SECTION NOTES – WEEK 0

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Pick and choose what you want to cover. There is about 70 minutes of material here for a 50-minute discussion section (those with early discussions can reject/verify my estimate).

0.a. **News.**

(1) Congrats on signing up for EECS 376!
(2) First worksheet is up soon (or already!). If you have questions, the textbook is your friend. It is fantastic. Get it.
(3) Office hours and other such info should be posted soon, if they haven't been yet.
(4) Read the syllabus.
(5) Submit stuff in on time. We aren't going to be lenient on late submissions. No "computer died" or "I was at the airport" sort of stuff. You should be submitting several hours or a day in advance.

0.b. **Intro to Topics.** Writing proofs is difficult. The order in which statements are written in a proof often doesn't match the order in which you figure them out. For now, we would like to explain the rote aspects of a proof, those you should be able to write out regardless of whether you have a complete solution or not.

Note, failing in this will preclude you from solving most problems in this course — it truly it necessary that you understand this.

Additional documents have been posted to the class' Google Drive if you need more background. For instance, basics of handling sets are posted there, which we won't cover today. We will also be happy to help in office hours.

1. Basic Connectives

Let X be a set, and let P, Q be propositions with respect to elements of X.

Example: $X = \mathbb{N}$ and, given $x \in X$, P is the proposition that x is even.

We will not sketch out the "generic structure" of proofs of the sorts of claims we will see in this course.

Proposition 1.1. *$\forall x \in X$, P holds.*

Proof. Fix $x \in X$. Then, use the properties of x to show that P holds. □

Exercise 1. Let $X = \{n \in \mathbb{N} : 6|n\}$. Show that $\forall x \in X$, X is not prime.

Proof. Fix $x \in X$. Note that $2|6$, so $2|x$, but $x \neq 2$ since $6 \nmid 2$, so x is not prime. □

Proposition 1.2. *$\exists x \in X$ such that P holds.*

Proof. Give an explicit description for such an x, and show that

(1) x is indeed an element of X, and
(2) P holds.

□

We often describe existence proofs in two parts, as a description of a contruction, and as a proof of correctness (of the contruction).

Existence proofs appear frequently throughout the course. A number of definitions we use will make claims about the existence of some "associated object".

For instance, a *language* L is a set of strings over an alphabet. We say L is *regular* provided that there exists a DFA D such that $L = L(D)$. (definitions are expanded and clarified in the reading due Monday).

Exercise 2. Let X be the set of prime numbers. Show there is $x \in X$ such that x is even.

Now we consider implication, for which there are multiple approaches towards proofs.

Here only, suppose P, Q are *unquantified* propositions.

Proposition 1.3. *$P \Rightarrow Q$.*

Proof. (Direct proof) Suppose P holds, then show Q holds, using the structure given by the assumption of P. □

Proof. (Contrapositive) Suppose $\neg Q$. Then show $\neg P$. □

Note that we are proving $\neg Q \Rightarrow \neg P$, which is logical contrapositive of the original claim.

Proof. (Contradiction) Suppose P holds and $\neg Q$ holds. Then show that R and $\neg R$ holds, for some proposition R. □

Note that contradiction in some sense gives you the most assumptions to work with, but then it is often difficult to figure out what R should be. When $R = P$, it sometimes possible to simplify the proof to a contrapositive claim. This is obvious when the claim P isn't used at all in the main argument. Since clarity of writing is part of the grading, you should attempt to write clear, cogent arguments.

Exercise 3. Suppose $n \in \mathbb{N}$. Prove the following by contradiction:
If n^2 is odd, then n is odd.

Proof. Suppose n is not odd. Then n is even, so $n = 2k$ for some $k \in \mathbb{N}$. Then $2 \cdot 2k^2 = 2^2k^2 = (2k)^2 = n^2$, so n^2 is even, so n^2 is not odd. □

We also have conjunction and disjunction:

Proposition 1.4. *$P \wedge Q$.*

Proof. Need to show that both P and Q hold. □

Proposition 1.5. *$P \vee Q$.*

Proof. Need to show that either of P or Q holds. This can be done by showing $\neg P \Rightarrow Q$ or $\neg Q \Rightarrow P$. □

Now we consider negation (already mentioned above).

Proposition 1.6. *$\neg \forall x \in X, P$.*

Proof. This is logically equivalent to $\exists x \in X$ such that $\neg P$. □

Proposition 1.7. $\neg\exists x \in X, P$.

Proof. This is logically equivalent to $\forall x \in X, \neg P$. □

Proposition 1.8. $\neg(P \Rightarrow Q)$.

Proof. This is logically equivalent to $P \wedge \neg Q$. □

Finally, note that propositions compose according to the connectives.

Exercise 4. Let X, Y, Z, W be sets. Negate the following proposition:

$$\forall x \in X, \exists y \in Y, \forall z \in Z, \exists w \in W \text{such that} P$$

We apply negation rules by the outermost connective above, alternating quantifiers to get:

$$\exists x \in X, \forall y \in Y, \exists z \in Z, \forall w \in W \text{such that} \neg P$$

2. Proof Tools

2.a. **Pidgeonhole principle.** Let X, Y be (finite) sets.

Definition 2.1. A function $f : X \to Y$ is *injective* provided that $\forall x, x' \in X$, $f(x) = f(x') \Rightarrow x = x'$.

Then we have

Theorem 2.1. *If a function $f : X \to Y$ is injective, then $|X| \leq |Y|$.*

Exercise 5. Write out the logical contrapositive of the pidgeonhole principle.

If $|X| > |Y|$, then for any function $f : X \to Y$, f is not injective. That is, there are some $x, x' \in X$ with $x \neq x'$ so that $f(x) = f(x')$.

Exercise 6. Fix 5 distinct points on a sphere. Then there is a hemisphere (including its boundary) containing 4 points.

Proof. Draw a line through the first two points that divides the sphere into two hemispheres. The remaining three points must be in at least one of the hemispheres, so by the pidgeonhole principle, one of the hemispheres contains at least two of the remaining points. This hemisphere also contains the initial two points since they lie on its boundary, so this hemisphere contains 4 points. □

(If you have more time)

Exercise 7. Let $S \subset \{1, 2, \ldots, 20\}$ with $|S| = 11$. Then there exists $x, y \in S$ such that $x - y = 2$.

Proof. Consider the 10 sets $\{1, 3\}, \{5, 7\}, \{9, 11\}, \{13, 15\}, \{17, 19\}, \{2, 4\}, \{6, 8\}$, $\{10, 12\}, \{14, 16\}, \{18, 20\}$ which partition $\{1, 2, \ldots, 20\}$, by the pidgeonhole principle, there is some subset with 2 elements. Let x be the greater of the two elements of the subset, and y the smaller of the two. □

2.b. **Induction.** Suppose we have some claim of the form $\forall n \in \mathbb{N}, P$ holds. Then it is enough to show

(1) P holds when $n = 1$, and
(2) If P holds for some n, then then P holds for $n + 1$.

A proof of this sort is a proof by (weak) induction. If, for the second part, we suppose instead that P holds for all $m \leq n$ and show P holds for $n + 1$ we still prove the original claim. Tis called a proof by *(strong) induction*.

(If you have time to spare – do this exercise. Otherwise do the next one instead.)

Exercise 8. Let $G = (V, E)$ be a planar graph with n vertices, m edges, f faces, and c connected components. Then

$$n - c + f - m = 1.$$

Proof. We argue by induction on the number of edges of G. If $m = 0$, then G has n vertices for some $n \in \mathbb{N}$, 0 edges, 1 face, and n connected components.

Suppose the claim holds for some $m \in \mathbb{N}$ number of edges. Let $G = (V, E)$ be a graph with $m + 1$ edges. Pick $e \in E$. Let $G' = (V, E')$ with $E' = E \setminus \{e\}$. G' has n vertices, m edges, f faces, and c connected components. Moreover, by the induction hypothesis the desired equality holds for G'. Note that G has n vertices and $m + 1$ edges. The edge e either connects two disconnected components or not. If it connects two disconnected components, then the number of connected components in G is $c - 1$ and the number of faces is f. Otherwise, the number of connected components is c while e bisects some face, so G has $f + 1$ faces. In either case, equality holds. $\square$

(If you are short on time)

Exercise 9. Let $G = (V, E)$ be a tree. Then G has $|V| - 1$ edges.

Proof. We argue by induction on the number of vertices in G. The case $|V| = 1$ is trivial (the only tree has no edges).

Suppose the claim holds for all graphs with $|V| \leq n$ for some $n \in \mathbb{N}$. Fix $G = (V, E)$ a tree with $|V| = n + 1$. Note that E is non-empty or else G would be completely disconnected, contradiction that G is a tree. So, fix $e \in E$. Let $G' = (V, E')$ with $E' = E \setminus \{e\}$. Then G' is composed of two connected components with k, l vertices, respectively, which are trees. Note that $k + l = n + 1$ and $k, l \leq n$. By the induction hypothesis, the connected components have $k - 1, l - 1$ edges. Adding e back in, we have that G has $1 + k - 1 + l - 1 = 1 + n + 1 - 2 = n$ edges. $\square$

3. Tuples

Sometimes we have a set with some associate structure. For instance, a graph G is a pair of vertices V and edges E as we have seen above.

A weighted graph is a triple (V, E, f) where $f : E \to \mathbb{R}$ associates to each edge a weight and is called a *weight function*.

Similarly, we can color the vertices of a graph. Let C be a set of colors. Then $G = (V, E, f, g)$ where f is a weight function and $g : V \to C$ is a coloring function. For "pairs" with more than k components to the pair, for $k \geq 3$ we simply resign ourselves to calling the "pair" a *k-tuple*.

Here is another one, which you will encounter in the reading:

Definition 3.1. A deterministic finite automaton (DFA) is $D = (Q, \Sigma, \delta, q_0, F)$ with

(1) Q a finite set, which we call the *state set*,
(2) Σ a finite set of characters, which we call the *alphabet*,
(3) q_0 is the *start state*,
(4) $F \subset Q$ is called *accept states*

To the description of D there is an associated definition of execution.

4. Constructive Proofs

Exercise 10. Show there are a, b irrational numbers such that a^b is rational.

Proof. Either $\alpha := \sqrt{2}^{\sqrt{2}}$ is rational or not. If it is, we are done. Otherwise $\alpha^{\sqrt{2}} = \sqrt{2}^2 = 2$, so setting $a = \alpha, b = \sqrt{2}$ gives the desired result. □

Note this proof crucially used the *law of excluded middle*: for any proposition P, either P holds or $\neg P$ holds. Proofs of this sort are called non-constructive, because we didn't actually give an explicit way to make such a, b.