Design via Root Locus

(Textbook - Nise's, Ch.9)

Introduction

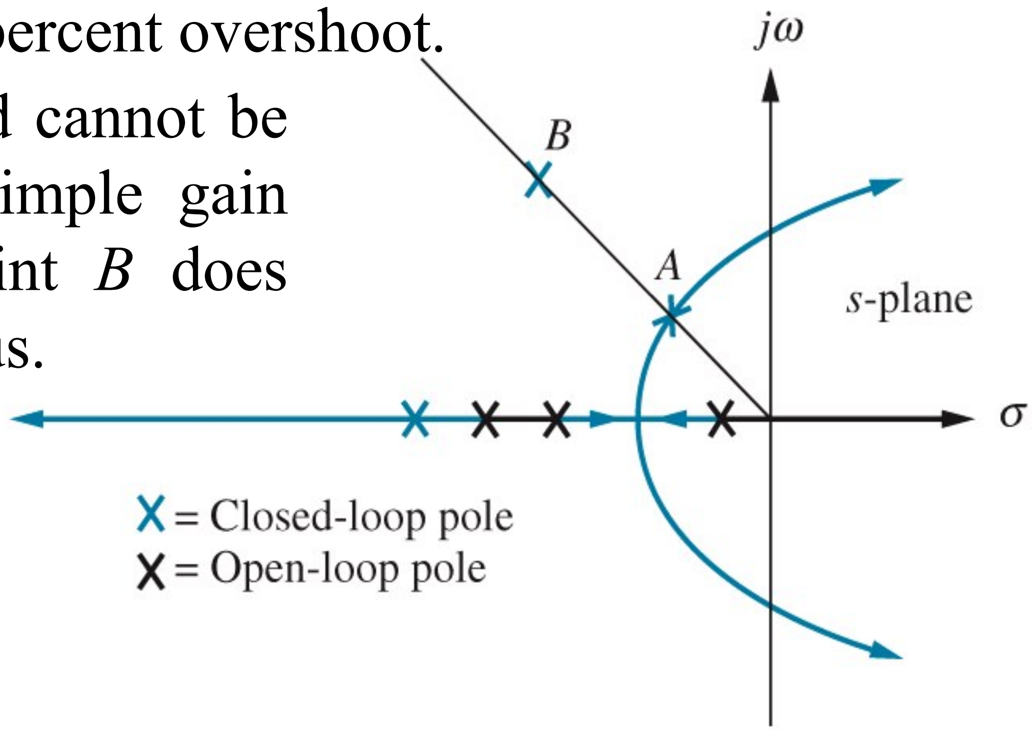
- The root locus typically allows us to choose the proper loop gain to meet a transient response specification.

- As the gain is varied, we move through different regions of response. Setting the gain at a particular value yields the transient response dictated by the poles at that point on the root locus.

- Thus, we are limited to those responses that exist along the root locus.

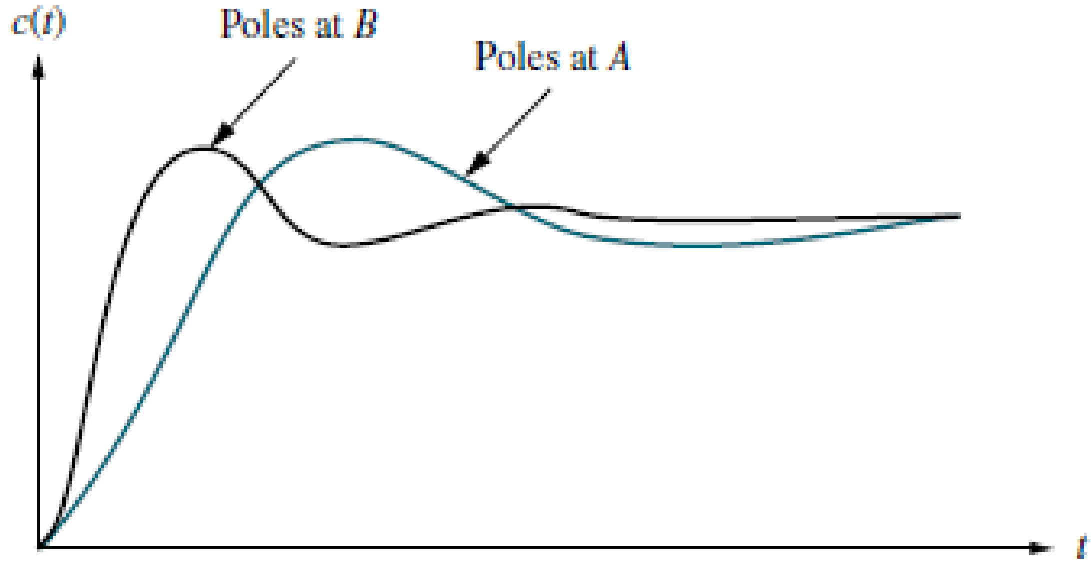
Introduction…

- Assume that the desired transient response, defined by percent overshoot and settling time, is represented by point *B*.
- Unfortunately, on the current root locus at the specified percent overshoot, we only can obtain the settling time represented by point *A* after a simple gain adjustment.
- Thus, our goal is to speed up the response at *A* to that of *B*, without affecting the percent overshoot.
- This increase in speed cannot be accomplished by a simple gain adjustment, since point *B* does not lie on the root locus.



Introduction…

- The figure illustrates the improvement in the transient response we seek: The faster response has the same percent overshoot as the slower response.



- One way to solve our problem is to augment, or compensate, the system with **additional poles and zeros**, so that the compensated system has a root locus that goes through the desired pole location for some value of gain.
- A possible disadvantage of compensating a system with additional open-loop poles and zeros is that the **system order can increase, with a subsequent effect on the desired response**.

Improving Steady-State Error

- Compensators are not only used to **improve the transient** response of a system; they are also used independently to **improve the steady-state error** characteristics.
- Previously, when the system gain was adjusted to meet the transient response specification, steady-state error performance deteriorated, since both the transient response and the static error constant were related to the gain. **The higher the gain, the smaller the steady-state error, but the larger the percent overshoot**.
- On the other hand, reducing gain to reduce overshoot increased the steady-state error. If we use **dynamic compensators**, compensating networks can be designed that will allow us to meet transient and steady-state error specifications simultaneously.
- In summary, **transient response is improved with the addition of differentiation, and steady-state error is improved with the addition of integration in the forward path**.

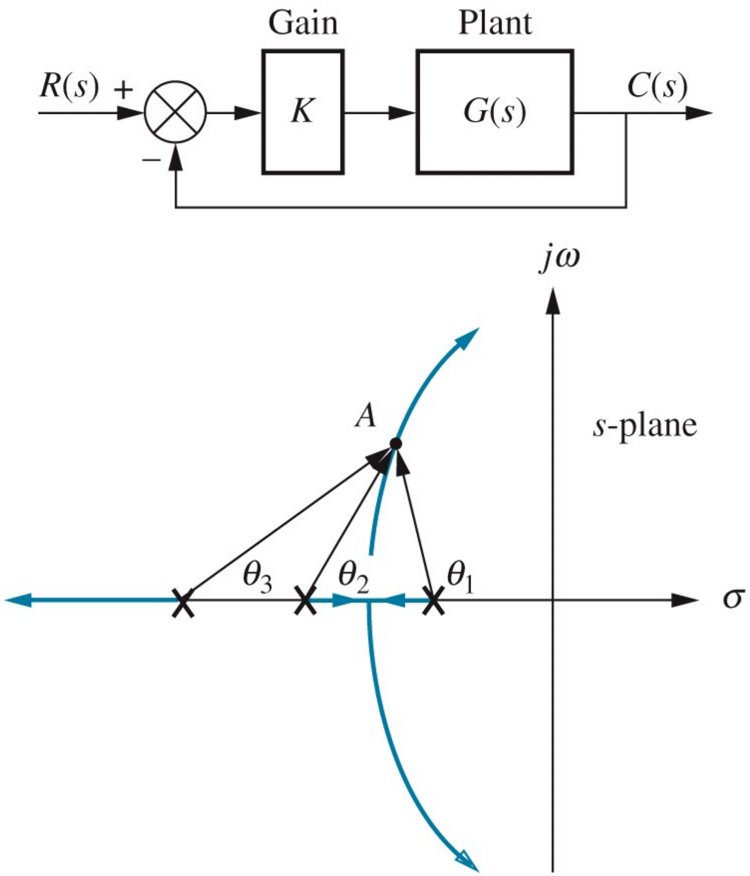
Ideal Integral Compensation (PI)

- Steady-state error can be improved by placing an open-loop pole at the origin, because this increases the system type by one. For example, a Type 0 system responding to a step input with a finite error responds with zero error if the system type is increased by one.

- Here we have a system operating with a desirable transient response generated by the closed-loop poles at *A*. If we add a pole at the origin to increase the system type, the angular contribution of the open-loop poles at point *A* is no longer 180^{o}, and the root locus no longer goes through point *A*, as shown in the figure presented on the next slide.

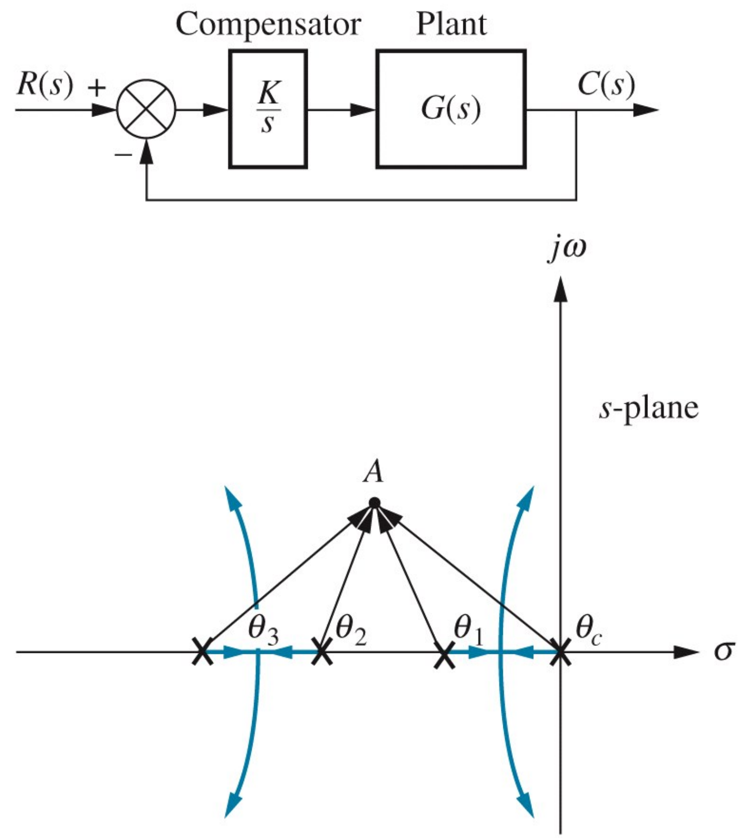
Ideal Integral Compensation (PI)…

e_{ss} becomes zero by placing an open-loop pole at the origin, however in this case we cannot get the same transient characteristics. Since the root locus changes drastically, see **(*a*)** and **(*b*)**.



$$-\theta_1 - \theta_2 - \theta_3 = (2k+1)180^\circ$$

(*a*)

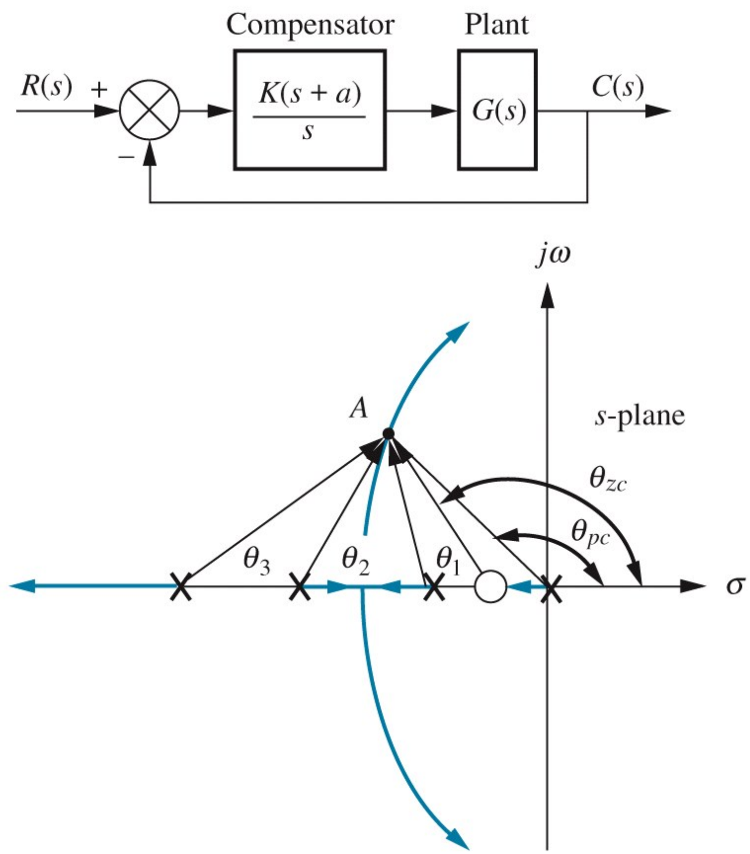


$$-\theta_1 - \theta_2 - \theta_3 - \theta_c \neq (2k+1)180^\circ$$

(*b*)

Ideal Integral Compensation (PI)…

- To solve the problem, we also add a zero close to the pole at the origin, as shown in **(*c*)**. Now the angular contribution of the compensator zero and compensator pole cancel out, point *A* is still on the root locus, and the system type has been increased.
- The required gain at the dominant pole is about the same as before compensation, since the ratio of lengths from the compensator pole and the compensator zero is approximately unity.
- We have improved the s-s error without appreciably affecting the transient response. **A compensator with a pole at the origin and a zero close to the pole is called an** **ideal integral compensator**.

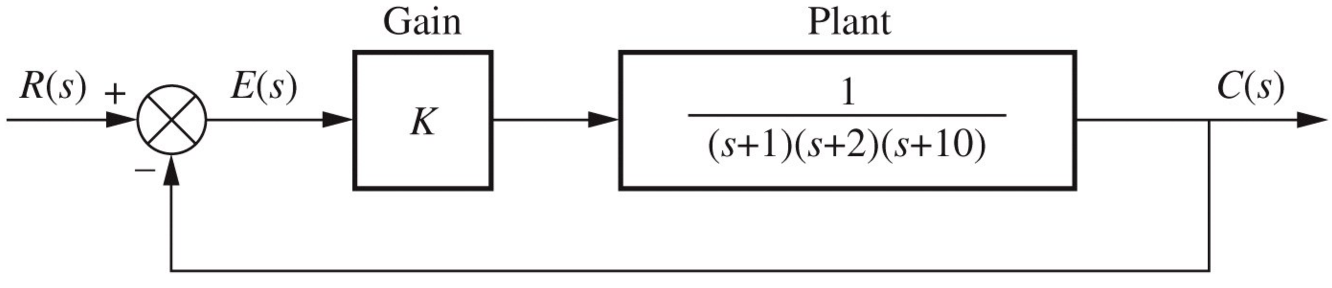


$$-\theta_1 - \theta_2 - \theta_3 - \theta_{pc} + \theta_{zc} \cong (2k+1)180^\circ$$

(*c*)

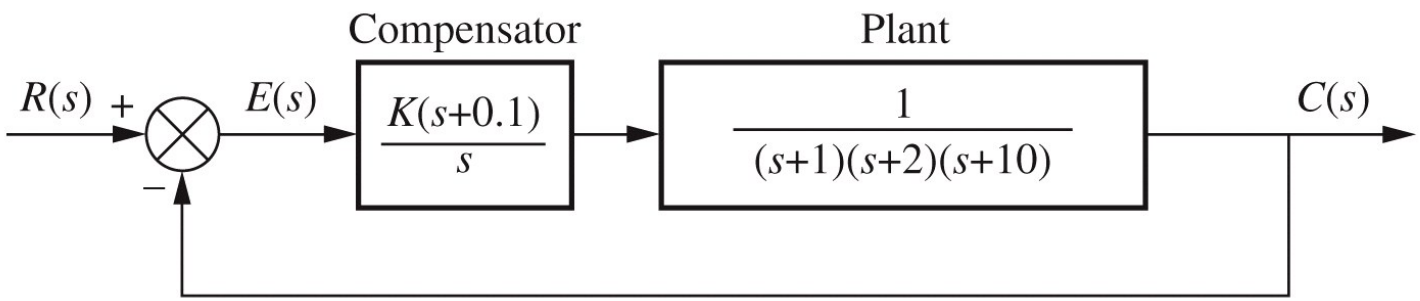
Example

Given the system, operating with a damping ratio of 0.174, show that the addition of the ideal integral compensator shown in **(*a*)** reduces the steady-state error to zero for a step input without appreciably affecting transient response.



(*a*)

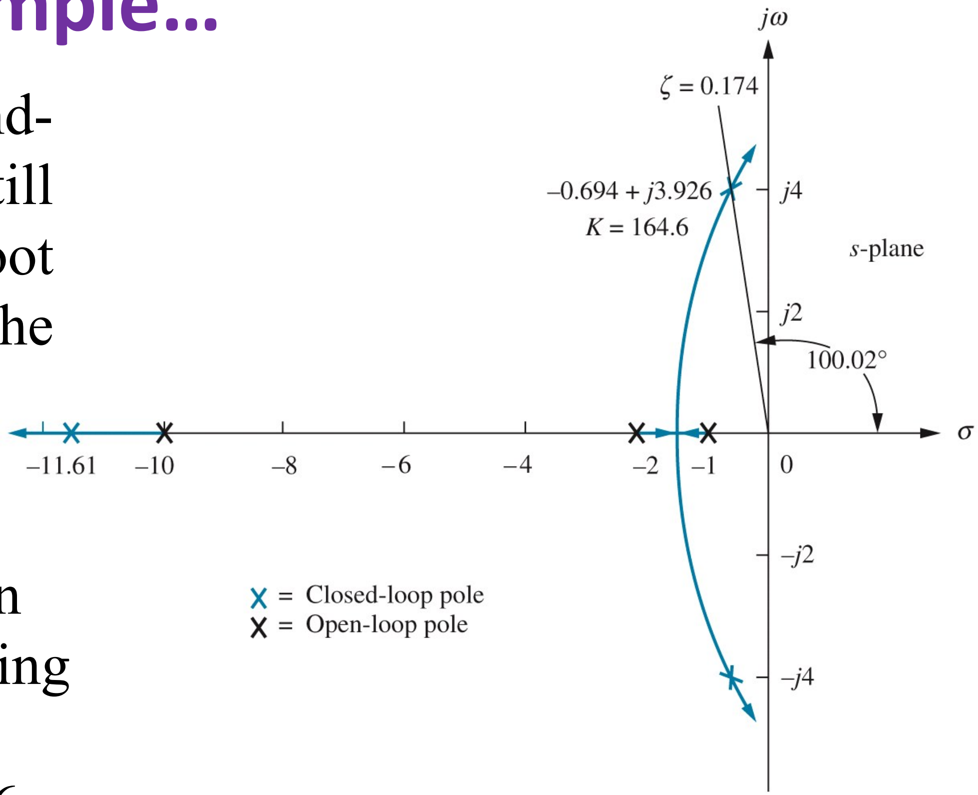
Solution: The compensating network is chosen with a pole at the origin to increase the system type and a zero at -0.1, close to the compensator pole, so that the angular contribution of the compensator evaluated at the original, dominant, 2nd poles is approximately zero, see (***b***).



(*b*)

Example...

- The original, dominant, second-order closed-loop poles are still approximately on the new root locus. The root locus for the uncompensated system is shown.



- A damping ratio of 0.174 is represented by a radial line drawn on the s-plane at 100.02°. Searching along this line, we find that the dominant poles are $-0.694 \pm j3.926$ for a gain, K, of 164.6.

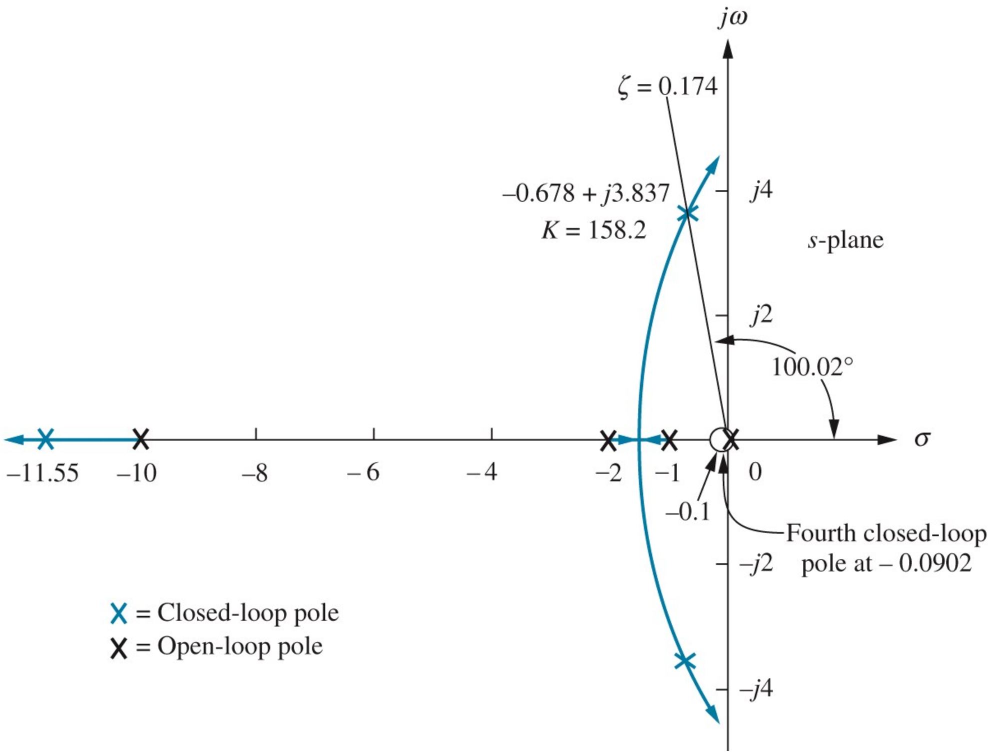
- Now look for the third pole on the root locus beyond -10 on the real axis. Searching for the same gain as that of the dominant pair, $K = 164.6$, we find that the third pole is approximately at -11.61. This gain yields $K_p = 8.23$.

Example…

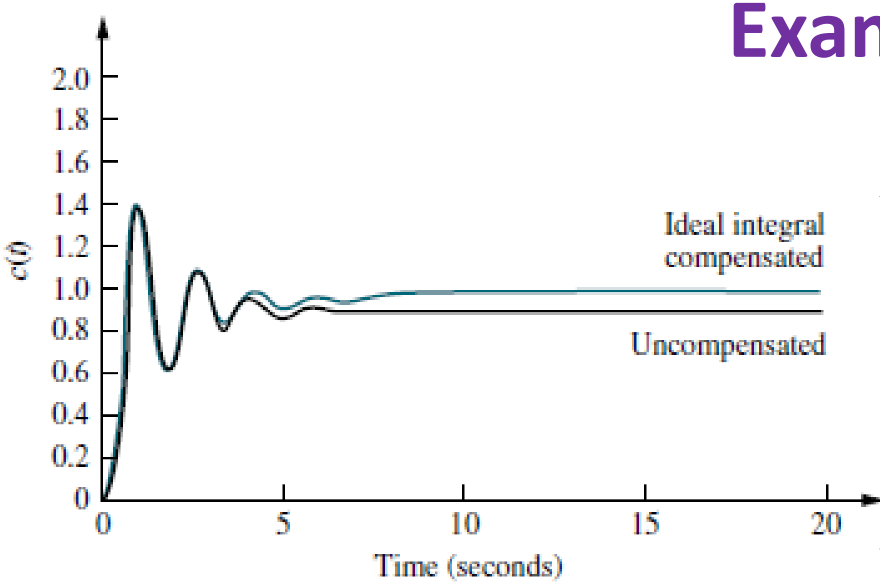
The steady-state error is $e(\infty) = \dfrac{1}{1+K_p} = \dfrac{1}{1+8.23} = 0.108$

Adding an ideal integral compensator with a zero at -0.1 we obtain the root locus shown in

The dominant second-order poles, the third pole beyond -10, and the gain are approximately the same as for the uncompensated system. Another section of the compensated root locus is between the origin and -0.1. Searching this region for the same gain at the dominant pair, $K = 158.2$, the fourth closed-loop pole is found at -0.0902, close enough to the zero to cause pole-zero cancellation.



Example…

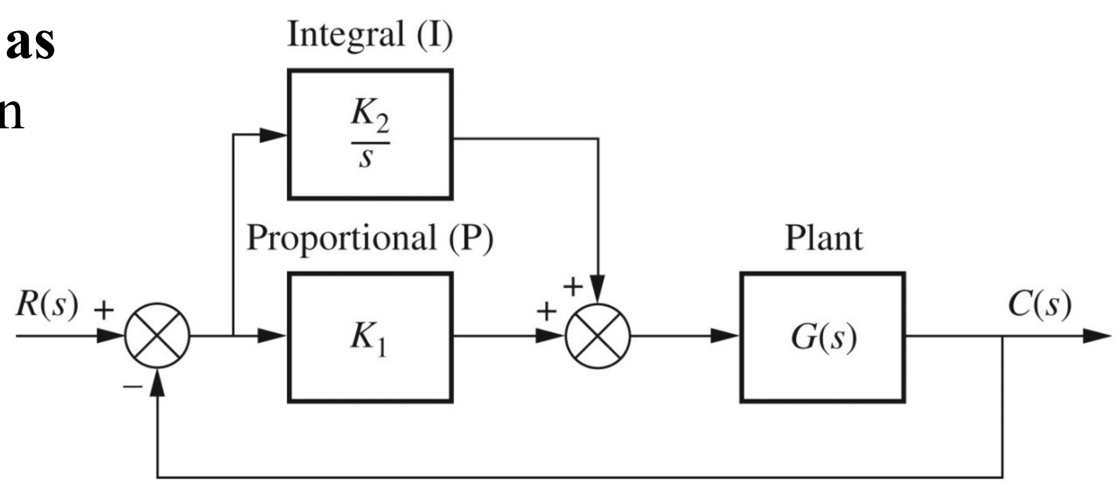


The figure compares the uncompensated response with the ideal integral compensated response. The step response of the ideal integral compensated system approaches unity in the steady state, while the uncompensated system approaches 0.892.

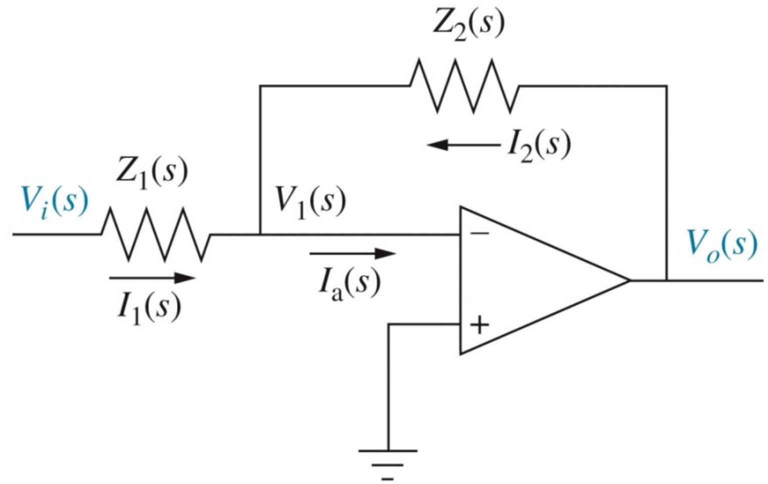
A method of implementing an ideal integral compensator as a block diagram is shown on the right.

The compensating network precedes $G(s)$ and is an ideal integral compensator:

$$G_c(s) = K_1 + \frac{K_2}{s} = \frac{K_1\left(s + \frac{K_2}{K_1}\right)}{s}$$



Physical Realization of Controllers

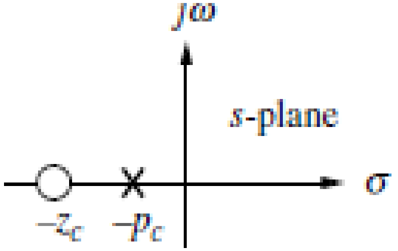


Function	$Z_1(s)$	$Z_2(s)$	$G_c(s) = -\frac{Z_2(s)}{Z_1(s)}$
Gain	R_1	R_2	$-\frac{R_2}{R_1}$
Integration	R	C	$-\frac{\frac{1}{RC}}{s}$
Differentiation	C	R	$-RCs$
PI controller	R_1	R_2 C	$-\frac{R_2}{R_1}\frac{\left(s+\frac{1}{R_2C}\right)}{s}$
PD controller	C R_1	R_2	$-R_2C\left(s+\frac{1}{R_1C}\right)$
PID controller	C_1 R_1	R_2 C_2	$-\left[\left(\frac{R_2}{R_1}+\frac{C_1}{C_2}\right)+R_2C_1s+\frac{\frac{1}{R_1C_2}}{s}\right]$

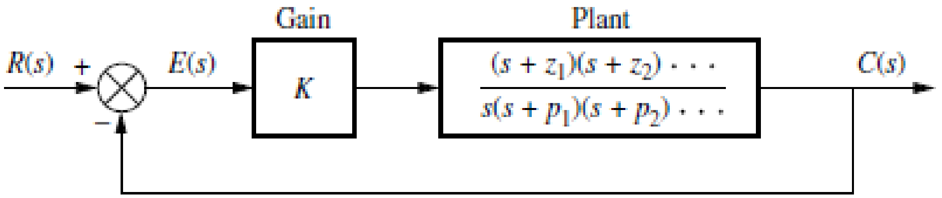
Lag Compensation

If we use passive networks, the pole and zero are moved to the left, close to the origin, as shown in

$$G_C(s) = \frac{(s + z_c)}{(s + p_c)}$$

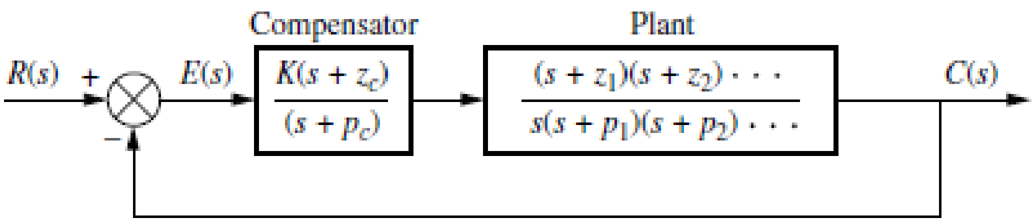


Assume the uncompensated system



The static error constant, K_{vo}, for the system is $$K_{vo} = \frac{K\, z_1\, z_2 \cdots}{p_1 p_2 \cdots}$$

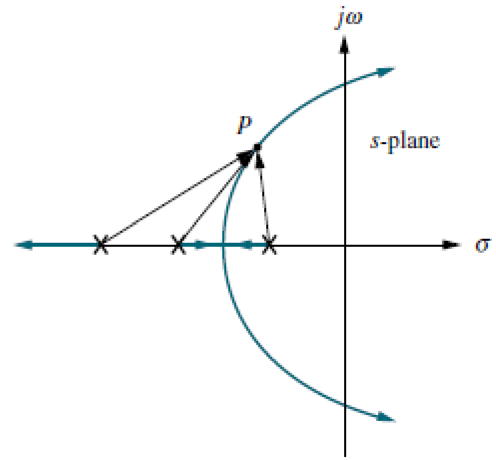
Assuming the lag compensator



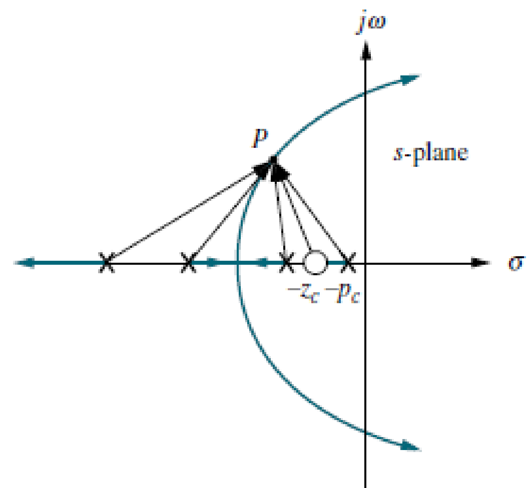
The new static error constant is $$K_{vN} = \frac{(K\, z_1\, z_2 \cdots)(z_c)}{(p_1 p_2 \cdots)(p_c)}$$

Lag Compensation…

The uncompensated system's root locus is given where point P is assumed to be the dominant pole. If the lag compensator pole and zero are close together, the angular contribution of the compensator to point P is approximately zero degrees.



The compensator has been added, point P is still at approximately the same location on the compensated root locus.



After inserting the compensator, we find that K is virtually the same for the uncompensated and compensated systems, since the lengths of the vectors drawn from the lag compensator are approximately equal and all other vectors have not changed.

Lag Compensation…

- What improvement can we expect in the steady-state error? Since we established that the gain, K, is about the same for the uncompensated and compensated systems, we can obtain

$$K_{v_N} = K_{v_O}\frac{z_c}{p_c} > K_{v_O}$$

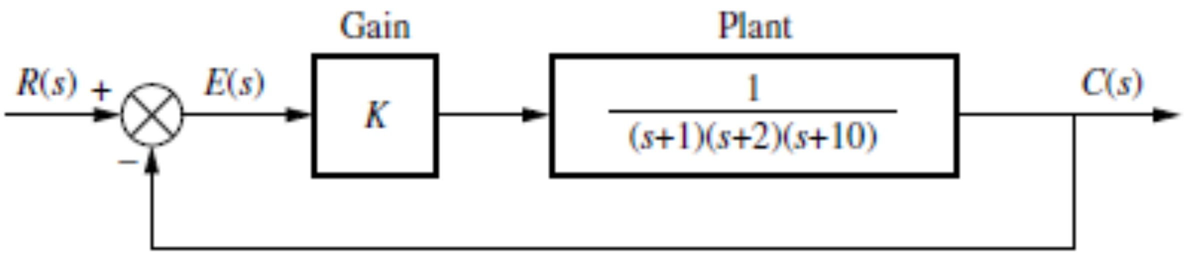
which shows that the improvement in the compensated system's K_v over the uncompensated system's K_v is equal to the ratio of the magnitude of the compensator zero to the compensator pole.

- In order to keep the transient response unchanged, we know the compensator pole and zero must be close to each other. The only way the ratio of z_c to p_c can be large in order to yield an appreciable improvement in steady-state error and simultaneously have the compensator's pole and zero close to each other to minimize the angular contribution is to place the compensator's pole-zero pair close to the origin.

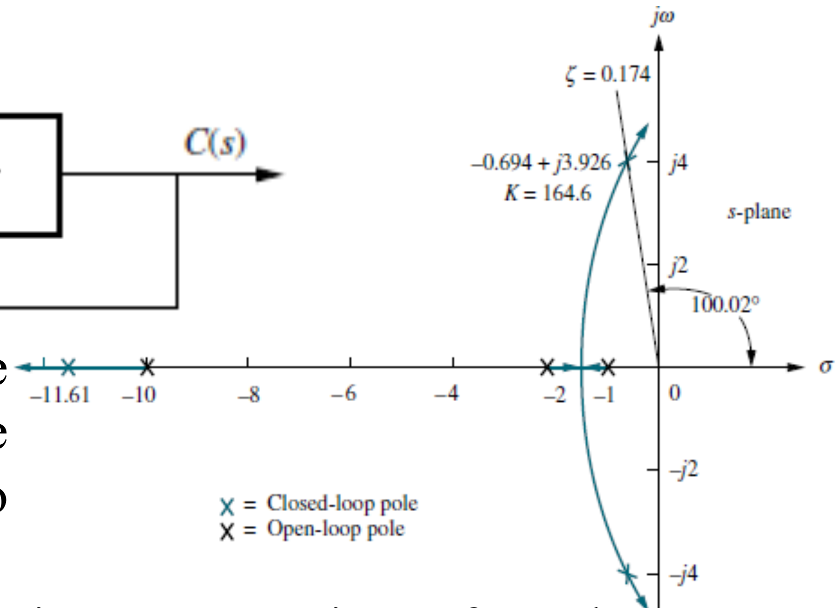
- For example, the ratio of z_c to p_c can be equal to 10 if the pole is at -0.001 and the zero is at -0.01. Thus, the ratio is 10, yet the pole and zero are very close, and the angular contribution of the compensator is small.

Example

Compensate the system of



whose root locus is shown to improve the steady-state error by a factor of 10 if the system is operating with a damping ratio of 0.174.



The uncompensated system error was 0.108 with K_P=8.23. A tenfold improvement means a steady-state error of

$$e(\infty) = \frac{0.108}{10} = 0.0108$$

$$e(\infty) = \frac{1}{1+K_p} = 0.0108$$

$$K_p = \frac{1-e(\infty)}{e(\infty)} = \frac{1-0.0108}{0.0108} = 91.59$$

The improvement in K_P from the uncompensated system to the compensated system is the required ratio of the compensator zero to the compensator pole, or

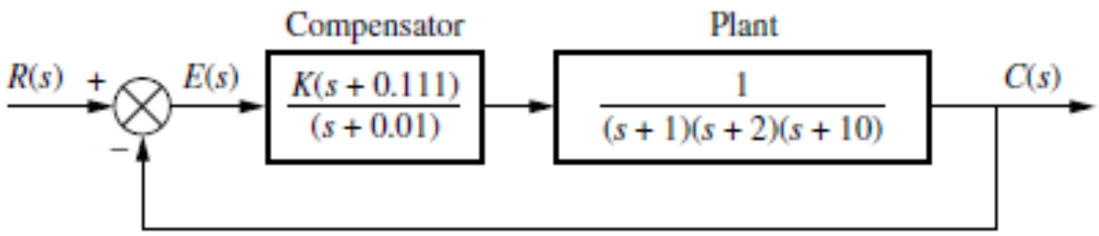
$$\frac{z_c}{p_c} = \frac{K_{p_N}}{K_{p_O}} = \frac{91.59}{8.23} = 11.13$$

$$p_c = 0.01$$

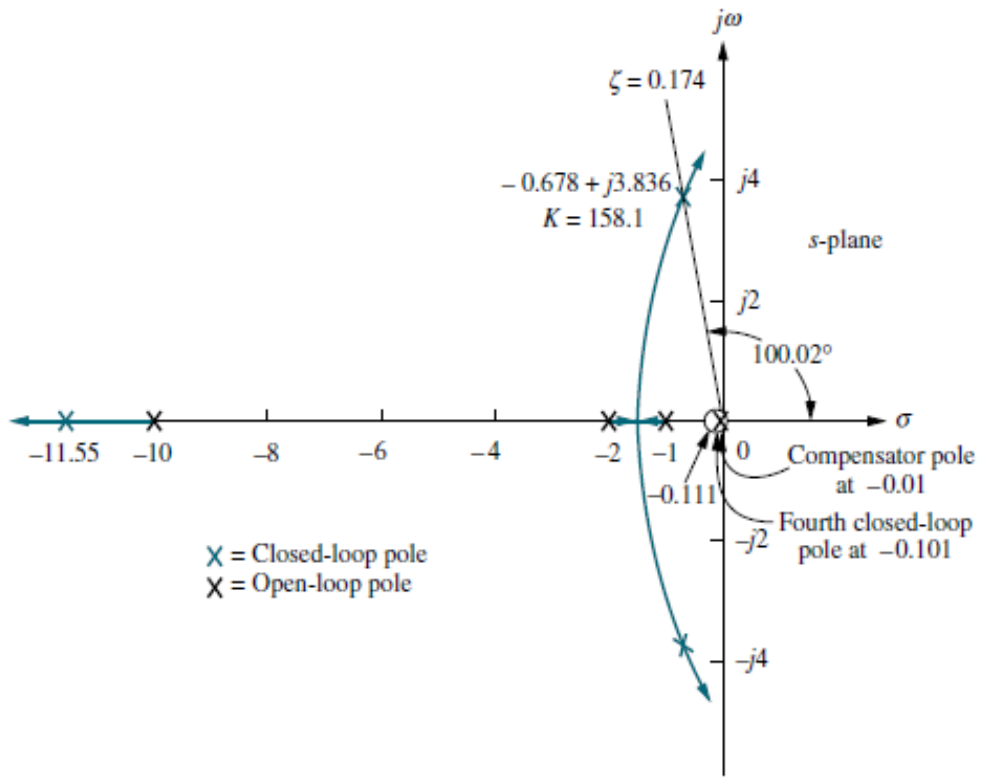
$$z_c = 11.13 p_c \approx 0.111$$

Example...

➢ Let us now compare the compensated system, shown in



with the uncompensated system. First sketch the root locus of the compensated system, as shown in

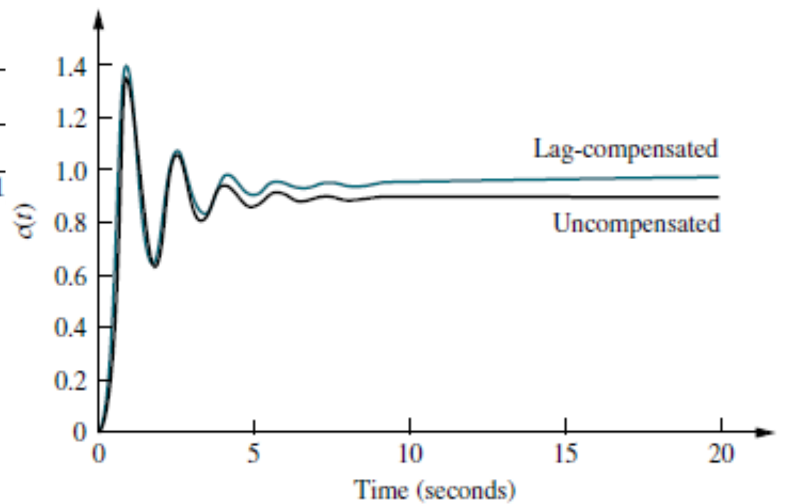


Example…

Next search along the ζ=0.174 line for a multiple of 180 and find that the second-order dominant poles are at -0.678±j3.836 with a gain, K, of 158.1. The third and fourth closed-loop poles are at 11.55 and 0.101, respectively, and are found by searching the real axis for a gain equal to that of the dominant poles.

The fourth pole of the compensated system cancels its zero. This leaves the remaining three closed-loop poles of the compensated system very close in value to the three closed-loop poles of the uncompensated system. Hence, the transient response of both systems is approximately the same, as is the system gain, but notice that the steady-state error of the compensated system is 1/9.818 that of the uncompensated system and is close to the design specification of a tenfold improvement.

Parameter	Uncompensated	Lag-compensated
Plant and compensator	$\frac{K}{(s+1)(s+2)(s+10)}$	$\frac{K(s+0.111)}{(s+1)(s+2)(s+10)(s+0.01}$
K	164.6	158.1
K_p	8.23	87.75
$e(\infty)$	0.108	0.011
Dominant second-order poles	$-0.694 \pm j3.926$	$-0.678 \pm j3.836$
Third pole	-11.61	-11.55
Fourth pole	None	-0.101
Zero	None	-0.111



Improving Transient Response via Cascade Compensation

- Since we have solved the problem of improving the steady-state error without affecting the transient response, let us now improve the transient response itself.

- In this section, we discuss two ways to improve the transient response of a feedback control system by using cascade compensation.

- Typically, the objective is to design a response that has a desirable percent overshoot and a shorter settling time than the uncompensated system.

Ideal Derivative Compensation (PD)

- The transient response of a system can be selected by choosing an appropriate closed-loop pole location on the *s*-plane.

- If this point is on the root locus, then a simple gain adjustment is all that is required in order to meet the transient response specification. If the closed-loop pole location is not on the root locus, then the root locus must be reshaped so that the compensated (new) root locus goes through the selected closed-loop pole location.

- In order to accomplish the latter task, poles and zeros can be added in the forward path to produce a new open-loop function whose root locus goes through the design point on the *s*-plane.

- One way to speed up the original system that generally works is to **add a single zero** to the forward path.

Ideal Derivative Compensation (PD)…

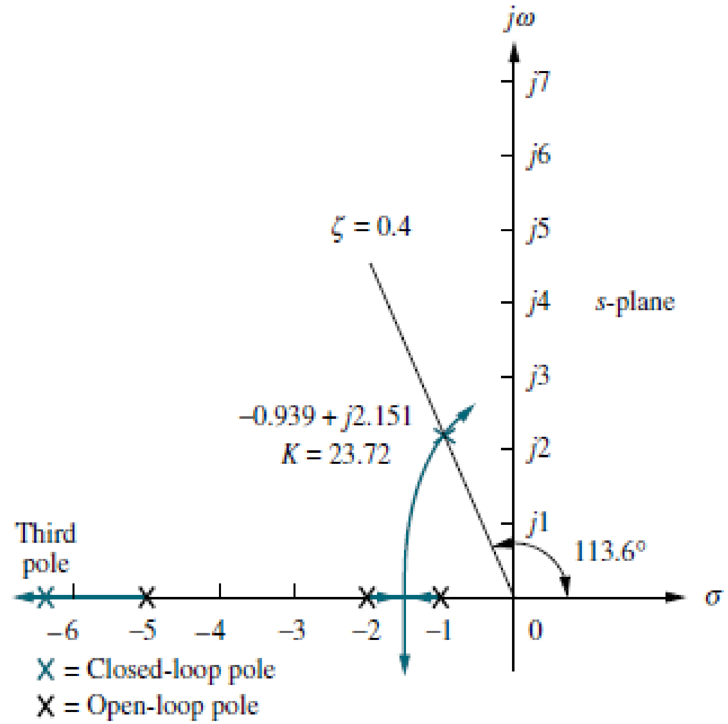
This zero can be represented by a compensator whose transfer function is

$$G_c(s) = K(s + z_c)$$

This function, the sum of a differentiator and a pure gain, is called an ideal derivative, or PD controller.

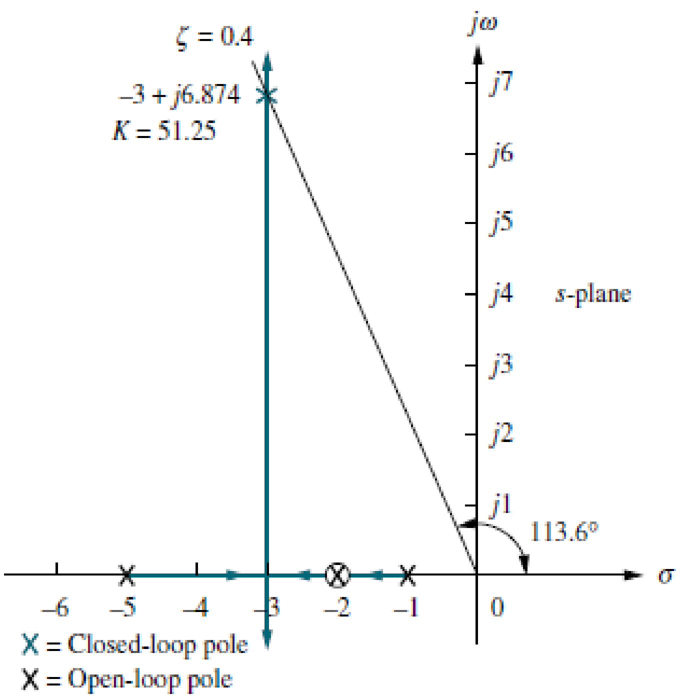
We now show that ideal derivative compensation speeds up the response of a system.

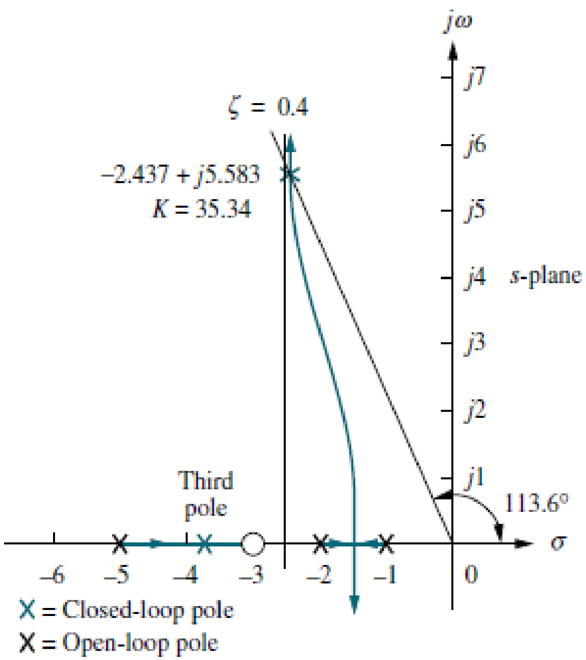
Several simple examples are shown where the uncompensated system given operating with a damping ratio of 0.4, becomes a compensated system by the addition of a compensating zero at –2, –3, and –4

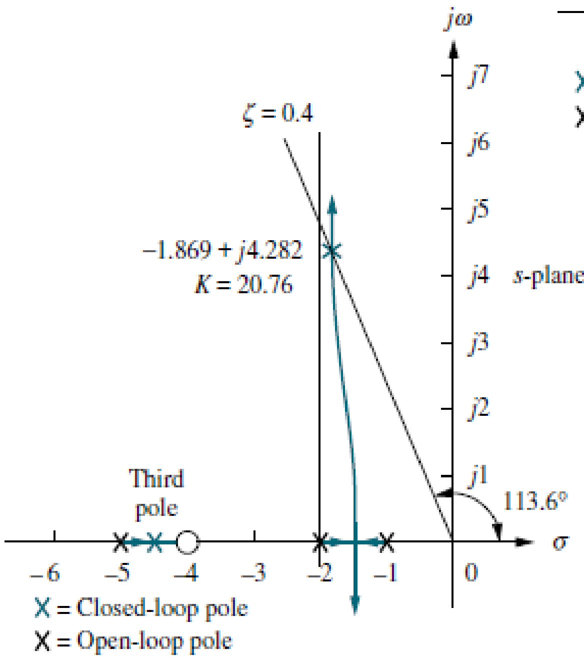


Ideal Derivative Compensation (PD)…

- In each design, the zero is moved to a different position, and the root locus is shown.
- For each compensated case, the dominant, second-order poles are further out along the 0.4 damping ratio line than the uncompensated system.
- Each of the compensated cases has dominant poles with the same damping ratio as the uncompensated case. We predict that the percent overshoot will be the same for each case.







Ideal Derivative Compensation (PD)…

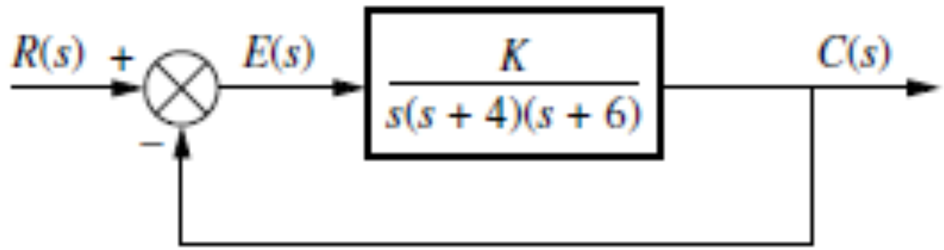
	Uncompensated	Compensation b	Compensation c	Compensation d
Plant and compensator	$\frac{K}{(s+1)(s+2)(s+5)}$	$\frac{K(s+2)}{(s+1)(s+2)(s+5)}$	$\frac{K(s+3)}{(s+1)(s+2)(s+5)}$	$\frac{K(s+4)}{(s+1)(s+2)(s+5)}$
Dom, poles	$-0.939 \pm j2.151$	$-3 \pm j6.874$	$-2.437 \pm j5.583$	$-1.869 \pm j4.282$
K	23.72	51.25	35.34	20.76
ζ	0.4	0.4	0.4	0.4
ω_n	2.347	7.5	6.091	4.673
$\%OS$	25.38	25.38	25.38	25.38
T_s	4.26	1.33	1.64	2.14
T_p	1.46	0.46	0.56	0.733
K_p	2.372	10.25	10.6	8.304
$e(\infty)$	0.297	0.089	0.086	0.107
Third pole	-6.123	None	-3.127	-4.262
Zero	None	None	-3	-4
Comments	Second-order approx. OK	Pure second-order	Second-order approx. OK	Second-order approx. OK

The table summarizes the results obtained from the root locus of each of the design cases. Although compensation methods **c** and **d** yield slower responses than method **b**, the addition of ideal derivative compensation shortened the response time in each case while keeping the percent overshoot the same.

This change can best be seen in the settling time and peak time, where there is at least a doubling of speed across all of the cases of compensation.

Example

Given the system of

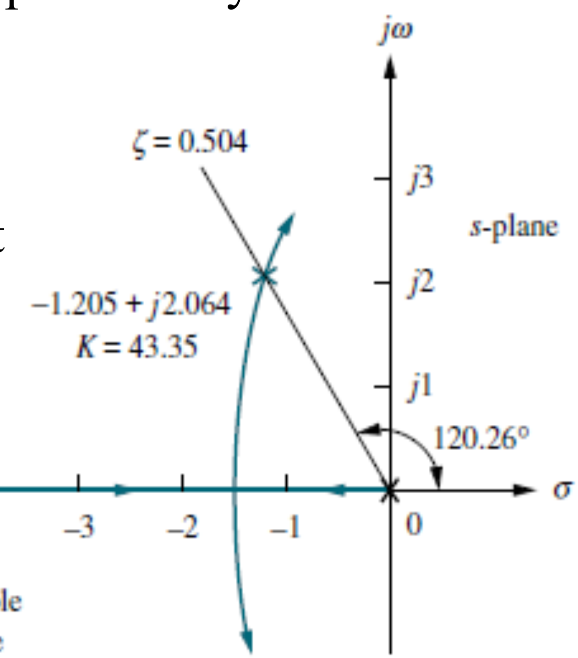


design an ideal derivative compensator to yield a 16% overshoot, with a threefold reduction in settling time.

Solution:

Let us first evaluate the performance of the uncompensated system operating with 16% overshoot. The root locus for the uncompensated system is

The dominant, second-order pair of poles is at -1.205±j2.064.



Example, *cont'd…*

- The settling time of the uncompensated system is $T_s = \frac{4}{\zeta\omega_n} = \frac{4}{1.205} = 3.320$
- Since our evaluation of percent overshoot and settling time is based upon a 2nd-order approximation, we must check the assumption by finding the third pole and justifying the 2nd-order approximation.
- Searching beyond –6 on the real axis for a gain equal to the gain of the dominant, 2nd pair, 43.35, we find a 3rd pole at –7.59, which is over six times as far from the $j\omega$-axis as the dominant, second-order pair.

	Uncompensated	**Simulation**	**Compensated**	**Simulation**
Plant and compensator	$\frac{K}{s(s+4)(s+6)}$		$\frac{K(s+3.006)}{s(s+4)(s+6)}$	
Dominant poles	$-1.205 \pm j2.064$		$-3.613 \pm j6.193$	
K	43.35		47.45	
ζ	0.504		0.504	
ω_n	2.39		7.17	
$\%OS$	16	14.8	16	11.8
T_s	3.320	3.6	1.107	1.2
T_p	1.522	1.7	0.507	0.5
K_v	1.806		5.94	
$e(\infty)$	0.554		0.168	
Third pole	-7.591		-2.775	
Zero	None		-3.006	
Comments	Second-order approx. OK		Pole-zero not canceling	

Example, *cont'd…*

The new settling time will be 1.107. Therefore, the real part of the compensated system's dominant, second-order pole is

$$\sigma = \frac{4}{T_s} = \frac{4}{1.107} = 3.613$$

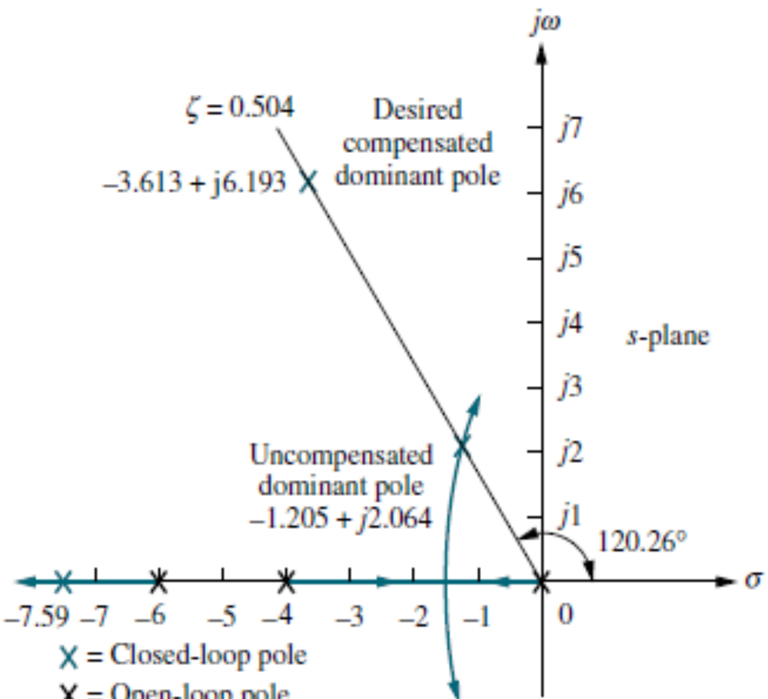


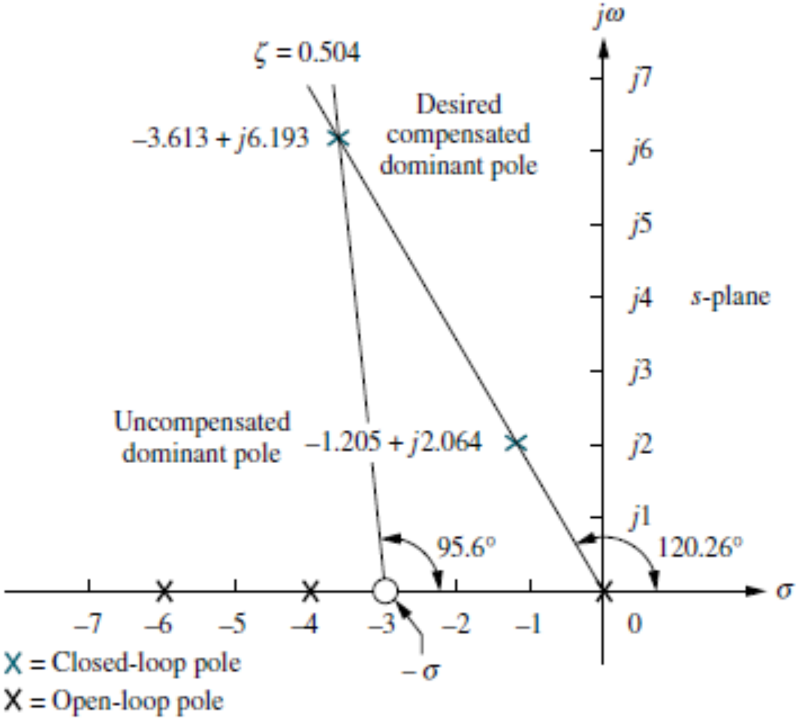
Figure shows the designed dominant, second-order pole, with a real part equal to -3.613 and an imaginary part of

$$\omega_d = 3.613 \tan(180^\circ - 120.26^\circ) = 6.193$$

The difference between the result obtained and 180 is the angular contribution required of the compensator zero. Using the open-loop poles shown and the test point, –3.613±j6.193, which is the desired dominant second-order pole, we obtain the sum of the angles as 275.6. Hence, the angular contribution required from the compensator zero for the test point to be on the root locus is 275.6–180 = 95.6

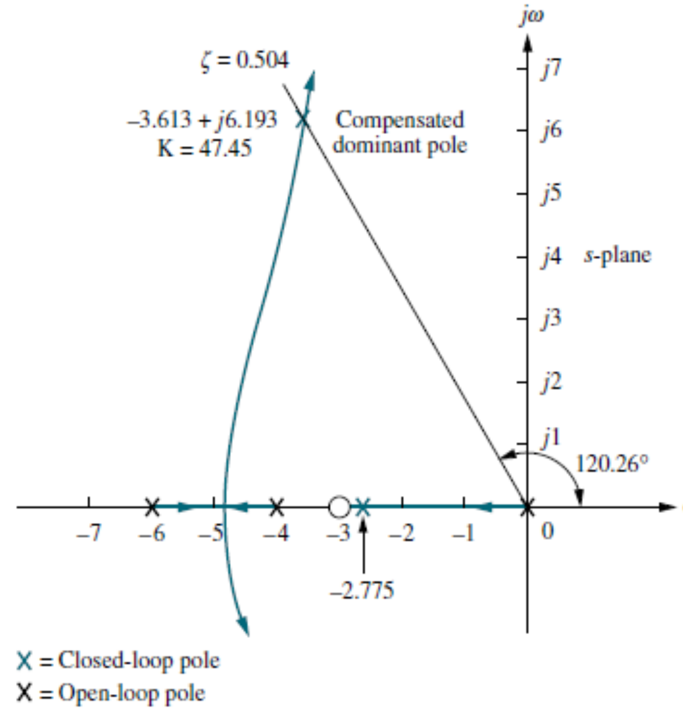
Example, *cont'd*…

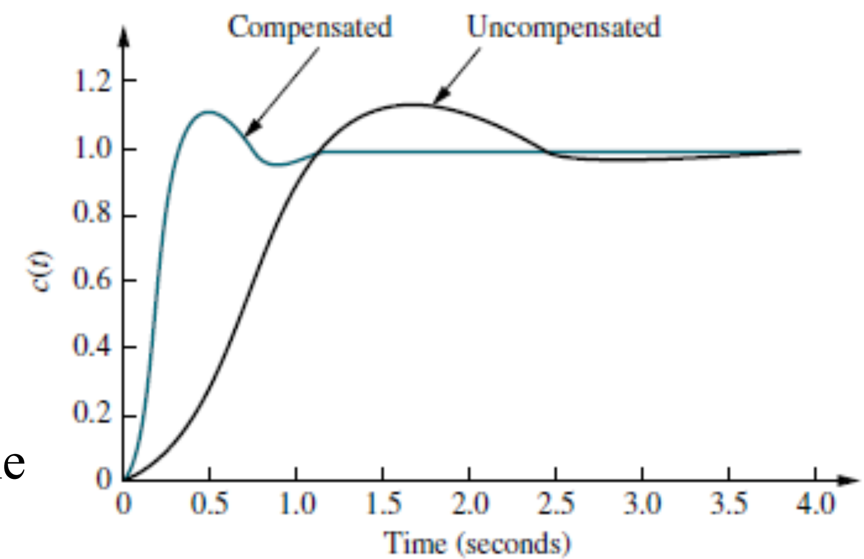
From the figure



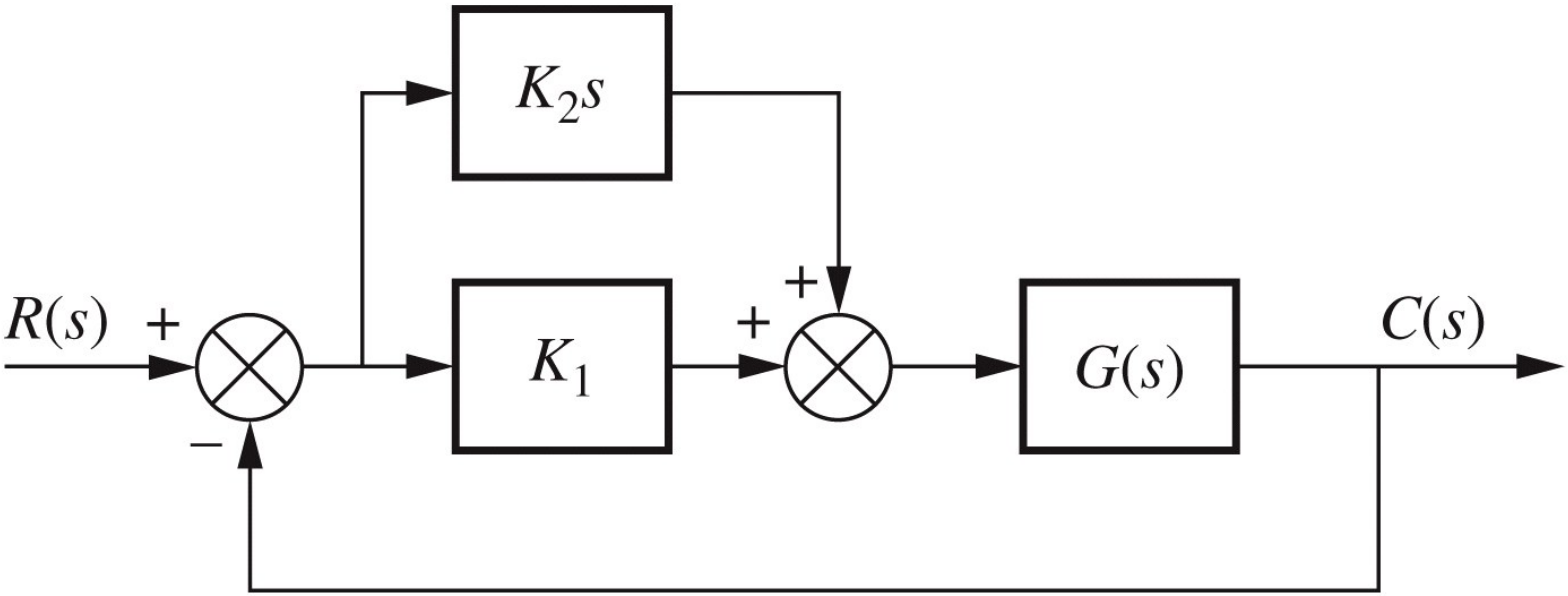
$$\frac{6.193}{3.613-\sigma}=\tan(180^\circ-95.6^\circ)$$

σ=3.006. The complete root locus for the compensated system is shown.





Block Diagram Realization of PD Control

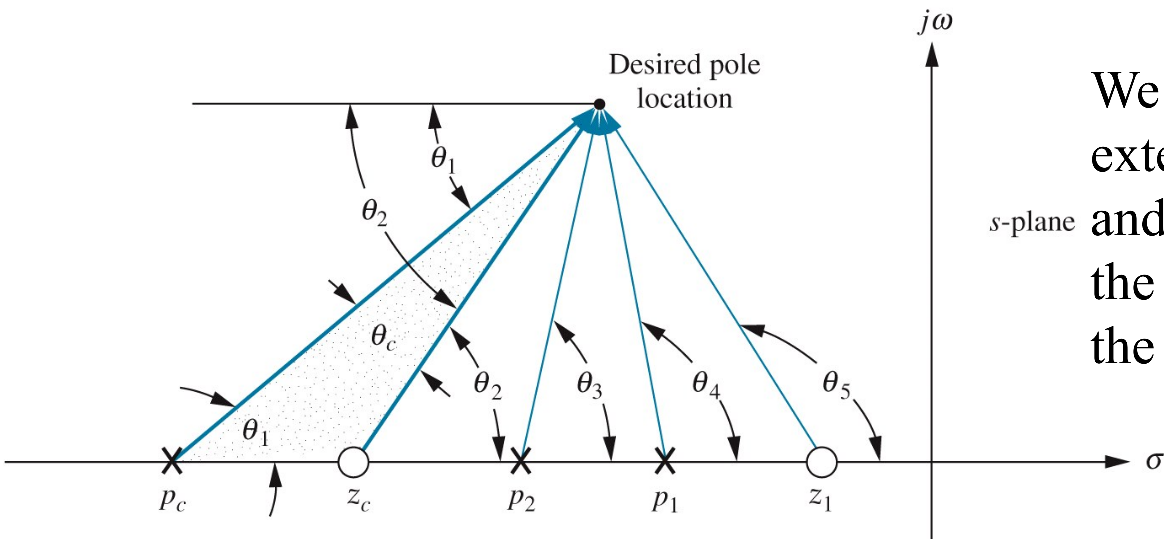


$$G_c(s) = G_{PD}(s) = K_1 + K_2 s = K_2\left(s + \frac{K_1}{K_2}\right)$$

$$G_{PD}(s) = K(s + z_c)$$

Lead Compensation

- Just as the active ideal integral compensator can be approximated with a passive lag network, an active ideal derivative compensator can be approximated with a passive lead compensator.
- If we select a desired dominant, second-order pole on the *s*-plane, the sum of the angles from the uncompensated system's poles and zeros to the design point can be found. The difference between 180 and the sum of the angles must be the angular contribution required of the compensator.



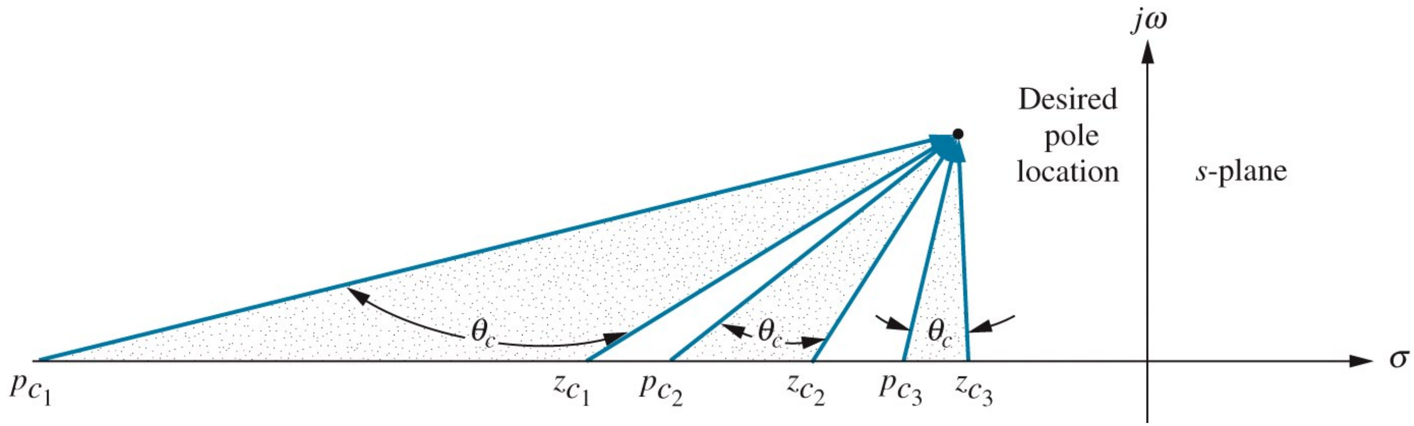
We see that θ_c is the angle of a ray extending from the design point and intersecting the real axis at the pole value and zero value of the compensator.

θ_2-θ_1-θ_3-θ_4+θ_5 = $(2k+1)\pi$

where θ_2-θ_1= θ_c is the angular contribution of the lead compensator.

Lead Compensation, *cont'd*…

➢ Now visualize this ray rotating about the desired closed-loop pole location and intersecting the real axis at the compensator pole and zero, as illustrated below.

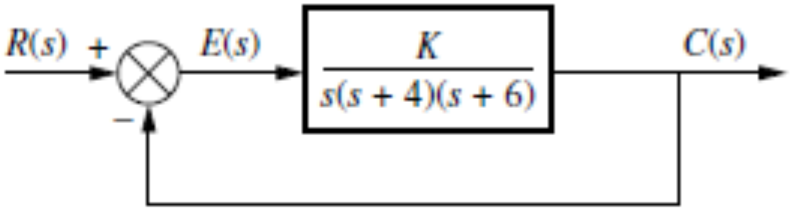


➢ We realize that an infinite number of lead compensators could be used to meet the transient response requirement.

➢ For design, we arbitrarily select either a lead compensator pole or zero and find the angular contribution at the design point of this pole or zero along with the system's open-loop poles and zeros.

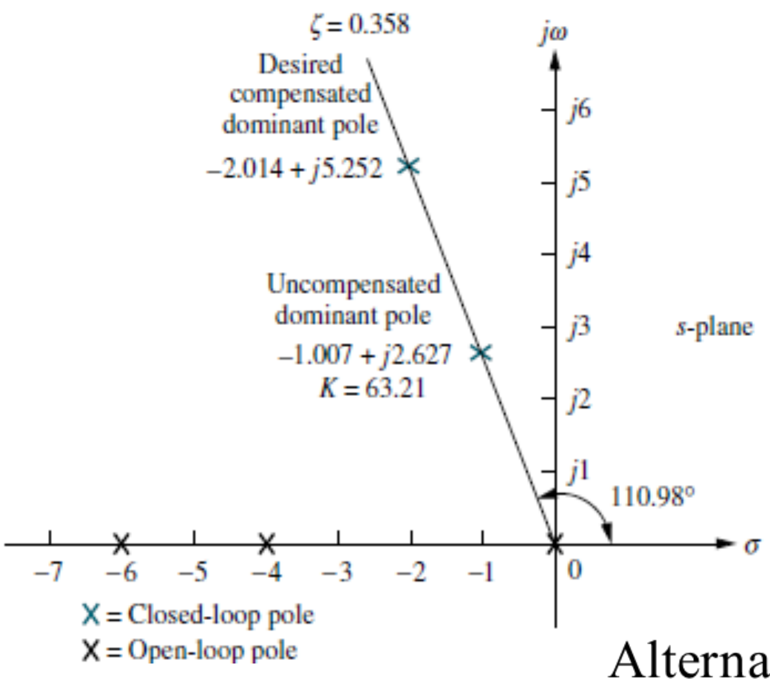
➢The difference between this angle and 180 is the required contribution of the remaining compensator pole or zero.

Example for Lead Design



Design three lead compensators for the system that will reduce the settling time by a factor of 2 while maintaining 30% overshoot. Compare the system characteristics between the three designs.

Solution: Since 30% overshoot is equivalent to a damping ratio of 0.358,



from the pole's real part, we calculate the uncompensated settling time as $T_s = 4/1.007 = 3.972$ sec.

Next we find the design point. A twofold reduction in settling time yields $T_s = 3.972/2 = 1.986$ seconds, from which the real part of the desired pole location is $-\zeta\omega_n = -4/T_s = -2.014$. The imaginary part is $\omega_d = -2.014\tan(110.98) = 5.252$.

Alternatively, the new design point is 2×old point ➔
$$s_n = 2s_o = 2(-1.007 + j2.627) = -2.014 + j5.252$$

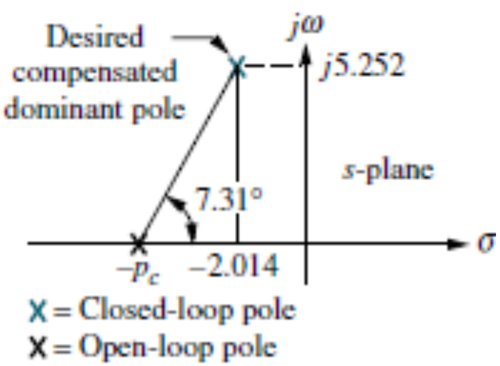
Example continues

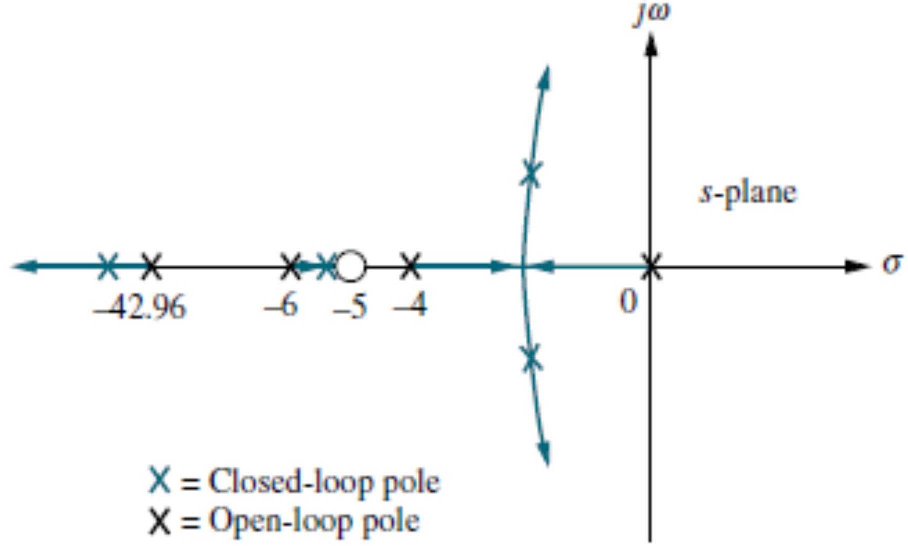
We continue by designing the lead compensator. Arbitrarily assume a compensator zero at –5 on the real axis as a possible solution. Sum the angles from both this zero and the uncompensated system's poles and zeros, using the design point as a test point.

The resulting angle is 172.69. The difference between this angle and 180 is the angular contribution required from the compensator pole in order to place the design point on the root locus. Hence, an angular contribution of 7.31^{o} is required from the compensator pole. The compensated system root locus is sketched blow.

$$\frac{5.252}{p_c - 2.014} = \tan 7.31^\circ$$

$$p_c = 42.96$$





Example continues

- In order to justify our estimates of $\%OS$ and T_s, we must show that the 2nd-order approximation is valid.
- To perform this validity check, we search for the 3rd and 4th closed-loop poles found beyond –42.96 and between –5 and –6.
- Searching these regions for the gain equal to that of the compensated dominant pole, 1423, we find that the 3rd and 4th poles are at –43.8 and –5.134, respectively.
- Since –43.8 is more than 20 times the real part of the dominant pole, the effect of the third closed-loop pole is negligible. Since the closed-loop pole at –5.134 is close to the zero at –5, we have pole-zero cancellation, and the second-order approximation is valid.

	Uncompensated	**Compensation a**	**Compensation b**	**Compensation c**
Plant and compensator	$\frac{K}{s(s+4)(s+6)}$	$\frac{K(s+5)}{s(s+4)(s+6)(s+42.96)}$	$\frac{K(s+4)}{s(s+4)(s+6)(s+20.09)}$	$\frac{K(s+2)}{s(s+4)(s+6)(s+8.971)}$
Dominant poles	$-1.007 \pm j2.627$	$-2.014 \pm j5.252$	$-2.014 \pm j5.252$	$-2.014 \pm j5.252$
K	63.21	1423	698.1	345.6
ζ	0.358	0.358	0.358	0.358
ω_n	2.813	5.625	5.625	5.625
$\%OS^*$	30 (28)	30 (30.7)	30 (28.2)	30 (14.5)
T_s^*	3.972 (4)	1.986 (2)	1.986 (2)	1.986 (1.7)
T_p^*	1.196 (1.3)	0.598 (0.6)	0.598 (0.6)	0.598 (0.7)
K_v	2.634	6.9	5.791	3.21
$e(\infty)$	0.380	0.145	0.173	0.312
Other poles	−7.986	−43.8, −5.134	−22.06	−13.3, −1.642
Zero	None	−5	None	−2
Comments	Second-order approx. OK	Second-order approx. OK	Second-order approx. OK	No pole-zero cancellation

*Simulation results are shown in parentheses.

Improving Steady-State Error <u>and</u> Transient Response

- We now combine the design techniques to obtain improvement in steady-state error and transient response independently.

- Basically, we first improve the transient response. Then we improve the steady-state error of this compensated system.

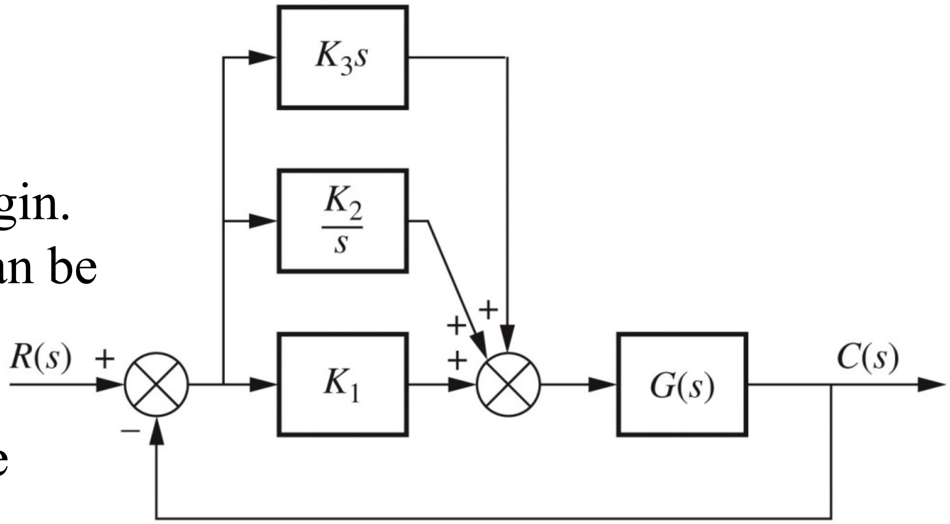
- A disadvantage of this approach is the slight decrease in the speed of the response when the steady-state error is improved.

- The design can use either active or passive compensators, as previously described. If we design an active PD controller followed by an active PI controller, the resulting compensator is called a proportional-plus-integral-plus-derivative (PID) controller.

- If we first design a passive lead compensator and then design a passive lag compensator, the resulting compensator is called a lag-lead compensator.

PID Controller Design

- The transfer function is given below.
- It has two zeros plus a pole at the origin.
- One zero and the pole at the origin can be designed as the ideal integral compensator;
- The other zero can be designed as the ideal derivative compensator. Hence, PD + PI = PID



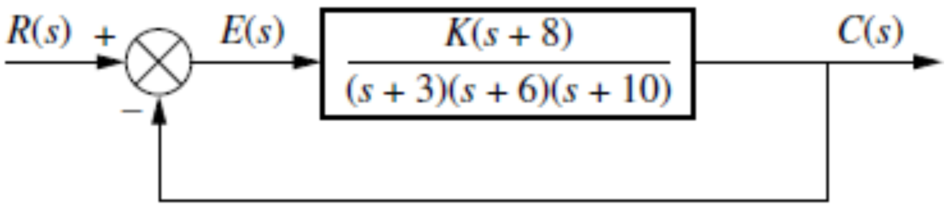
$$G_c(s) = G_{PID}(s) = K_1 + \frac{K_2}{s} + K_3 s = \frac{K_1 s + K_2 + K_3 s^2}{s} = \frac{K_3\left(s^2 + \frac{K_1}{K_3} s + \frac{K_2}{K_3}\right)}{s}$$

The design technique consists of the following steps:

- Evaluate the performance of the uncompensated system to determine how much improvement in transient response is required.
- Design the PD controller to meet the transient response specifications. The design includes the zero location and the loop gain.
- Simulate the system to be sure all requirements have been met.
- Redesign if the simulation shows that requirements have not been met.
- Design the PI controller to yield the required steady-state error.
- Determine the gains, K_1, K_2, and K_3.
- Simulate the system to be sure all requirements have been met.
- Redesign if simulation shows that requirements have not been met.

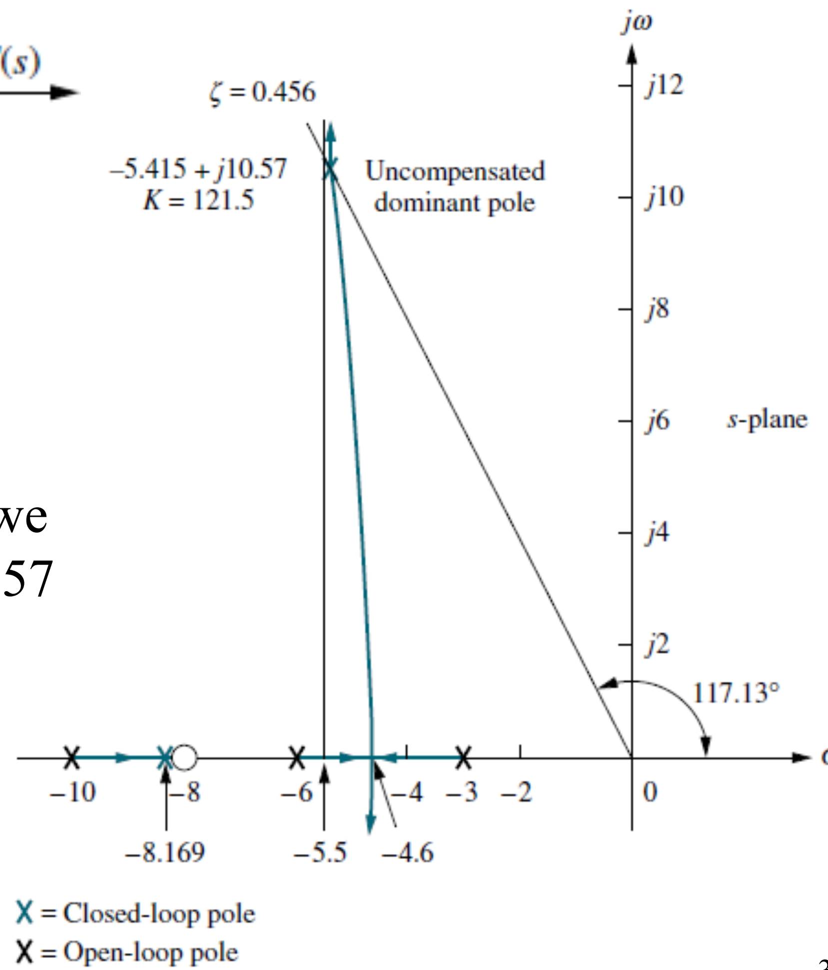
Example for PID Design

- Given the system, design a PID controller so that the system can operate with a peak time that is two-thirds that of the uncompensated system at 20% overshoot and with zero steady-state error for a step input.



- Let us first evaluate the uncompensated system operating at 20% overshoot.

- Searching along the 20% overshoot line, we find the dominant poles to be $-5.415 \pm j10.57$ with a gain of 121.5. A third pole, which exists at -8.169, is found by searching the region between -8 and -10 for a gain equivalent to that at the dominant poles.

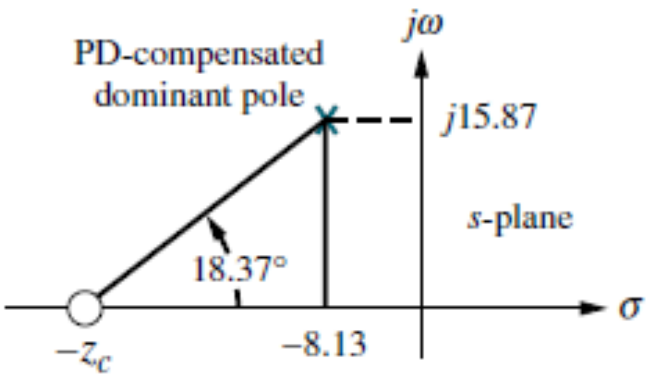


Example...

- To compensate the system to reduce the peak time to two-thirds of that of the uncompensated system, we must first find the compensated system's dominant pole location. The imaginary part of the compensated dominant pole is

$$\omega_d = \frac{\pi}{T_p} = \frac{\pi}{(2/3)(0.297)} = 15.87$$

- The real part of the compensated dominant pole is $\sigma = \frac{\omega_d}{\tan 117.13^\circ} = -8.13$



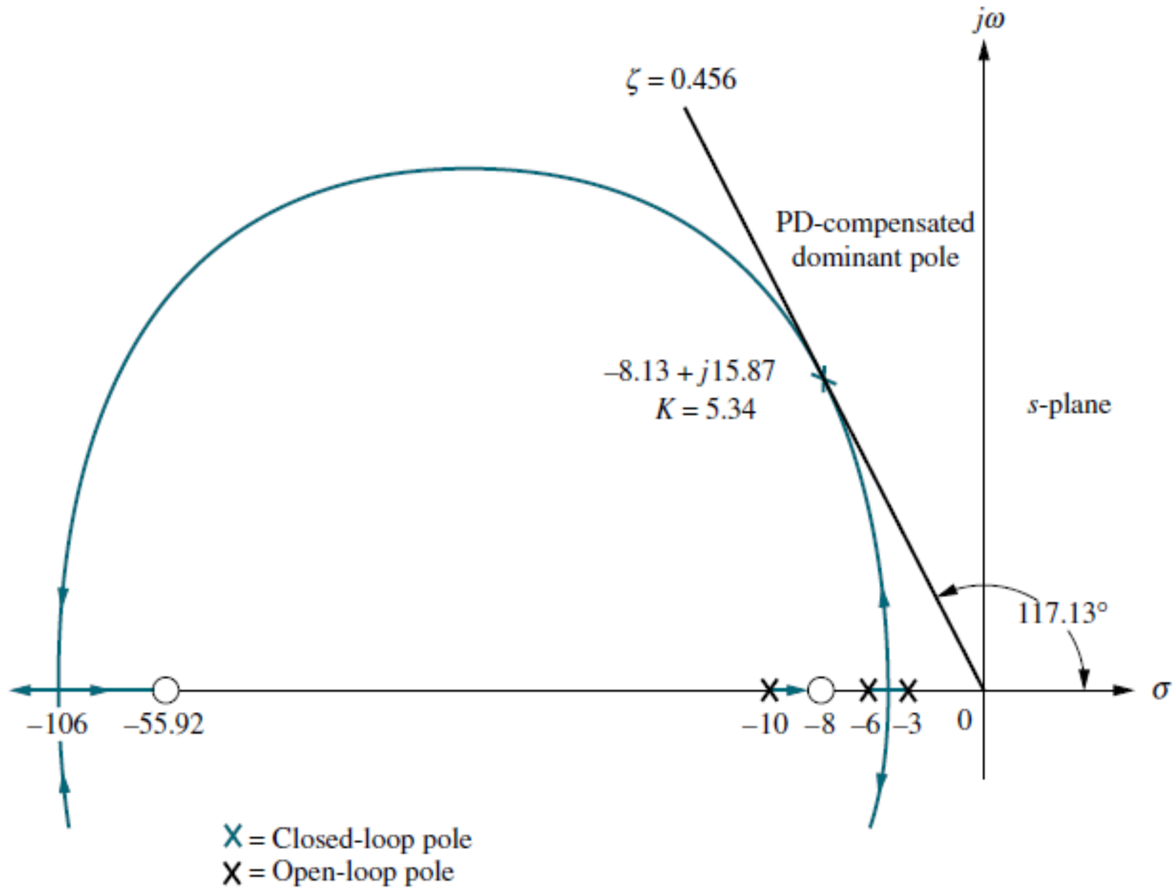
- Next we design the compensator. Using the geometry shown,
- We calculate the compensating zero's location. We find the sum of angles from the uncompensated system's poles and zeros to the desired compensated dominant pole to be 198.37.The contribution required from the compensator zero is 18.37°.
- Assume that the compensator zero is located at z_c,

$$\frac{15.87}{z_c - 8.13} = \tan 18.37^\circ$$

$$\rightarrow z_c = 55.92$$

$$G_{PD}(s) = (s + 55.92)$$

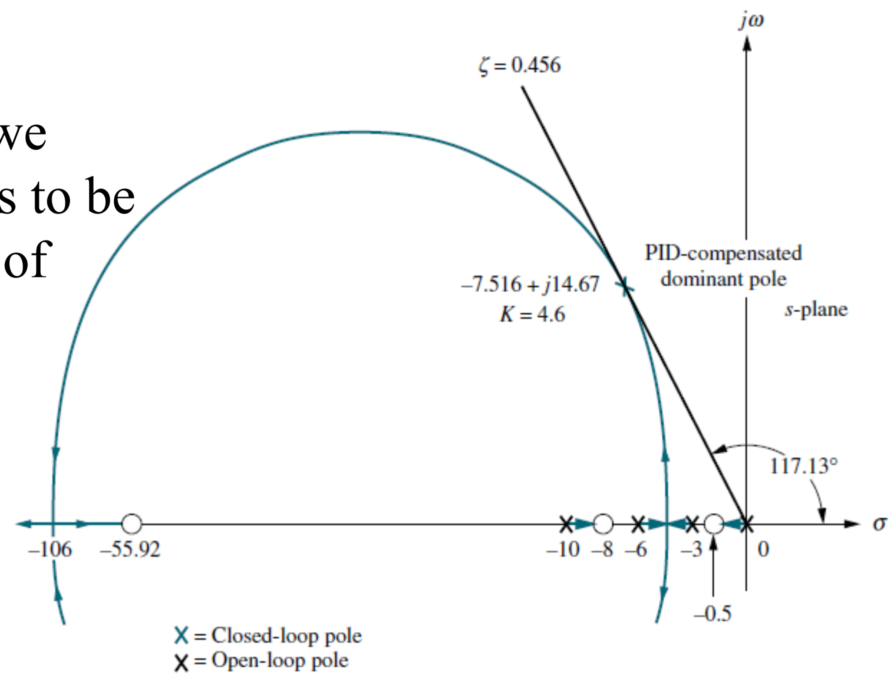
Example…



- The complete root locus for the PD-compensated system is sketched in the figure above. The gain at the design point is 5.34.
- After we design the PD controller, we design the ideal integral compensator to reduce the steady-state error to zero for a step input. Any ideal integral compensator zero will work, as long as the zero is placed close to the origin.

Example…

- Choosing the ideal integral compensator to be $G_{PI}(s) = \dfrac{s + 0.5}{s}$
- We sketch the root locus for the PID-compensated system.
- Searching the 0.456 damping ratio line, we find the dominant, the second-order poles to be –7.516 ± j14.67, with an associated gain of 4.6.



Now we determine the gains, K_1, K_2, and K_3.

$$G_{PID}(s) = \frac{K(s+55.92)(s+0.5)}{s} = \frac{4.6(s+55.92)(s+0.5)}{s}$$

$$= \frac{4.6(s^2 + 56.42s + 27.96)}{s}$$

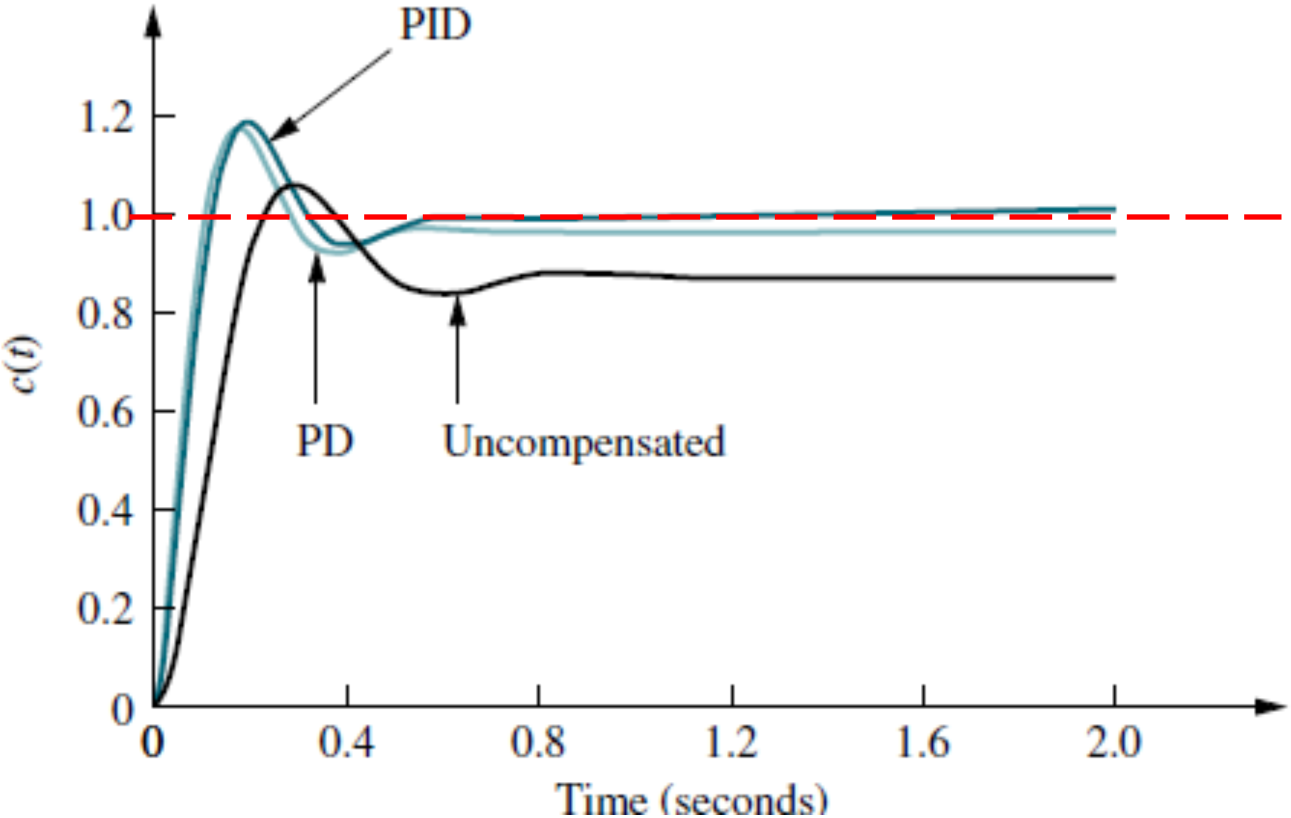
$$K_1 = 259.5,\ K_2 = 128.6,\ \text{and}\ K_3 = 4.6$$

Example… (Let's present the findings in a table)

	Uncompensated	PD-compensated	PID-compensated
Plant and compensator	$\dfrac{K(s+8)}{(s+3)(s+6)(s+10)}$	$\dfrac{K(s+8)(s+55.92)}{(s+3)(s+6)(s+10)}$	$\dfrac{K(s+8)(s+55.92)(s+0.5)}{(s+3)(s+6)(s+10)s}$
Dominant poles	$-5.415 \pm j10.57$	$-8.13 \pm j15.87$	$-7.516 \pm j14.67$
K	121.5	5.34	4.6
ζ	0.456	0.456	0.456
ω_n	11.88	17.83	16.49
$\%OS$	20	20	20
T_s	0.739	0.492	0.532
T_p	0.297	0.198	0.214
K_p	5.4	13.27	∞
$e(\infty)$	0.156	0.070	0
Other poles	-8.169	-8.079	-8.099, -0.468
Zeros	-8	-8, -55.92	-8, -55.92, -0.5
Comments	Second-order approx. OK	Second-order approx. OK	Zeros at -55.92 and -0.5 not canceled

- The complete performance of the uncompensated system is shown in the first,
- the PD-compensated in the second and
- the PID-compensated in the third column of the table above.

Example…



- ✓ PD compensation improved the transient response by decreasing the time required to reach the first peak as well as yielding some improvement in the steady-state error.
- ✓ The complete PID controller further improved the steady state error without appreciably changing the transient response designed with the PD controller.

Lag-Lead Compensator Design

- We improve both transient response and the steady-state error by using a lead compensator and a lag compensator similar to the ideal PID.

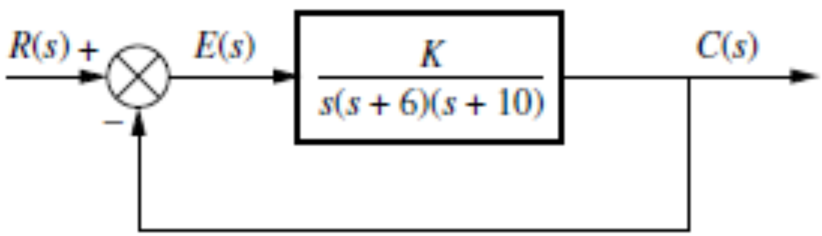
- We first design the lead compensator to improve the transient response. Next we evaluate the improvement in steady-state error that still required. Finally, we design the lag compensator to meet the steady-state error requirement.

The following steps summarize the design procedure:

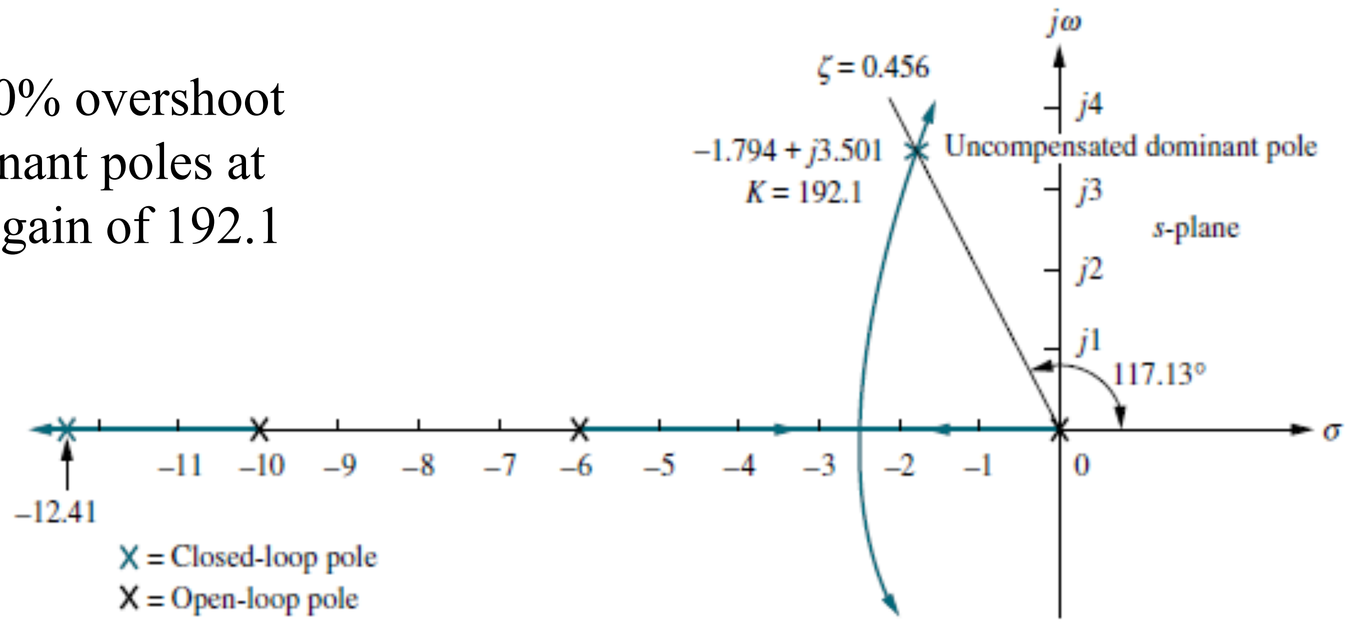
- Evaluate the performance of the uncompensated system to determine how much improvement in transient response is required.
- Design the lead compensator to meet the transient response specifications. The design includes the zero location, pole location, and the loop gain.
- Simulate the system to be sure all requirements have been met.
- Redesign if the simulation shows that requirements have not been met.
- Evaluate the steady-state error performance for the lead-compensated system to determine how much more improvement in steady-state error is required.
- Design the lag compensator to yield the required steady-state error.
- Simulate the system to be sure all requirements have been met.
- Redesign if the simulation shows that requirements have not been met.

Example

Design a lag-lead compensator for the system so that the system will operate with 20% overshoot and a twofold reduction in settling time. Further, the compensated system will exhibit a tenfold improvement in steady-state error for a ramp input.



Searching along the 20% overshoot line, we find the dominant poles at -1.794±j3.501, with a gain of 192.1



Example...

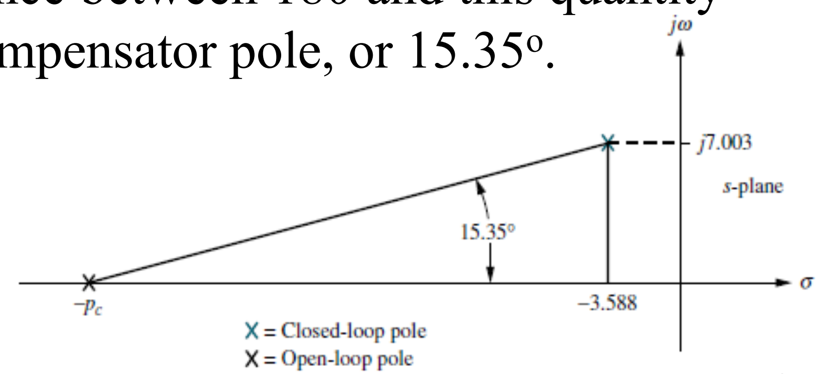
Next we begin the lead compensator design by selecting the location of the compensated system's dominant poles. In order to realize a twofold reduction in settling time, the real part of the dominant pole must be increased by a factor of 2, since the settling time is inversely proportional to the real part.

$$-\zeta\omega_n = -2(1.794) = -3.588$$

The imaginary part of the design point is $\omega_d = \zeta\omega_n \tan 117.13^\circ = 3.588 \tan 117.13^\circ = 7.003$

Now we design the lead compensator. Arbitrarily select a location for the lead compensator zero. For this example, we select the location of the compensator zero coincident with the open-loop pole at -6. This choice will eliminate a zero and leave the lead-compensated system with three poles. Sum the angles to the design point from the uncompensated system's poles and zeros and the compensator zero and get -164.65. The difference between 180 and this quantity is the angular contribution required from the compensator pole, or 15.35°.

$$\frac{7.003}{p_c - 3.588} = \tan 15.35^\circ$$



Example…

- The location of the compensator pole, p_c, is found to be -29.1. The gain setting at the design point is found to be 1977.

- Continue by designing the lag compensator to improve the steady-state error. Since the uncompensated system's open-loop transfer function is

$$G(s) = \frac{192.1}{s(s+6)(s+10)}$$

  the static error constant, K_v, which is inversely proportional to the steady state error, is 3.201. Since the open-loop transfer function of the lead compensated system is

$$G_{LC}(s) = \frac{1977}{s(s+10)(s+29.1)}$$

  the static error constant, K_v, which is inversely proportional to the steady state error, is 6.794. The addition of lead compensation has improved the steady-state error by a factor of 2.122. Since the requirements of the problem specified a tenfold improvement, the lag compensator must be designed to improve the steady-state error by a factor of 4.713 (10/2.122 = 4.713) over the lead-compensated system.

Example...

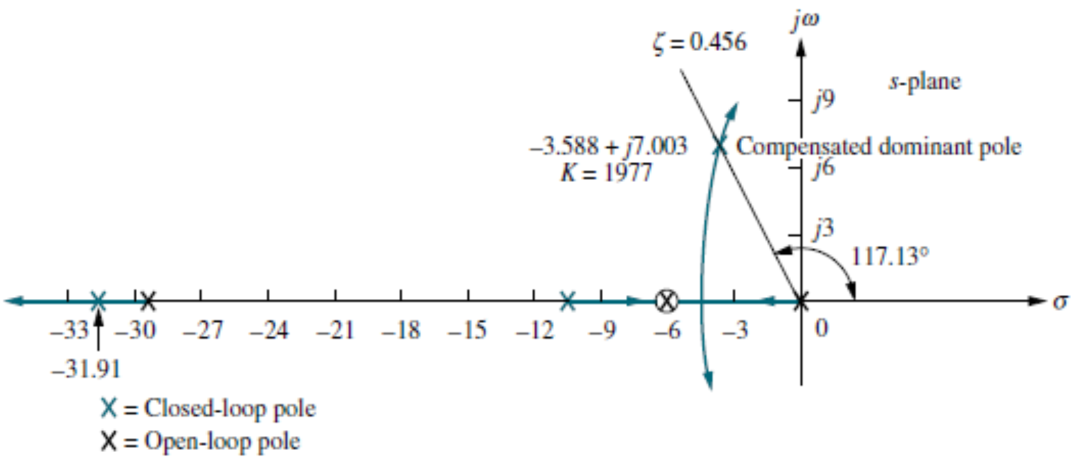
➢ We arbitrarily choose the lag compensator pole at 0.01, which then places the lag compensator zero at 0.04713, yielding

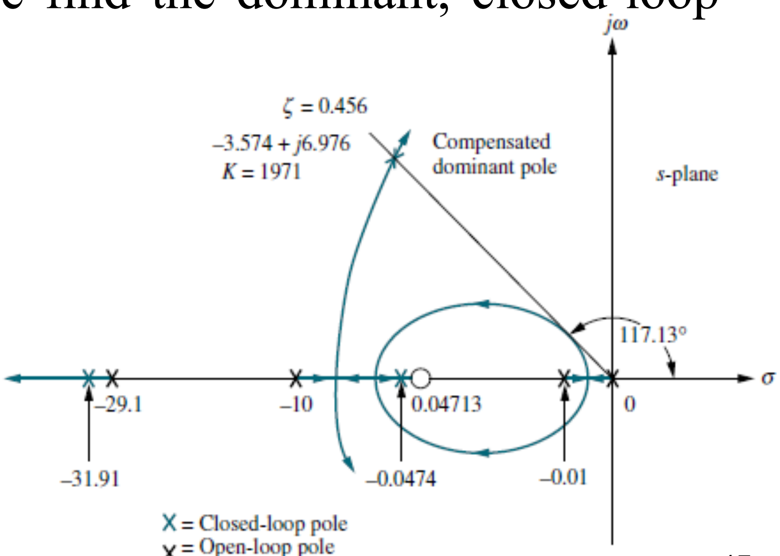
$$G_{\text{lag}}(s) = \frac{(s+0.04713)}{(s+0.01)}$$

as the lag compensator. The lag-lead-compensated system's open-loop transfer function is

$$G_{\text{LLC}}(s) = \frac{K(s+0.04713)}{s(s+10)(s+29.1)(s+0.01)}$$

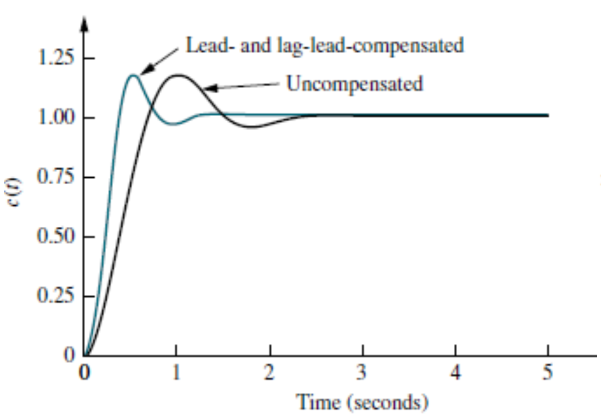
where the uncompensated system pole at -6 canceled the lead compensator zero at -6. By drawing the complete root locus for the lag-lead compensated system and by searching along the 0.456 damping ratio line, we find the dominant, closed-loop poles to be at -3.574±j6.976, with a gain of 1971.

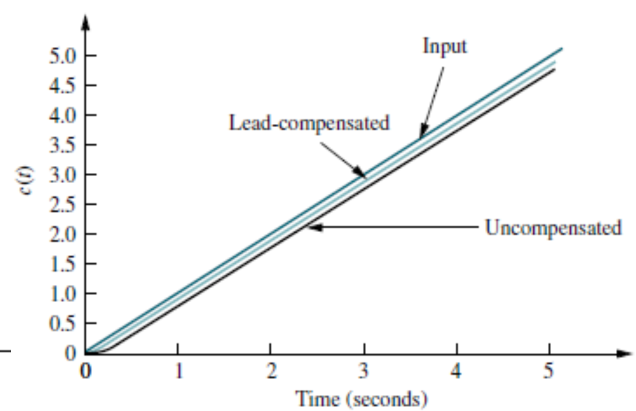


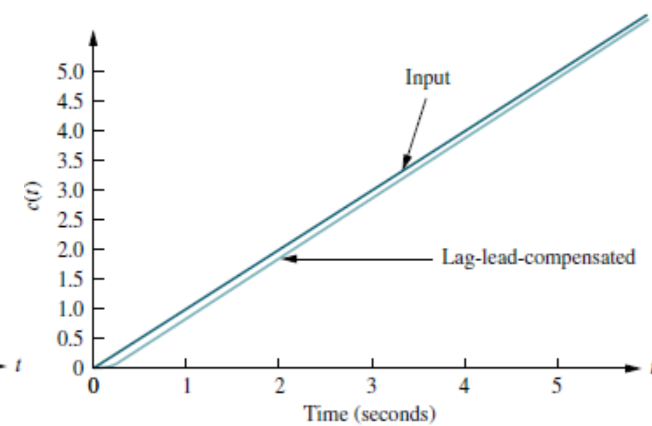


Example (summary)

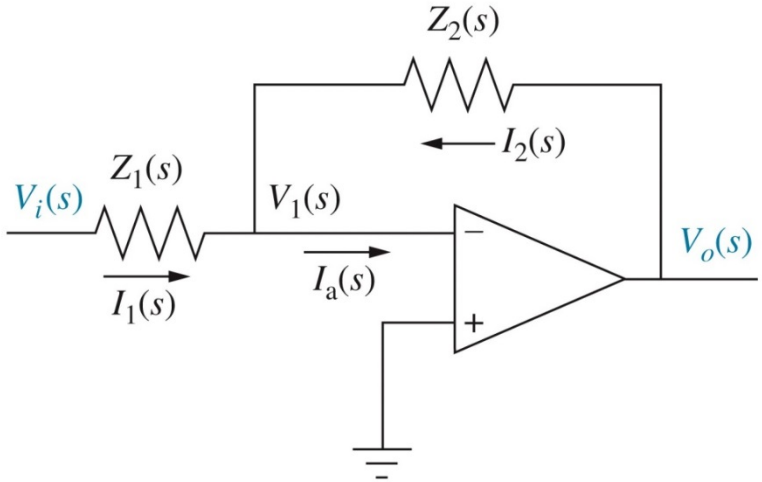
	Uncompensated	Lead-compensated	Lag-lead-compensated
Plant and compensator	$\dfrac{K}{s(s+6)(s+10)}$	$\dfrac{K}{s(s+10)(s+29.1)}$	$\dfrac{K(s+0.04713)}{s(s+10)(s+29.1)(s+0.01)}$
Dominant poles	$-1.794 \pm j3.501$	$-3.588 \pm j7.003$	$-3.574 \pm j6.976$
K	192.1	1977	1971
ζ	0.456	0.456	0.456
ω_n	3.934	7.869	7.838
$\%OS$	20	20	20
T_s	2.230	1.115	1.119
T_p	0.897	0.449	0.450
K_v	3.202	6.794	31.92
$e(\infty)$	0.312	0.147	0.0313
Third pole	-12.41	-31.92	-31.91, -0.0474
Zero	None	None	-0.04713
Comments	Second-order approx. OK	Second-order approx. OK	Second-order approx. OK



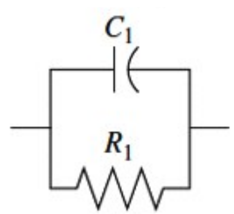


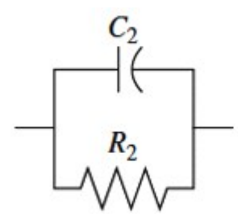


Physical Realization of Phase Compensators – via active networks



Lag compensation

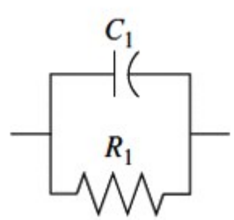


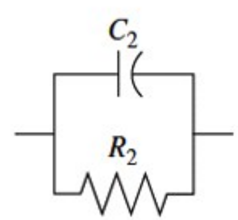


$$-\frac{C_1}{C_2}\frac{\left(s+\frac{1}{R_1C_1}\right)}{\left(s+\frac{1}{R_2C_2}\right)}$$

where $R_2C_2 > R_1C_1$

Lead compensation

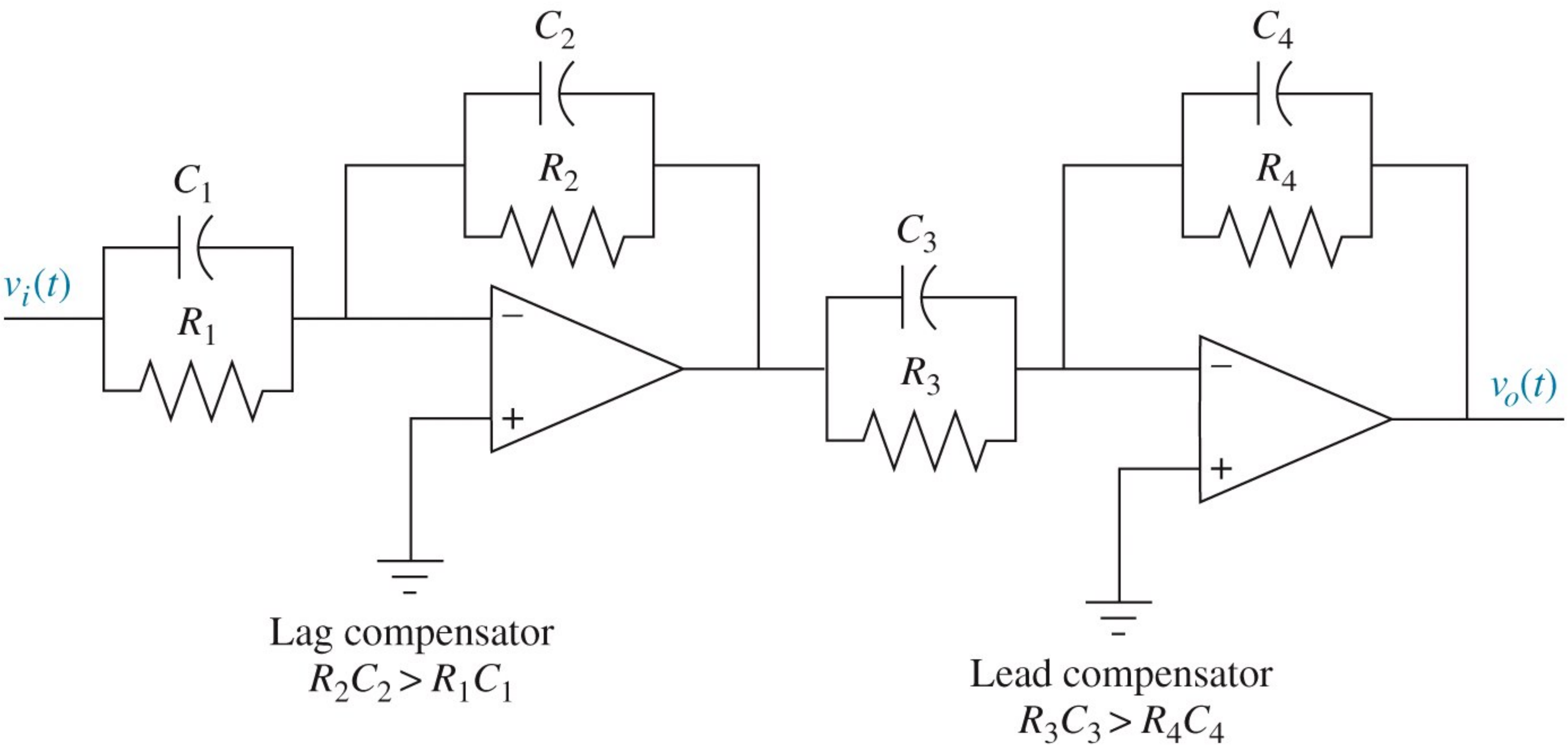




$$-\frac{C_1}{C_2}\frac{\left(s+\frac{1}{R_1C_1}\right)}{\left(s+\frac{1}{R_2C_2}\right)}$$

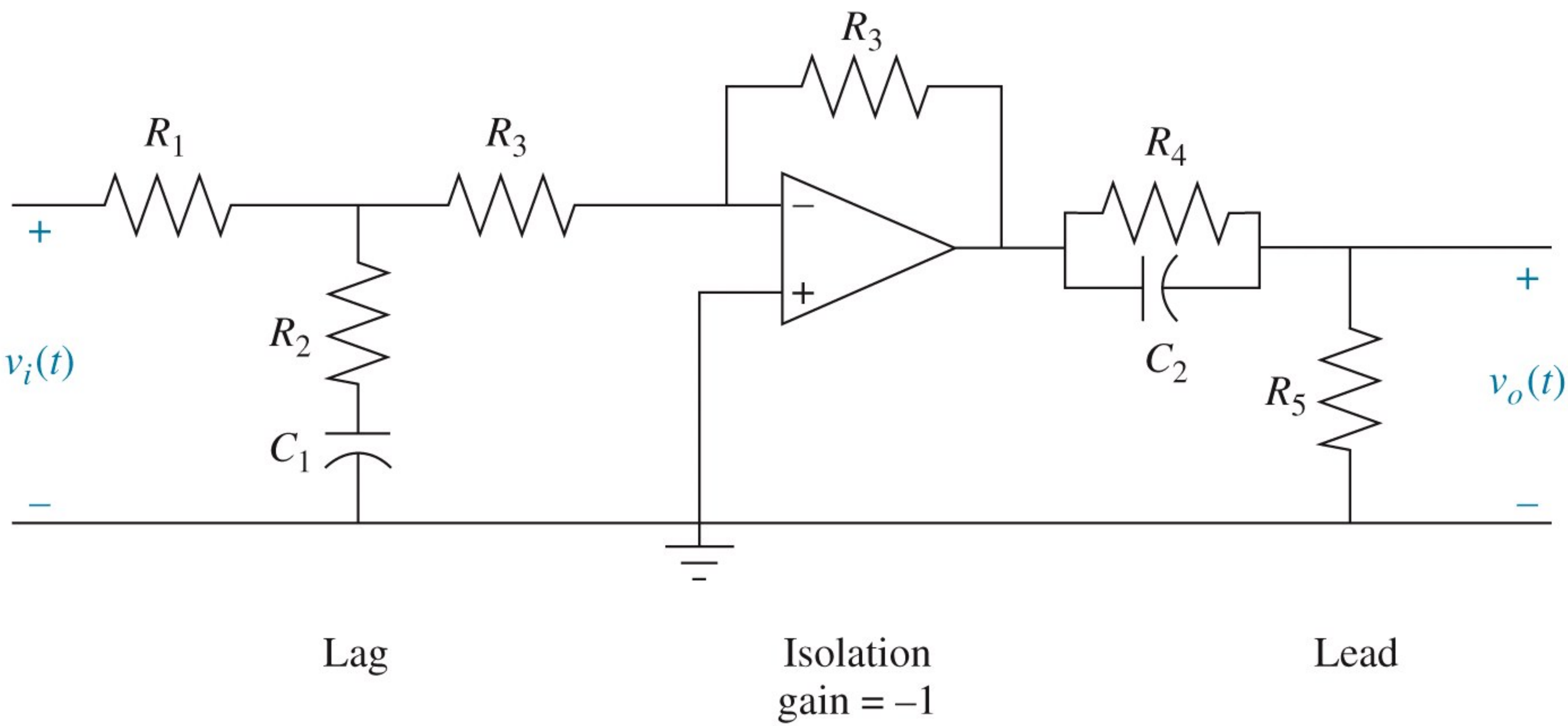
where $R_1C_1 > R_2C_2$

Physical Realization of Phase Compensators – via active networks…



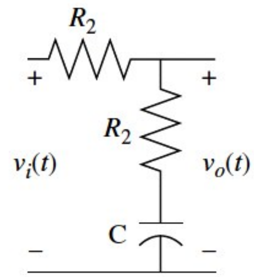
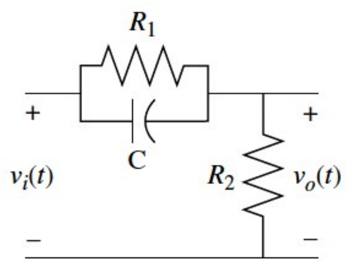
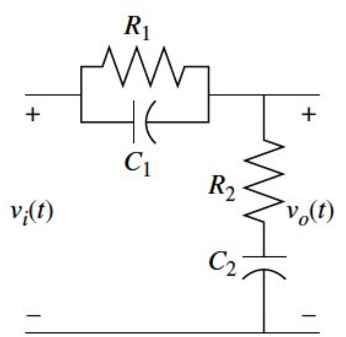
Lag-lead compensator implemented with operational amplifiers

Physical Realization of Phase Compensators – via active networks…



Lag-lead compensator implemented with cascaded lag and lead networks with isolation

Physical Realization of Phase Compensators – via passive networks

Function	Network	Transfer function, $\frac{V_o(s)}{V_i(s)}$
Lag compensation		$\frac{R_2}{R_1+R_2}\frac{s+\frac{1}{R_2C}}{s+\frac{1}{(R_1+R_2)C}}$
Lead compensation		$\frac{s+\frac{1}{R_1C}}{s+\frac{1}{R_1C}+\frac{1}{R_2C}}$
Lag-lead compensation		$\frac{\left(s+\frac{1}{R_1C_1}\right)\left(s+\frac{1}{R_2C_2}\right)}{s^2+\left(\frac{1}{R_1C_1}+\frac{1}{R_2C_2}+\frac{1}{R_2C_1}\right)s+\frac{1}{R_1R_2C_1C_2}}$