_m + j\beta\hat{\mathbf{z}} \times \mathbf{e}_m = j\omega\mu\mathbf{h}_m$$
$$\nabla_\perp \times \mathbf{h}_m + j\beta\hat{\mathbf{z}} \times \mathbf{h}_m = -j\omega\mu\varepsilon\mathbf{e}_m \qquad (29.95)$$

Re-arrange the terms is (29.95)

$$-\nabla_\perp \times \mathbf{e}_m + j\omega\mu\mathbf{h}_m = j\beta\hat{\mathbf{z}} \times \mathbf{e}_m$$
$$-\nabla_\perp \times \mathbf{h}_m - j\omega\varepsilon_0\varepsilon_r\mathbf{e}_m = j\beta\hat{\mathbf{z}} \times \mathbf{h}_m \qquad (29.96)$$

and multiply the first equation by $\mathbf{h}_m^*$ and the second one by $\mathbf{e}_m^*$ to obtain

$$-\left(\nabla_\perp \times \mathbf{e}\right)\cdot\mathbf{h}_m^* + j\omega\mu\mathbf{h}_m\cdot\mathbf{h}_m^* = j\beta\left(\hat{\mathbf{z}} \times \mathbf{e}_m\right)\cdot\mathbf{h}_m^*$$
$$-\left(\nabla_\perp \times \mathbf{h}_m\right)\cdot\mathbf{e}_m^* - j\omega\varepsilon_0\varepsilon_r\mathbf{e}_m\cdot\mathbf{e}_m^* = j\beta\left(\hat{\mathbf{z}} \times \mathbf{h}_m\right)\cdot\mathbf{e}_m^* \qquad (29.97)$$

Then we subtract the second equation in (29.97) from the first one

$$\left(\nabla_\perp \times \mathbf{h}_m + j\omega\varepsilon_0\varepsilon_r\mathbf{e}_m\right)\cdot\mathbf{e}_m^* - \left(\nabla_\perp \times \mathbf{e} - j\omega\mu\mathbf{h}_m\right)\cdot\mathbf{h}_m^* = j\beta\left[\left(\hat{\mathbf{z}} \times \mathbf{e}_m\right)\cdot\mathbf{h}_m^* - \left(\hat{\mathbf{z}} \times \mathbf{h}_m\right)\cdot\mathbf{e}_m^*\right] = j\beta\hat{\mathbf{z}}\cdot\left[\mathbf{e}_m \times \mathbf{h}_m^* + \mathbf{e}_m^* \times \mathbf{h}_m\right]$$
$$(29.98)$$

Now consider that dielectric constant is perturbed as $\varepsilon_r = \overline{\varepsilon}_r + \delta\varepsilon_r$ and propagation constant became $\beta_m + \delta\beta_m$.

$$\begin{aligned}&\left(\nabla_\perp \times \mathbf{h}_m + j\omega\varepsilon_0\overline{\varepsilon}_r\mathbf{e}_m\right)\cdot\mathbf{e}_m^* + j\omega\varepsilon_0\delta\varepsilon_r\mathbf{e}_m\cdot\mathbf{e}_m^* - \left(\nabla_\perp \times \mathbf{e} - j\omega\mu\mathbf{h}_m\right)\cdot\mathbf{h}_m^* = \\ &= j\beta\hat{\mathbf{z}}\cdot\left[\mathbf{e}_m^*\times\mathbf{h}^* + \mathbf{e}_m^*\times\mathbf{h}_m\right] + j\delta\beta\hat{\mathbf{z}}\cdot\left[\mathbf{e}_m\times\mathbf{h}_m^* + \mathbf{e}_m^*\times\mathbf{h}_m\right]\end{aligned} \tag{29.99}$$

Cancelling the first and third terms on the l.h.s. and the first term on the r.h.s. according to (29.98), we have just two terms remaining

$$\omega\varepsilon_0\delta\varepsilon_r\mathbf{e}_m\cdot\mathbf{e}_m^* = \delta\beta\hat{\mathbf{z}}\cdot\left[\mathbf{e}_m\times\mathbf{h}_m^* + \mathbf{e}_m^*\times\mathbf{h}_m\right] \tag{29.100}$$

Integrating it over the area of waveguide and invoking the normalization condition (29.90) we obtain

$$\delta\beta = \frac{\omega\varepsilon_0\iint\delta\varepsilon_r(x,y)\mathbf{e}_m\cdot\mathbf{e}_m^* dxdy}{\iint\hat{\mathbf{z}}\cdot\left[\mathbf{e}_m\times\mathbf{h}_m^* + \mathbf{e}_m^*\times\mathbf{h}_m\right]dxdy} = \frac{\omega}{4c\eta_0}\iint\delta\varepsilon_r(x,y)\mathbf{e}_m\cdot\mathbf{e}_m^* dxdy \tag{29.101}$$

For TE –like mode using (29.70) we obtain

$$\delta\beta = \frac{\omega}{2cn_{eff}}\frac{\iint\delta\varepsilon_r(x,y)\mathbf{e}_m\cdot\mathbf{e}_m^* dxdy}{\iint\mathbf{e}_m\cdot\mathbf{e}_m^* dxdy} \sim \frac{\omega}{2cn_{eff}}\delta\varepsilon_r\Gamma_m \sim \frac{\omega}{c}\delta n\Gamma_m \tag{29.102}$$

Where in the last step we assumed that dielectric constant only changed in the core of waveguide and Γ_m is the confinement factor, similar to (29.49),

$$\Gamma_m = \frac{\iint_{core}\mathbf{e}_m\cdot\mathbf{e}_m^* dxdy}{\iint\mathbf{e}_m\cdot\mathbf{e}_m^* dxdy} \sim \tag{29.103}$$

For TM wave we obtain essentially the same expression

$$\delta\beta = \frac{\omega}{2cn_{eff}}\frac{\iint\delta\varepsilon_r(x,y)\mathbf{e}_m\cdot\mathbf{e}_m^* dxdy}{\iint\left[n(x,y)/n_{eff}\right]^2\mathbf{e}_m\cdot\mathbf{e}_m^* dxdy} \sim \frac{\omega}{2cn_{eff}}\delta\varepsilon_r\Gamma_m \sim \frac{\omega}{c}\delta n\Gamma_m \tag{29.104}$$

Note that since both real and imaginary parts of permittivity can be perturbed one can change both propagating constant and extinction coefficient.

Now we use a different approach to the same problem Instead of changing the propagation constant, we assume that the amplitude of the wave changes as A(z) as shown in Fig.29.20

$$\begin{aligned}\mathbf{E} &= \frac{1}{2}A(z)e^{j\beta_m z}\mathbf{e}_m(x,y) \\ \mathbf{H} &= \frac{1}{2}A(z)e^{j\beta_m z}\mathbf{h}_m(x,y)\end{aligned} \tag{29.105}$$

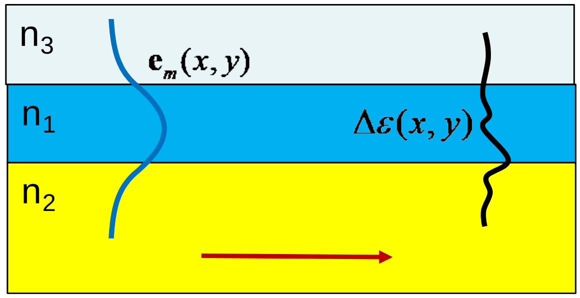

Figure 29.20 Change of amplitude of a wave in a Waveguide with a perturbation of permittivity (and refractive index)

Substituting (29.105) into wave equation we obtain (dropping index m as we only consider one mode)

$$\begin{aligned} A\left(\nabla_\perp \times \mathbf{e} + jA\beta\hat{\mathbf{z}}\times\mathbf{e}\right) + \frac{dA}{dz}\hat{\mathbf{z}}\times\mathbf{e} &= jA\omega\mu\mathbf{h} \\ A\left(\nabla_\perp \times \mathbf{h} + jA\beta\hat{\mathbf{z}}\times\mathbf{h}\right) + \frac{dA}{dz}\hat{\mathbf{z}}\times\mathbf{h} &= -jA\omega\varepsilon_0\varepsilon_r\mathbf{e} \end{aligned} \tag{29.106}$$

Dividing by the amplitude A(z), Rearranging the terms and dot-multiplying by $\mathbf{h}^*$ and $\mathbf{e}^*$ we obtain

$$\begin{aligned} -\left(\nabla_\perp \times \mathbf{e}\right)\cdot\mathbf{h}^* + j\omega\mu\mathbf{h}\cdot\mathbf{h}^* &= j\beta\left(\hat{\mathbf{z}}\times\mathbf{e}\right)\cdot\mathbf{h}^* + A^{-1}\frac{dA}{dz}\left(\hat{\mathbf{z}}\times\mathbf{e}\right)\cdot\mathbf{h}^* \\ -\left(\nabla_\perp \times \mathbf{h}\right)\cdot\mathbf{e}^* - j\omega\varepsilon_0\varepsilon_r\mathbf{e}\cdot\mathbf{e}^* &= j\beta\left(\hat{\mathbf{z}}\times\mathbf{h}\right)\cdot\mathbf{e}^* + A^{-1}\frac{dA}{dz}\left(\hat{\mathbf{z}}\times\mathbf{h}\right)\cdot\mathbf{e}^* \end{aligned} \tag{29.107}$$

Subtracting the second equation from the first one we obtain

$$\left(\nabla_\perp \times \mathbf{h} + j\omega\varepsilon_0\varepsilon_r\mathbf{e}\right)\cdot\mathbf{e}^* - \left(\nabla_\perp \times \mathbf{e} - j\omega\mu\mathbf{h}\right)\cdot\mathbf{h}^* = j\beta\hat{\mathbf{z}}\cdot\left[\mathbf{e}\times\mathbf{h}^* + \mathbf{e}^*\times\mathbf{h}\right] + A^{-1}\frac{dA}{dz}\hat{\mathbf{z}}\cdot\left[\mathbf{e}\times\mathbf{h}^* + \mathbf{e}^*\times\mathbf{h}\right]$$

(29.108)

Now consider that dielectric permittivity is perturbed as $\varepsilon_r = \varepsilon_r + \delta\varepsilon_r$, then

$$\left(\nabla_\perp \times \mathbf{h} + j\omega\varepsilon_0\bar{\varepsilon}_r\mathbf{e}\right)\cdot\mathbf{e}^* + j\omega\varepsilon_0\delta\varepsilon_r\mathbf{e}\cdot\mathbf{e}^* - \left(\nabla_\perp \times \mathbf{e} - j\omega\mu\mathbf{h}\right)\cdot\mathbf{h}^* = j\bar{\beta}\hat{\mathbf{z}}\cdot\left[\mathbf{e}\times\mathbf{h}^* + \mathbf{e}^*\times\mathbf{h}\right] + A^{-1}\frac{dA}{dz}\hat{\mathbf{z}}\cdot\left[\mathbf{e}\times\mathbf{h}^* + \mathbf{e}^*\times\mathbf{h}\right]$$

(29.109)

Now, Cancelling the first and third terms on the l.h.s. and the first term on the r.h.s. according to (29.98) , we have just two terms remaining

$$j\omega\varepsilon_0\delta\varepsilon_r\mathbf{e}\cdot\mathbf{e}^* = A^{-1}\frac{dA}{dz}\hat{\mathbf{z}}\cdot\left[\mathbf{e}\times\mathbf{h}^* + \mathbf{e}^*\times\mathbf{h}\right] \tag{29.110}$$

Once again integrating over the area of waveguide we obtain

$$\frac{dA}{dz}=\frac{j\omega\varepsilon_0\iint\delta\varepsilon_r(x,y)\mathbf{e}\cdot\mathbf{e}^*dxdy}{\iint\hat{\mathbf{z}}\cdot\left[\mathbf{e}\times\mathbf{h}^*+\mathbf{e}^*\times\mathbf{h}\right]dxdy}A=A\frac{j\omega}{4c\eta_0}\iint\delta\varepsilon_r(x,y)\mathbf{e}\cdot\mathbf{e}^*dxdy=jA\frac{\omega}{c}\delta n_{eff} \qquad (29.111)$$

Where we have used $\varepsilon_r=n^2,\ \delta\varepsilon_r(x,y)=2n(x,y)\delta n(x,y)$. Foe TE wave

$$\delta n_{eff}=\frac{\iint\left[n(x,y)/n_{eff}\right]\delta n(x,y)\delta\varepsilon_r(x,y)\mathbf{e}_m\cdot\mathbf{e}_m^*dxdy}{\iint\mathbf{e}_m\cdot\mathbf{e}_m^*dxdy}\sim\delta n\Gamma_m \qquad (29.112)$$

And for TM wave

$$\delta n_{eff}=\frac{\iint\delta n(x,y)\delta\varepsilon_r(x,y)\mathbf{e}_m\cdot\mathbf{e}_m^*dxdy}{\iint\left[n(x,y)/n_{eff}\right]\mathbf{e}_m\cdot\mathbf{e}_m^*dxdy}\sim\delta n\Gamma_m \qquad (29.113)$$

Obviously, if the change of dielectric constant is due to absorption, it is easy to see that

$$\alpha_{wg}=\delta\alpha\Gamma_m \qquad (29.114)$$

Note that even though we did not use the slow variable approach, all our results are still only applicable for a relatively small index perturbations, because we assumed that the mode shapes $\mathbf{e}_m$ remain constant, which is, of course, no longer true for strong perturbations.