IST1990 Probability And Statistics

- Lecture 6

- Mean (Expected Value) and Variance for Discrete R.V.'s
- Mean (Expected Value) and Variance for Cont. R.V's
- Properties of Mean and Variance

MEAN AND VARIANCE OF A DISCRETE RANDOM VARIABLE

- There are two simple «summaries» of the distribution of a random variable X.
- The mean (expected value) is a measure of the center or middle of the probability distribution;
- and the variance is a measure of the dispersion, or variability in the distribution.

Definition

The **mean** or **expected value** of the discrete random variable X, denoted as μ or $E(X)$, is

$$\mu = E(X) = \sum_x x f(x)$$

The **variance** of X, denoted as σ^2 or $V(X)$, is

$$\sigma^2 = V(X) = E(X-\mu)^2 = \sum_x (x-\mu)^2 f(x) = \sum_x x^2 f(x) - \mu^2$$

The **standard deviation** of X is $\sigma = \sqrt{\sigma^2}$.

Note that: Standart deviation is the positive square root of variance.
Variance is always non-negative.

- The **mean** is a weighted average of the possible values of X, where the weights are their corresponding probabilities.

Therefore, the mean describes the «center» of the distribution. In other words, X takes values around its mean.

- The **variance** measures the dispersion of X around the mean.

A **small variance** indicates that the data points tend to be very close to the mean, and to each other.

A high **variance** indicates that the data points are very spread out from the mean, and from one another.

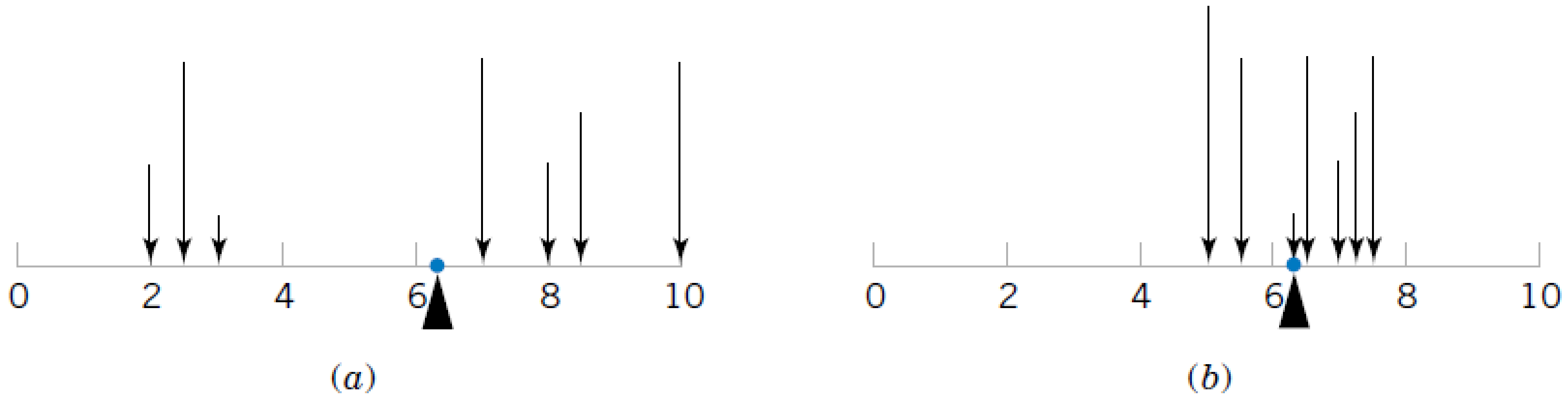


Figure. A probability distribution can be viewed as a loading with the mean equal to the balance point.
Parts (a) and (b) illustrate equal means, but Part (a) illustrates a larger variance.

Example

For the experiment of flipping three coins with the random variable number of heads, find the expected value, variance and standart deviation of X.

Let's first construct the **probability distribution** of the random variable X :

Number of Heads (x_i)	$f(x_i)=P(X=x_i)$
0	$\frac{\binom{3}{0}}{8} = 0.125$
1	$\frac{\binom{3}{1}}{8} = 0.375$
2	$\frac{\binom{3}{2}}{8} = 0.375$
3	$\frac{\binom{3}{3}}{8} = 0.125$

Then, the pmf of X is:

$$f(x) = \begin{cases} \dfrac{\binom{3}{x}}{2^3} & x = 0,1,2,3 \\ 0 & \text{otherwise} \end{cases}$$

1th way:

X=x	f(x)	x.f(x)	[x-E(X)]²	[x-E(X)]².f(x)
0	1/8	0	$(0-3/2)^2=9/4$	9/32
1	3/8	3/8	$(1-3/2)^2=1/4$	3/32
2	3/8	6/8	$(2-3/2)^2=1/4$	3/32
3	1/8	3/8	$(3-3/2)^2=9/4$	9/32
E(X)=12/8=3/2			$\sigma^2x=24/32=3/4$	

$$Var(X)=\sigma^2=E[(X-\mu)^2]=\sum_{i=0}^{3}(x_i-E(X))^2.f(x_i)=\frac{24}{32}=\frac{3}{4}$$

$$\sigma_X=\sqrt{\frac{3}{4}}=\frac{\sqrt{3}}{2}$$

2nd way:

X=x	f(x)=P(X=x)	x.f(x)	x^2.f(x)
0	1/8	0	0^2.1/8=0
1	3/8	3/8	1^2.3/8=3/8
2	3/8	6/8	2^2.3/8=12/8
3	1/8	3/8	3^2.1/8=9/8
		E(X)=3/2	E(X^2)=24/8=3

$$Var(X) = \sigma^2 = E[(X-\mu)^2] = \sum_{i=0}^{3} x_i^2 f(x_i) - [E(X)]^2 = 3 - \left(\frac{3}{2}\right)^2 = \frac{3}{4}$$

$$\sigma_X = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$

Example – Rolling 2 Dice

Suppose that the random variable Y denotes the sum of the up faces of the two dice. Table gives value of y for all elements in sample space S. Totally, we have 36 elements in S:

$1^{st}\backslash 2^{nd}$	**1**	**2**	**3**	**4**	**5**	**6**
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

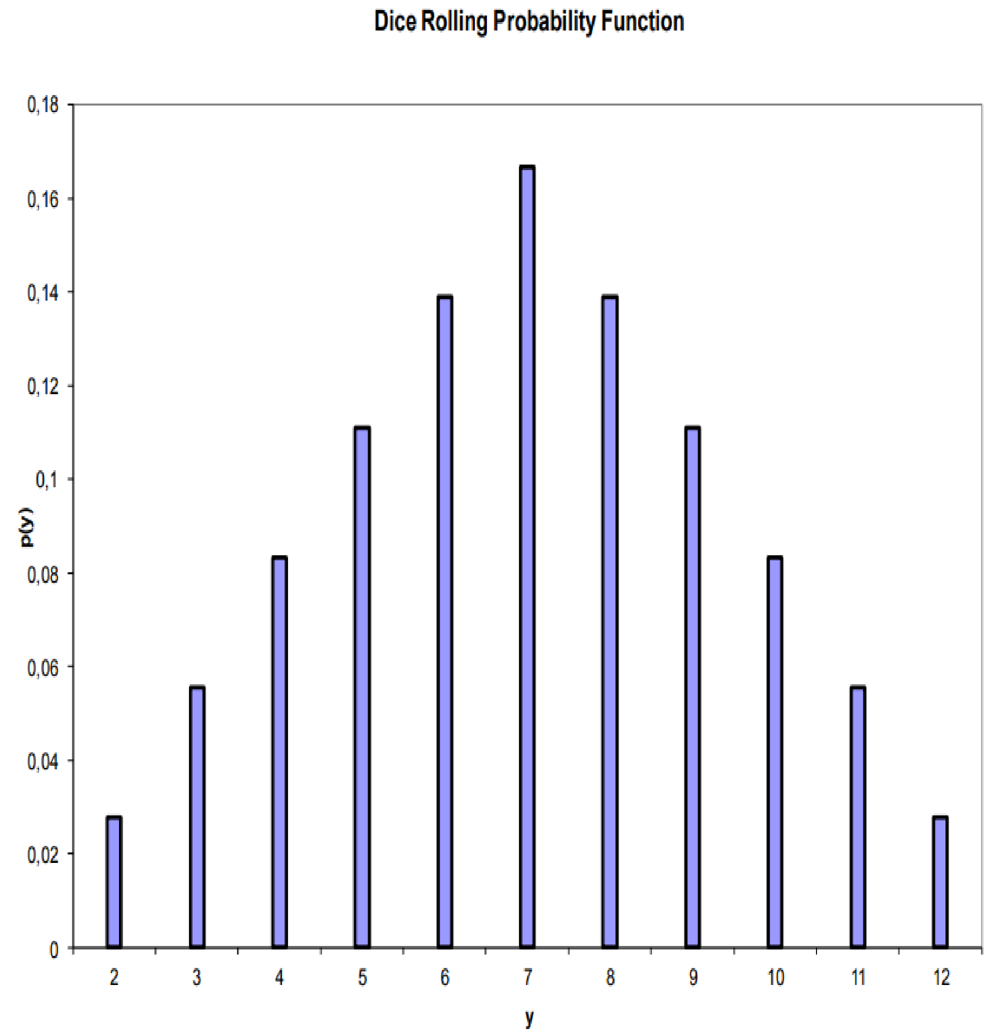
PMF and CDF

y	p(y)	F(y)
2	1/36	1/36
3	2/36	3/36
4	3/36	6/36
5	4/36	10/36
6	5/36	15/36
7	6/36	21/36
8	5/36	26/36
9	4/36	30/36
10	3/36	33/36
11	2/36	35/36
12	1/36	36/36

$$p(y) = \frac{\#\text{ of ways 2 die can sum to } y}{\#\text{ of ways 2 die can result in}}$$

$$F(y) = \sum_{t=2}^{y} p(t)$$

PMF-Graph



y	p(y)	yp(y)	y^2p(y)
2	1/36	2/36	4/36
3	2/36	6/36	18/36
4	3/36	12/36	48/36
5	4/36	20/36	100/36
6	5/36	30/36	180/36
7	6/36	42/36	294/36
8	5/36	40/36	320/36
9	4/36	36/36	324/36
10	3/36	30/36	300/36
11	2/36	22/36	242/36
12	1/36	12/36	144/36
Sum	36/36= 1.00	252/36 =7.00	1974/36=5 4.833

$$\mu = E(Y) = \sum_{y=2}^{12} yp(y) = 7.0$$

$$\sigma^2 = E[(Y-\mu)^2] = \sum_{y=2}^{12} y^2 p(y) - \mu^2$$

$$= 54.8333 - (7.0)^2 = 5.8333$$

$$\sigma = \sqrt{5.8333} = 2.4152$$

Example

The number of messages sent per hour over a computer network has the following distribution:

x = number of messages	10	11	12	13	14	15
$f(x)$	0.08	0.15	0.30	0.20	0.20	0.07

Determine the mean and standard deviation of the number of messages sent per hour.

$$E(X) = 10(0.08) + 11(0.15) + \cdots + 15(0.07) = 12.5$$

$$V(X) = 10^2(0.08) + 11^2(0.15) + \cdots + 15^2(0.07) - 12.5^2 = 1.85$$

$$\sigma = \sqrt{V(X)} = \sqrt{1.85} = 1.36$$

The variance of a random variable X can be considered to be the expected value of a specific function of X, namely, $h(X) = (X - \mu)^2$. In general, the expected value of any function $h(X)$ of a discrete random variable is defined in a similar manner.

Functions of Random Variables

Functions of random variables are ALSO random variables.

Let X be a random variable, and let $h : \mathbb{R} \to \mathbb{R}$ be an arbitrary function. Then $Y = h(X)$ is also a random variable.

Example: Let $X \in \{0, 1, 2\}$ be a random variable with p.m.f.

$$f(x) = P(X = x) = \begin{cases} 0.9 & \text{if } x = 0 \\ 0.08 & \text{if } x = 1 \\ 0.02 & \text{if } x = 2 \end{cases}$$

Let $Y = (X-1)^2$ be another random variable. What is the p.m.f. of Y? What is the expected value of Y?

If $X = 0, 2$ then $Y = 1$. If $X = 1$ then $Y = 0$. Therefore $Y \in \{0, 1\}$ and

$$P(Y = 0) = 0.08 \qquad \text{and} \qquad P(Y = 1) = 0.92 \; .$$

Therefore $\mathbb{E}[Y] = 0 \cdot 0.08 + 1 \cdot 0.92 = 0.92$.

There is an easier way to get this result…

Expected Value of a Function of a Discrete Random Variable

Proposition:
Let X be a random variable, and let $h : \mathbb{R} \to \mathbb{R}$ be an arbitrary function. Then $Y = h(X)$ is also a random variable. If X is discrete and takes values $\{x_1, x_2, \ldots\}$ then

$$\mathbb{E}[Y] = \mathbb{E}[h(X)] = \sum_i h(x_i) f(x_i)$$

Example: Let $X \in \{0, 1, 2\}$ be a random variable with p.m.f.

$$f(x) = P(X = x) = \begin{cases} 0.9 & \text{if } x = 0 \\ 0.08 & \text{if } x = 1 \\ 0.02 & \text{if } x = 2 \end{cases}$$

Let $Y = (X - 1)^2$ be another random variable. What is the expected value of Y?

$$\mathbb{E}[Y] = (0-1)^2 \cdot 0.9 + (1-1)^2 \cdot 0.08 + (2-1)^2 \cdot 0.02 = 0.92.$$

MEAN AND VARIANCE OF A CONTINUOUS RANDOM VARIABLE

The mean and variance of a continuous random variable are defined similarly to a discrete random variable. Integration replaces summation in the definitions.

Definition

Suppose X is a continuous random variable with probability density function $f(x)$. The **mean** or **expected value** of X, denoted as μ or $E(X)$, is

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\, dx \tag{4-4}$$

The **variance** of X, denoted as $V(X)$ or σ^2, is

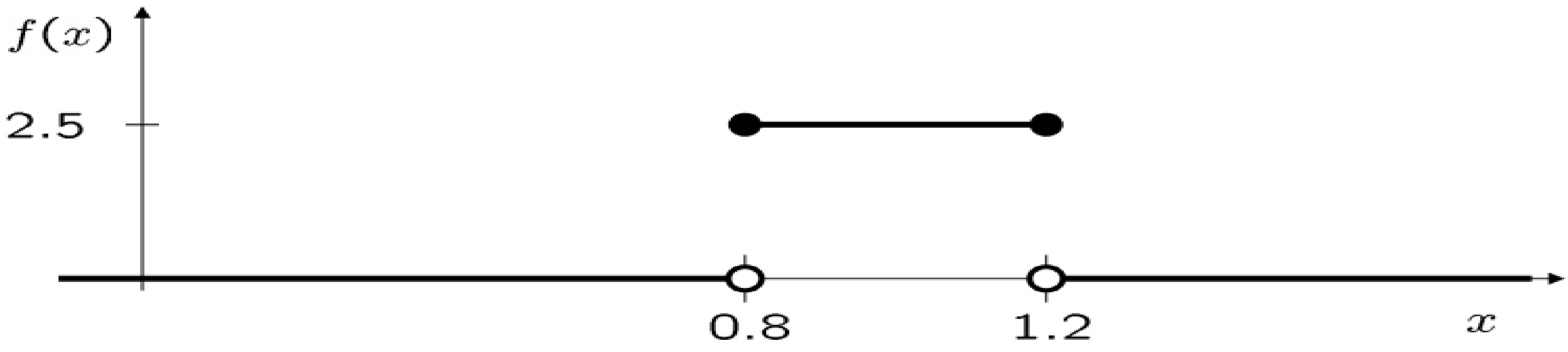
$$\sigma^2 = V(X) = \int_{-\infty}^{\infty} (x - \mu)^2 f(x)\, dx = \int_{-\infty}^{\infty} x^2 f(x)\, dx - \mu^2$$

The **standard deviation** of X is $\sigma = \sqrt{\sigma^2}$.

Example

Suppose you want to model the clock frequency (in GHz) of a certain mobile device processor. This is a random quantity, and in some cases it can be well modeled by a continuous random variable with density

$$f(x) = \begin{cases} 2.5 & \text{if } 0.8 \leq x \leq 1.2 \\ 0 & \text{otherwise} \end{cases}$$



Let's study this random variable in more detail...

Example cont.

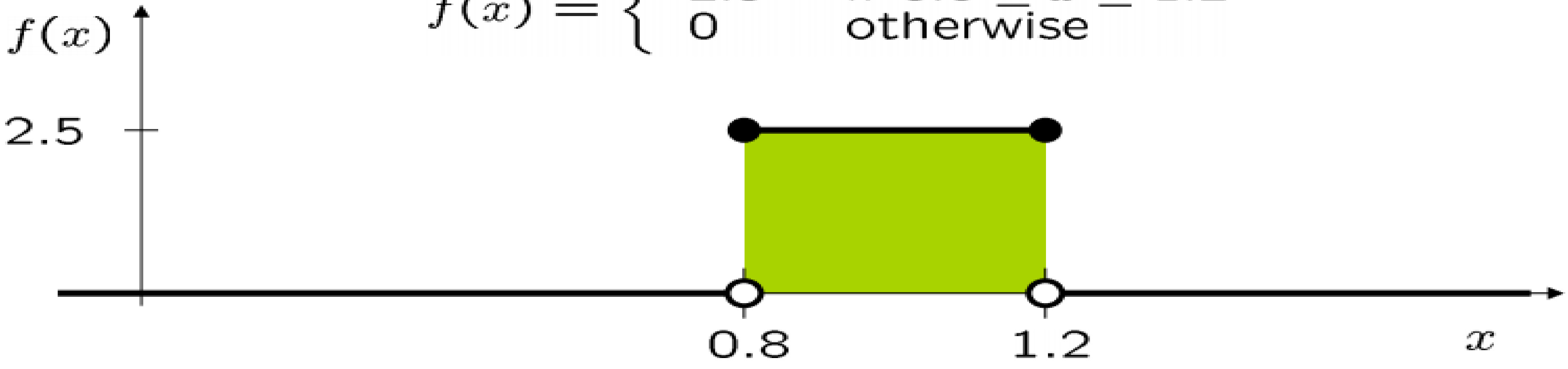
Let's first check this is a valid probability density function:

A valid **probability density function** $f(x)$ must satisfy

(i) $f(x) \geq 0$ for all $x \in \mathbb{R}$ ✔

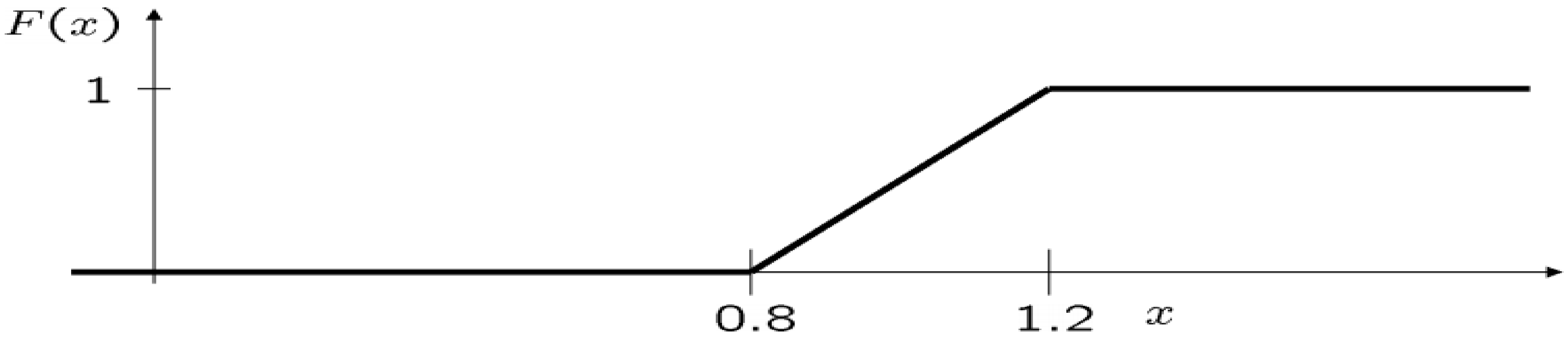
(ii) $\int_{-\infty}^{\infty} f(x)dx = 1$ ✔

$$f(x) = \begin{cases} 2.5 & \text{if } 0.8 \leq x \leq 1.2 \\ 0 & \text{otherwise} \end{cases}$$



Example cont.

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)dt = \begin{cases} 0 & \text{if } x < 0.8 \\ 2.5(x - 0.8) & \text{if } 0.8 \le x < 1.2 \\ 1 & \text{if } x \ge 1.2 \end{cases}$$



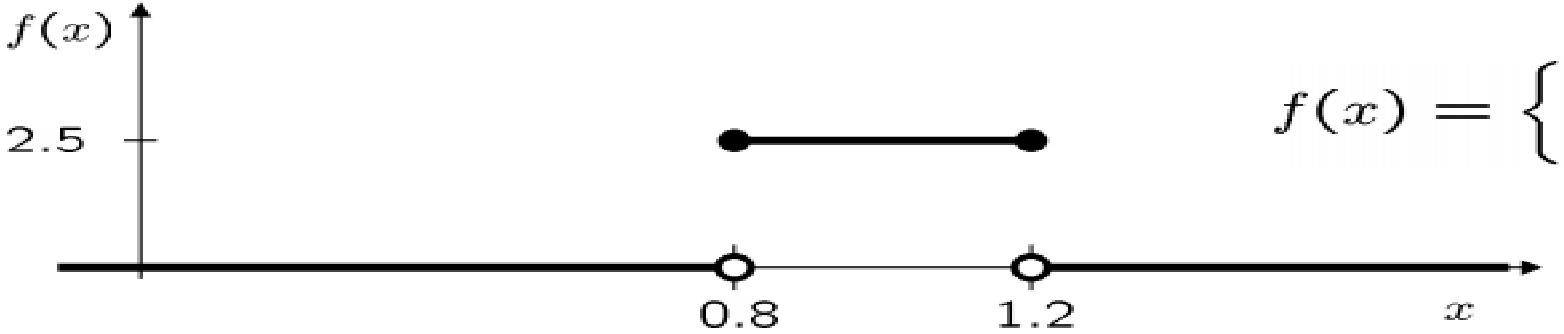
$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f(x)dx = \int_{0.8}^{1.2} 2.5x dx = \cdots = 1 \quad \text{(GHz)}$$

$$\mathsf{V}[X] = \int_{-\infty}^{\infty} (x - \mu_X)^2 f(x)dx = \int_{0.8}^{1.2} 2.5(x-1)^2 dx = \cdots = 0.01333(3)\ldots$$

$$\sqrt{\mathsf{V}[X]} = \sigma \approx 0.1155 \text{ (GHz)}$$

Example

Suppose you replace the old processor with a better one, with the same average speed, but much more steady...

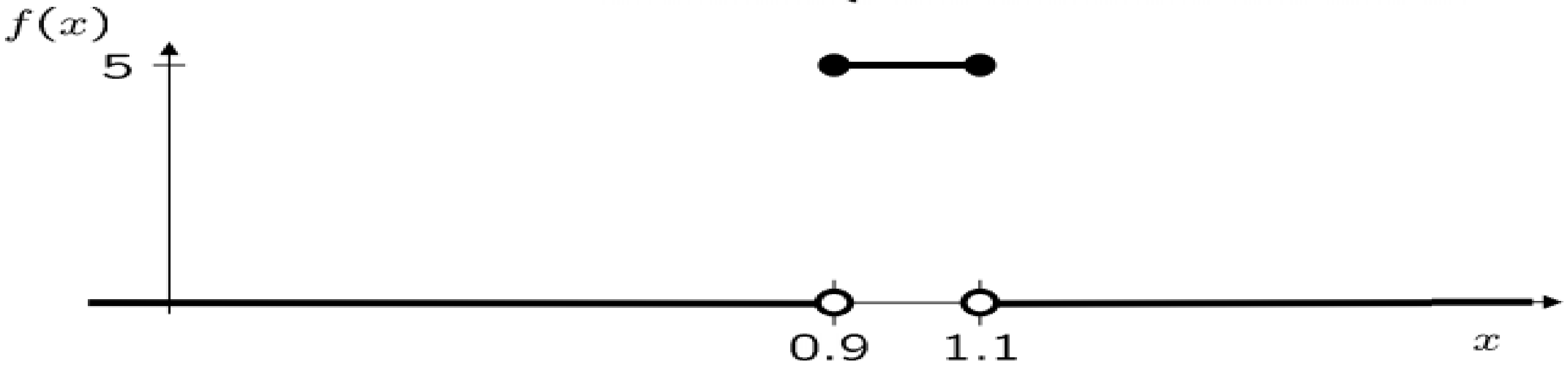


$$f(x) = \begin{cases} 2.5 & \text{if } 0.8 \leq x \leq 1.2 \\ 0 & \text{otherwise} \end{cases}$$

Before

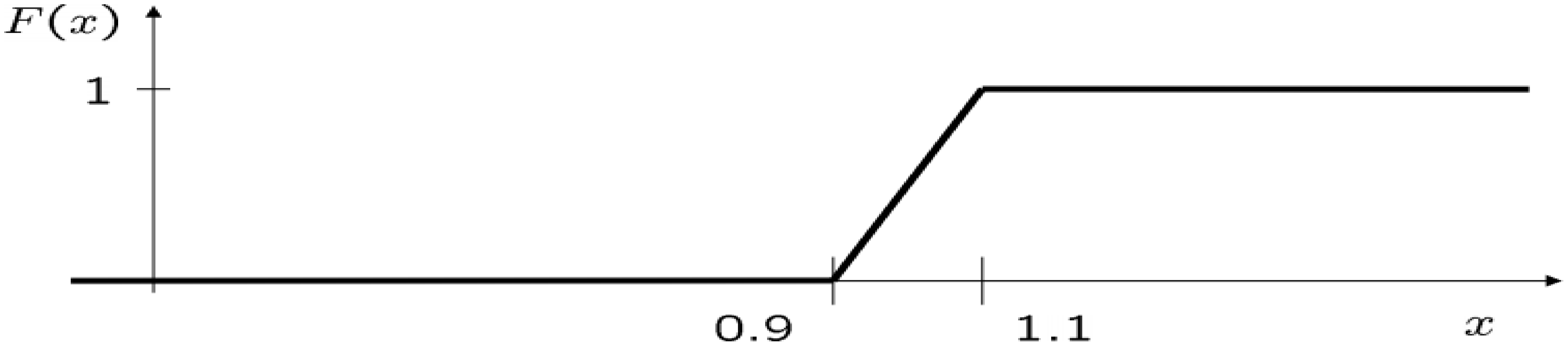
After

$$f(x) = \begin{cases} 5 & \text{if } 0.9 \leq x \leq 1.1 \\ 0 & \text{otherwise} \end{cases}$$



Example

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)dt = \begin{cases} 0 & \text{if } x < 0.9 \\ 5(x-0.9) & \text{if } 0.9 \le x < 1.1 \\ 1 & \text{if } x \ge 1.1 \end{cases}$$



$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f(x) dx = \int_{0.9}^{1.1} 5x dx = \cdots = 1 \quad \text{(GHz)}$$

$$\mathsf{V}[X] = \int_{-\infty}^{\infty} (x-\mu_X)^2 f(x) dx = \int_{0.9}^{1.1} 5(x-1)^2 dx = \cdots = 0.00333(3)\ldots$$

$$\sqrt{\mathsf{V}[X]} = \sigma \approx 0.0577 \text{ (GHz)}$$

Expected Value of a Function of a Continuous Random Variable

Functions of Random Variables

Let X be a random variable, and let $h : \mathbb{R} \to \mathbb{R}$ be an arbitrary function. Then $Y = h(X)$ is also a random variable.

Proposition: (law of the unconscious statistician) If X is a continuous random variable then

$$\mathbb{E}[Y] = \mathbb{E}[h(X)] = \int_{-\infty}^{\infty} h(x)f(x)dx$$

Note that the variance is the expected value of the random variable $Y = (X - \mu_X)^2$ (in this case $h(\cdot) = (\cdot - \mu_X)^2$):

$$\mathbb{E}[Y] = \mathbb{E}[h(X)] = \int_{-\infty}^{\infty} h(x)f(x)dx = \int_{-\infty}^{\infty} (x - \mu_X)^2 f(x)dx$$

Properties of the Mean and Variance

Properties:

Let X be a random variable. Let $a, b \in \mathbb{R}$. Then

(i) $\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$

(ii) $\mathrm{V}(X) = \mathbb{E}[(X - \mu_X)^2] = \mathbb{E}[X^2] - \mu_X^2$

(iii) $\mathrm{V}(aX + b) = a^2\ V(X)$

(iv) $\sqrt{\mathrm{V}(aX + b)} = |a|\sqrt{V(X)}$

(v)

If $X_1, \ldots, X_n$ random variables then

$$\mathbb{E}[X_1 + \cdots + X_n] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n]\ .$$

(vi)

Furthermore, if these are jointly independent then

$$\mathbb{E}[X_1 \times \cdots \times X_n] = \mathbb{E}[X_1] \times \cdots \times \mathbb{E}[X_n]$$

$$\mathbb{V}[X_1 + \cdots + X_n] = \mathbb{V}[X_1] + \cdots + \mathbb{V}[X_n]\ .$$

Note: The properties of mean and variance given in (i) – (vi) are for both discrete and continuous random variables.

WARNING!!!

$$E[X^2] \neq (E[X])^2 \text{ or } E[\sqrt{X}] \neq \sqrt{E[X]}$$

Independence of Random Variables

Earlier we talked about independent events. Actually the concept of independence extends naturally to random variables, and plays a crucial role both in probability and statistics.

Definition: Independence of Multiple Random Variables

Let $X_1, X_2, \ldots, X_n$ be denote n random variables.

We say $X_1, X_2, \ldots, X_n$ are **jointly independent** if for **any** sets $A_1, A_2, \ldots, A_n$ we have

$$\begin{aligned} & P(X_1 \in A_1, X_2 \in A_2, \ldots, X_n \in A_n) \\ = \quad & P(X_1 \in A_1) P(X_2 \in A_2) \cdots P(X_n \in A_n) \end{aligned}$$

Example

Suppose that you are playing a machine game. The machine gives a prize of €100 with probability 0.02. To play you need to pay €5..

Let X be a random variable taking values $\{0, 1\}$, where 1 indicates the machine gave a prize. Therefore $P(X = 1) = 0.02$ and $P(X = 0) = 0.98$.

Let Y be your profit after playing the game. Clearly

$$Y = 100X - 5$$

What is the mean and variance of the profit?

$$\mathbb{E}[X] = P(X = 1) = 0.02$$

$$\begin{aligned} \mathrm{V}(X) &= E[X^2] - (E[X])^2 \\ &= P(X = 1) - (P(X = 1))^2 \\ &= P(X = 1)(1 - P(X = 1)) = 0.0196 \end{aligned}$$

$$\Longrightarrow$$

$$\mathbb{E}[Y] = 100E[X] - 5 = -3€$$

$$\mathrm{V}(Y) = 100^2 * 0.0196 = 196$$

$$\sigma = \sqrt{\mathrm{V}(Y)} = 14€$$

Example

Let X be a discrete random variable with the following probability distribution:

X	-1	0	2
f(x)	1/6	1/3	1/2

$$E(X) = (-1)\left(\frac{1}{6}\right) + (0)\left(\frac{1}{3}\right) + (2)\left(\frac{1}{2}\right) = \frac{5}{6}$$

$$E(X^2) = (-1)^2\left(\frac{1}{6}\right) + (0)^2\left(\frac{1}{3}\right) + (2)^2\left(\frac{1}{2}\right) = \frac{13}{6}$$

$$E(3X-2) = 3E(X) - 2 = 3\left(\frac{5}{6}\right) - 2 = \frac{1}{2}$$

$$E(-X+4) = -E(X) + 4 = -\left(\frac{5}{6}\right) + 4 = \frac{19}{6}$$

$$E((X-1)^2) = E(X^2 - 2X + 1) = E(X^2) - 2E(X) + 1 = \frac{13}{6} - 2 * \frac{5}{6} + 1 = \frac{3}{2}$$

$$Var(X) = E(X^2) - E(X)^2 = \frac{13}{6} - \left(\frac{5}{6}\right)^2 = \frac{53}{36}$$

$$Var(2X+1) = (2)^2 Var(X) = 4 * \frac{53}{36} = \frac{53}{9}$$

$$Var(-3X+1) = (-3)^2 Var(X) = 9 * \frac{53}{36} = \frac{53}{4}$$

Example: Let X be a random variable with density function

$$f(x) = \begin{cases} \frac{x^2}{3}, & -1 < x < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

Find E(X) and E(4X + 3).

$$E(X) = \int_{-\infty}^{\infty} x\, f(x) dx = \int_{-1}^{2} x\, \frac{x^2}{3} dx = \int_{-1}^{2} \frac{x^3}{3} dx = \frac{x^4}{12} \Big|_{-1}^{2} = \frac{5}{4}$$

$$E(4X+3) = \int_{-1}^{2} \frac{(4x+3)x^2}{3}\, dx = \frac{1}{3}\int_{-1}^{2} (4x^3 + 3x^2)\, dx = 8.$$

or

$$E(4X+3) = 4\,E(X) + 3 = 4\left(\frac{5}{4}\right) + 3 = 8$$

Example: The weekly demand for a drinking-water product, in thousands of liters, from a local chain of efficiency stores is a continuous random variable X having the probability density

$$f(x) = \begin{cases} 2(x-1), & 1 < x < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the mean and variance of X.

$\mathrm{V}(X) = \mathbb{E}[(X-\mu_X)^2] = \mathbb{E}[X^2] - \mu_X^2$

$$\mu = E(X) = 2\int_1^2 x(x-1)\, dx = \frac{5}{3}$$

and

$$E(X^2) = 2\int_1^2 x^2(x-1)\, dx = \frac{17}{6}.$$

Therefore,

$$\sigma^2 = \frac{17}{6} - \left(\frac{5}{3}\right)^2 = \frac{1}{18}.$$

Example

Let X be a random variable with following pdf:

$$f(x) = \begin{cases} \frac{4}{3}(1-x^3) & 0 \leq x \leq 1 \\ 0 & diger \end{cases}$$

a) Find variance and standart deviation of X.

b) Var(2X+3)=?

a) $$E(X) = \frac{4}{3}\int_0^1 x(1-x^3)dx = \frac{4}{3}\left(\frac{x^2}{2} - \frac{x^5}{5}\right)\Bigg|_0^1 = \frac{2}{5}$$

$$E(X^2) = \frac{4}{3}\int_0^1 x^2(1-x^3)dx = \frac{4}{3}\left(\frac{x^3}{3} - \frac{x^6}{6}\right)\Bigg|_0^1 = \frac{2}{9}$$

$$Var(X) = E(X^2) - [E(X)]^2 = \frac{2}{9} - \left(\frac{2}{5}\right)^2 = \frac{14}{225} = 0,062$$

$$\sigma = \sqrt{Var(x)} \Rightarrow \sigma = \sqrt{0,06} \Rightarrow \sigma = 0,245$$

b) $$Var(2X+3) = 4Var(X) = 4 \times 0.062 = 0.248$$