KOM3712 Control Systems Design

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Design via State Space Methods – 4 of 4: *Summary, Examples & LQR*

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Textbooks followed mostly for the Design in the State Space are,

- **Modern Control Engineering** (5th ed.), Katsuhiko Ogata, Chap. 9
- **Control Systems Engineering** (7th ed.), Norman S. Nise, Chap. 12
- **Feedback Control of Dynamic Systems** (7th ed.), Gene F. Franklin, J. David Powell, Abbas Emami-Naeini, Chap 7

Example-1 – The previous example with initial states

$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- The system uses the state feedback control law, $u =- \mathbf{K}\mathbf{x}$.
- We may choose the desired closed-loop poles at

$$s = -2 + j4, \qquad s = -2 - j4, \qquad s = -10$$

- Determine the state feedback gain matrix K.

```
>> A=[0  1  0;  0  0  1; -1  -5  -6]; B=[0; 0; 1];
>> P=[-2+4*j  -2-4*j  -10];
>>  K=acker(A, B, P); % or K=place(A, B, P);
        K =   199    55     8
```

Response to Initial Condition:

- Suppose that the initial conditions of the states are $\mathbf{x}(0) = [1 \;\; 0 \;\; 0]^T$
- Let us obtain the plots of the states of the controlled system vs time

Example-1 – response to initial condition – *continues*-1:

- In order to obtain the response to the given initial condition $\mathbf{x}(0) = [1 \;\; 0 \;\; 0]^T$, substitute $u = -\mathbf{Kx}$ into the plant equation and get $\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{BK})\mathbf{x}$
- To plot the response curves (x_1, x_2 and x_3 vs t), we may use the command, initial.
- If we first define the state-space equations for the system as,

  >> G1=ss(A-B*K,B,C,D); % now, the system matrix is A-BK
- We get a SISO and 3rd order (three states) system as checked by,

  >> size(G1)

  State-space model with 1 outputs, 1 inputs, and 3 states.
- We need to use the initial command as follows:

  >> t = 0:0.01:4;

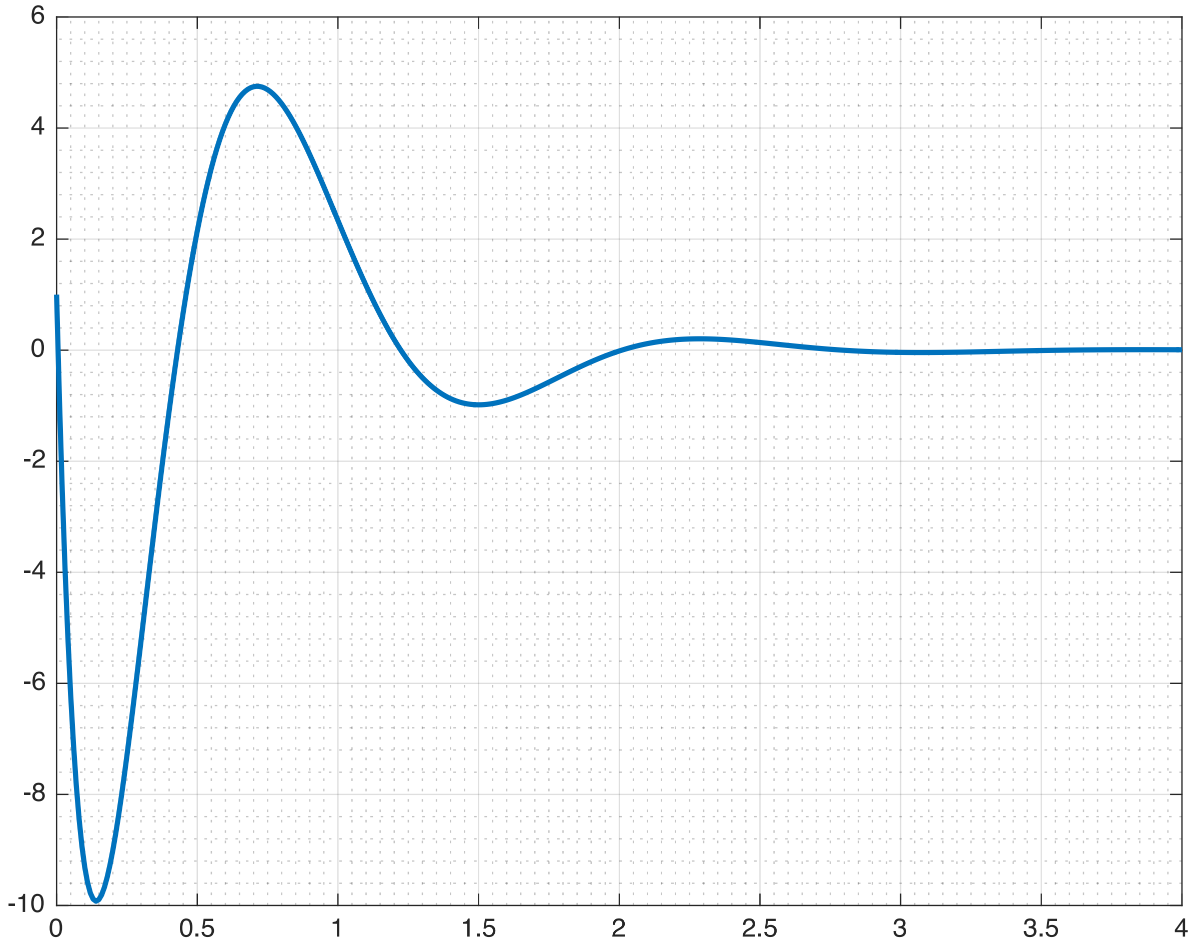
  >> x = initial(G1, [1;0;0],t); >> size(x) (ans = 401)

  where t is the time duration we want to use to obtain the output

  >> plot(t, x), grid on % It'll give us the plot of only one var.

Example-1 – response to initial condition – *continues*-2:

```
>> G1=ss(A-B*K,B,C,D); x = initial(G1, [1;0;0],t); plot(t, x),
```



Response to Initial Condition for a SISO and 3rd order System

Example-1 – response to initial condition – *continues*-3:

- $\mathbf{x}(0) = [1 \;\; 0 \;\; 0]^T$; $u = -\mathbf{Kx}$; and get $\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{BK})\mathbf{x}$
- To plot the response curves (x_1, x_2 and x_3 vs t), we may use the command, initial.
- This time we will convert the system to MIMO so that we can get three states at the output
- Now define the state-space equations for the system as,

  >> G2= ss(A - B*K, eye(3), eye(3), eye(3));
- Check the size of this new system, G2, with size(G2):

  State-space model with 3 outputs, 3 inputs, and 3 states.
- As seen, the system is now a MIMO and 3rd order (not SISO)

  >> t = 0:0.01:4;

  >> x = initial(G2, [1;0;0], t);

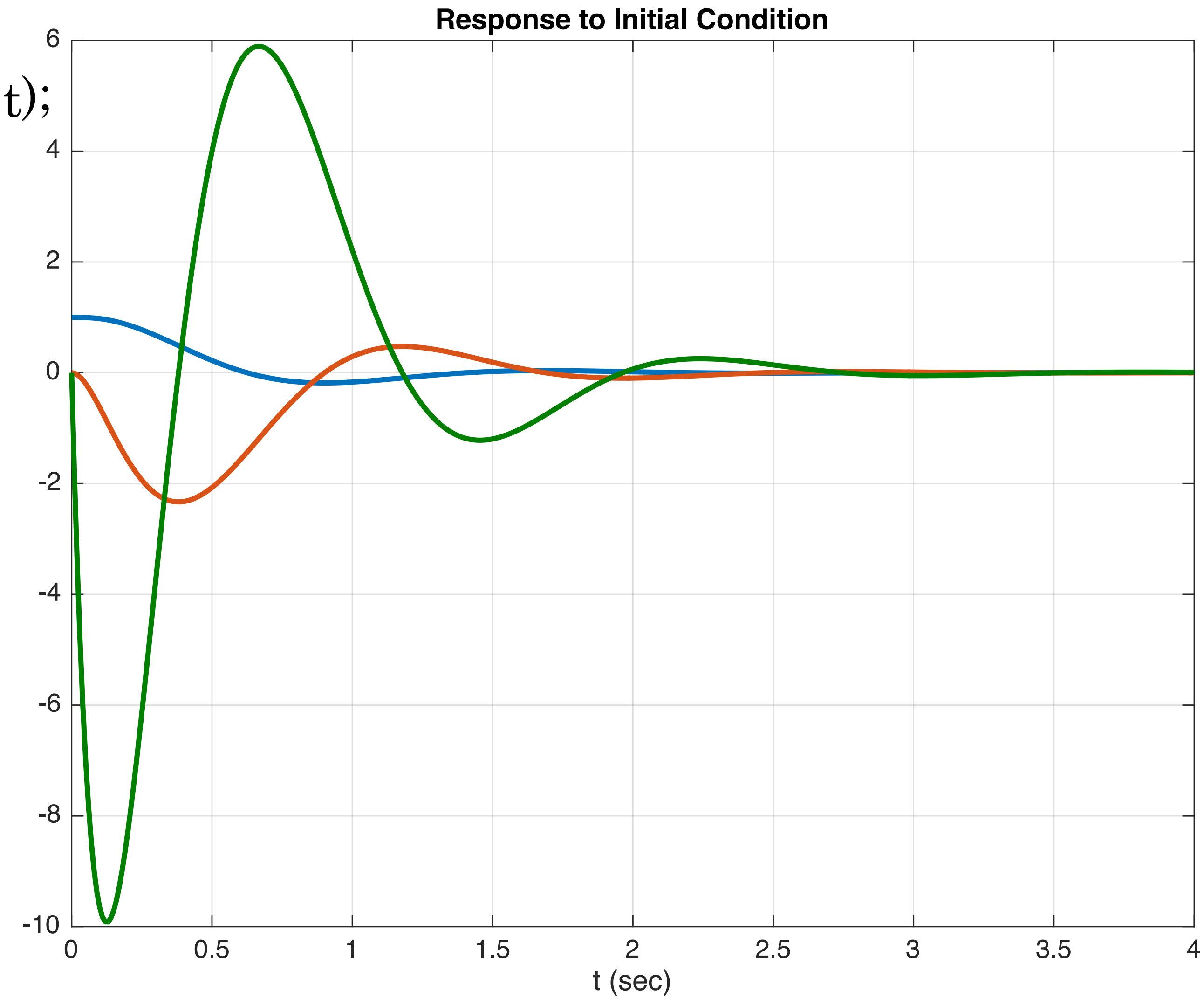
  >> size(x)

  ans = 401 3

Now let's plot the states at the output.

Example-1 – response to initial condition – *continues*-5:

```
>> A=[0  1  0;  0  0  1; -1  -5  -6]; B=[0; 0; 1];
>> P=[-2+4*j  -2-4*j  -10];  K=acker(A, B, P)        % K = 199  55  8
>> G2= ss(A - B*K, eye(3), eye(3), eye(3));
>> t = 0:0.01:4;
x = initial(G2, [1;0;0], t);
plot(t, x), grid
```



Response to Initial Condition for a MIMO and 3rd order System

Example-1 – response to initial condition – *cont.'s*-5: MATLAB - .m file

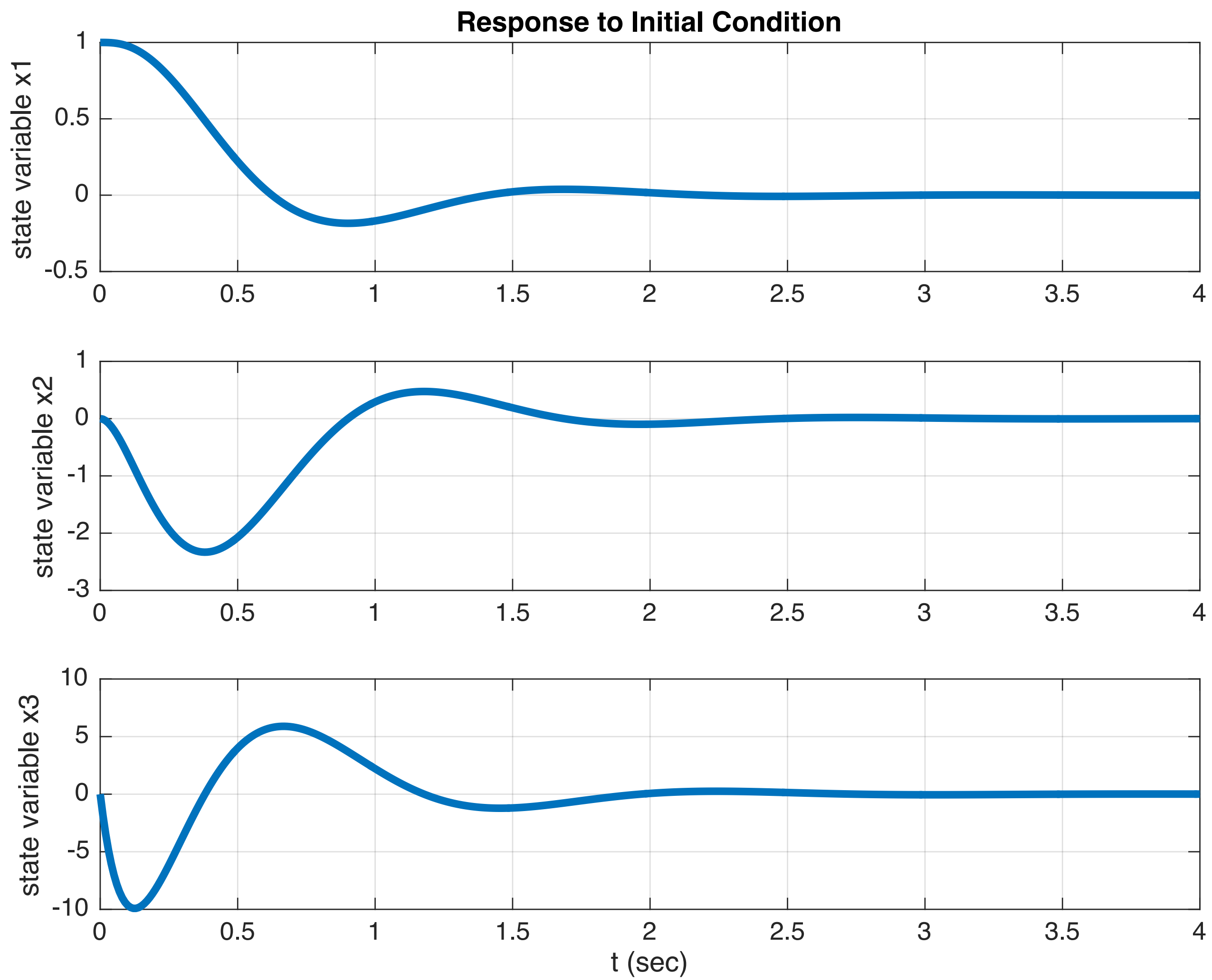
```
close all, clear, clc
A=[0  1  0;  0  0  1; -1  -5  -6]; B=[0; 0; 1];
P=[-2+4*j  -2-4*j  -10];
K=acker(A, B, P)    % K = 199  55  8
% G1= ss(A, eye(3), eye(3), eye(3));  % no control - MIMO
G2= ss(A - B*K, eye(3), eye(3), eye(3)); % with control
t = 0:0.01:4; x = initial(G2, [1;0;0], t);
x1=[1 0 0]*x'; x2=[0 1 0]*x'; x3=[0 0 1]*x';
```

```
subplot(3,1,1); plot(t,x1, 'linewidth', 3), grid,
title('Response to Initial Condition'),
ylabel('state variable x1')
subplot(3,1,2); plot(t,x2, 'linewidth', 3), grid,
ylabel('state variable x2')
subplot(3,1,3);
plot(t,x3, 'linewidth', 3), grid,
xlabel('t (sec)'),  ylabel('state variable x3')
figure(2)
subplot(2,1,1)
plot(t, x, 'linewidth', 3), grid on
title('Responses to Initial Condition'),
xlabel('t (sec)'), ylabel('state variables x1, x2, x3')
```

```
u=-K*x';
subplot(2,1,2)
plot(t, u, 'linewidth', 3), grid on
title('Control Signal to Initial Condition'),
xlabel('t (sec)'),
ylabel('Control Signal u')
```

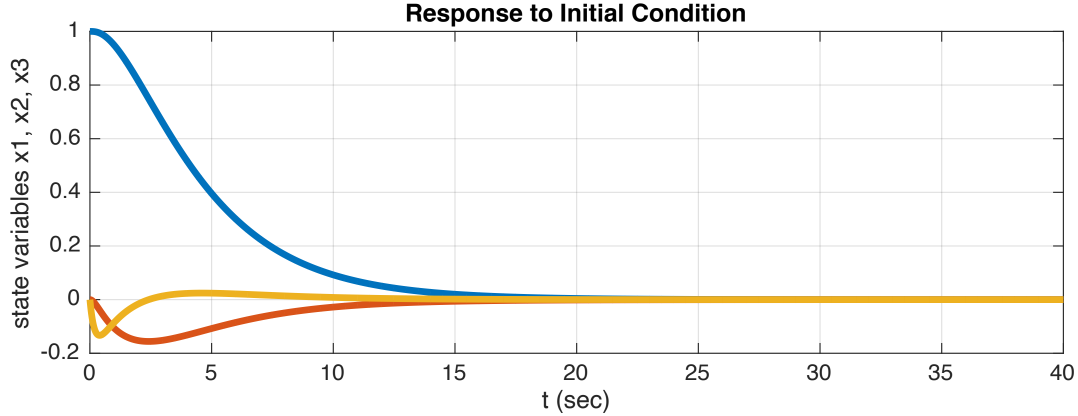
Example-1 – response to initial condition – *continues*-6:

```
>> G2= ss(A - B*K, eye(3), eye(3), eye(3));
>> t = 0:0.01:4; x = initial(G2, [1;0;0], t);
>> x1=[1 0 0]*x'; x2=[0 1 0]*x'; x3=[0 0 1]*x'; subplot(3,1,1); plot(t,x1) …
```

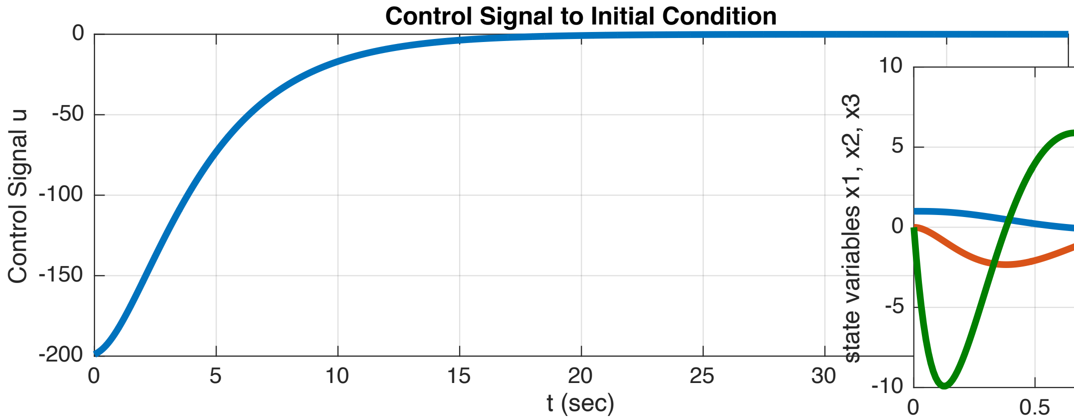


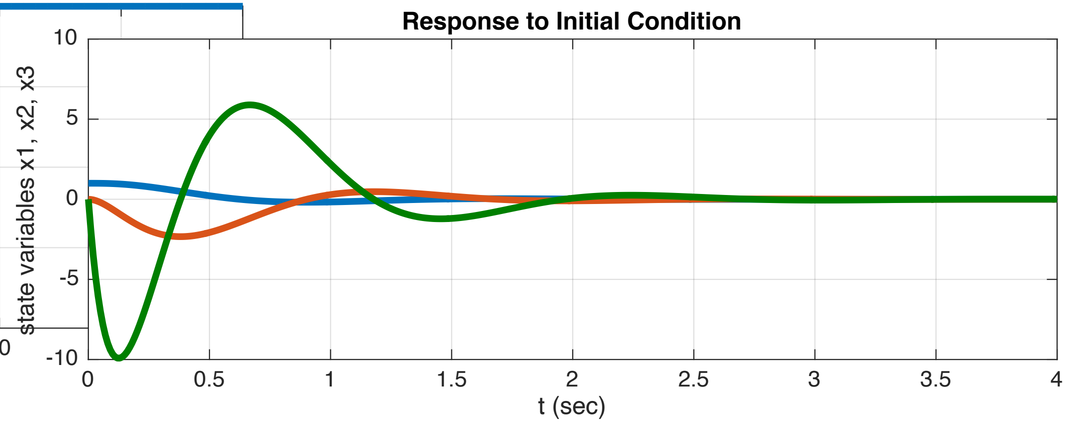
Response to Initial Condition for a MIMO and 3rd order System

Example-1 – response to initial condition – *continues*-7:

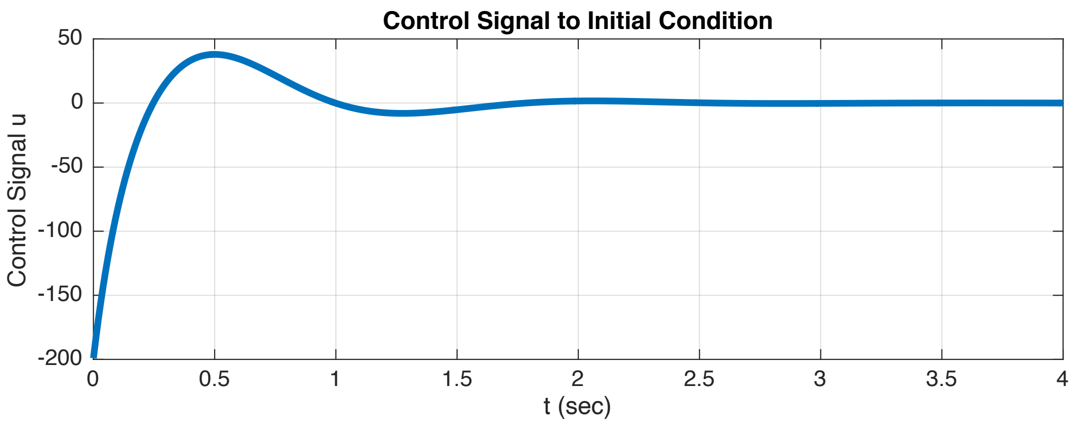


Response to Initial Condition for a MIMO with No Control and 3rd Order System and the Control Signal, u.





Response to Initial Condition for a MIMO and 3rd Order System and the Control Signal, u.

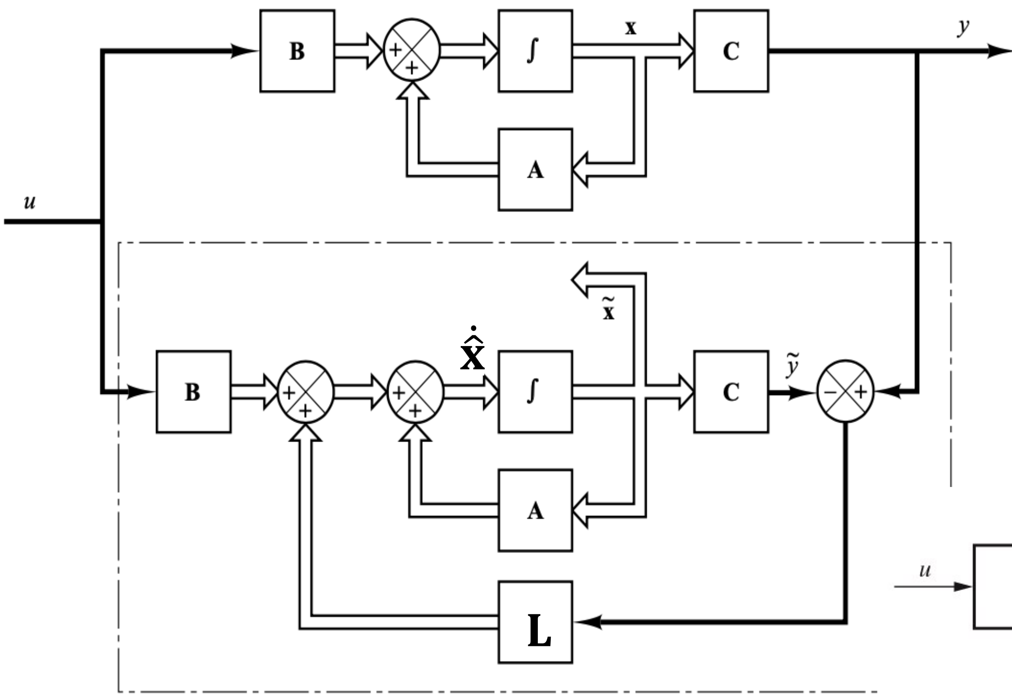


Full-order state observer - revisited

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u; \, y = \mathbf{C}\mathbf{x}$$
$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}u + \mathbf{L}(y - \hat{y})$$
$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}u + \mathbf{L}(y - \mathbf{C}\hat{\mathbf{x}})$$
$$\dot{\hat{\mathbf{x}}} = (\mathbf{A} - \mathbf{L}\mathbf{C})\hat{\mathbf{x}} + \mathbf{B}u + \mathbf{L}y$$

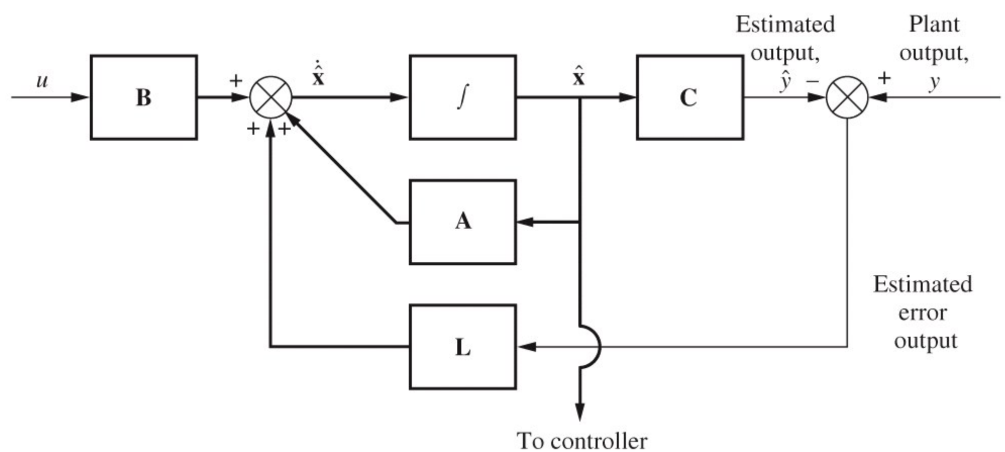
$$\dot{\mathbf{x}} - \dot{\hat{\mathbf{x}}} = \mathbf{A}(\mathbf{x} - \hat{\mathbf{x}}) - \mathbf{L}\mathbf{C}(\mathbf{x} - \hat{\mathbf{x}})$$

$$\dot{\boldsymbol{e}}_x = (\mathbf{A} - \mathbf{L}\mathbf{C})\boldsymbol{e}_x$$
$$y - \hat{y} = \mathbf{C}\boldsymbol{e}_x$$



Full-order state observer

A SISO system, where input u and output y are scalars.



(c)

Dual Problem – Pole Placement & Observer

- The problem of designing a full-order observer becomes that of determining the observer gain matrix $\mathbf{L}$ such that the error dynamics defined by $(\mathbf{A} - \mathbf{LC})$ are asymptotically stable with sufficient speed of response.
- Thus, the problem here becomes the same as the pole-placement problem we discussed before.
- In fact, the two problems are mathematically the same.
- This property is called **duality**.
- Consider the system defined by $\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{B}u;\ y = \mathbf{Cx}$
- In designing the full-order state observer, we may solve the dual problem, that is, solving the pole-placement problem for the dual system $\dot{\mathbf{z}} = \mathbf{A}^T\mathbf{z} + \mathbf{C}^T v;\ n = \mathbf{B}^T\mathbf{z}$
- Assuming the control signal v to be $v = -\mathbf{Kz}$
- If the dual system is completely state controllable, then the state feedback gain matrix $\mathbf{K}$ can be determined such that matrix $\mathbf{A}^T - \mathbf{C}^T\mathbf{K}$ will yield a set of the desired eigenvalues, $\mu_1, \mu_2, \ldots\ \mu_n$.

Dual Problem – Pole Placement & Observer, *cont.s*...

- The desired eigenvalues of the state-feedback gain matrix of the dual system,

$$|s\mathbf{I} - \mathbf{A}^T + \mathbf{C}^T\mathbf{K}| = (s-\mu_1)(s-\mu_2)\dots(s-\mu_3)$$

- Note that the eigen values of $\mathbf{A}^T - \mathbf{C}^T\mathbf{K}$ and that of $\mathbf{A} - \mathbf{K}^T\mathbf{C}$, are the same,

$$|s\mathbf{I} - \mathbf{A}^T + \mathbf{C}^T\mathbf{K}| = |s\mathbf{I} - \mathbf{A} + \mathbf{K}^T\mathbf{C}| = |s\mathbf{I} - (\mathbf{A} - \mathbf{K}^T\mathbf{C})|$$

- Comparing the right hand side of the characteristic equation with that of the observer's, $\dot{\hat{\mathbf{x}}} = (\mathbf{A} - \mathbf{LC})\hat{\mathbf{x}} + \mathbf{B}u + \mathbf{L}y$, we see that $\mathbf{L}$ and $\mathbf{K}$ are related:

$$\boxed{\mathbf{L} = \mathbf{K}^T}$$

- Thus we can make use of Ackermann's formula.

Necessary and Sufficient Condition for State Observation

- As discussed, a necessary and sufficient condition for the determination of the observer gain matrix $\mathbf{L}$ for the desired eigenvalues of $\mathbf{A} - \mathbf{LC}$ is that the dual of the original system

$$\dot{\mathbf{z}} = \mathbf{A}^T\mathbf{z} + \mathbf{C}^T v; \quad n = \mathbf{B}^T\mathbf{z}$$

be completely state controllable. That is the complete state controllability condition for this dual system is that the rank of observability matrix must be full:

$$\mathbf{O}^T = \left[\mathbf{C} \;\; \mathbf{CA} \;\; \mathbf{CA}^2 \;\; \dots \;\; \mathbf{CA}^{n-1}\right]$$

Observer Design via Ackermann

- Consider the system defined by $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u;\ y = \mathbf{C}\mathbf{x}$
- Ackermann's formula for pole placement for this system was,

$$\mathbf{K} = [0 \quad 0 \quad \dots \quad 0 \ \ 1][\mathbf{B} \ \ \mathbf{AB} \ \ \dots \ \ \mathbf{A}^{n-1}\mathbf{B}]^{-1}\phi(\mathbf{A})$$

- The dual of the system, defined by $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$, was,

$$\dot{\mathbf{z}} = \mathbf{A}^T\mathbf{z} + \mathbf{C}^T v;\ \ n = \mathbf{B}^T\mathbf{z}$$

- The preceding Ackermann's formula for pole placement is now modified to
- $\mathbf{K} = [0 \quad 0 \quad \dots \quad 0 \ \ 1][\mathbf{C}^T \ \ \mathbf{A}^T\mathbf{C}^T \ \dots \ \ (\mathbf{A}^T)^{n-1}\mathbf{C}^T]^{-1}\phi(\mathbf{A}^T)$
- As stated earlier, the state observer gain matrix $\mathbf{L}$ is given by $\mathbf{K}^T$ or $\mathbf{K}^*$, where $\mathbf{K}$ is given by

$$\mathbf{L} = \mathbf{K}^T = \phi(\mathbf{A}^T)^T \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \\ \vdots \\ \mathbf{CA}^{n-2} \\ \mathbf{CA}^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = \phi(\mathbf{A}) \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \\ \vdots \\ \mathbf{CA}^{n-2} \\ \mathbf{CA}^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

where $\phi(\mathbf{A})$ is the desired characteristic polynomial for the state observer,

Comments on Selecting the Best $\mathbf{L}$ - 1

- Referring to the figure for full-state observer, notice that the feedback signal through the observer gain matrix $\mathbf{L}$ serves as a correction signal to the plant model to account for the unknowns in the plant.
- If significant unknowns are involved, the feedback signal through the matrix $\mathbf{L}$ should be relatively large.
- However, if the output signal is contaminated significantly by disturbances and measurement noises, then the output y is not reliable and the feedback signal through the matrix $\mathbf{L}$ should be relatively small.
- In determining the matrix $\mathbf{L}$, we should carefully examine the effects of disturbances and noise involved in the output y.
- Remember that the observer gain matrix $\mathbf{L}$ depends on the desired characteristic equation,

$$(s-\mu_1)(s-\mu_2)\ldots(s-\mu_3)=0$$

- The choice of a set of $\mu_1, \mu_2, \ldots\ \mu_n$ is, in many instances, not unique.

Comments on Selecting the Best **L** - 2

- As a general rule, however, the observer poles must be two to five times faster than the controller poles to make sure the observation error (estimation error) converges to zero quickly.
- This means that the observer estimation error decays two to five times faster than does the state vector $\mathbf{x}$.
- Such faster decay of the observer error compared with the desired dynamics makes the controller poles dominate the system response.
- It is important to note that if sensor noise is considerable, we may choose the observer poles to be slower than two times the controller poles, so that the bandwidth of the system will become lower and can smooth the noise.
- In this case the system response will be strongly influenced by the observer poles.
- The selection of the best matrix **L** boils down to a compromise between speedy response and sensitivity to disturbances and noises.

Quadratic Optimal Regulator Systems *or simply* Linear Quadratic Optimal Regulator

- The quadratic optimal control method provides a systematic way of computing the state feedback control gain matrix.
- Consider the system defined by $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$, where the optimal control vector $\mathbf{K}$ is determined by the control law $\mathbf{u} = -\mathbf{K}\mathbf{x}$ so as to **minimize** the performance index or **cost function** J,

$$J = \int_0^\infty (\mathbf{x}^*\mathbf{Q}\mathbf{x} + \mathbf{u}^*\mathbf{R}\mathbf{u})\,dt$$

- Or, since we work with system matrices carrying only real elements,

$$J = \int_0^\infty (\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u})\,dt$$

- Here, $\mathbf{Q}$ is a positive-definite (or positive-semidefinite) Hermitian or real symmetric matrix and $\mathbf{R}$ is a positive-definite Hermitian or real symmetric matrix.
- Note, $\mathbf{x}$ and $\mathbf{u}$ are functions of time: $\mathbf{x}(t)$ and $\mathbf{u}(t)$

Quadratic Optimal Regulator Systems *or simply* Linear Quadratic Optimal Regulator – *cont.'s*…

- The cost function,

$$J = \int_0^{\infty} (\mathbf{x}^T \mathbf{Q}\mathbf{x} + \mathbf{u}^T \mathbf{R}\mathbf{u})\, dt$$

- Note that the second term on the right-hand side of the equation accounts for the expenditure of the energy of the control signals.
- The firs term is associated with the importance of the states.
- The matrices $\mathbf{Q}$ and $\mathbf{R}$ determine the relative importance of the states and the expenditure energy needed.
- We assume that the control vector $\mathbf{u}(t)$ is unconstrained.
- The linear control law given by $\mathbf{u} = -\mathbf{K}\mathbf{x}$ to be the optimal control law.
- Therefore, if the unknown elements of the matrix $\mathbf{K}$ are determined so as to minimize the performance index, or the cost function, then $\mathbf{u}(\mathrm{t}) = -\mathbf{K}\mathbf{x}(t)$ is optimal for any initial state $\mathbf{x}(0)$.

Linear Quadratic Optimal Regulator – *cont.'s…*

- The cost function as was given,

$$J = \int_0^{\infty} (\mathbf{x}^T \mathbf{Q}\mathbf{x} + \mathbf{u}^T \mathbf{R}\mathbf{u})\, dt$$

- Now let us solve the optimization problem. Making use of the control law, $\mathbf{u} = -\mathbf{K}\mathbf{x}$, the state equation becomes $\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}$
- We assume that the matrix $\mathbf{A} - \mathbf{B}\mathbf{K}$ is stable, i.e., eigenvalues of $\mathbf{A} - \mathbf{B}\mathbf{K}$ have negative real parts. Substitute this into J,

$$J = \int_0^{\infty} (\mathbf{x}^T \mathbf{Q}\mathbf{x} + \mathbf{x}^T \mathbf{K}^T \mathbf{R}\mathbf{K}\mathbf{x})\, dt$$

$$= \int_0^{\infty} \mathbf{x}^T (\mathbf{Q} + \mathbf{K}^T \mathbf{R}\mathbf{K})\mathbf{x}\, dt$$

- Now, let us set $\mathbf{x}^T (\mathbf{Q} + \mathbf{K}^T \mathbf{R}\mathbf{K})\mathbf{x} = -\frac{d}{dt}(\mathbf{x}^T \mathbf{P}\mathbf{x})$

  where $\mathbf{P}$ is a positive-definite Hermitian or real symmetric matrix. Then we obtain,

$$\mathbf{x}^T \left(\mathbf{Q} + \mathbf{K}^T \mathbf{R}\mathbf{K}\right)\mathbf{x} = -\dot{\mathbf{x}}^T \mathbf{P}\mathbf{x} - \mathbf{x}^T \mathbf{P}\dot{\mathbf{x}} = -\mathbf{x}^T [(\mathbf{A} - \mathbf{B}\mathbf{K})^T \mathbf{P} + \mathbf{P}(\mathbf{A} - \mathbf{B}\mathbf{K})]\mathbf{x}$$

Linear Quadratic Optimal Regulator – *cont.'s*…

- Let's re-write the last equation below,

$$\mathbf{x}^{T}\left(\mathbf{Q}+\mathbf{K}^{T}\mathbf{R}\mathbf{K}\right)\mathbf{x}=-\dot{\mathbf{x}}^{T}\mathbf{P}\mathbf{x}-\mathbf{x}^{T}\mathbf{P}\dot{\mathbf{x}}=-\mathbf{x}^{T}\left[(\mathbf{A}-\mathbf{B}\mathbf{K})^{T}\mathbf{P}+\mathbf{P}(\mathbf{A}-\mathbf{B}\mathbf{K})\right]\mathbf{x}$$

- Comparing both sides of this last equation and noting that this equation must hold true for any $\mathbf{x}$, it is required that

$$(\mathbf{A}-\mathbf{B}\mathbf{K})^{T}\mathbf{P}+\mathbf{P}(\mathbf{A}-\mathbf{B}\mathbf{K})=-(\mathbf{Q}+\mathbf{K}^{T}\mathbf{R}\mathbf{K})$$

- It can be proved that if $\mathbf{A} - \mathbf{B}\mathbf{K}$ is a stable matrix, there exists a positive-definite matrix $\mathbf{P}$ that satisfies the equation above.
- Hence our procedure is to determine the elements of $\mathbf{P}$ from the equation above and see if it is positive definite.
- Note that more than one matrix $\mathbf{P}$ may satisfy this equation. If the system is stable, there always exists one positive-definite matrix $\mathbf{P}$ to satisfy this equation. This means that, if we solve this equation and find one positive-definite matrix $\mathbf{P}$, the system is stable.
- Other $\mathbf{P}$ matrices that satisfy this equation are not positive definite and must be discarded.

Linear Quadratic Optimal Regulator – *cont.'s*...

- The performance index J can be evaluated as

$$J = \int_0^\infty \mathbf{x}^T(\mathbf{Q} + \mathbf{K}^T\mathbf{R}\mathbf{K})\mathbf{x}\,dt = -\mathbf{x}^T\mathbf{P}\mathbf{x}\Big|_0^\infty = -\mathbf{x}^T(\infty)\mathbf{P}\mathbf{x}(\infty) + \mathbf{x}^T(0)\mathbf{P}\mathbf{x}(0)$$

- Since all eigenvalues of $\mathbf{A} - \mathbf{B}\mathbf{K}$ are assumed to have negative real parts, $\mathbf{x}(\infty) \rightarrow \mathbf{0}$. Therefore, we obtain

$$J = \mathbf{x}^T(0)\mathbf{P}\mathbf{x}(0)$$

- Thus, the performance index or cost function J can be obtained in terms of the initial condition $\mathbf{x}(0)$ and $\mathbf{P}$.
- To obtain the solution to the quadratic optimal control problem, we proceed as follows: Since $\mathbf{R}$ has been assumed to be a positive-definite Hermitian or real symmetric matrix, we can write

$$\mathbf{R} = \mathbf{T}^T\mathbf{T}$$

  where $\mathbf{T}$ is a nonsingular matrix.
- The equation $(\mathbf{A} - \mathbf{B}\mathbf{K})^T\mathbf{P} + \mathbf{P}(\mathbf{A} - \mathbf{B}\mathbf{K}) = -(\mathbf{Q} + \mathbf{K}^T\mathbf{R}\mathbf{K})$ can be written as,

Linear Quadratic Optimal Regulator – *cont.'s…*

$$(\mathbf{A}^T - \mathbf{K}^T\mathbf{B}^T)\mathbf{P} \ + \ \mathbf{P}(\mathbf{A} - \mathbf{B}\mathbf{K}) + \mathbf{Q} + \mathbf{K}^T\mathbf{T}^T\mathbf{T}\mathbf{K} \ = \ \mathbf{0}$$

which can be rewritten as

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} + [\mathbf{T}\mathbf{K} - (\mathbf{T}^T)^{-1}\mathbf{B}^T\mathbf{P}]^T[\mathbf{T}\mathbf{K} \ - \ (\mathbf{T}^T)^{-1}\mathbf{B}^T\mathbf{P}] - \ \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} \ + \mathbf{Q} = \mathbf{0}$$

The minimization of J with respect to $\mathbf{K}$ requires the minimization of

$$\mathbf{x}^T[\mathbf{T}\mathbf{K} \ - \ (\mathbf{T}^T)^{-1}\mathbf{B}^T\mathbf{P}]^T[\mathbf{T}\mathbf{K} \ - \ (\mathbf{T}^T)^{-1}\mathbf{B}^T\mathbf{P}]\mathbf{x}$$

with respect to $\mathbf{K}$.

Since this last expression is nonnegative, the minimum occurs when it is zero, or when

$$\mathbf{T}\mathbf{K} = (\mathbf{T}^T)^{-1}\mathbf{B}^T\mathbf{P}$$
$$\mathbf{K} = \mathbf{T}^{-1}(\mathbf{T}^T)^{-1}\mathbf{B}^T\mathbf{P} \ = \ \mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}$$

This equation gives the optimal matrix $\mathbf{K}$.

Thus, the optimal control law to the LQR control problem when the cost function $J = \int_0^\infty (\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u})\, dt$ is linear and is given by

$$\mathbf{u}(t) = -\mathbf{K}\mathbf{x}(t) = -\boldsymbol{R}^{-1}\mathbf{B}^T\mathbf{P}\mathbf{x}(t)$$

Linear Quadratic Optimal Regulator – *cont.'s…*

The matrix $\mathbf{P}$ in

$$\mathbf{K} = \mathbf{T}^{-1}(\mathbf{T}^T)^{-1}\mathbf{B}^T\mathbf{P} \;\; \rightarrow \;\; \mathbf{K} = \mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}$$

must satisfy →

$$(\mathbf{A} - \mathbf{BK})^T\mathbf{P} + \mathbf{P}(\mathbf{A} - \mathbf{BK}) = -(\mathbf{Q} + \mathbf{K}^T\mathbf{RK})$$

or the following reduced equation:

$$\mathbf{A}^T\mathbf{P} + \mathbf{PA} - \mathbf{PBR}^{-1}\mathbf{B}^T\mathbf{P} + \mathbf{Q} = \mathbf{0} \quad (\mathbf{ARE})$$

- This last equation is called the **reduced-matrix Riccati equation** or **ARE**.

The design steps may be stated as follows:

1. Solve the the reduced-matrix Riccati equation for the matrix $\mathbf{P}$

   (If a positive-definite matrix $\mathbf{P}$ exists (certain systems may not have a positive-definite matrix $\mathbf{P}$), the system is stable, or matrix $\mathbf{A} - \mathbf{BK}$ is stable).
2. Substitute this matrix $\mathbf{P}$ into $\mathbf{K} = \mathbf{T}^{-1}(\mathbf{T}^T)^{-1}\mathbf{B}^T\mathbf{P} \; = \; \mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}$

   The resulting matrix $\mathbf{K}$ is the optimal matrix.

Summary of LQR Design

- The cost function to minimize $J = \int_0^\infty (\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u})\, dt$
- Given the plant with matrices $\mathbf{A}$ and $\mathbf{B}$, we need to decide
  - ➢ how much do we care about the states, $\mathbf{x}$ (even each state separately) and
  - ➢ how much do we care about the control effort, $\mathbf{u}$
- Choose $\mathbf{Q}$ and $\mathbf{R}$, accordingly.
- We have to solve the Algebraic Riccati Equation (ARE) for $\mathbf{P}$.
- Compute the gain matrix $\mathbf{K} = \ \mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}$
- We can find more than one $\mathbf{P}$ and hence obtain more than one $\mathbf{K}$. We need to choose $\mathbf{K}$ to yield a stable solution.
- In MATLAB it is so easy. First enter the matrices A,B,Q,R then,

```
>> [K P E] = lqr(A,B,Q,R)
```

K: Full-State Feedback Gain Matrix (FSFB gains)

P: Solution to ARE

E: Eigen values of FSFB controller that is eig(A-BK)

LQR Design in MATLAB - Example Mass/Spring/Damper system

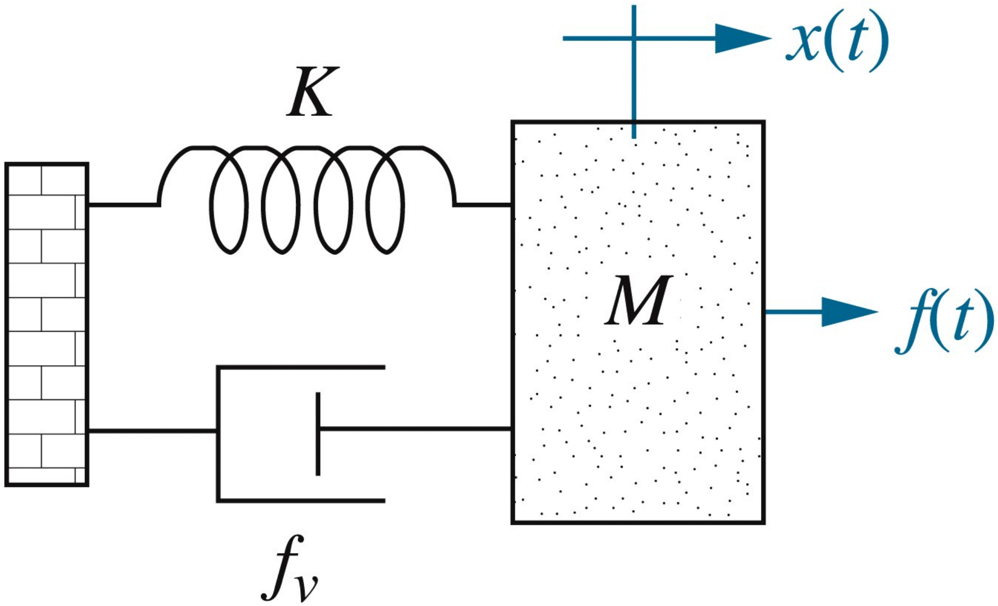
Problem: For the given simple translational mechanical system,

a) **Write the equation of motion** using Newton's first law (as a differential equation).

b) **Transfer function representation** $G(s) = X(s)/F(s)$.

c) **State space representation** of the same system with $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$.

d) **Design a FSFB Controller with LQR** for the values of mass, coefficient of viscous friction and spring constant are M=1kg, f_v=0.2N-s/m and K= 0 N/m (assuming the body is rigid), respectively.



LQR Design in MATLAB – Example, *cont.'s*…1

a) The equation of motion

$$\sum F = Ma \;\Rightarrow\; -f_v \frac{dx(t)}{dt} - Kx(t) + f(t) = M\frac{d^2x(t)}{dt^2} \;\Rightarrow$$

$$M\frac{d^2x}{dt^2} + f_v\frac{dx}{dt} + Kx = f(t)$$

b) The transfer function representation,

Taking the Laplace transform of each side of the above equation for zero initial conditions,

$$(Ms^2 + f_v s + K)X(s) = F(s)$$

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{Ms^2 + f_v s + K} = \frac{\frac{1}{M}}{s^2 + \frac{f_v}{M}s + \frac{K}{M}}$$

c) The state space representation,

The states could be selected as the position and velocity and the output as the position

$x_1 = x, x_2 = v = \dot{x}_1$ and $y = x_1 = x$

LQR Design in MATLAB – Example, *cont.'s*…2

c) The state space representation,

The states are position and velocity, the output position

$x_1 = x, x_2 = v = \dot{x}_1$ and $y = x_1 = x$

The equation of motion $-f_v \frac{dx(t)}{dt} - Kx(t) + f(t) = M \frac{d^2x(t)}{dt^2}$ now can be written in terms of the state variables selected,

$x_1 = x, x_2 = v = \dot{x}_1$ and $y = x_1 = x; f(t) = u(t)$

$-f_v v - Kx + u(t) = M\dot{v}$ ➔ $-f_v x_2 - Kx_1 + u(t) = M\ddot{x}_2$

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{K}{M} & -\frac{f_v}{M} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} u(t)$$

$$y(t) = [1 \;\; 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Using the numerical values we get the $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ matrices as,

```
>> A=[0  1; 0  -0.2]; B=[0;  1];
```

LQR Design in MATLAB – Example, *cont.'s*…3

- **Step-1:** >> A=[0 1; 0 -0.2]; B=[0; 1];
- **Step-2:** Choosing $\mathbf{Q}$ and $\mathbf{R}$: >> Q=[1 0; 0 1]; R=[0.01];
- **Step-3:** Solve ARE for P
- **Step-4:** Solve K using **[K P E] = lqr(A,B,Q,R)**

```
>> A=[0  1; 0  -0.2]; B=[0;  1];
>> Q=[1  0; 0  1]; R=[0.01];
>> [K P E] = lqr(A,B,Q,R)
        K =   10.0000   10.7563
        P =    1.0956    0.1000    0.1000    0.1076
        E =   -1.0049   -9.9514
```

- **Responses to initial conditions,** $\mathbf{x(0)} = [\pi,\ -2]$. The aim is to regulate the initial conditions to zero.

LQR Design in MATLAB – Example, *cont.'s*…3

- **Scenarios:**

Scenario	Description	Q	R	K, E	Comment
1	Control is cheap, nonzero state is expensive	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$[0.01]$	$[10 \quad 10.765]$ $[-1.0049 - 9.95]$	Similar gains
2	Control is expensive, nonzero state is cheap	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$[1000]$	$[0.0316\ 0.123]$ $-0.16 \pm j\, 0.075$	Smaller gains
3	Only nonzero velocity state is expensive	$\begin{bmatrix} .001 & 0 \\ 0 & 10 \end{bmatrix}$	$[1]$	$[0.0316 \quad 2.978]$ $[-0.0100 - 3.1686]$	Second gain is larger