Lecture 22 Light Propagation in Anisotropic medium.

Isotropic medium is the medium in which susceptibility and dielectric constant are scalars, hence according to the material equations

$$\begin{aligned} \mathbf{P} &= \varepsilon_0 \chi \mathbf{E} \\ \mathbf{D} &= \varepsilon_0 \mathbf{E} + \mathbf{P} = \varepsilon_0 \varepsilon_r \mathbf{E} \end{aligned} \tag{22.1}$$

vectors $\mathbf{E}, \mathbf{P}$, and $\mathbf{D}$ are all parallel to each other. According to Maxwell's equations for the monochromatic wave

$$\begin{aligned} \mathbf{k} \times \mathbf{H} &= -\omega \mathbf{D} \\ \mathbf{k} \times \mathbf{E} &= \omega \mathbf{B} \\ \mathbf{k} \cdot \mathbf{D} &= 0 \\ \mathbf{k} \cdot \mathbf{B} &= 0 \end{aligned} \tag{22.2}$$

The wave is transverse, as shown in Fig.22.1. Furthermore, since Poynting vector is $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ it is collinear with wave vector $\mathbf{k}$.

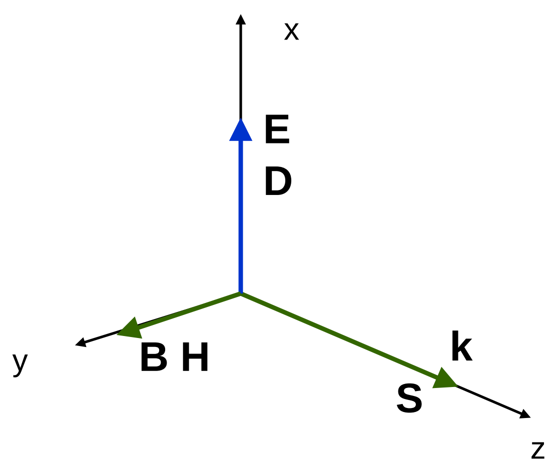

Figure.22.1 Fields and propagation vectors in isotropic medium.

Let us now consider the origin of anisotropy. First take a look at the simple model isotropic medium shown in Fig.22.2 (a) . The electron finds itself in a symmetric environment (symmetric potential) – hence whether it is moved along x,y, or z axis it will experience the same restoring force, with the same restoring force $K_x = K_y = K_z$. It is easy to show that no matter along which direction we apply the electric field we get the same response.

Obviously, in the anisotropic medium shown in Fig. 22.2.(b) the environment is no longer isotropic – it may be easier to move the electron cloud along ,let us say x direction than along z and y. In general, $K_x \neq K_y \neq K_z$. Thus anisotropic medium's response depends on the direction of applied field $\mathbf{E}$ (i.e. polarization of optical wave) .

When electric field $\mathbf{E} = E_x\hat{\mathbf{x}} + E_y\hat{\mathbf{y}} + E_z\hat{\mathbf{z}}$ is applied we can write three equations of motion

$$
\begin{aligned}
m_0 \frac{d^2x}{dt^2} &= -K_x x - \gamma_x m_0 \frac{dx}{dt} - eE_x(t) \\
m_0 \frac{d^2y}{dt^2} &= -K_y y - \gamma_y m_0 \frac{dy}{dt} - eE_y(t) \\
m_0 \frac{d^2z}{dt^2} &= -K_z z - \gamma_z m_0 \frac{dz}{dt} - eE_z(t)
\end{aligned}
\tag{22.3}
$$

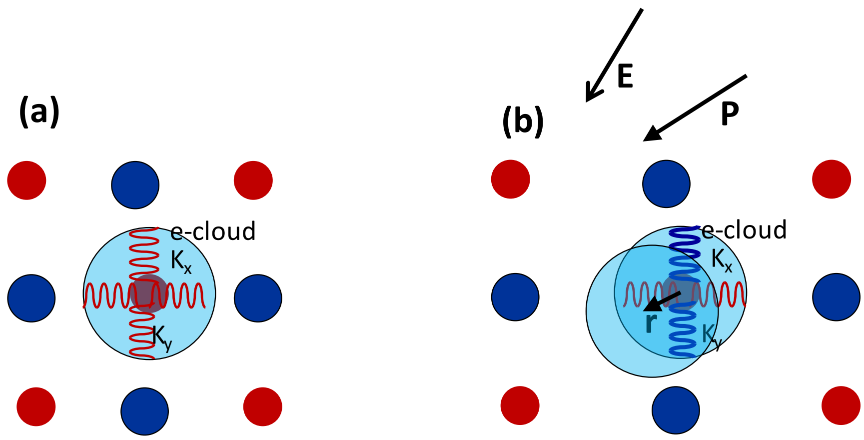

Figure 22.2 Models of (a) isotropic and (b) anisotropic polarizable media

We can introduce three distinct resonance frequencies $\omega_{0i} = \sqrt{K_i / m_0};\ i = x, y, z$ and for harmonic field $E(t) = Ee^{-j\omega t}$ we obtain three different projections of charge displacement

$$
\begin{aligned}
x_\omega &= \frac{-eE_x / m_0}{\omega_{0x}^2 - \omega^2 - j\omega\gamma_x} \\
y_\omega &= \frac{-eE_y / m_0}{\omega_{0y}^2 - \omega^2 - j\omega\gamma_y} \\
z_\omega &= \frac{-eE_z / m_0}{\omega_{0z}^2 - \omega^2 - j\omega\gamma_z}
\end{aligned}
\tag{22.4}
$$

Clearly the charge movement $\mathbf{r}$ Is not collinear with the electric field. Three projections of the polarization vector can be found as

$$\begin{aligned}
P_{x\omega} &= -Nex_{\omega} = \frac{Ne^2/m_0}{\omega_{0x}^2-\omega^2-j\omega\gamma_x}E_x = \varepsilon_0\chi_{xx}E_x \\
P_{y\omega} &= -Ney_{\omega} = \frac{Ne^2/m_0}{\omega_{0y}^2-\omega^2-j\omega\gamma_y}E_y = \varepsilon_0\chi_{yy}E_y \\
P_{z\omega} &= -Nez_{\omega} = \frac{Ne^2/m_0}{\omega_{0z}^2-\omega^2-j\omega\gamma_z}E_z = \varepsilon_0\chi_{zz}E_z
\end{aligned} \tag{22.5}$$

where we have introduced three diagonal elements of the susceptibility tensor. The main result of it is that the polarization vector $\mathbf{P}_{\omega} = P_{x\omega}\hat{\mathbf{x}} + P_{y\omega}\hat{\mathbf{y}} + P_{z\omega}\hat{\mathbf{z}}$ is not collinear with the electric field $\mathbf{E}$.

Susceptibility tensor

In general, the system of coordinates is not necessarily aligned with axis of crystal and one can write a general relation

$$\begin{aligned}
P_{x\omega} &= \varepsilon_0\chi_{11}E_x + \varepsilon_0\chi_{12}E_y + \varepsilon_0\chi_{13}E_z \\
P_{y\omega} &= \varepsilon_0\chi_{21}E_x + \varepsilon_0\chi_{22}E_y + \varepsilon_0\chi_{23}E_z \\
P_{z\omega} &= \varepsilon_0\chi_{31}E_x + \varepsilon_0\chi_{32}E_y + \varepsilon_0\chi_{33}E_z
\end{aligned} \tag{22.6}$$

The tensor of susceptibility is represented by a matrix

$$\chi(\omega) = \begin{pmatrix} \chi_{11} & \chi_{12} & \chi_{13} \\ \chi_{21} & \chi_{22} & \chi_{23} \\ \chi_{31} & \chi_{32} & \chi_{33} \end{pmatrix} \tag{22.7}$$

Then (22.6) becomes

$$\mathbf{P}(\omega) = \varepsilon_0 \begin{pmatrix} \chi_{11} & \chi_{12} & \chi_{13} \\ \chi_{21} & \chi_{22} & \chi_{23} \\ \chi_{31} & \chi_{32} & \chi_{33} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = \varepsilon_0 \chi(\omega)\mathbf{E}(\omega) \tag{22.8}$$

Next we introduce tensor of dielectric constant

$$\varepsilon_r(\omega) = 1 + \chi(\omega) = \begin{pmatrix} 1+\chi_{11} & \chi_{12} & \chi_{13} \\ \chi_{21} & 1+\chi_{22} & \chi_{23} \\ \chi_{31} & \chi_{32} & 1+\chi_{33} \end{pmatrix} = \begin{pmatrix} \varepsilon_{r,11} & \varepsilon_{r,12} & \varepsilon_{r,13} \\ \varepsilon_{r,21} & \varepsilon_{r,22} & \varepsilon_{r,23} \\ \varepsilon_{r,31} & \varepsilon_{r,32} & \varepsilon_{r,33} \end{pmatrix} \tag{22.9}$$

So that

$$\begin{aligned}
\mathbf{P} &= \varepsilon_0 \chi \mathbf{E} \\
\mathbf{D} &= \varepsilon_0 \varepsilon_r \mathbf{E}
\end{aligned} \tag{22.10}$$

which formally looks exactly as (22.1) , but χ and ε_r are tensors rather than scalars. Of course in isotropic media only three diagonal elements are present and they are equal to each other, so the tensor is just a scalar multiplied by a unity tensor. So, in anisotropic media displacement is not collinear with electric field. According to Maxwell's equations (22.2) the displacement vector $\mathbf{D}$ is normal to $\mathbf{k},\mathbf{B}$ and therefore to $\mathbf{H}$ as shown in Fig.22.3 Electric field $\mathbf{E}$ is normal to $\mathbf{B}$ and $\mathbf{H}$ but not to wavevector and lies in *xz* plane. Obviously polarization vector $\mathbf{P}$ is also in xz plane. Poynting vector $\mathbf{S}=\mathbf{E}\times\mathbf{H}$ being perpendicular to the magnetic field also lies in xz plane, but is not collinear with wavevector $\mathbf{k}$ so phase and energy do not propagate in the same direction.

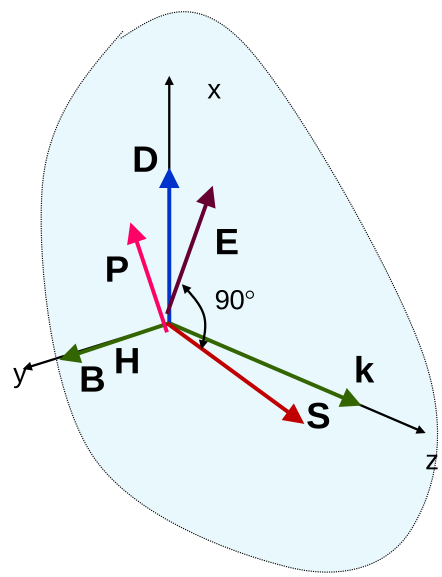

Figure 22.3 Plane wav ein anisotropic medium

Index ellipsoid

For any tensor it is possible to choose the coordinate system in such a way that it becomes diagonal –this system of coordinates is called principal system and is related to crystal symmetry. So we have

$$\varepsilon_r(\omega)=1+\chi(\omega)=\begin{pmatrix}\varepsilon_{r,11} & 0 & 0\\ 0 & \varepsilon_{r,22} & 0\\ 0 & 0 & \varepsilon_{r,33}\end{pmatrix}=\begin{pmatrix}n_x^2 & 0 & 0\\ 0 & n_y^2 & 0\\ 0 & 0 & n_z^2\end{pmatrix} \tag{22.11}$$

We introduce the *index ellipsoid* as

$$\frac{x^2}{n_x^2}+\frac{y^2}{n_y^2}+\frac{z^2}{n_z^2}=1 \tag{22.12}$$

as shown in Fig. 22.4(a). If all three refractive indices are different the crystal is called biaxial which is the most general case. Consider simpler cases.

Isotropic medium (which can be amorphous as glass or have cubic symmetry like Si or GaAs. In isotropic medium

$$\varepsilon_{r,11} = \varepsilon_{r,22} = \varepsilon_{r,33} = \varepsilon_r = n^2 \tag{22.13}$$

and

$$\varepsilon_r(\omega) = \begin{pmatrix} n^2 & 0 & 0 \\ 0 & n^2 & 0 \\ 0 & 0 & n^2 \end{pmatrix} = n^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{22.14}$$

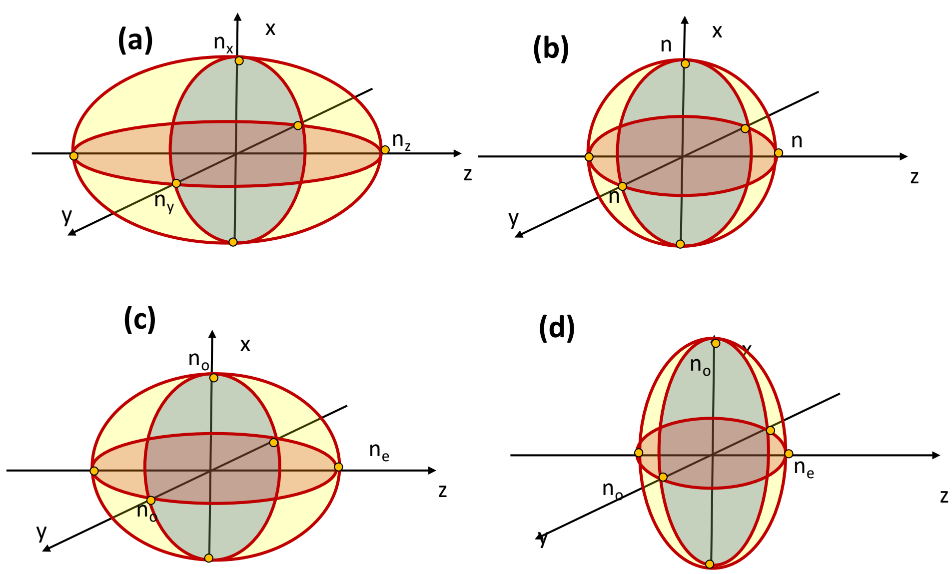

Figure 22.4 Index Ellipsoid for (a) biaxial crystal (b) isotropic medium (c) uniaxial positive crystal (d) uniaxial negative crystal

Dielectric constant is a scalar and the index ellipsoid is a sphere shown in Fig 22.4(b).

$$x^2 + y^2 + z^2 = n^2 \tag{22.15}$$

Next case are so –called uniaxial crystals in which two indices are equal and the third is different.

$$\varepsilon_{r,11} = \varepsilon_{r,22} = \varepsilon_{r,\|} \neq \varepsilon_{r,33} = \varepsilon_{\perp} \tag{22.16}$$

We introduce *ordinary refractive index a*s

$$n_x = n_y = \sqrt{\varepsilon_{r,\|}} = n_o \tag{22.17}$$

And *extraordinary refractive index* as

$$n_z = \sqrt{\varepsilon_{r,\perp}} = n_e \tag{22.18}$$

So that

$$\varepsilon_r(\omega) = \begin{pmatrix} n_0^2 & 0 & 0 \\ 0 & n_o^2 & 0 \\ 0 & 0 & n_e^2 \end{pmatrix} \tag{22.19}$$

The index ellipsoid is an ellipsoid of revolution around axis z which is called *optical axis*

$$\frac{x^2+y^2}{n_o^2}+\frac{z^2}{n_e^2}=1 \tag{22.20}$$

There exist two possibilities. (a) $n_e > n_o$ - such crystal is called positive and its index ellipsoid is prolate and looks like a football as shown in Fig.22.4(c) (b) $n_e < n_o$ such crystal is called negative and its index ellipsoid is oblate and looks like a muffin as shown in Fig.22.4(d)

Examples of positive crystals are quartz (SiO_2) n_o=1.5427 n_e=1.5518 and Rutile (TiO_2) n_o=2.616, n_e=2.903. Examples of negative crystals are Calcite ($CaCO_3$) n_o =1.6629, n_e =1.4885 and sodium nitrate ($NaNO_3$) n_o=1.5854 n_e=1. 3369.

Light propagation in uniaxial crystals – wave equation

Consider the light propagating at the arbitrary angle θ to the optical axis z as in Fig. 22.5. Since $\mathbf{D}$ and $\mathbf{E}$ are not collinear and angle between them is β the same angle is between $\mathbf{S}$ and $\mathbf{k}$, and it is called a *walk-off angle*

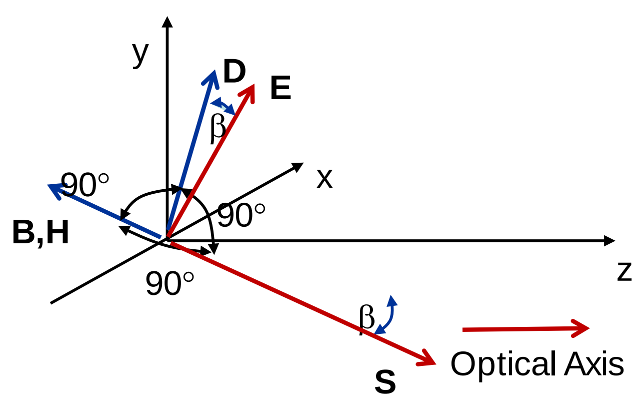

Figure 22.5 Fields and directions of the plane wave propagating in uniaxial crystal

We start as always with the first two Maxwell's equations

$$\begin{aligned} \mathbf{k}\times\mathbf{H} &= -\omega\mathbf{D} = -\omega\varepsilon_0\varepsilon_r\mathbf{E} \\ \mathbf{k}\times\mathbf{E} &= \omega\mathbf{B} = \omega\mu\mathbf{H} \end{aligned} \tag{22.21}$$

Performing vector multiplication of the second equation of (22.21) by $\mathbf{k}$ and using the first equation we obtain

$$\mathbf{k}\times\mathbf{k}\times\mathbf{E}=\omega\mu\mathbf{k}\times\mathbf{H}=-\omega^2\mu\varepsilon_0\varepsilon_r\mathbf{E} \tag{22.22}$$

Now,

$$\mathbf{k}\times\mathbf{k}\times\mathbf{E}=\mathbf{k}(\mathbf{k}\cdot\mathbf{E})-\mathbf{E}(\mathbf{k}\cdot\mathbf{k}) \tag{22.23}$$

And the wave equation becomes

$$\mathbf{k}(\mathbf{k}\cdot\mathbf{E})-\mathbf{E}(\mathbf{k}\cdot\mathbf{k})=-\omega^2\mu\varepsilon_0\varepsilon_r\mathbf{E} \tag{22.24}$$

In isotropic medium the dot product $\mathbf{k}\cdot\mathbf{E}=0$ but in anisotropic medium it is no longer true and we write explicitly

$$\mathbf{k}(\mathbf{k}\cdot\mathbf{E})=\begin{pmatrix}k_x\\k_y\\k_z\end{pmatrix}\left(k_xE_x+k_yE_y+k_zE_z\right)=\begin{pmatrix}k_x^2E_x+k_xk_yE_y+k_xk_zE_z\\k_yk_xE_x+k_y^2E_y+k_yk_zE_z\\k_zk_xE_x+k_zk_yE_y+k_z^2E_z\end{pmatrix}=\begin{pmatrix}k_x^2 & k_xk_y & k_xk_z\\k_yk_x & k_y^2 & k_yk_z\\k_zk_x & k_zk_y & k_z^2\end{pmatrix}\begin{pmatrix}E_x\\E_y\\E_z\end{pmatrix}$$

(22.25)

Introduce the outer product of two vectors

$$\mathbf{a}\otimes\mathbf{b}=\begin{pmatrix}a_xb_x & a_xb_y & a_xb_z\\a_yb_x & a_yb_y & a_yb_z\\a_zb_x & a_zb_y & a_zb_z\end{pmatrix} \tag{22.26}$$

Then

$$\mathbf{k}(\mathbf{k}\cdot\mathbf{E})=\left(\mathbf{k}\otimes\mathbf{k}\right)\mathbf{E} \tag{22.27}$$

Wave equation is now

$$\left(\mathbf{k}\otimes\mathbf{k}\right)\mathbf{E}-k^2\mathbf{E}+\omega^2\mu\varepsilon_0\varepsilon_r\mathbf{E}=0 \tag{22.28}$$

Next

$$\omega^2\mu\varepsilon_0\varepsilon_r\mathbf{E}=\omega^2\mu\varepsilon_0\begin{pmatrix}n_o^2 & 0 & 0\\0 & n_o^2 & 0\\0 & 0 & n_e^2\end{pmatrix}\begin{pmatrix}E_x\\E_y\\E_z\end{pmatrix}=\begin{pmatrix}k_o^2 & 0 & 0\\0 & k_o^2 & 0\\0 & 0 & k_e^2\end{pmatrix}\begin{pmatrix}E_x\\E_y\\E_z\end{pmatrix} \tag{22.29}$$

Where we have introduced ordinary and extraordinary wavevectors

$$\begin{aligned}k_o&=\omega(\mu_0\varepsilon_0)^{1/2}n_o=\frac{\omega}{c}n_o\\k_e&=\omega(\mu_0\varepsilon_0)^{1/2}n_e=\frac{\omega}{c}n_e\end{aligned} \tag{22.30}$$

Finally

$$\mathbf{E}(\mathbf{k}\cdot\mathbf{k}) = \begin{pmatrix} k^2 & 0 & 0 \\ 0 & k^2 & 0 \\ 0 & 0 & k^2 \end{pmatrix}\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} \tag{22.31}$$

Substituting (22.25),(22.29)and (22.31) into (22.28) we obtain

$$\begin{pmatrix} k_o^2 - k_y^2 - k_z^2 & k_x k_y & k_x k_z \\ k_y k_x & k_o^2 - k_x^2 - k_z^2 & k_y k_z \\ k_z k_x & k_z k_y & k_e^2 - k_x^2 - k_y^2 \end{pmatrix}\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0 \tag{22.32}$$

This is our wave equation for vector E, or, rather 3 equations for its 3 projections, and this equation has non-trivial solution if the determinant of the matrix is equal to 0 . So we end up with a characteristic, or dispersion, equation for the 3 components of wavevector k

$$\begin{vmatrix} k_o^2 - k_y^2 - k_z^2 & k_x k_y & k_x k_z \\ k_y k_x & k_o^2 - k_x^2 - k_z^2 & k_y k_z \\ k_z k_x & k_z k_y & k_e^2 - k_x^2 - k_y^2 \end{vmatrix} = 0 \tag{22.33}$$

In uniaxial crystals we can assume without the loss of generality that $k_x = 0$ and $k^2 = k_y^2 + k_z^2$ Then the wave equation (22.32) becomes

$$\begin{pmatrix} k_o^2 - k^2 & 0 & 0 \\ 0 & k_o^2 - k_z^2 & k_y k_z \\ 0 & k_z k_y & k_e^2 - k_y^2 \end{pmatrix}\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0 \tag{22.34}$$

and it can be separated into 2 separate equations – one for E_x -

$$\left(k_o^2 - k^2\right)E_x = 0 \tag{22.35}$$

and the other for two remaining components –

$$\begin{pmatrix} k_o^2 - k_z^2 & k_y k_z \\ k_z k_y & k_e^2 - k_y^2 \end{pmatrix}\begin{pmatrix} E_y \\ E_z \end{pmatrix} = 0 \tag{22.36}$$

Consider the wave that has non-zero E_x - to satisfy (22.35) we obtain

$$k_{ord} = k_o = \frac{\omega}{c} n_o \tag{22.37}$$

Substituting (22.37) into (22.36) we obtain

$$\begin{pmatrix} k_y^2 & k_y\sqrt{k_o^2-k_y^2} \\ k_y\sqrt{k_o^2-k_y^2} & k_e^2-k_y^2 \end{pmatrix}\begin{pmatrix} E_y \\ E_z \end{pmatrix}=0 \tag{22.38}$$

The determinant

$$\begin{vmatrix} k_y^2 & k_y\sqrt{k_o^2-k_y^2} \\ k_y\sqrt{k_o^2-k_y^2} & k_e^2-k_y^2 \end{vmatrix}=k_y^2\left(k_e^2-k_y^2\right)-k_y^2\left(k_o^2-k_y^2\right)=k_y^2\left(k_e^2-k_o^2\right)\neq 0 \tag{22.39}$$

Therefore only trivial solution $E_y=E_z=0$ is allowed. Thus if $E_x\neq 0$ then $E_y=E_z=0$ - the wave is polarized along x, i.e. normal to the optical axis and is called *ordinary wave.* The value of wavevector of this wave defined by (22.37) does not depend on the propagation direction. The fields in the ordinary wave are shown in Fig. 22.6

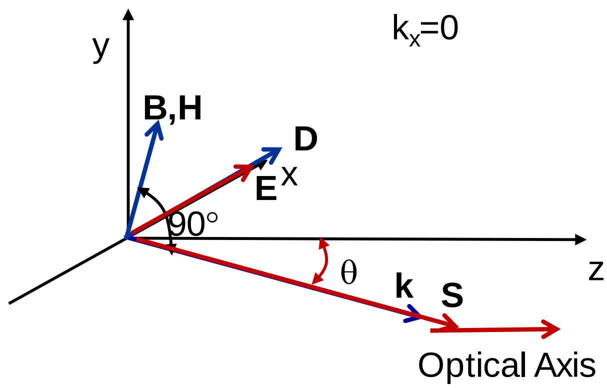

Figure 22.6 Fields and directions in ordinary wave

Note that since $E_y=E_z=0$ the displacement vector

$$\mathbf{D}=\varepsilon_0\varepsilon_r\mathbf{E}=\varepsilon_0\begin{pmatrix} n_o^2 & 0 & 0 \\ 0 & n_o^2 & 0 \\ 0 & 0 & n_e^2 \end{pmatrix}\begin{pmatrix} E_x \\ 0 \\ 0 \end{pmatrix}=\varepsilon_0\begin{pmatrix} n_o^2E_x \\ 0 \\ 0 \end{pmatrix} \tag{22.40}$$

is parallel to $\mathbf{E}$ and then $\mathbf{k}||\mathbf{S}$. There is no walk-off and the ordinary wave behaves as if it was propagating in isotropic medium,

Next consider the wave with non-zero E_y and E_z - to satisfy (22.36) we need the determinant equal to zero, i.e.

$$\begin{vmatrix} k_o^2-k_z^2 & k_yk_z \\ k_zk_y & k_e^2-k_y^2 \end{vmatrix}=k_o^2k_e^2-k_z^2k_e^2-k_y^2k_o^2=0 \tag{22.41}$$

Divide it by $k_o^2k_e^2$ and obtain

$$\frac{k_z^2}{k_o^2}+\frac{k_y^2}{k_e^2}=1 \tag{22.42}$$

This is an equation of ellipse with half axes k_o and k_e .shown in Fig. 22.7 a. We can describe it in polar coordinates,

$$\begin{aligned} k_z &= k\cos\theta; \\ k_y &= k\sin\theta; \end{aligned} \tag{22.43}$$

So that

$$\frac{1}{k_{ext}(\theta)^2}=\frac{\cos^2\theta}{k_o^2}+\frac{\sin^2\theta}{k_e^2}=\frac{c^2}{\omega^2}\left(\frac{\cos^2\theta}{n_o^2}+\frac{\sin^2\theta}{n_e^2}\right) \tag{22.44}$$

What about E_x? Since $k \neq k_o$ (unless a very special case of $\theta = 0$) the term in parenthesis of (22.35) is not zero and therefore $E_x = 0$. This is an extraordinary wave whose wavevector depends on the angle of propagation.

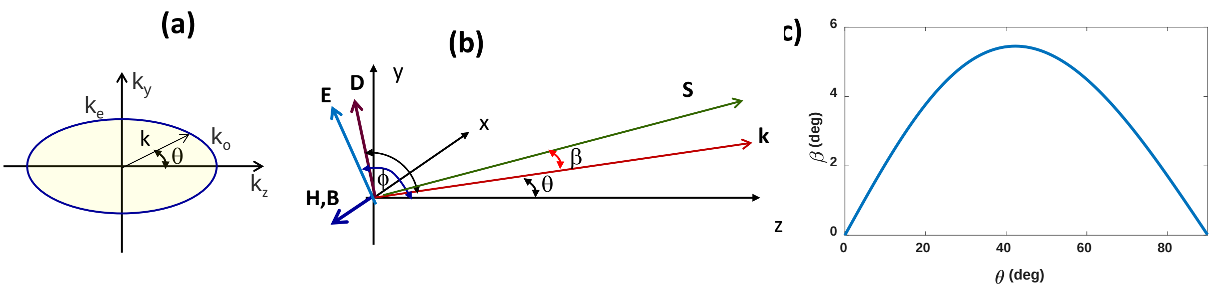

Figure 22.7 Extraordinary wave (a) determining wavevector (b) Fields and walk-off (c) Walk off vs propagation angle

Let us find polarization of the extraordinary wave. (we know that it is in yz plane…but what is the polarization, i.e. angle of $\mathbf{E}$ relative to optical axis z shown as φ in Fig.22.7 (b) . ? It is not simple because $\mathbf{E}$ is not normal to $\mathbf{k}$. Using the third Maxwell equation in (22.2) we get

$$\mathbf{k}\cdot\mathbf{D} = \varepsilon_0\mathbf{k}\cdot\varepsilon_r\mathbf{E} = \begin{pmatrix} k_y & k_z \end{pmatrix}\begin{pmatrix} n_o^2 & 0 \\ 0 & n_e^2 \end{pmatrix}\begin{pmatrix} E_y \\ E_z \end{pmatrix} = k_y n_o^2 E_y + k_z n_e^2 E_z = 0 \tag{22.45}$$

And therefore

$$\frac{n_o^2}{n_e^2}k_y E_y = -k_z E_z \tag{22.46}$$

Now, according to (22.43)

$$\frac{k_z}{k_y}=\tan\left(\frac{\pi}{2}-\theta\right) \tag{22.47}$$

Then

$$\tan\varphi = \frac{E_y}{E_z} = -\frac{n_e^2}{n_o^2}\frac{k_z}{k_y} = -\frac{n_e^2}{n_o^2}\tan\left(\frac{\pi}{2}-\theta\right) = -\frac{n_e^2}{n_o^2}\frac{1}{\tan\theta} \tag{22.48}$$

Finally, we find walk of angle. Since $\beta+\theta=\varphi-\pi/2$;

$$\begin{gathered}\tan(\beta+\theta) = -\frac{1}{\tan\phi} = \frac{n_o^2}{n_e^2}\tan\theta \\ \beta = \tan^{-1}\left(\frac{n_o^2}{n_e^2}\tan\theta\right)-\theta\end{gathered} \tag{22.49}$$

As shown in Fig. 22.7 © Consider now two special cases

(a) Light propagates along the optical axis $\theta=0$ then $\varphi=\pi/2$ and electric field is normal to wavevector. $\mathbf{E}\perp\mathbf{k}$ and $\mathbf{E}\parallel\mathbf{D}$. Obviously $k_{ext}=k_o$ and $\beta=0$. There is no walk off.

(b) Light propagates normal to the optical axis. $\theta=\pi/2$ then $\varphi=0$ and once again the electric field is normal to wavevector. $\mathbf{E}\perp\mathbf{k}$ and $\mathbf{E}\parallel\mathbf{D}$. Obviously $k_{ext}=k_e$ and $\beta=0$. There is no walk off.

Normal surface

Normal surface, shown in Fig.22.8 is convenient and instructive graphical method of obtaining refractive indices for ordinary and extraordinary waves.

Effective refractive index of the ordinary wave according to (22.37) is

$$n_{ord} = \frac{ck_{ord}}{\omega} = n_o \tag{22.50}$$

For the extraordinary wave according to (22.44)

$$\frac{1}{n_{ext}^2(\theta)} = \frac{\cos^2\theta}{n_o^2}+\frac{\sin^2\theta}{n_e^2} \tag{22.51}$$

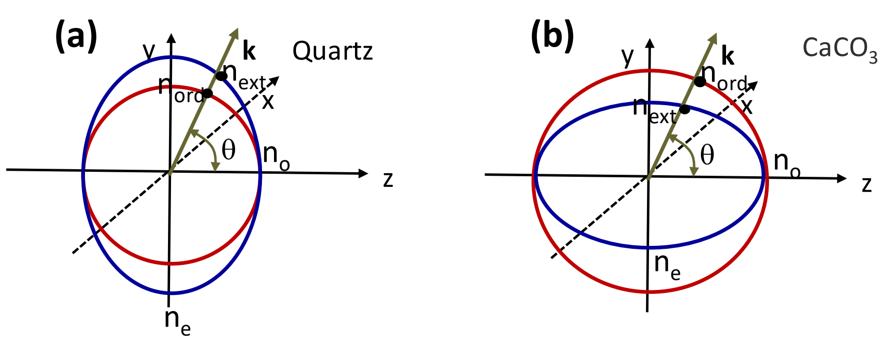

Figure 22.8 Normal surfaces for (a) positive and (b) negative uniaxial crystals

Now, for the ordinary wave we can plot a sphere

$$\frac{x^2}{n_o^2}+\frac{y^2}{n_o^2}+\frac{z^2}{n_o^2}=1 \qquad (22.52)$$

and for extraordinary wave an ellipsoid

$$\frac{x^2}{n_e^2}+\frac{y^2}{n_e^2}+\frac{z^2}{n_o^2}=1 \qquad (22.53)$$

Note that this ellipsoid is different from index ellipsoid (22.20) – the indices n_o and n_e are transposed! Then, we draw a direction of propagation $\mathbf{k}$ at an angle θ the intersection with sphere and ellipsoid will yield refractive indices for the ordinary and extraordinary waves respectively. Note that normal surface for positive crystal (Fig.22.8 a) is a sphere inside a muffin, while normal surface for negative (Fig.22.8 b) crystal –a football inside a sphere.

Consider two special cases.

(1) $\theta=0$ - then $n_{ext}=n_{ord}=n_o$ Light propagates parallel to the optical axis, therefore no matter how it is polarized and the effective the index is n_o. There is no walk off as shown in Fig. 22.9 (a)

(2) $\theta=\pi/2$ then $n_{ext}=n_e, n_{ord}=n_o$ Light propagates normal to the optical axis, therefore there are two possible directions of polarization – either normal to the optical axis(ordinary) or parallel to it (extraordinary). Again there is no walk off (Fig.22.9.b)

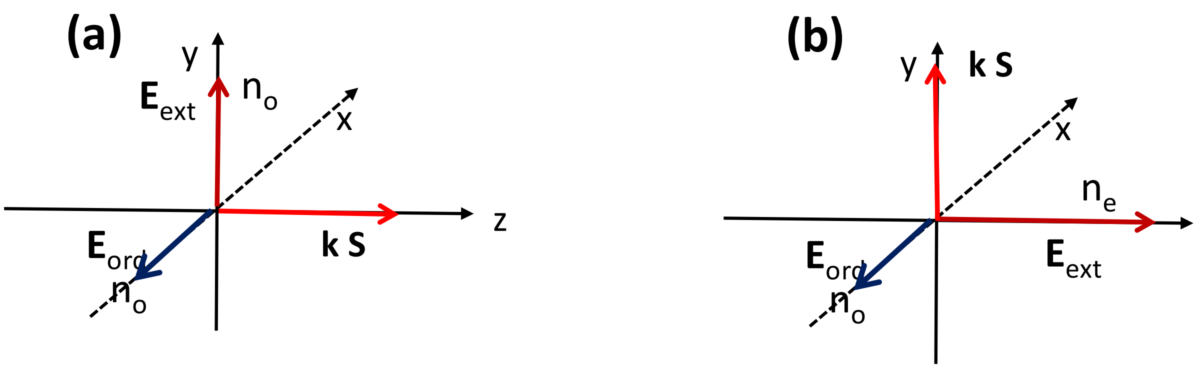

Figure 22.9 Fields of light propagating in uniaxial crystals (a) parallel and (b) normal to the optical axis.

By far the easiest way to find the effective refractive indices for two waves is illustrated in Fig. 22.10.

If we plot index ellipsoid (22.20) and direction of wavevector and the consider the intersection of the plane normal to $\mathbf{k}$ with the ellipsoid it will be an ellipse with half axes n_o in horizontal direction y and $n_{ext}(\theta)$ as defined by (22.51) in orthogonal direction x'. Thus two half-axes are equal to two indices and also define two orthogonal directions of displacement vector

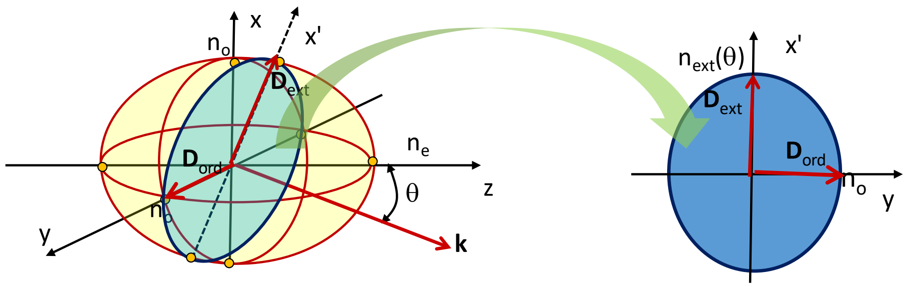

Figure 22.10 Finding effective refractive indices

Phase and group velocities.

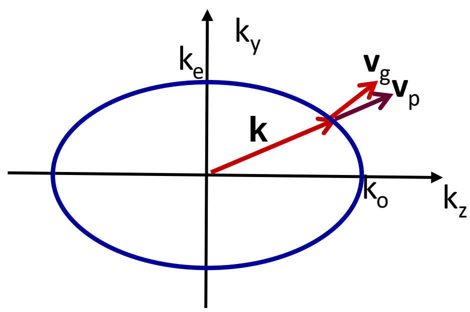

Figure 22.11 Isofrequency surface and group and phase velocities

Let us draw the normal surface as

$$\frac{k_x^2}{k_e^2}+\frac{k_y^2}{k_e^2}+\frac{k_z^2}{k_o^2}=1 \tag{22.54}$$

(Fig.22.11) which can be re-worked as

$$\frac{k_x^2}{n_e^2}+\frac{k_y^2}{n_e^2}+\frac{k_z^2}{n_0^2}=\frac{\omega^2}{c^2} \tag{22.55}$$

And eventually

$$\omega^2(\mathbf{k})=\frac{c^2k_x^2}{n_e^2}+\frac{c^2k_y^2}{n_e^2}+\frac{c^2k_z^2}{n_0^2} \tag{22.56}$$

This is the equation of the surface of constant frequency (isofrequency surface) as all points on it correspond to the same frequency. Phase velocity can be found as

$$\mathbf{v}_p=\frac{\omega}{k}\hat{\mathbf{k}} \tag{22.57}$$

And group velocity is defined as

$$\mathbf{v}_g = \frac{\partial \omega}{\partial \mathbf{k}} = \nabla_{\mathbf{k}} \omega(\mathbf{k}) \equiv \frac{\partial \omega}{\partial k_x}\hat{\mathbf{x}} + \frac{\partial \omega}{\partial k_y}\hat{\mathbf{y}} + \frac{\partial \omega}{\partial k_z}\hat{\mathbf{z}} \tag{22.58}$$

Obviously, being a gradient, group velocity is normal to the surface of constant frequency, and if that surface is not a sphere (extraordinary wave) the group velocity is different from phase velocity (both magnitude and direction) just because of the birefringence, even if the medium Is not dispersive. Next we need to figure out the energy velocity and how it relates to the group velocity.

We start, as always, with a definition of energy density

$$U = \frac{1}{2}\left(\mathbf{E}\cdot\mathbf{D} + \mathbf{H}\cdot\mathbf{B}\right) = \frac{1}{2}\left(\varepsilon_0 \mathbf{E}\cdot\varepsilon_r \mathbf{E} + \mu_0 \mathbf{H}\cdot\mu_r \mathbf{H}\right) \tag{22.59}$$

and Poynting vector $\mathbf{S} = \mathbf{E}\times\mathbf{H}$ and Maxwell's equations (22.2)

$$\begin{aligned} \mathbf{k}\times\mathbf{H} &= -\omega \mathbf{D} = -\omega\varepsilon_0\varepsilon_r \mathbf{E} \\ \mathbf{k}\times\mathbf{E} &= \omega \mathbf{B} = \omega\mu_0\mu_r \mathbf{H} \end{aligned} \tag{22.60}$$

Let us differentiate them. If frequency changes by small amount dω then all other variable, except ε and μ also change, so we obtain

$$\begin{aligned} \delta\mathbf{k}\times\mathbf{H} + \mathbf{k}\times\delta\mathbf{H} &= -\delta\omega\varepsilon_0\varepsilon_r\mathbf{E} - \omega\varepsilon_0\varepsilon_r\delta\mathbf{E} \\ \delta\mathbf{k}\times\mathbf{E} + \mathbf{k}\times\delta\mathbf{E} &= \delta\omega\mu_0\mu_r\mathbf{H} + \omega\mu_0\mu_r\delta\mathbf{H} \end{aligned} \tag{22.61}$$

Dot multiply these equations by $\mathbf{E}$ and $\mathbf{D}$ respectively

$$\begin{aligned} \mathbf{E}\cdot\left(\delta\mathbf{k}\times\mathbf{H}\right) + \mathbf{E}\cdot\left(\mathbf{k}\times\delta\mathbf{H}\right) &= -\delta\omega\varepsilon_0\varepsilon_r\mathbf{E}\cdot\mathbf{E} - \omega\varepsilon_0\varepsilon_r\delta\mathbf{E}\cdot\mathbf{E} \\ \mathbf{H}\cdot\left(\delta\mathbf{k}\times\mathbf{E}\right) + \mathbf{H}\cdot\left(\mathbf{k}\times\delta\mathbf{E}\right) &= \delta\omega\mu_0\mu_r\mathbf{H}\cdot\mathbf{H} + \omega\mu_0\mu_r\delta\mathbf{H}\cdot\mathbf{H} \end{aligned} \tag{22.62}$$

Use the rules $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})$ as well as $\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$

$$\begin{aligned} -\delta\mathbf{k}\cdot\left(\mathbf{E}\times\mathbf{H}\right) - \delta\mathbf{H}\cdot\left(\mathbf{k}\times\mathbf{E}\right) &= -\delta\omega\varepsilon_0\mathbf{E}\cdot\varepsilon_r\mathbf{E} - \mathbf{E}\cdot\omega\varepsilon_0\varepsilon_r\delta\mathbf{E} \\ \delta\mathbf{k}\cdot\left(\mathbf{E}\times\mathbf{H}\right) - \delta\mathbf{E}\cdot\left(\mathbf{k}\times\mathbf{H}\right) &= \delta\omega\mu_0\mathbf{H}\cdot\mu_r\mathbf{H} + \mathbf{H}\cdot\omega\mu_0\mu_r\delta\mathbf{H} \end{aligned} \tag{22.63}$$

Tensors ε and μ are symmetric, i.e. $\varepsilon_{r,ij} = \varepsilon^*_{r,ji}$ and $\mu_{r,ij} = \mu^*_{r,ji}$ therefore (assuming these tensors are real

$$\begin{aligned} \mathbf{E}\cdot\omega\varepsilon_0\varepsilon_r\delta\mathbf{E} &= \delta\mathbf{E}\cdot\omega\varepsilon_0\varepsilon_r\mathbf{E} \\ \mathbf{H}\cdot\omega\mu_0\mu_r\delta\mathbf{H} &= \delta\mathbf{H}\cdot\omega\mu_0\mu_r\mathbf{H} \end{aligned} \tag{22.64}$$

and (22.63) becomes

$$\begin{aligned} -\delta\mathbf{k}\cdot\left(\mathbf{E}\times\mathbf{H}\right) - \delta\mathbf{H}\cdot\left(\mathbf{k}\times\mathbf{E}\right) &= -\delta\omega\varepsilon_0\mathbf{E}\cdot\varepsilon_r\mathbf{E} - \delta\mathbf{E}\cdot\omega\varepsilon_0\varepsilon_r\mathbf{E} \\ \delta\mathbf{k}\cdot\left(\mathbf{E}\times\mathbf{H}\right) - \delta\mathbf{E}\cdot\left(\mathbf{k}\times\mathbf{H}\right) &= \delta\omega\mu_0\mathbf{H}\cdot\mu_r\mathbf{H} + \delta\mathbf{H}\cdot\omega\mu_0\mu_r\mathbf{H} \end{aligned} \tag{22.65}$$

Now subtract the first equation from the second one and obtain

$$
\begin{aligned}
2\delta\mathbf{k}\cdot(\mathbf{E}\times\mathbf{H})-\delta\omega(\mu_0\mathbf{H}\cdot\mu_r\mathbf{H}+\varepsilon_0\mathbf{E}\cdot\varepsilon_r\mathbf{E})= \\
=\delta\mathbf{H}\cdot(\omega\mu_0\mu_r\mathbf{H}-\mathbf{k}\times\mathbf{E})+\delta\mathbf{E}(\omega\varepsilon_0\varepsilon_r\mathbf{E}+\mathbf{k}\times\mathbf{H})
\end{aligned}
\quad (22.66)
$$

The terms inside parentheses on the r.h.s. are zeroes according to Maxwell's equations (22.60). The first term in parenthesis on the l.h.s. is the Poynting vector and the term in the second parentheses is, according to (22.59) twice the energy density. Hence we are left with

$$\delta\mathbf{k}\cdot\mathbf{S}=\delta\omega U \quad (22.67)$$

or

$$\mathbf{S}=\frac{\delta\omega}{\delta\mathbf{k}}U\rightarrow\frac{\partial\omega}{\partial\mathbf{k}}U=\mathbf{v}_g U \quad (22.68)$$

Therefore, energy moves with the group velocity and ray vector is normal to the isofreqeuncy surface.

Double refraction at the boundary of anisotropic medium

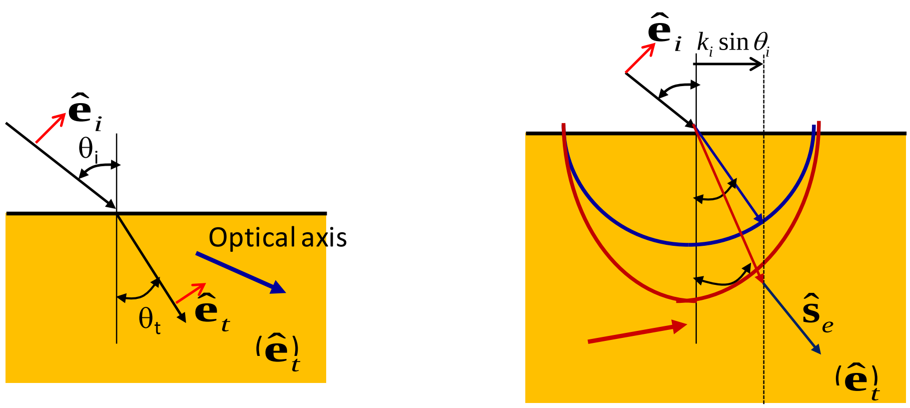

Figure 22.12 (a) General picture of refraction at the boundary with anisotropic medium (b) Finding directions of transmitted rays for uniaxial medium

One can write the expression for the Snell's law for anisotropic medium as shown in Fig.22.12(a) , but one should remember that effective refractive index in general depends on the direction of polarization $\hat{\mathbf{e}}$ hence

$$n_1\sin\theta_i=n_2\left[\hat{\mathbf{e}}_t(\theta_t)\right]\sin\theta_t \quad (22.69)$$

Index depends on polarization which depends on transmission angle that in its turn depends on index and so on resulting in nonlinear transcendental equation, which can only be solved iteratively. However, for uniaxial crystals it is easy to find graphic solution as shown in Fig.22.12 (b)

Expect two separate solutions –ordinary and extraordinary draw the intersections of normal surfaces with the plane of incidence according to (22.54). Snell's law can be written as

$$k_i\sin\theta_i=k_o\sin\theta_{to}=k_{ext}(\theta_{te})\sin\theta_{te} \quad (22.70)$$

Intersection of vertical line $k_x = k_i \sin\theta_i$ with normal surfaces yields the wavevectors of ordinary and extraordinary rays and then normal to the surface yields the direction of group velocity and energy flow – ray vector $\hat{\mathbf{s}}$.

The situation is more complicated when it comes to determining Fresnel coefficients because outside we have s and p waves and inside ordinary and extraordinary waves – two different systems of coordinates – hence there are no two independent solutions for reflectivity and transmission – solution is an elliptically polarized wave. But solutions can be separated for special cases, as shown in Fig.22.13

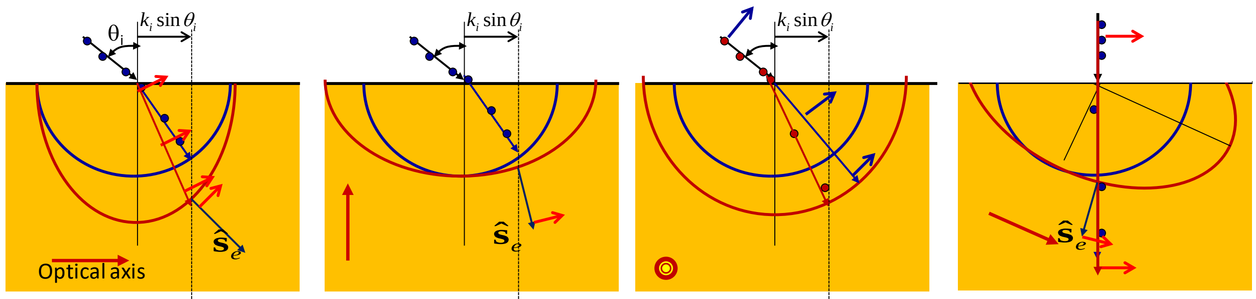

Figure 22.13 Special cases of refraction at interface with a uniaxial medium (a) Optical axis is in the interface plane and plane of incidence (b) Optical axis is normal to the interface (c) Optical axis is normal to the plane of incidence (d) Normal incidence

Case (a) Optical axis is in the interface plane and plane of incidence - s-wave outside becomes ordinary inside and p-wave outside becomes extraordinary inside. Notice that p wave is bent less (ray vector is what you see)

Case (b) Optical axis is normal to the interface plane and plane of incidence - s-wave outside becomes ordinary inside and p-wave outside becomes extraordinary inside. Notice that p wave is bent more (ray vector is what you see)

Case (c) Optical axis is normal to the plane of incidence - s-wave outside becomes ordinary inside and p-wave outside becomes extraordinary inside. There is no walk-off

Case (d) Normal incidence. Optical axis lies in the plane of figure. s-wave outside becomes ordinary inside and p-wave outside becomes extraordinary inside. Since angle of incidence is 0 both waves have their k-vectors along the normal. However, the ray vector of e-wave is directed away from the normal –double refraction –as shown in Fig. 22.14 for the case of calcite (Icelandic spar)

Figure 22.14 Double refraction in the calcite crystal

Polarized light and double refraction for navigation

Double refraction has been used 1000 years ago by Vikings to determine position of the sun on the cloudy day. Consider the sunlight propagating through the clouds and scattered by the water particles as shown in Fig. 22.15. Lght is a transverse wave, hence it is polarized normal to the direction of propagation. Light scattered forward keeps the polarization so it appears non-polarized to the observer. But the light scattered at 90 degrees is polarized along the vertical direction as horizontally polarized light would be the one initially polarized along the light ray, and such ray does not exist. One can use a polarization filter on the camera lens to improve contrast by dimming the sky why leaving other objects unaffected. The light scattered at other angles is partially polarized vertically.

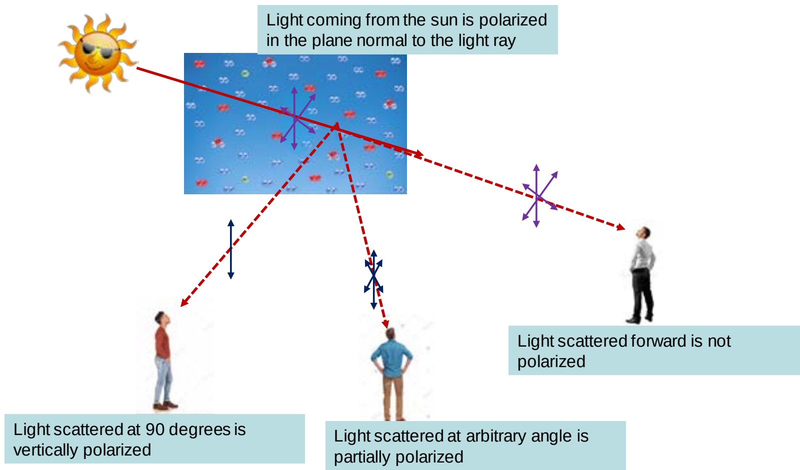

Figure 22.15 Polarization of the light scattered by atmosphere or clouds

Now, one can use a crystal of calcite (also called Icelandic Spar) to obtain a double image of some object , typically an aperture as shown in Fig.22.16 (a)

Figure 22.16 (a) double refraction in calcite (b) change of the relative brightness of two images with the angle (c) Replica of the Vikings' navigational device.

In general, because the scattered sunlight coming from the sky is partially polarized two images have different brightness, as shown in Fig.22.16 (b) but when the direction normal to the face of the crystal points to the sun, the incoming line is not polarized and two images have equal brightness (middle frame)