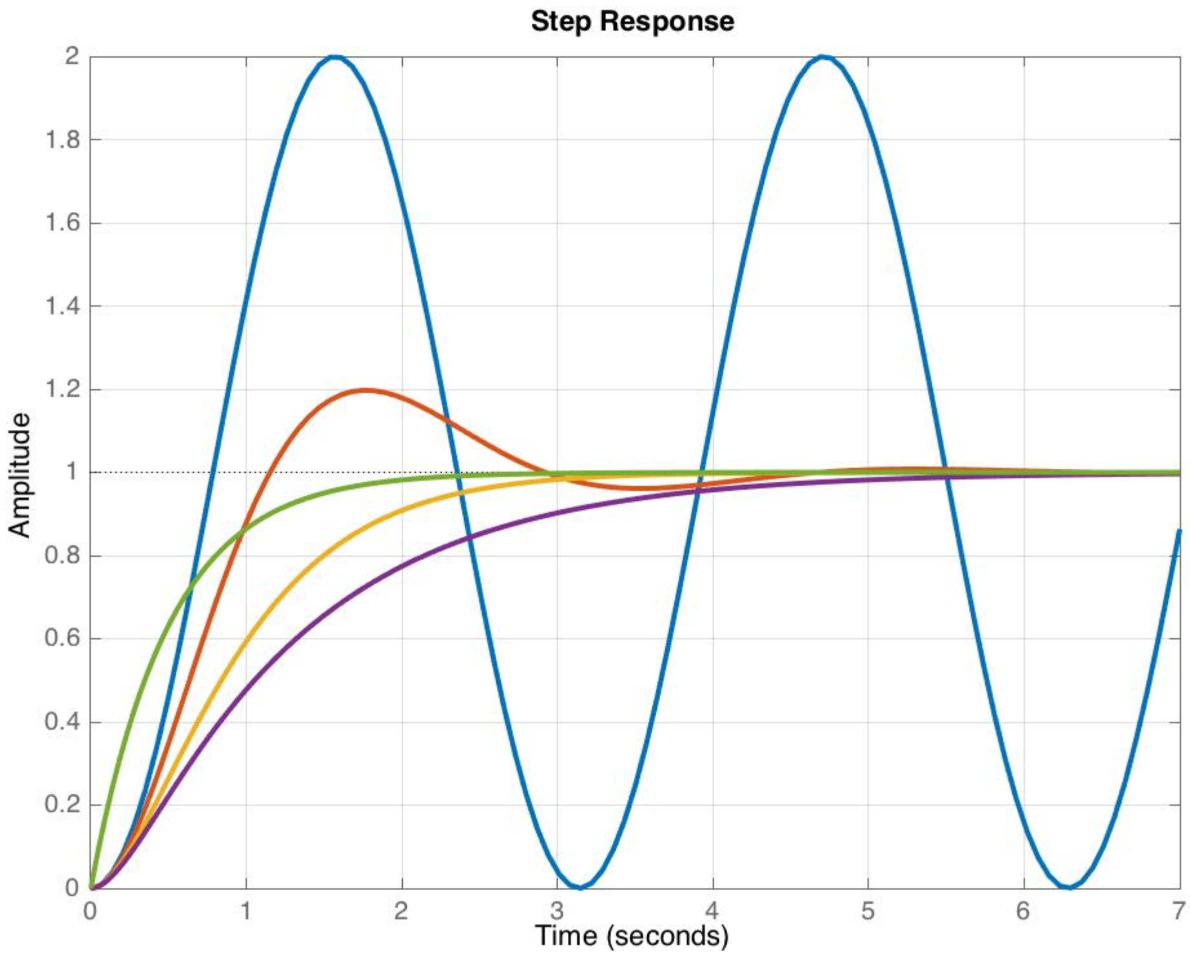
TIME RESPONSE



Textbook: Control Systems Engineering, Norman S. Nise, CSPU, CA, US
Instructor: Şeref Naci Engin, YTU & Bilgi Univ., Istanbul, TR

Learning Outcomes from Time Response

After reviewing this chapter the student will be able to:

- Use poles and zeros of transfer functions to determine the time response of a control system
- Describe quantitatively the transient response of **first-order systems**
- Write the general response of **second-order systems** given the pole location
- Find the damping ratio, ζ, and natural frequency, ω_n, of a second-order system
- Find the settling time, peak time, percent overshoot, and rise time for an underdamped second-order system
- Approximate higher-order systems and systems with zeros as first- or second-order systems

Poles, Zeros and System Response

- The output response of a system is the sum of two parts:
  - ✓ the *forced response* and
  - ✓ the *natural response*.
- The forced response is also called the *steady-state response* or *particular solution*. The *natural response* is also called the *homogeneous solution*.
- Although many techniques, such as solving a diff. eqn. or taking the inverse Laplace transform, enable us to evaluate this output response, they are laborious and time-consuming.
- The use of **poles** and **zeros** and their relationship to the time response of a system is a technique which allows us to simplify the evaluation of a system's response.

Definition of Poles and Zeros

<u>Poles of a Transfer Function</u>: The poles of a transfer function are

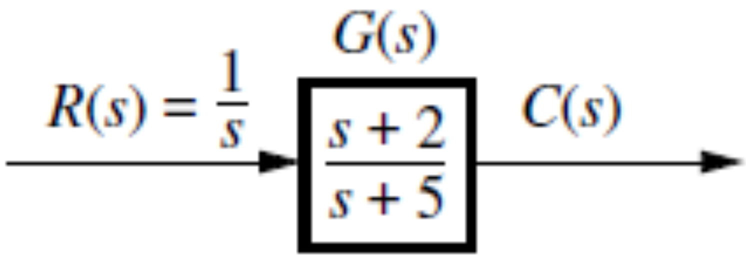
1. the values of the Laplace transform variable, s, that cause the transfer function to become infinite or
2. any roots of the denominator of the transfer function that are common to roots of the numerator.

<u>Zeros of a Transfer Function</u>: The zeros of a transfer function are

1. the values of the Laplace transform variable, s, that cause the transfer function to become zero, or
2. any roots of the numerator of the transfer function that are common to roots of the denominator.

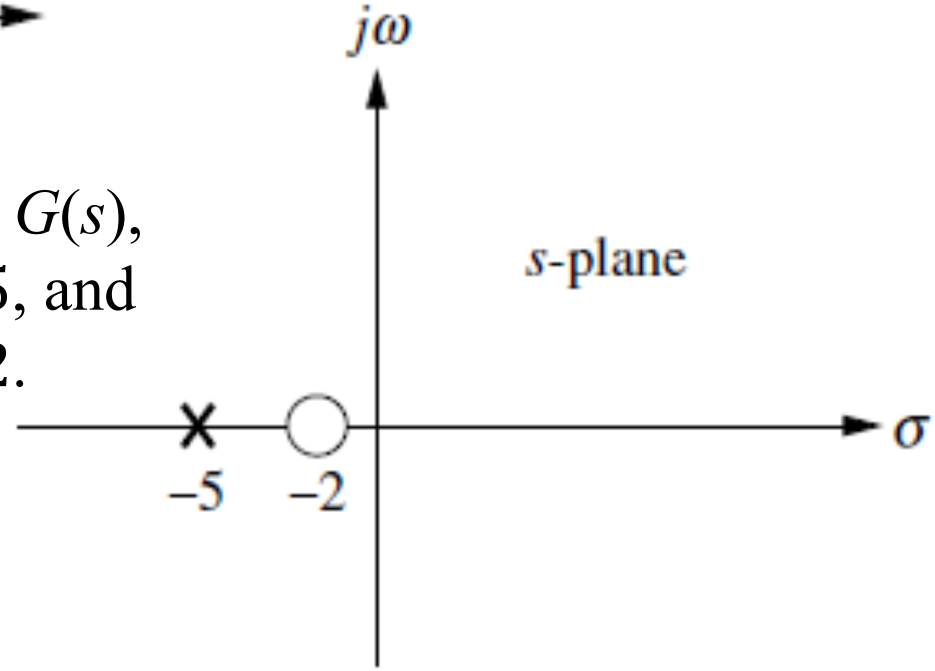
Example - 1 (first order system)

Consider the first order system given below.



Given the transfer function $G(s)$,

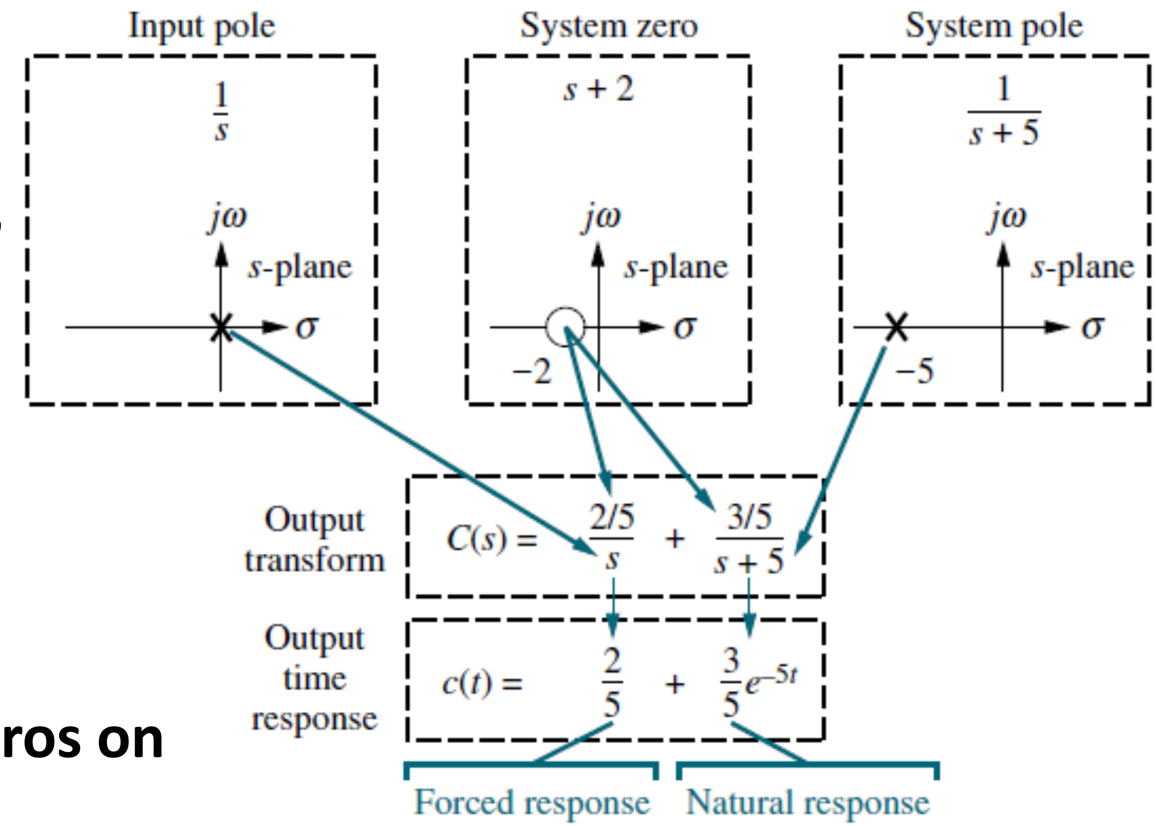
- a pole exists at $s = -5$, and
- a zero exists at $s = -2$.



The unit step response of the system can be determined as

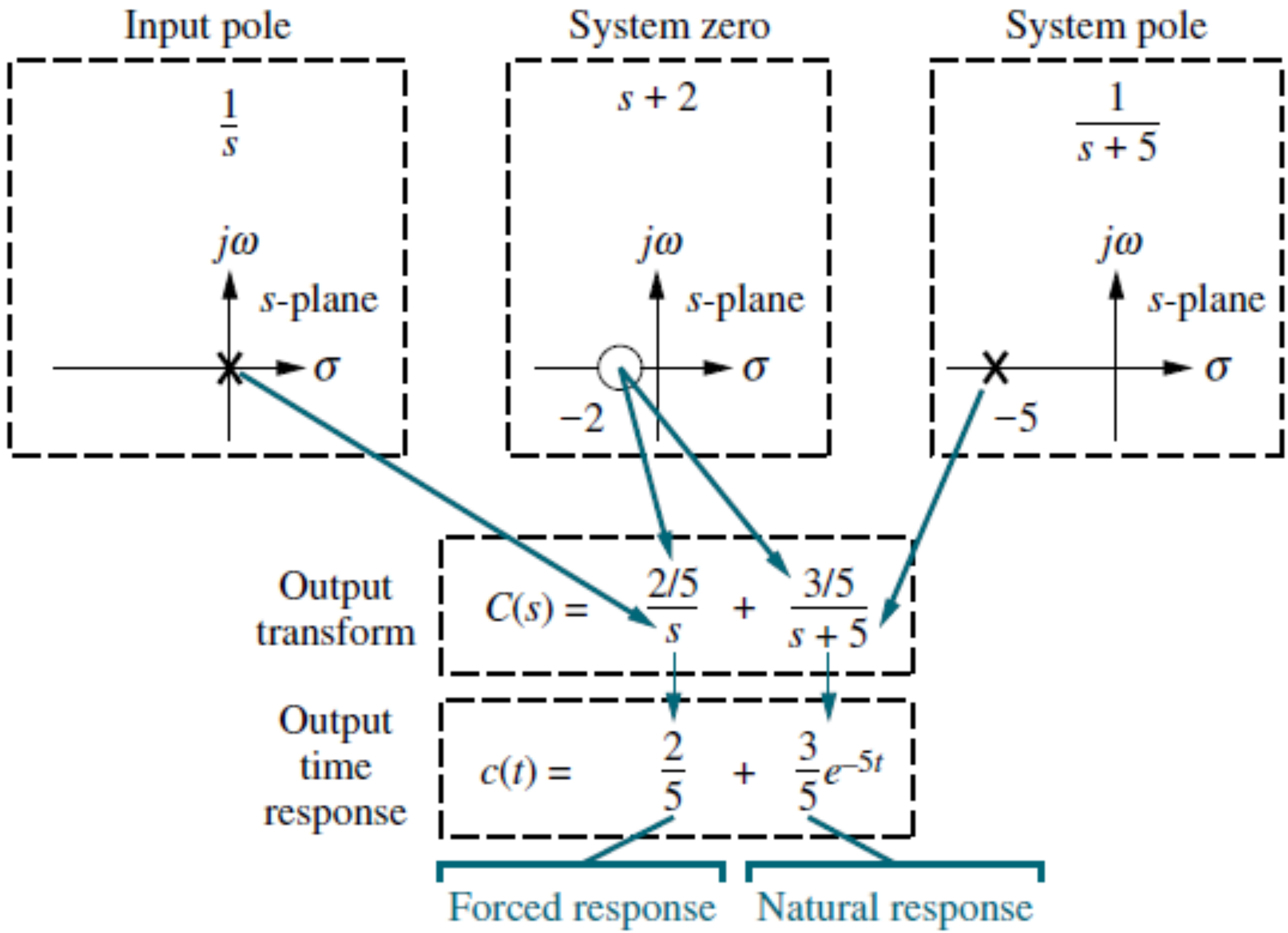
$$c(t) = \frac{2}{5} + \frac{3}{5}e^{-5t}$$

Example - 1
first order system, *cnt'd*.



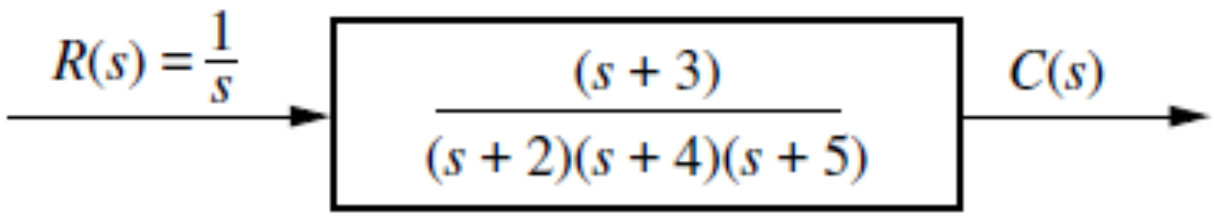
Impacts of poles and zeros on time response:

1. A pole of the input function generates the form of the *forced response,*
2. A pole of the trans. function generates the form of the *natural response,*
3. A pole on the real axis generates an *exponential* response.
   Thus, the further to the left a pole is on the negative real axis, the faster the exponential transient response will decay to zero.
4. The zeros and poles generate the *amplitudes* for both the forced and natural responses.


Input pole
$\frac{1}{s}$
$j\omega$
s-plane
σ
System zero
$s+2$
$j\omega$
s-plane
σ
-2
System pole
$\frac{1}{s+5}$
$j\omega$
s-plane
σ
-5
Output transform
$C(s) = \frac{2/5}{s} + \frac{3/5}{s+5}$
Output time response
$c(t) = \frac{2}{5} + \frac{3}{5}e^{-5t}$
Forced response
Natural response

Example - 2 (evaluating response using poles)

Given the system below, write the output, $c(t)$, in general terms. Specify the forced and natural parts of the solution.



Each system pole generates an exponential as part of the *natural response*. The input's pole generates the *forced response*. Thus,

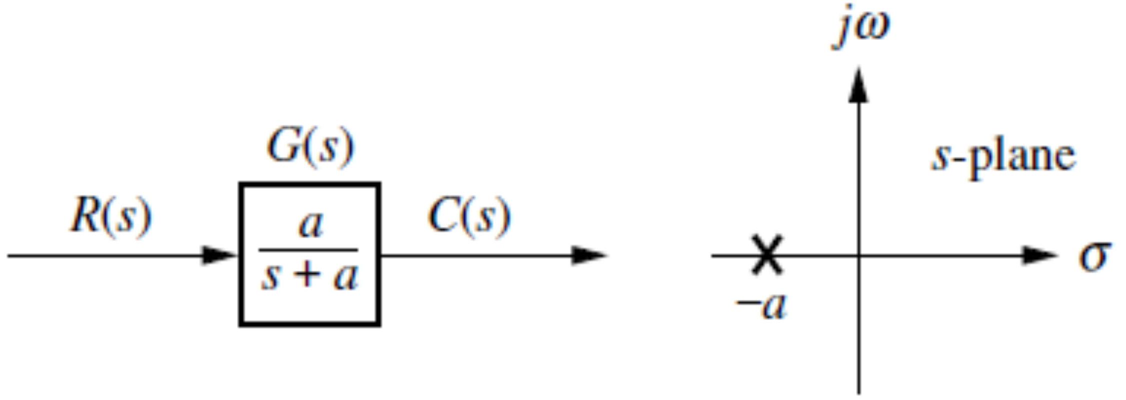
$$C(s) \equiv \underbrace{\frac{K_1}{s}}_{\text{Forced response}} + \underbrace{\frac{K_2}{s+2} + \frac{K_3}{s+4} + \frac{K_4}{s+5}}_{\text{Natural response}}$$

Taking the inverse Laplace transform, we get

$$c(t) = \underbrace{K_1}_{\text{Forced response}} + \underbrace{K_2 e^{-2t} + K_3 e^{-4t} + K_4 e^{-5t}}_{\text{Natural response}}$$

FIRST ORDER SYSTEMS

We now discuss first-order systems without zeros to define a performance specification for such a system.



If the input is a unit step, $R(s)=1/s$, the Laplace transform of the step response $C(s)$ is,

$$C(s) = R(s)G(s) = \frac{a}{a(s+a)}$$

Taking the inverse transform, the step response is given by

$$c(t) = c_f(t) + c_n(t) = 1 - e^{-at}$$

FIRST ORDER SYSTEMS *cont.'s*.

$$c(t) = c_f(t) + c_n(t) = 1 - e^{-at}$$

- Here, the input pole at the origin generates the forced response, and
- the system pole at $-a$ generates the natural response.
- Thus, the only parameter needed to describe the transient response is a.
- When $t = 1/a$,

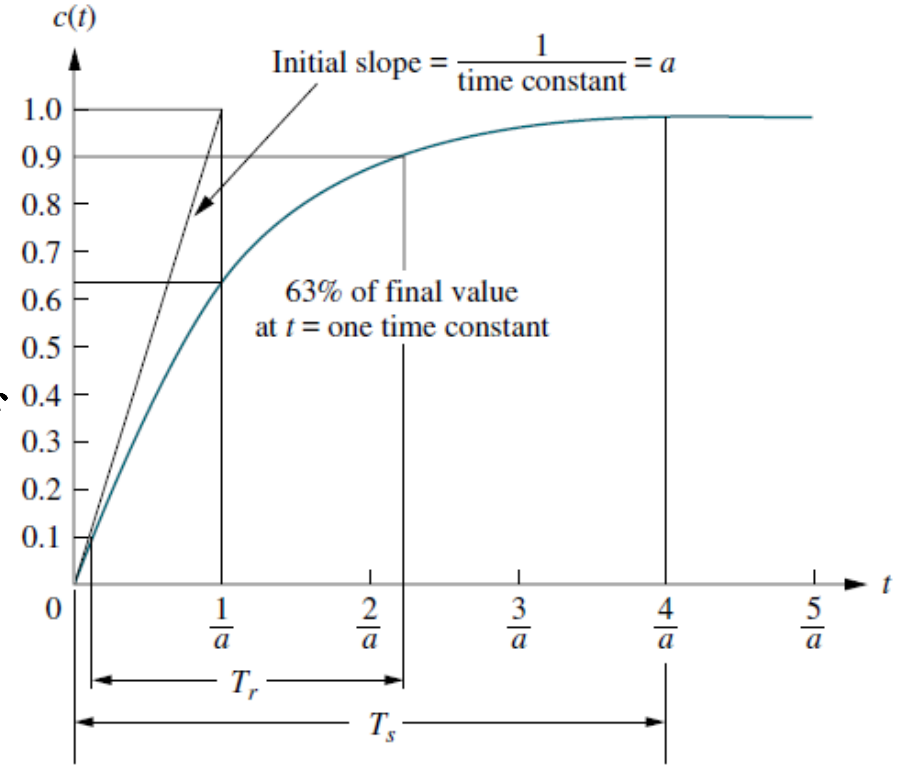
$$e^{-at}\big|_{t=1/a} = e^{-1} = 0.37$$

$$c(t)\big|_{t=1/a} = 1 - e^{-at}\big|_{t=1/a} = 1 - 0.37 = 0.63$$

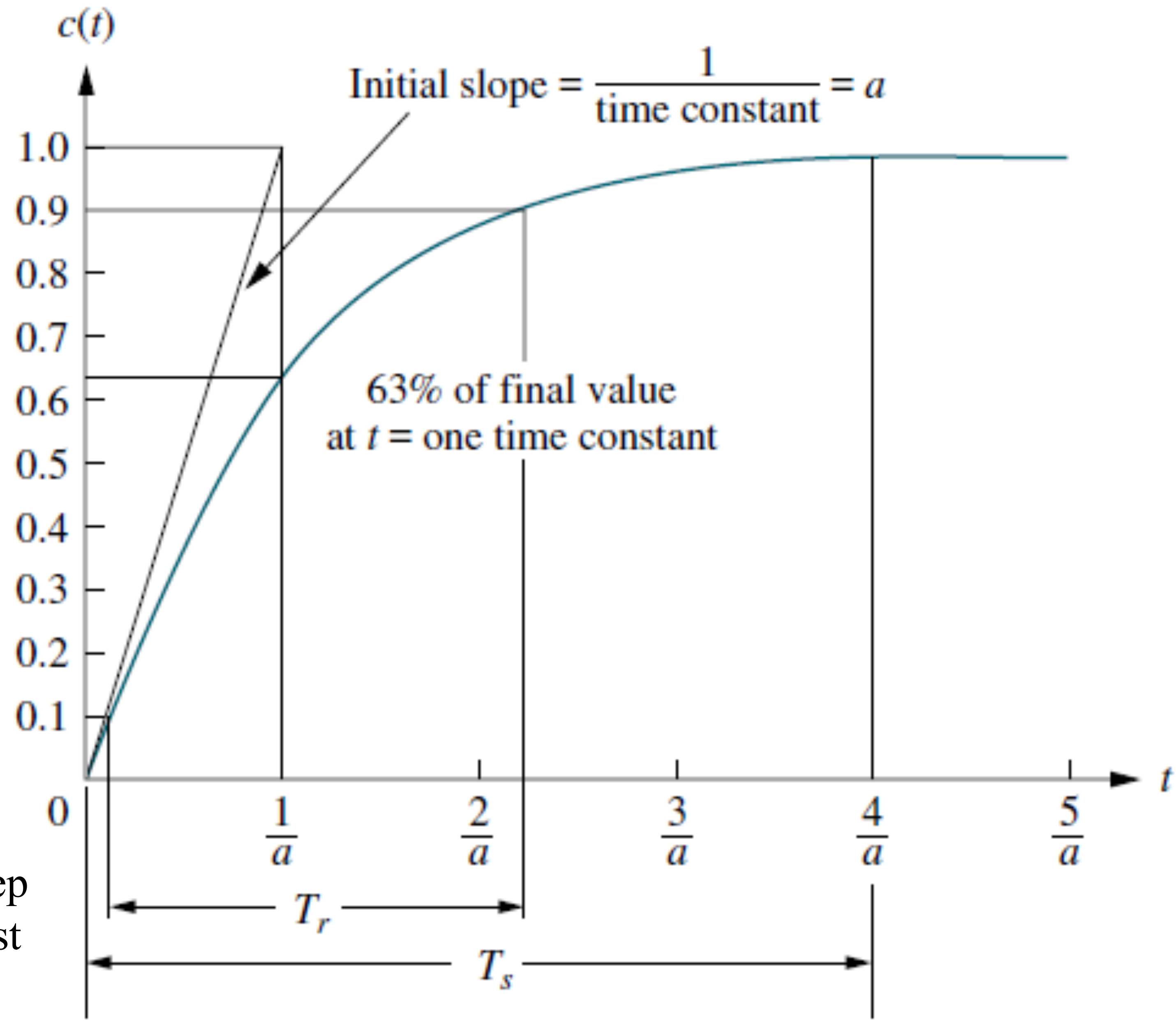
FIRST ORDER SYSTEMS *cont.'s*.

Time Constant (T_c or τ): $1/a$ is called the *time constant*. Thus, the time constant can be described as the time for e^{-at} to decay to 37% of its initial value. Alternately, the time constant is the time it takes for the step response to rise to 63% of its final value.

- The reciprocal of the time constant has the units (1/sec), or frequency. Thus, a is called the *exponential frequency*.
- a is the initial rate of change of the exponential at $t = 0$.
- The time constant can be considered a transient response specification for a first-order system, since it is related to the speed at which the system responds to a step input.



FIRST ORDER SYSTEMS *cont.'s*.



A typical unit step response of a first order system.

FIRST ORDER SYSTEMS *cont.'s*.

Rise Time: *Rise time* is defined as the time for the step response to go from 0.1 to 0.9 of its final value.
Rise time is found by solving $c(t) = 1 - e^{-at}$ for the difference in time at $c(t)$=0.9 and $c(t)$=0.1. It is found as,

$$\boxed{T_r = \frac{2.31}{a} - \frac{0.11}{a} = \frac{2.2}{a}}$$

Settling Time: *Settling time* is defined as the time for the response to reach, and stay within, 2% of its final value. Letting $c(t) = 0.98$ and solving for t, we find the settling time as,

$$\boxed{T_s = \frac{4}{a}} \quad \text{or} \quad \boxed{T_s = 4T_c}$$

First-Order Transfer Functions via Testing

- Often it is not possible or practical to obtain a system's transfer function analytically. Perhaps the system is closed, and the component parts are not easily identifiable.
- Since the transfer function is a representation of the system from input to output, *the system's step response can lead to a representation* even though the inner construction is not known.
- With a step input, we can measure the time constant and the steady-state value, from which the transfer function can be calculated.
- Consider a simple first-order system $G(s) = \frac{K}{s+a}$
- Its step response would be

$$C(s) = \frac{K}{s(s+a)} = \frac{K/a}{s} - \frac{K/a}{(s+a)}$$

- If we can identify K and a from laboratory testing, we can obtain the transfer function of the system.
- So the time response would be in the following form,

$$c(t) = \frac{K}{a}(1 - e^{-at})$$

First-Order Transfer Functions via Testing

- Often it is not possible or practical to obtain a system's transfer function analytically. The system might be closed and the components are not easily identifiable.
- Since the transfer function is a representation of the system from input to output, the system's **step response** can be used **to represent the system** even though the inner construction is not known.
- We can measure the time constant and the steady-state value from the step response, then the transfer function can be estimated.
- Consider a simple first order system,

$$C(s) = \frac{K}{s(s+a)} = \frac{K/a}{s} - \frac{K/a}{(s+a)}$$

- If we can identify K and a from laboratory testing, we can obtain the transfer function of the system.
- So the time response would be in the following form,

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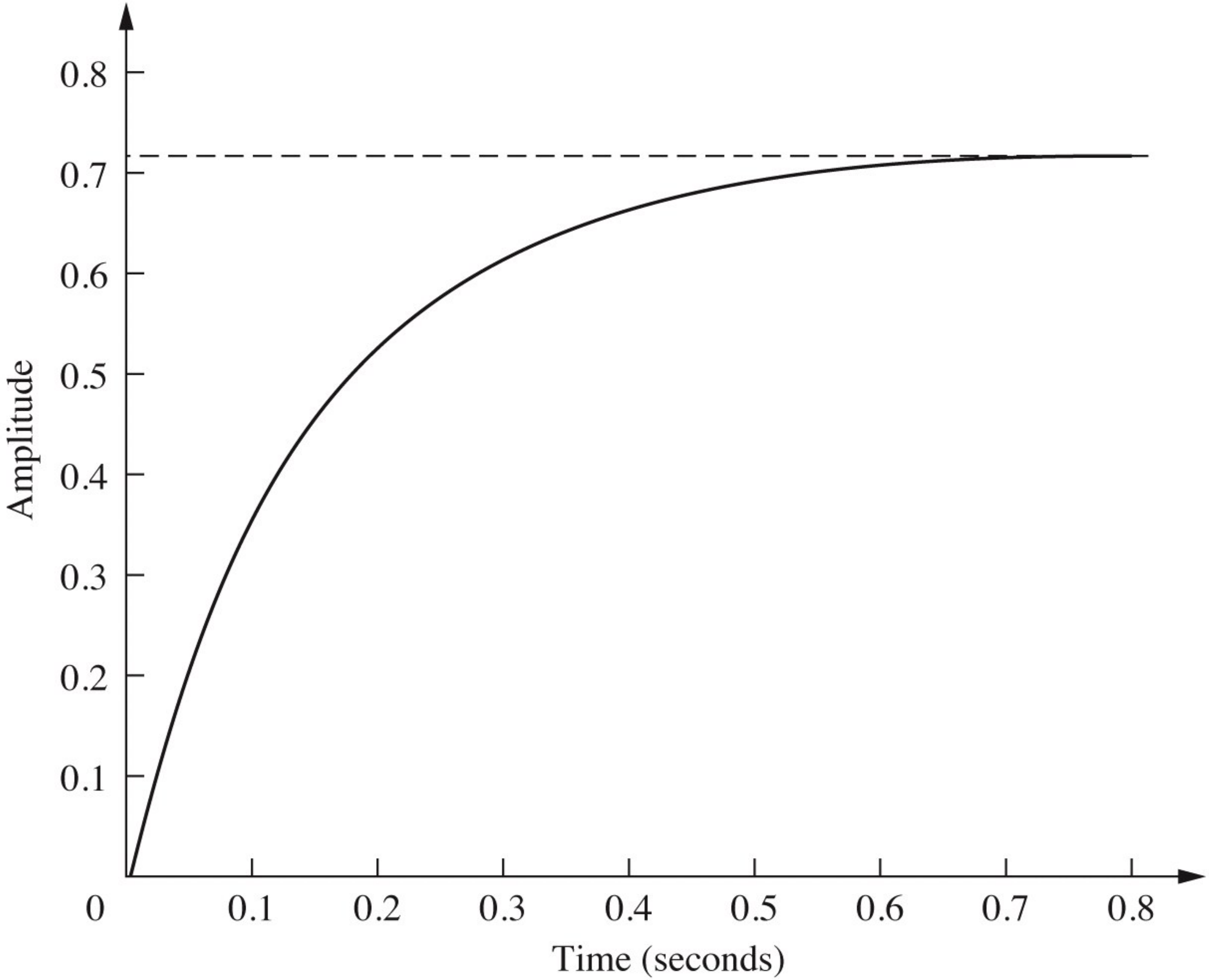
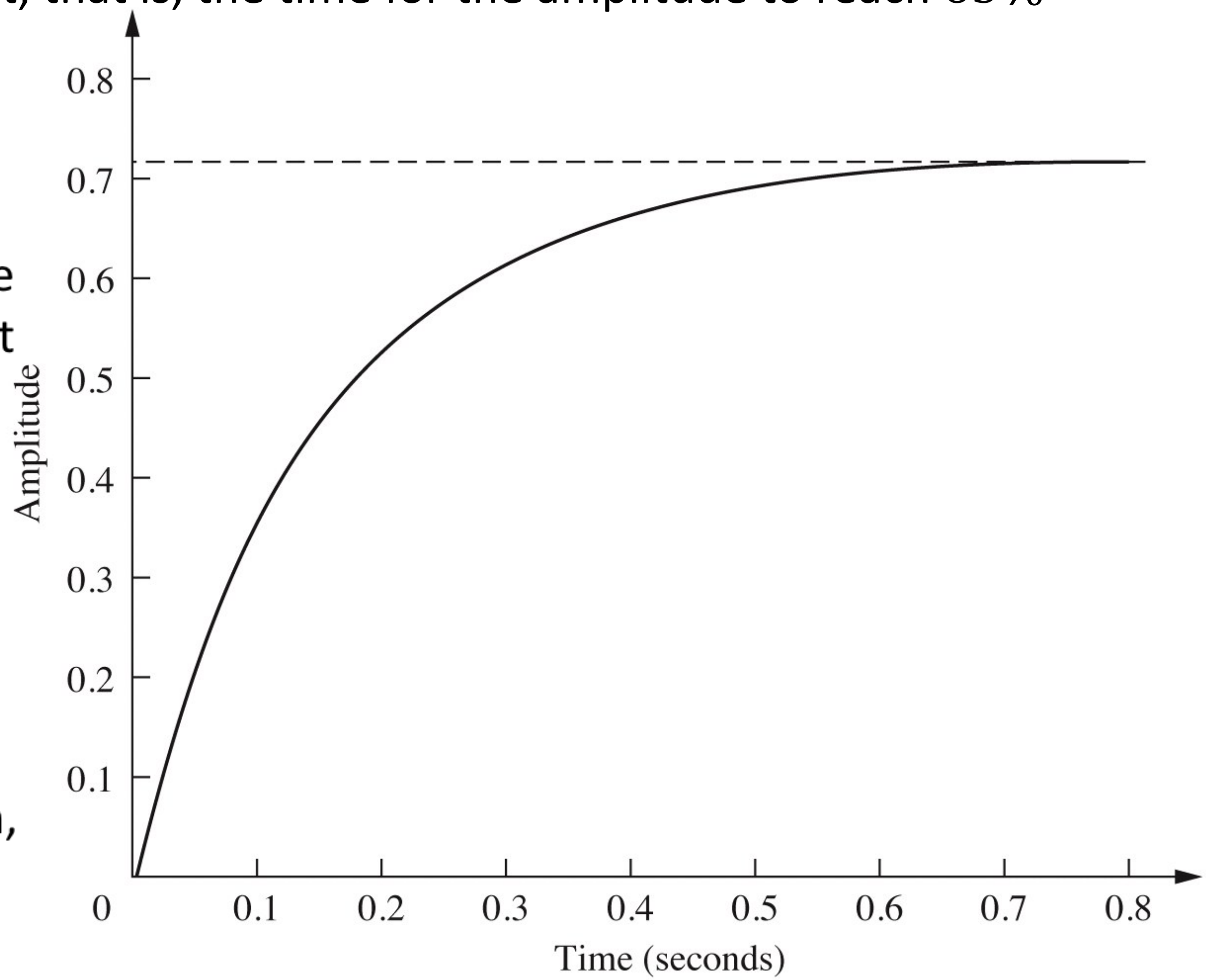
Example: Assume the unit step response for a system is recorded at the lab. Construct a transfer function for this system.



Example: Assume the unit step response for a system is recorded at the lab. Construct a transfer function for this system.

Solution: It seems it has the first-order characteristics we have seen so far, such as no overshoot and nonzero initial slope. From the response, we measure the time constant, that is, the time for the amplitude to reach 63% of its final value.



Since the final value is 0.72, the time constant is evaluated where the curve reaches 0.63x0.72 =0.45 at about 0.13 sec.

Hence $a = \frac{1}{0.13} = 7.7.$

The final value is

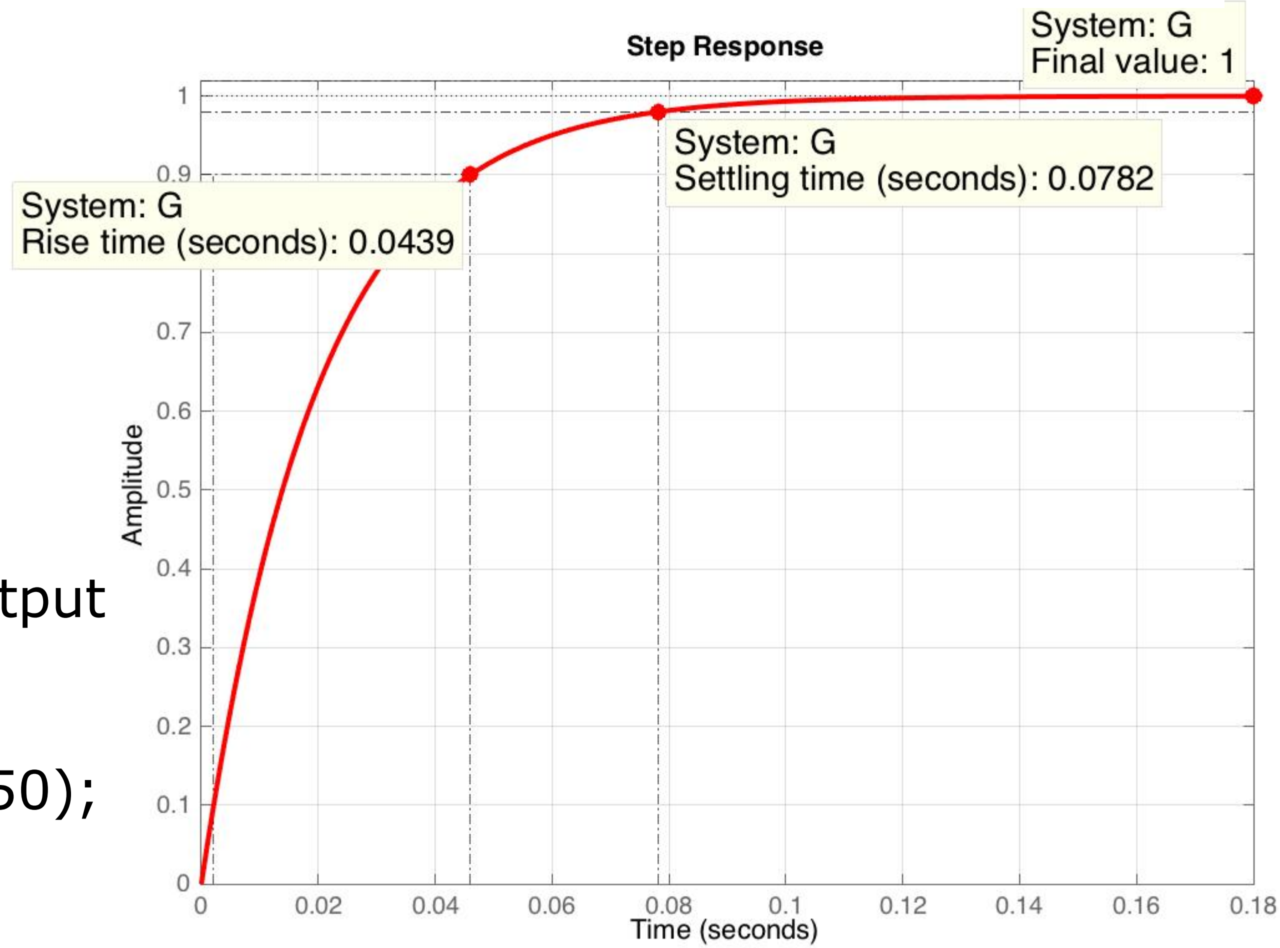
$\frac{K}{a} = 0.72 \rightarrow K = 5.54.$

Then the transfer function,

$$\boldsymbol{G(s) = \frac{5.54}{s + 7.7}}$$

PROBLEM: A system has a transfer function, $G(s) = \frac{50}{s+50}$. Find the time constant, T_c, settling time, T_s, and rise time, T_r.

ANSWER: $T_c = 0.02\ \text{s}$, $T_s = 0.08\ \text{s}$, and $T_r = 0.044\ \text{s}$.

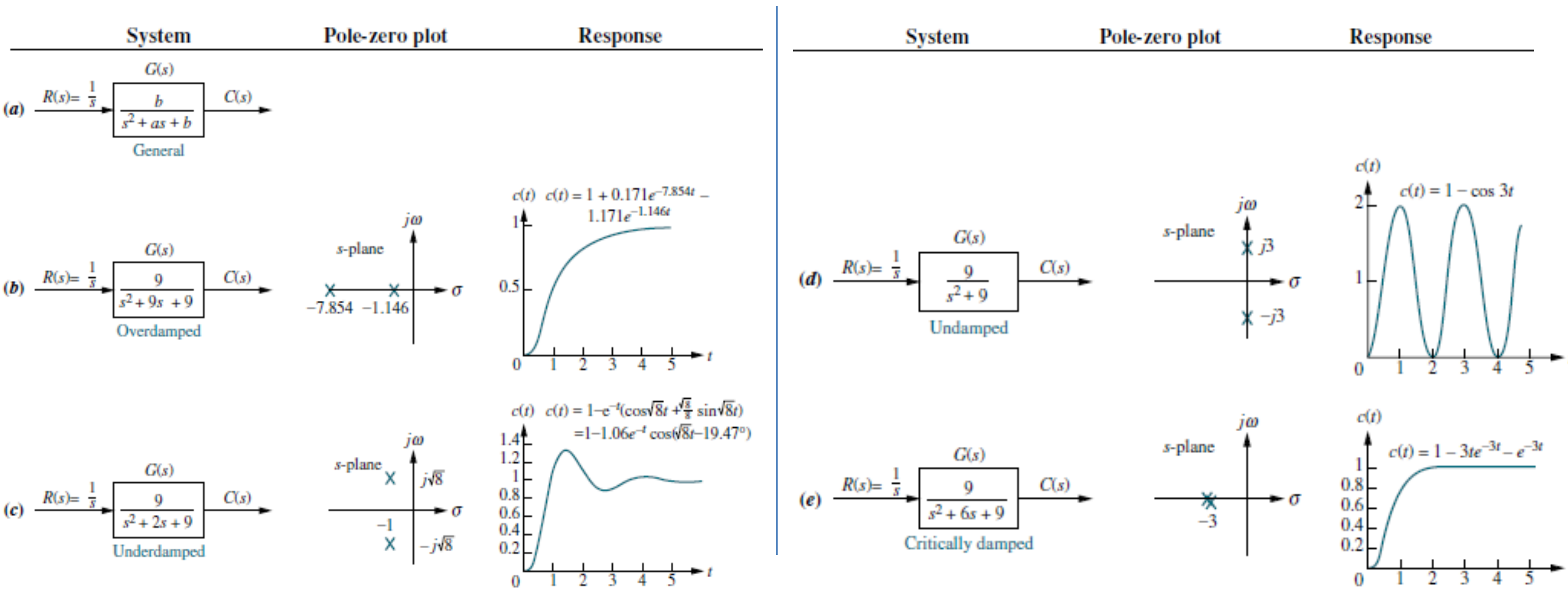


Plotting the output in MATLAB

```
>> s=tf('s');
>> G=50/(s+50);
>> step(G)
```

SECOND ORDER SYSTEMS - *Introduction*

- Compared to the simplicity of a first-order system, a second-order system exhibits a wide range of responses that must be analyzed and described.
- Changes in the parameters of a second-order system can change the form of the response.



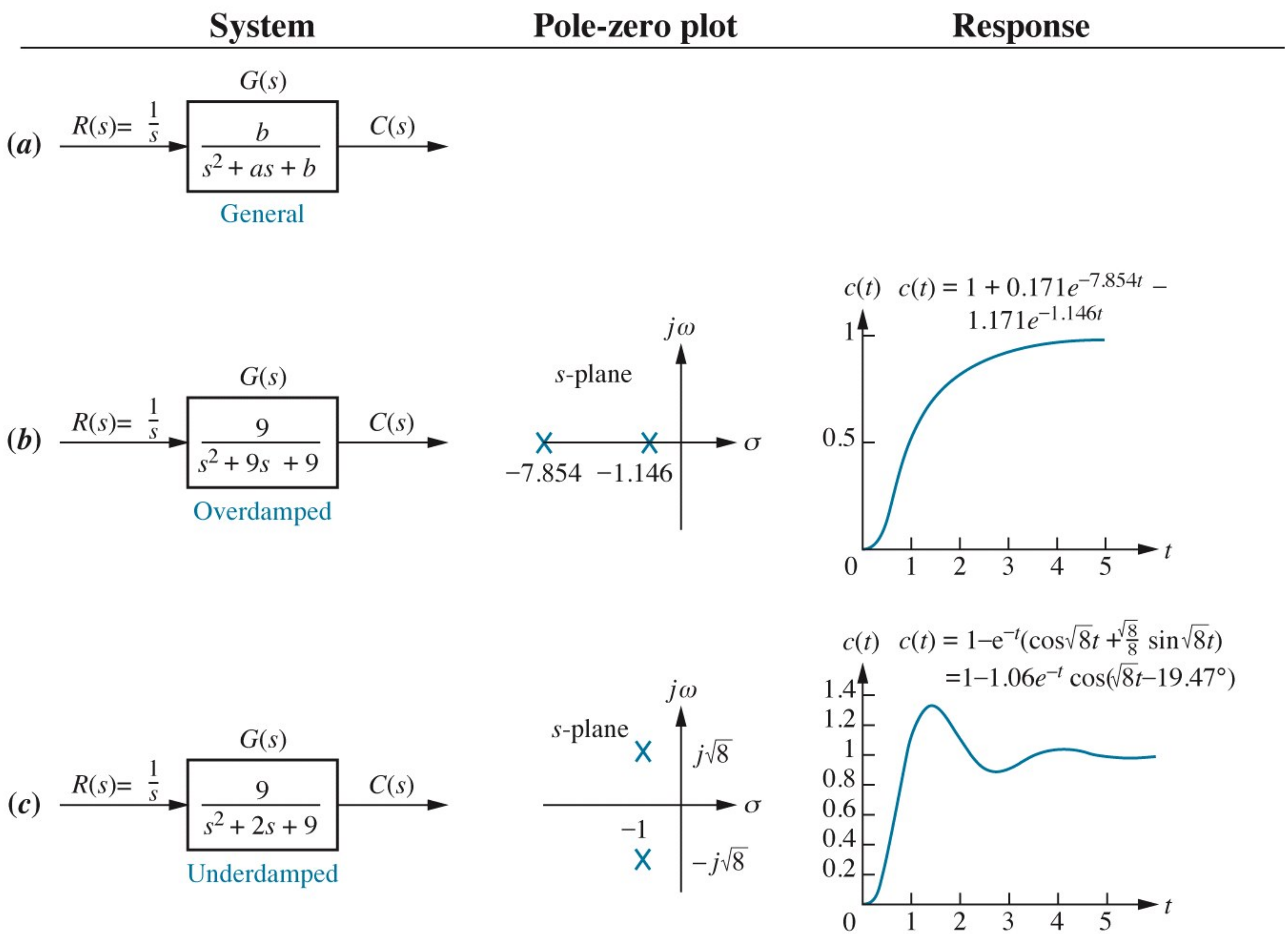


Figure 4.7abc

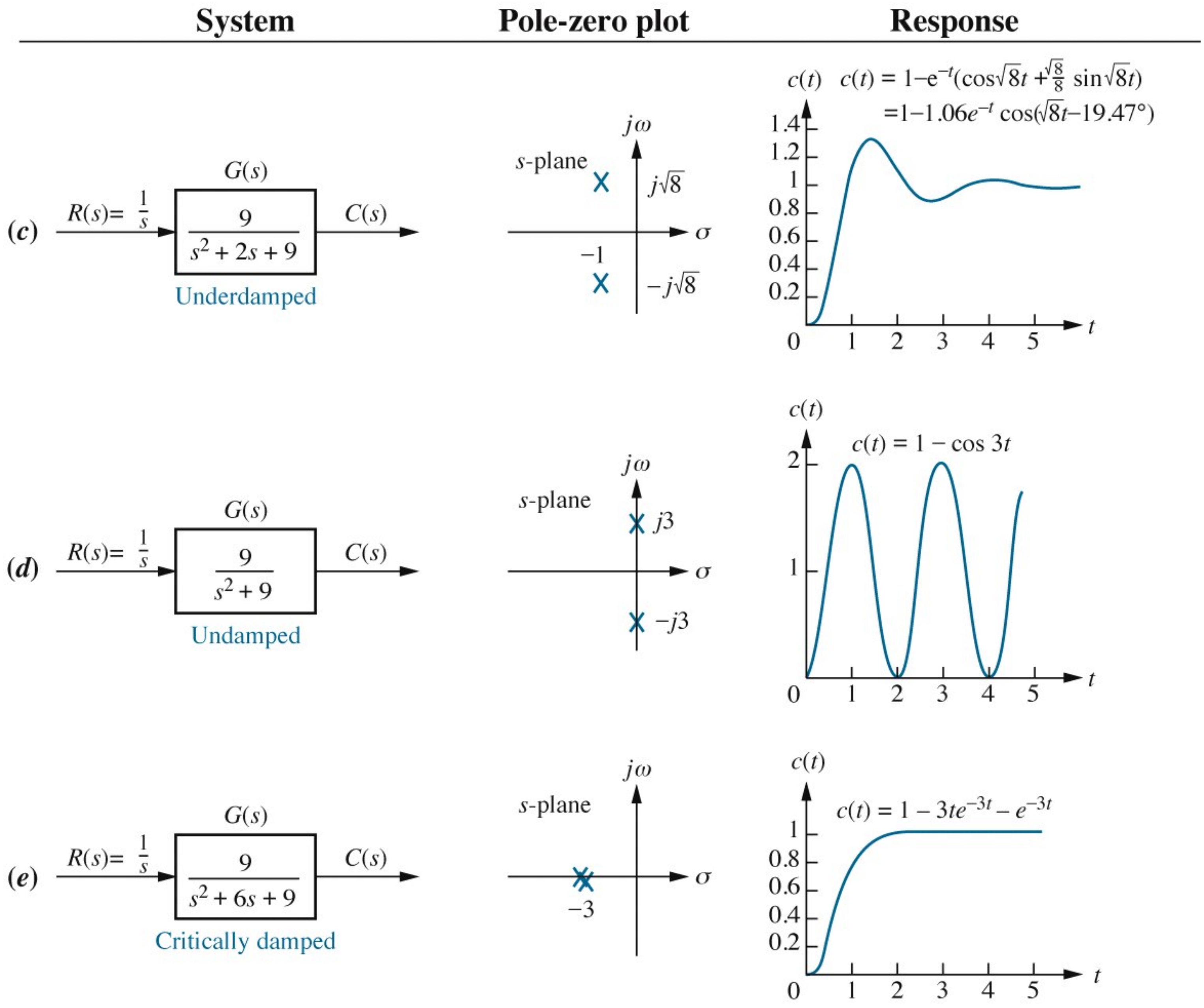
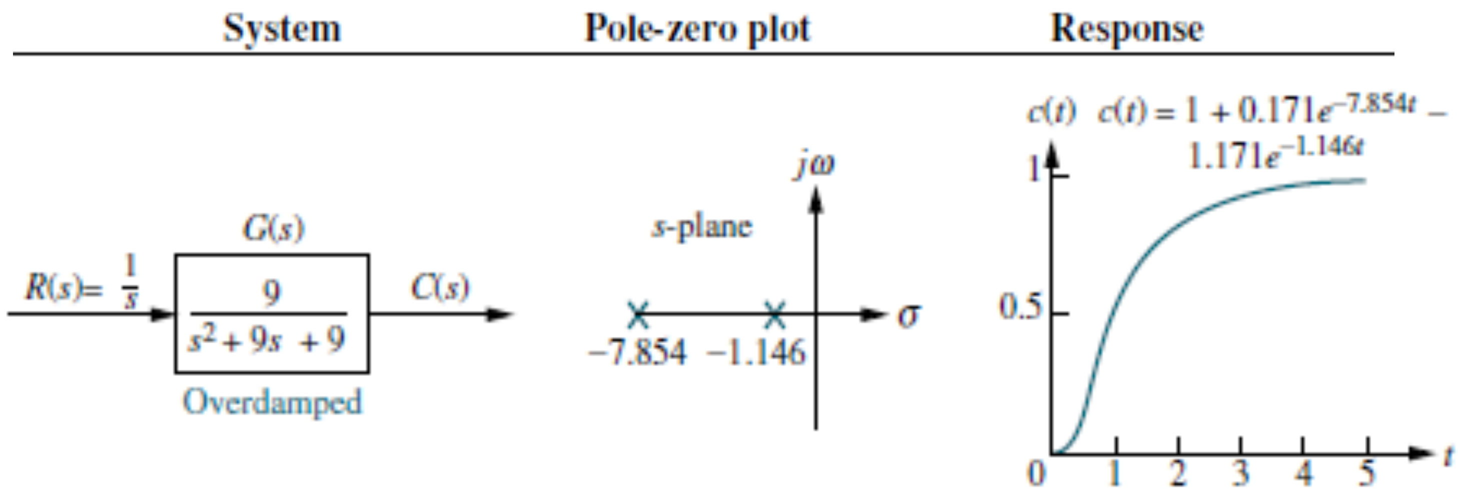


Figure 4.7cde

Overdamped Response

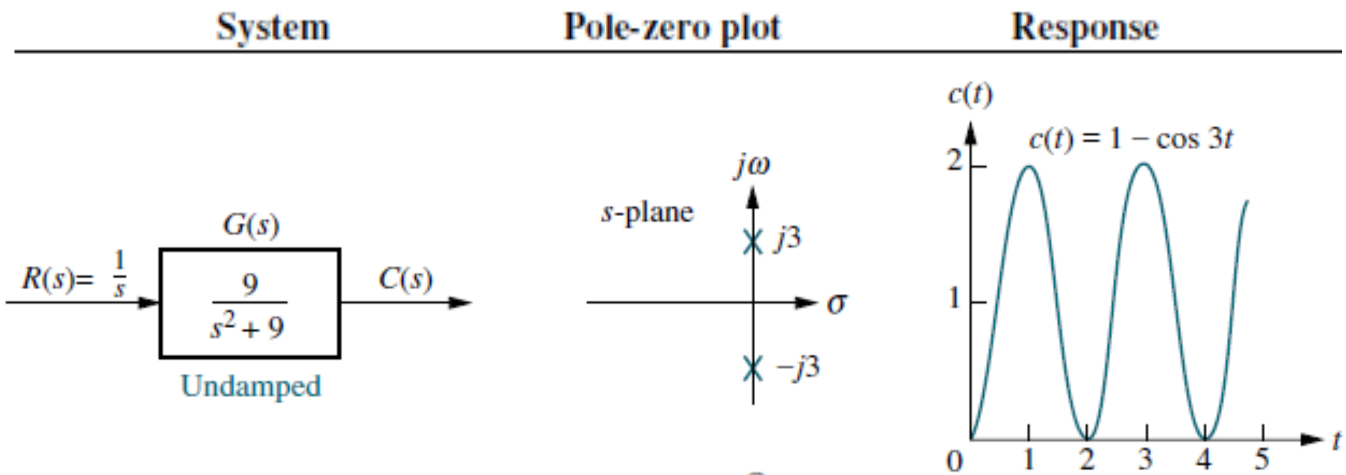


$$C(s) = \frac{9}{s(s^2 + 9s + 9)} = \frac{9}{s(s + 7.854)(s + 1.146)}$$

- The estimation of the output: $c(t) = K_1 + K_2 e^{-7.854t} + K_3 e^{-1.146t}$

- The input pole at the origin generates the constant forced response; each of the two system poles on the real axis generates an exponential natural response whose exponential frequency is equal to the pole location.

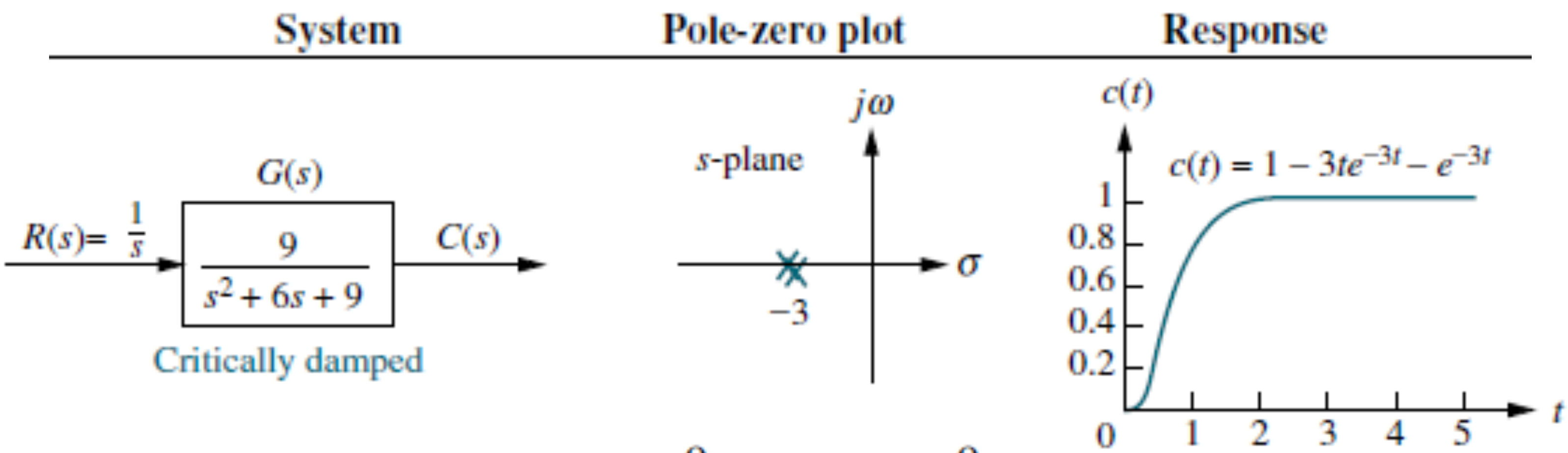
Undamped Response



$$C(s) = \frac{9}{s(s^2+9)}$$

- The estimation of the output: $c(t) = K_1 + K_4 \cos(3t - \varphi)$
- The input pole at the origin generates the constant forced response, and
- the two system poles on the imaginary axis at $\pm j3$ generate a sinusoidal natural response whose frequency is equal to the location of the imaginary poles.

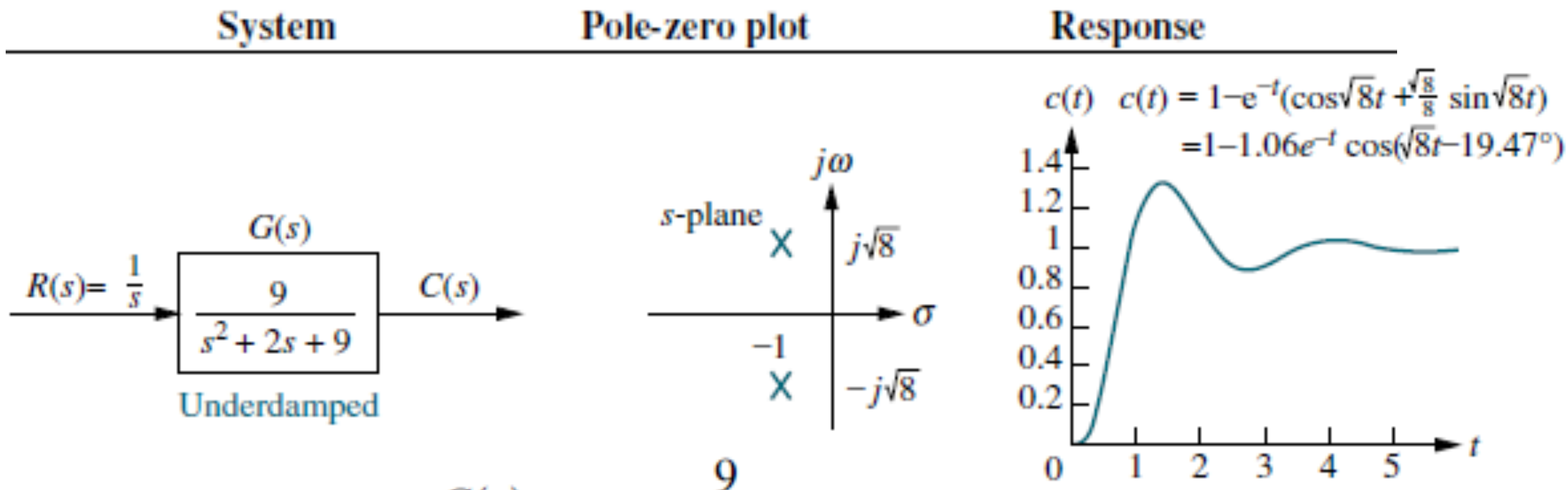
Critically Damped Response



$$C(s) = \frac{9}{s(s^2+6s+9)} = \frac{9}{s(s+3)^2}$$

- The estimation of the output: $c(t) = K_1 + K_2e^{-3t} + K_3te^{-3t}$.
- The input pole at the origin generates the constant forced response, and
- The two poles on the real axis at -3 generate a natural response consisting of an exponential and an exponential multiplied by time, where the exponential freq. is equal to the magnitude of the real poles.
- *Critically damped responses are the fastest possible without the overshoot.*

Underdamped Response



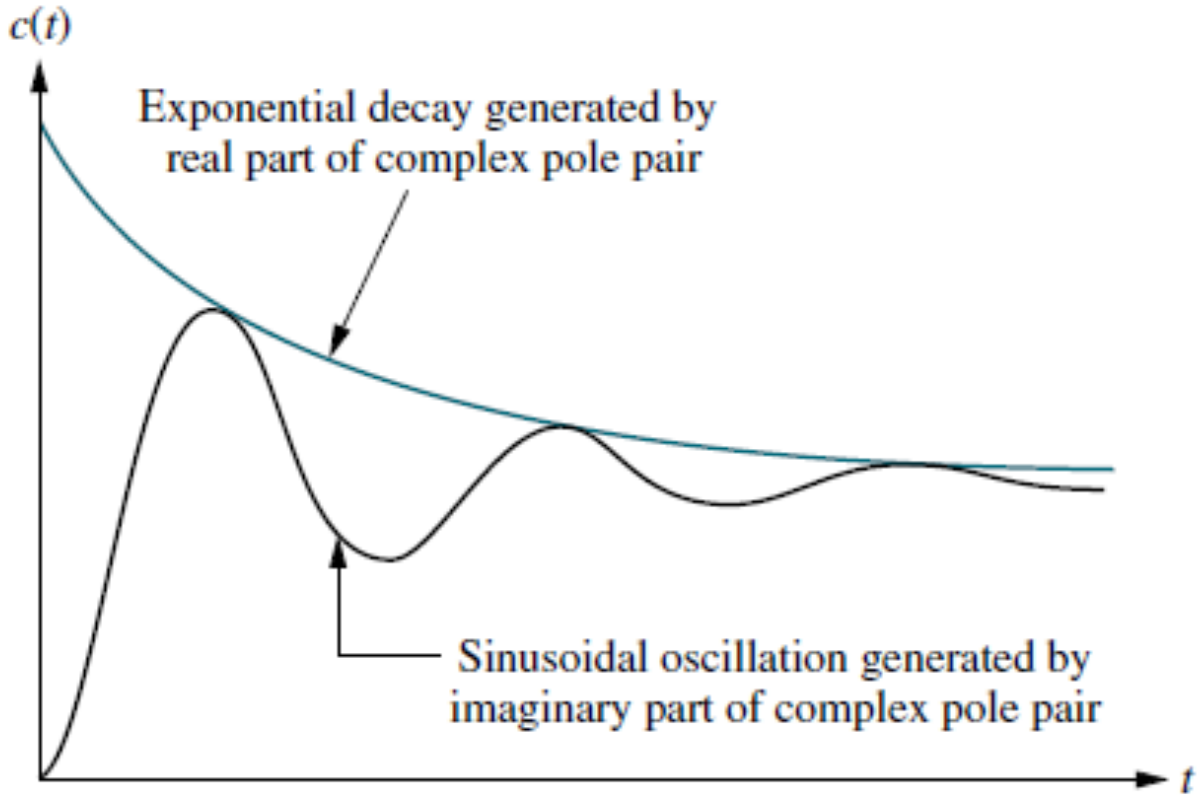
$$C(s) = \frac{9}{s(s^2+2s+9)}$$

- This function has a pole at the origin that comes from the unit step input and two complex poles that come from the system.
- The poles that generate the natural response are at $s = -1 \mp \sqrt{8}$
- We see that the real part of the pole matches the *exponential decay frequency* of the sinusoid's amplitude, while the imaginary part of the pole matches the frequency of the sinusoidal oscillation.

$$c(t) = K_1 + e^{-t}\left(K_2\cos\left(\sqrt{8}t\right) + K_3\sin\left(\sqrt{8}t\right)\right) = K_1 + K_4 e^{-t}\cos(\sqrt{8}t - \phi)$$

where $\phi = \tan^{-1}K_3/K_2$, $K_4 = \sqrt{K_2^2 + K_3^2}$, $\boldsymbol{c(t) = 1 - 1.06e^{-t}\cos(\sqrt{8}t - 19.47°)}$

Underdamped Response ctd.



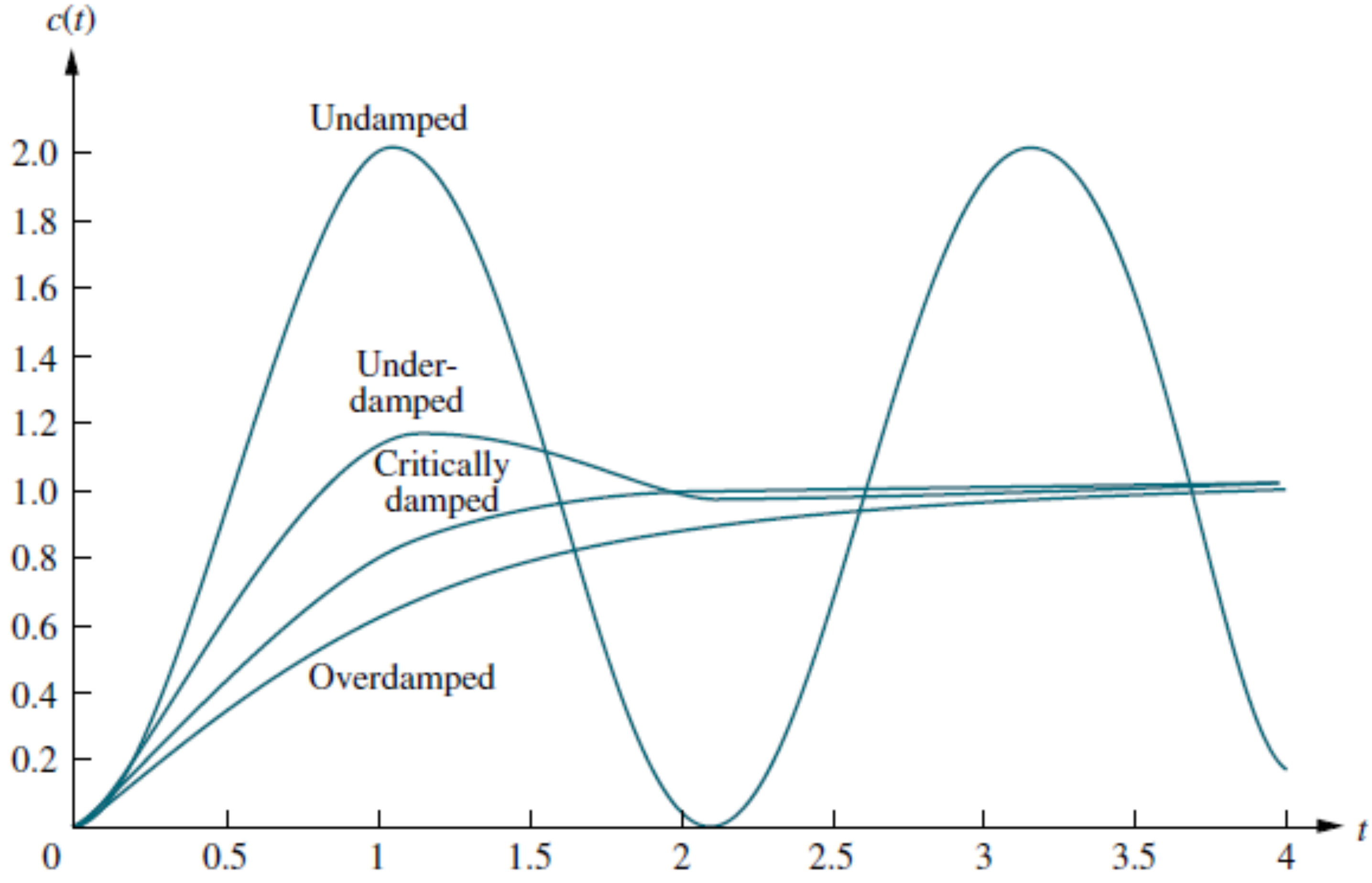
- The transient response consists of an exponentially decaying amplitude generated by the real part of the system pole times a sinusoidal waveform generated by the imaginary part of the system pole.
- The time constant of the exponential decay is equal to the reciprocal of the real part of the system pole. The value of the imaginary part is the actual frequency of the sinusoid, as depicted in the figure.
- This sinusoidal frequency is given the name *damped frequency of oscillation* (ω_d).
- Finally, the steady-state response (unit step) was generated by the input pole located at the origin.

Summary

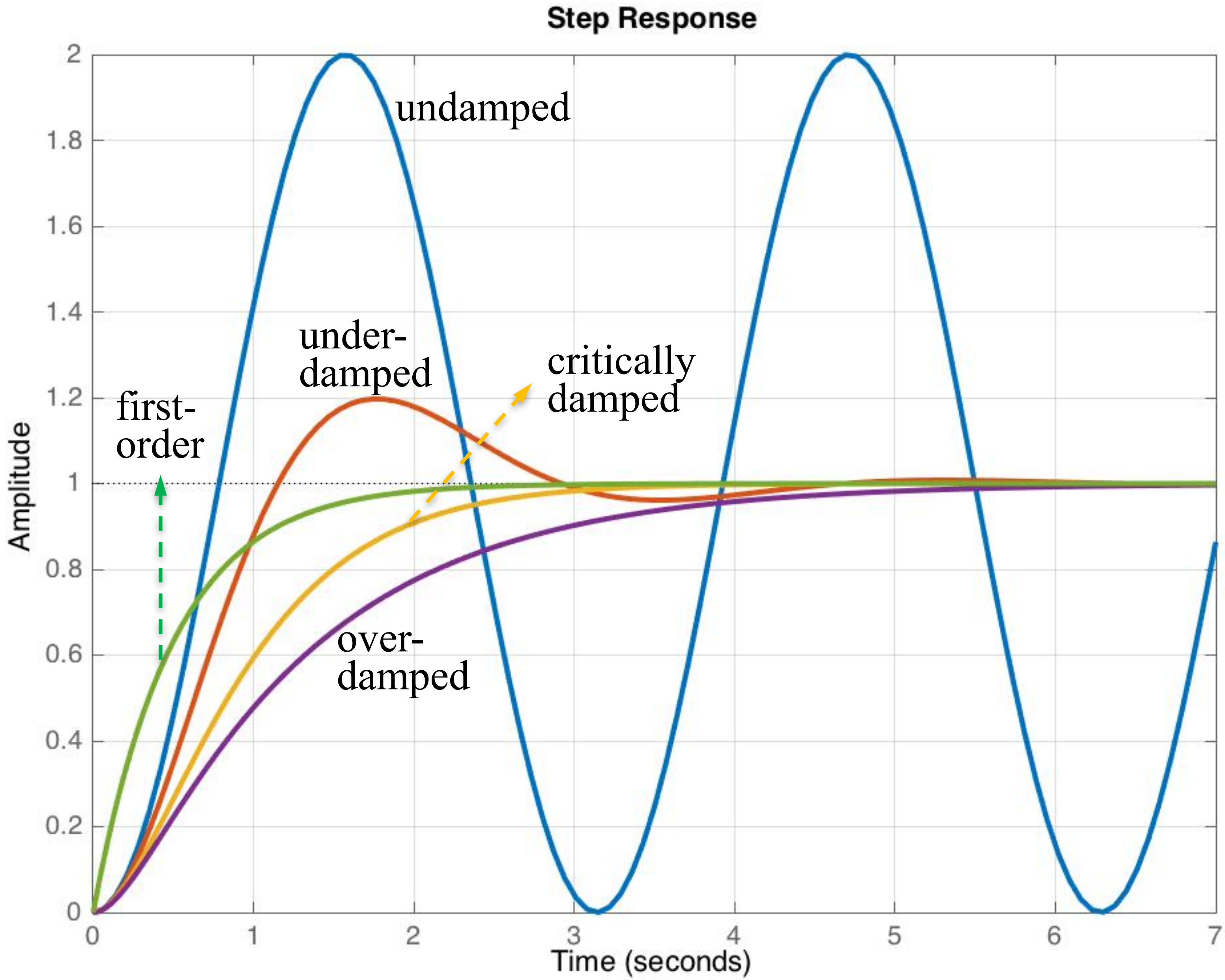
Overdamped Responses	**Poles:**	Two real at σ_1 and σ_2
	Natural Response:	$c(t) = K_1 e^{-\sigma_1 t} + K_2 e^{-\sigma_2 t}$
Underdamped Responses	**Poles:**	$-\sigma_d \mp j\omega_d$
	Natural Response:	$c(t) = Ae^{-\sigma_d t} \cos(\omega_d t - \phi)$
Undamped Responses	**Poles:**	$\mp j\omega_1$
	Natural Response:	$c(t) = A \cos(\omega_1 t - \phi)$
Critically Damped Responses	**Poles:**	Two real at $-\sigma_1$
	Natural Response:	$c(t) = K_1 e^{-\sigma_1 t} + K_2 t e^{-\sigma_1 t}$

Summary *cont.'s*.

Step responses for second-order system damping cases:



Step responses for 1st-order and also 2nd-order system damping cases:



The General Second Order System

- We define two physically meaningful specifications for second-order systems which can be used to describe the characteristics of the second-order transient response just as time constants describe the first-order system response.
- The two quantities are called *natural frequency* and *damping ratio*.

Natural frequency (ω_n): The *natural frequency* of a 2^{nd}-order system is the frequency of oscillation of the system without damping.

Damping ratio (ζ): The *damping ratio* of a second-order system is the ratio of the exponential decay frequency to the natural frequency. The damping ratio is also proportional to the ratio of the natural period to the exponential time constant.

$$\zeta = \frac{\text{Exponential decay frequency}}{\text{Natural frequency (rad/second)}} = \frac{1}{2\pi}\frac{\text{Natural period (seconds)}}{\text{Exponential time constant}}$$

Note that, the damping ratio is constant regardless of the time scale.

The General Second Order System *cont.'s*.

- The general second-order system can be transformed to show the quantities ζ and ω_n. Consider the general system,

$$G(s) = \frac{b}{s^2 + as + b}$$

- Without damping, the poles would be on the $j\omega$-axis, and the response would be an undamped sinusoid. If the poles to be purely imaginary, $a = 0$.

$$G(s) = \frac{b}{s^2 + b}$$

- By definition, the natural frequency ω_n, is the frequency of oscillation of this system; since the poles of this system are on the imaginary axis: $\pm jb$

$$\omega_n = \sqrt{b} \Rightarrow b = \omega_n^2$$

The General Second Order System *cont.'s*.

- Assuming an underdamped system, the complex poles have a real part, σ, equal to $-a/2$. The magnitude of this value is then the exponential decay frequency.

$$\zeta = \frac{\text{Exponential decay frequency}}{\text{Natural frequency (rad/second)}} = \frac{|\sigma|}{\omega_n} = \frac{a/2}{\omega_n} \rightarrow \quad a = 2\zeta\omega_n$$

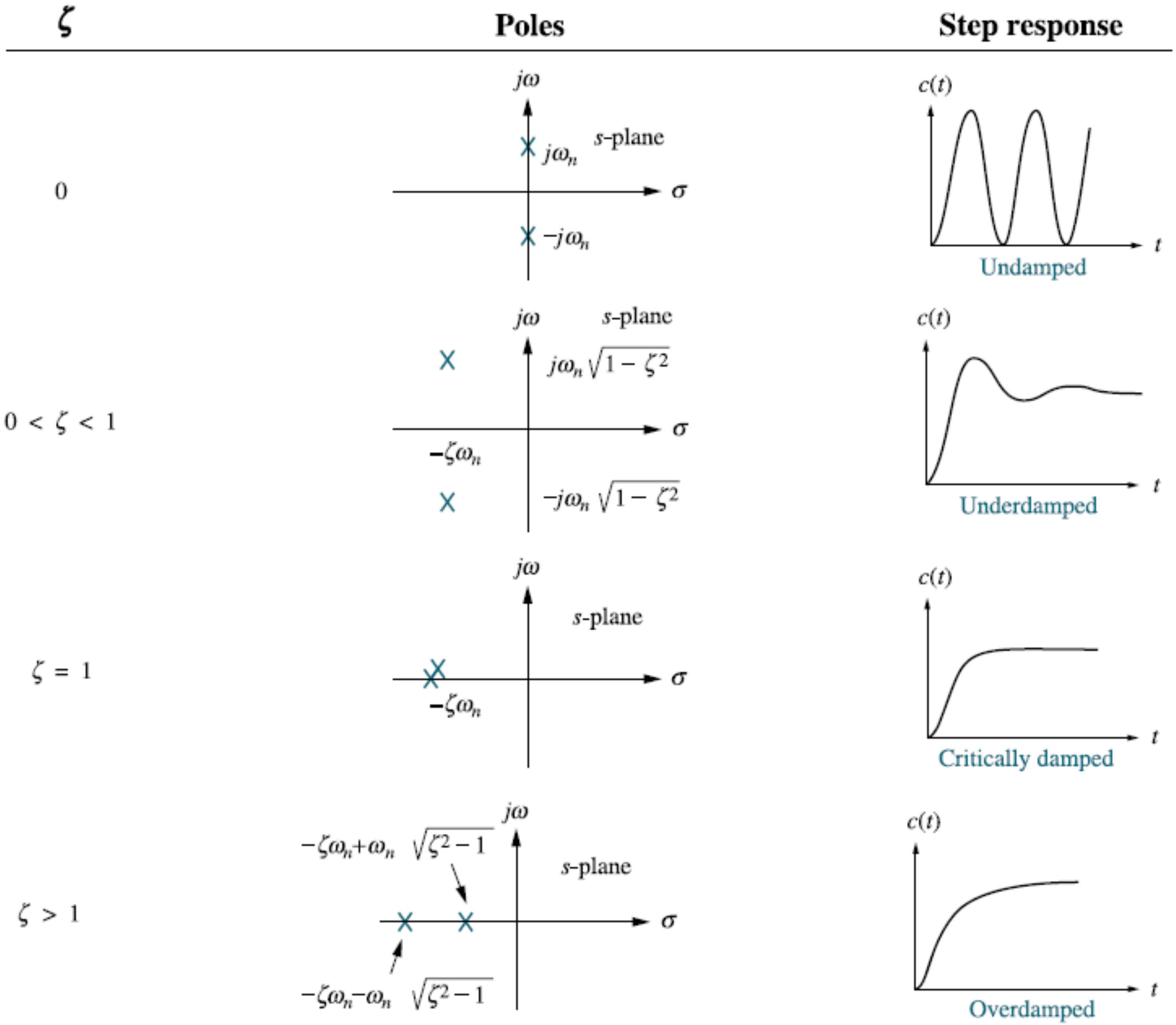
- The general second-order transfer function finally looks like,

$$\boxed{G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}}$$

- Solving for the poles of the general transfer function yields,

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

Second-order response as a function of damping ratio



Underdamped Second-Order Systems

Now, we have generalized the second-order transfer function in terms of ζ and ω_n.

$$\boxed{G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}}$$

We now going to analyze the step response of an underdamped second-order system which is a common model for physical problems and has a unique behavior. Let's begin by finding the step response.

The transform of the response, $C(s)$, is the transform of the input times the transfer function, or (assuming that ζ <1, the underdamped case)

$$C(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} = \frac{K_1}{s} + \frac{K_2 s + K_3}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Underdamped Second-Order Systems *cont.'s*.

$$C(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} = \frac{K_1}{s} + \frac{K_2 s + K_3}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Expanding by partial fractions, and taking the inverse Laplace transform produces,

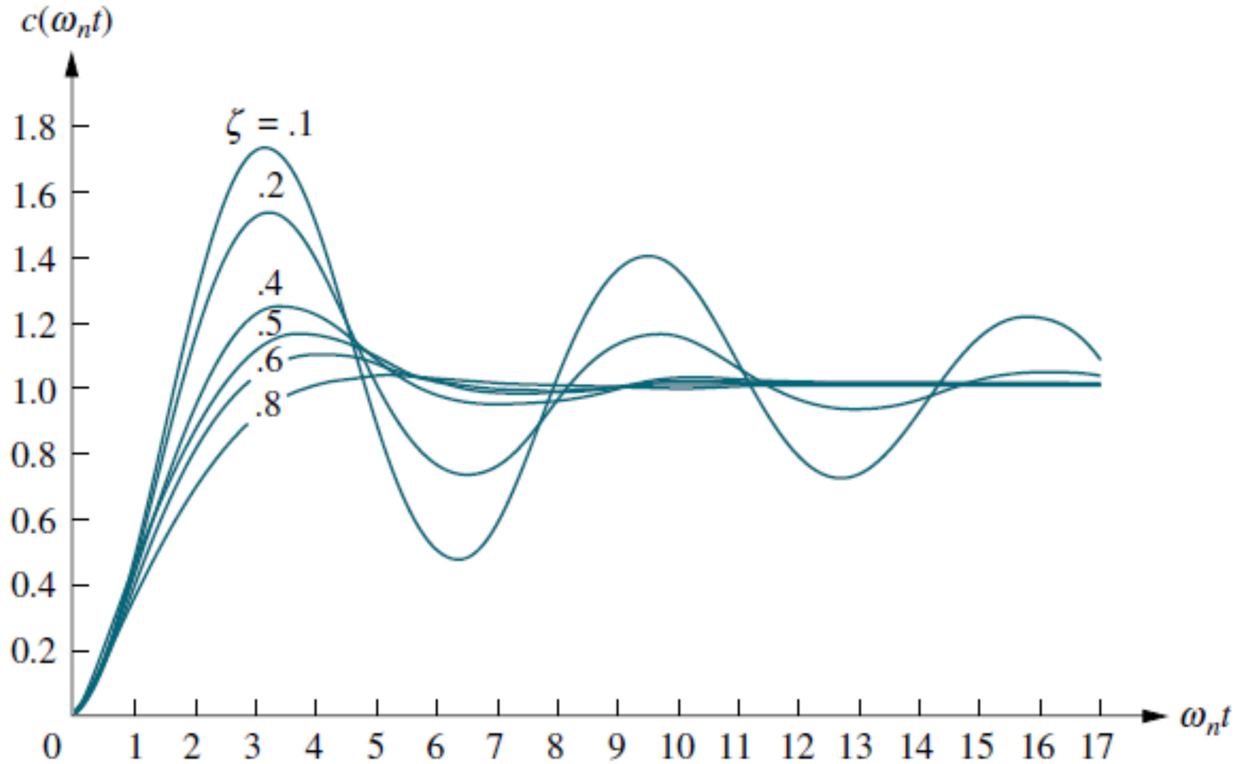
$$c(t) = 1 - e^{-\zeta\omega_n t}\left(\cos \omega_n\sqrt{1-\zeta^2}t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin \omega_n\sqrt{1-\zeta^2}t\right)$$

$$= 1 - \frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\cos(\omega_n\sqrt{1-\zeta^2}t - \phi)$$

where $\phi = \tan^{-1}(\frac{\zeta}{\sqrt{1-\zeta^2}})$

Underdamped Second-Order Systems *cont.'s*.

A plot of this response appears in the following figure, for various values of ζ, plotted along a time axis normalized to the ω_n.



We now see the relationship between the value of ζ and the type of response obtained:

The lower the value of ζ, the more oscillatory the response.

Underdamped Second-Order Systems *cont.'s*.

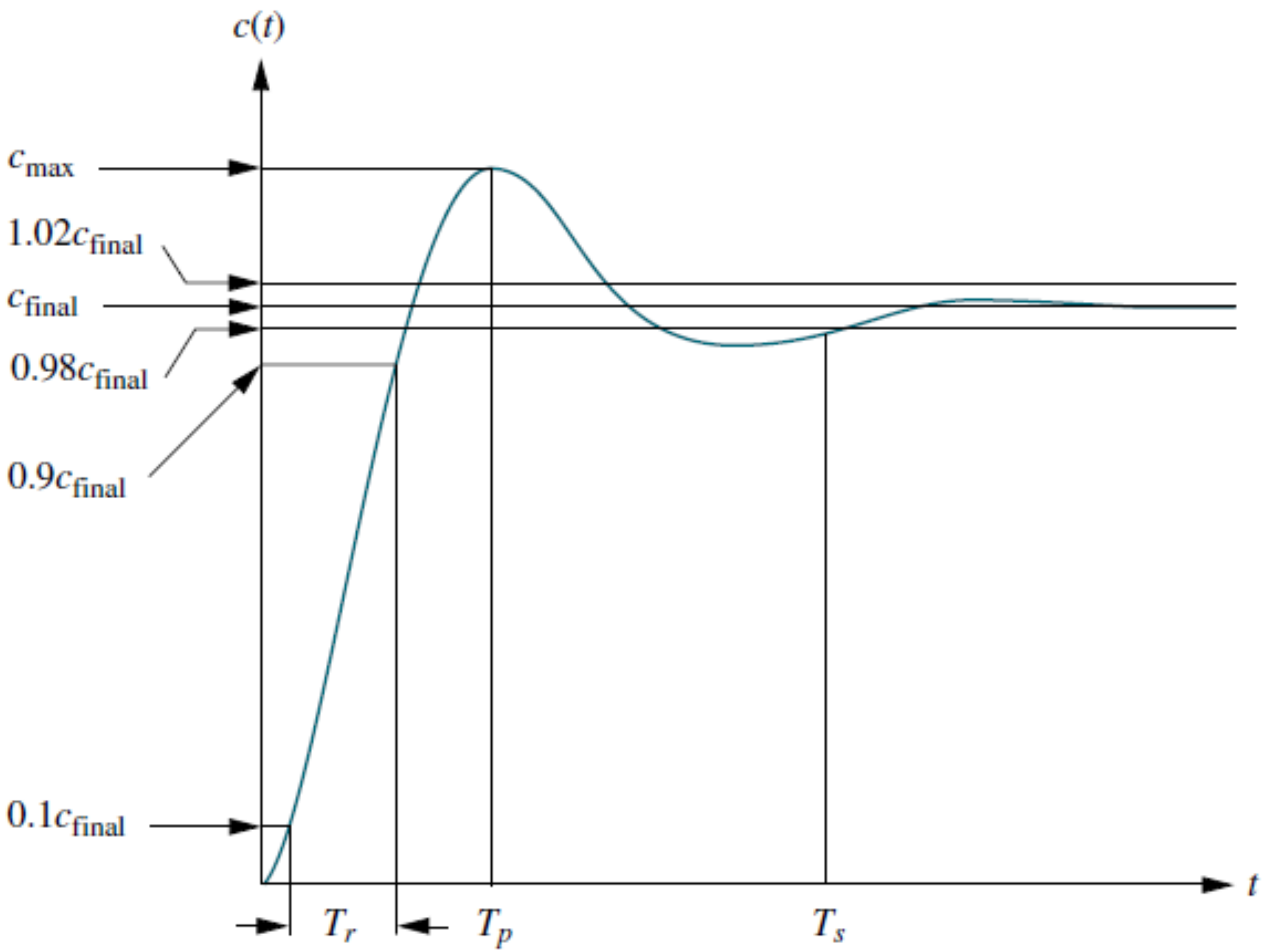
1. ***Rise time, T_r:***
   The time required for the response to go from 0.1 of the final value to 0.9 of the final value.

2. ***Peak time, T_p:***
   The time required to reach the first, or maximum, peak.

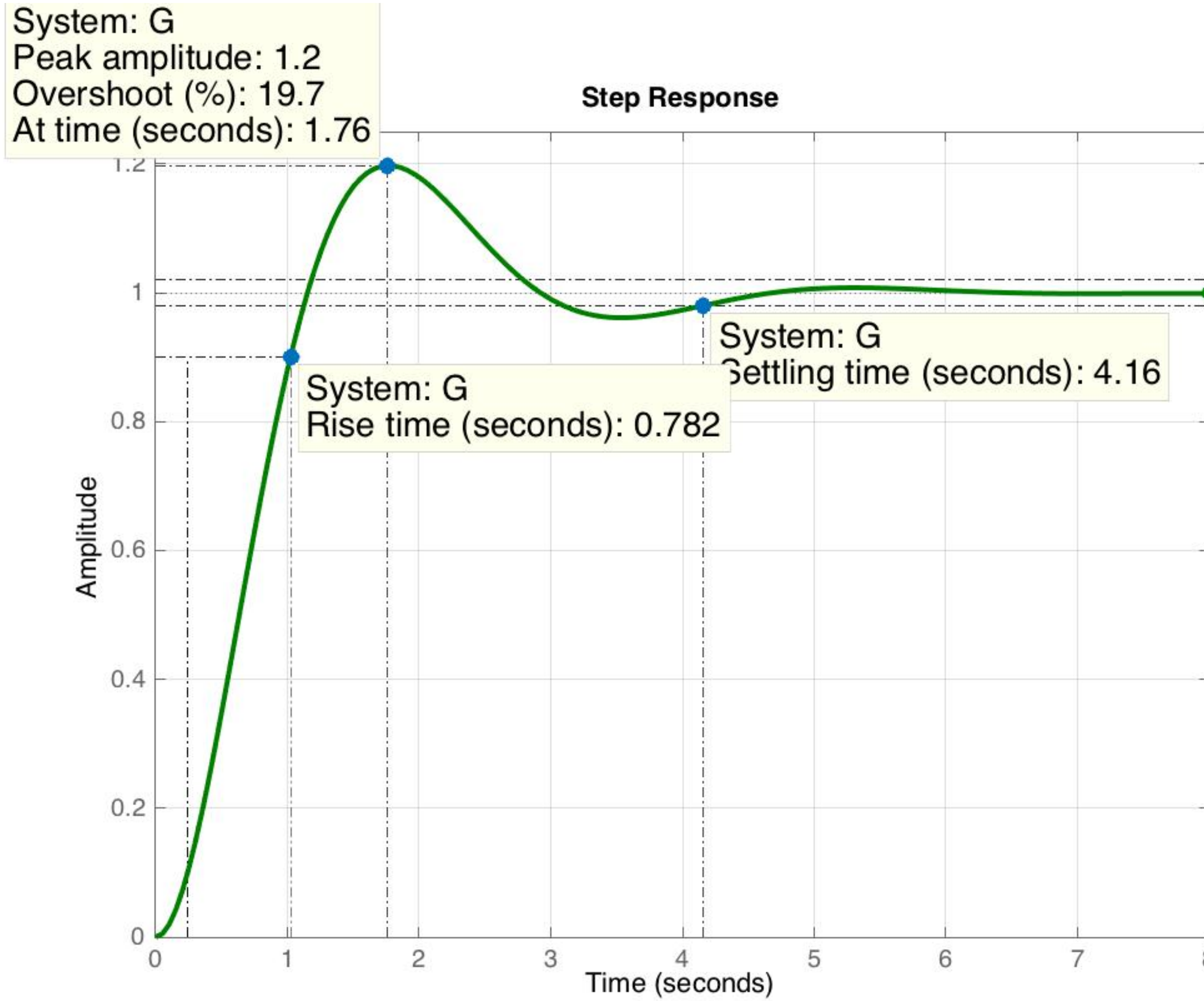
3. ***Percent overshoot, %OS:***
   The amount that the waveform overshoots the steady state, or final, value at the peak time, expressed as a percentage of the steady-state value.

4. ***Settling time, T_s:***
   The time required for the transient's damped oscillations to reach and stay within ±2% of the steady-state value.

Underdamped Second-Order Systems *cont.'s*.



Underdamped Second-Order Systems *cont.'s*.



Evaluation of T_p for Underdamped Systems

T_p is found by differentiating $c(t)$ and finding the first zero crossing after $t = 0$. We can differentiate the output in the frequency domain,

$$\mathscr{L}[\dot{c}(t)] = sC(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Completing squares in the denominator, we have

$$\mathscr{L}[\dot{c}(t)] = \frac{\omega_n^2}{(s+\zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)} = \frac{\frac{\omega_n}{\sqrt{1-\zeta^2}}\omega_n\sqrt{1-\zeta^2}}{(s+\zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)}$$

$$\dot{c}(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\omega_n\sqrt{1-\zeta^2}\, t$$

Setting the derivative equal to zero yields $\omega_n\sqrt{1-\zeta^2}\, t = n\pi$
Each value of n yields the time for local maxima or minima. Letting n=0 yields t=0, the first point on the curve that has zero slope. The first peak, which occurs at the peak time, T_p, is found by letting $n = 1$.

$$\boxed{T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}}$$

Evaluation of %*OS* for Underdamped Systems

The percent overshoot is given by

$$\%OS = \frac{c_{\max} - c_{\text{final}}}{c_{\text{final}}} \times 100$$

By means of the following two equations,

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

$$c(t) = 1 - e^{-\zeta\omega_n t}\left(\cos\omega_n\sqrt{1-\zeta^2}t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_n\sqrt{1-\zeta^2}t\right)$$
$$= 1 - \frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\cos(\omega_n\sqrt{1-\zeta^2}t - \phi)$$

the term $c_{\max}$ is found by evaluating $c(t)$ at the peak time, $c(T_p)$

$$c_{\max} = c(T_p) = 1 - e^{-(\zeta\pi/\sqrt{1-\zeta^2})}\left(\cos\pi + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\pi\right)$$
$$= 1 + e^{-(\zeta\pi/\sqrt{1-\zeta^2})}$$

For the unit step input, we know $c_{final} = 1$. Substituting these results, we get

$$\%OS = 100.e^{-\zeta\pi/\sqrt{1-\zeta^2}}$$

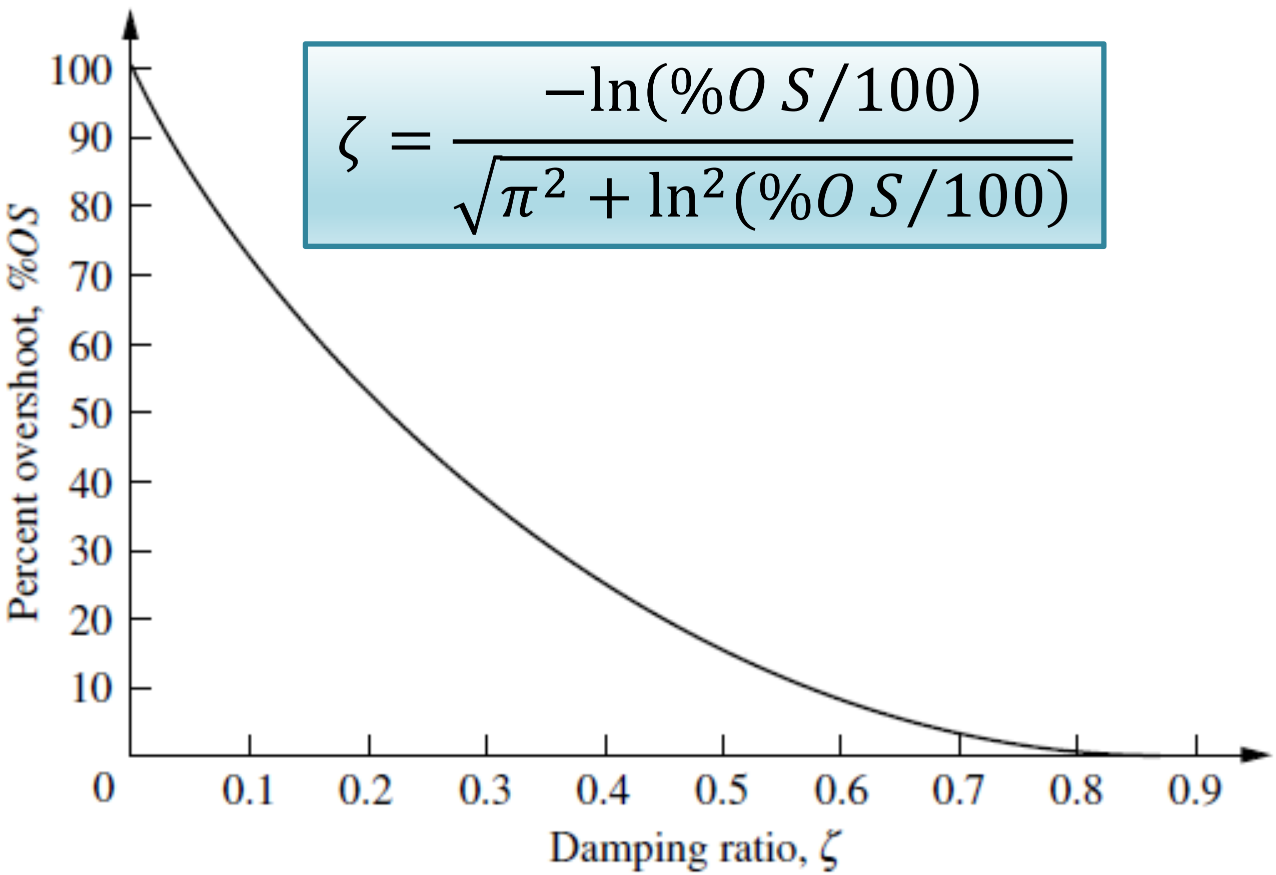
Evaluation of %OS *cont.'s*.

Notice that the % overshoot is a function only of the damping ratio,

$$\%OS = 100.\, e^{-\zeta\pi/\sqrt{1-\zeta^2}}$$

- It allows us to find $\%OS$ for a given ζ,
- The inverse of the equation allows us to solve for ζ for given $\%OS$.

$$\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}}$$



Evaluation of T_s for Underdamped Systems

$$c(t) = 1 - e^{-\zeta\omega_n t}\left(\cos \omega_n\sqrt{1-\zeta^2}t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin \omega_n\sqrt{1-\zeta^2}t\right)$$

$$= 1 - \frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\cos(\omega_n\sqrt{1-\zeta^2}t - \phi)$$

Using the definition, the settling time is the time it takes for the amplitude of the decaying sinusoid in $c(t)$ to reach 0.02,

$$\boxed{\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\pi t} = 0.02}$$

- This equation is a conservative estimate, since we are assuming that $\cos(\cdot) = 1$ at the settling time. Solving for t gives,

$$T_s = \frac{-\ln\left(0.02\sqrt{1-\zeta^2}\right)}{\zeta\omega_n}$$

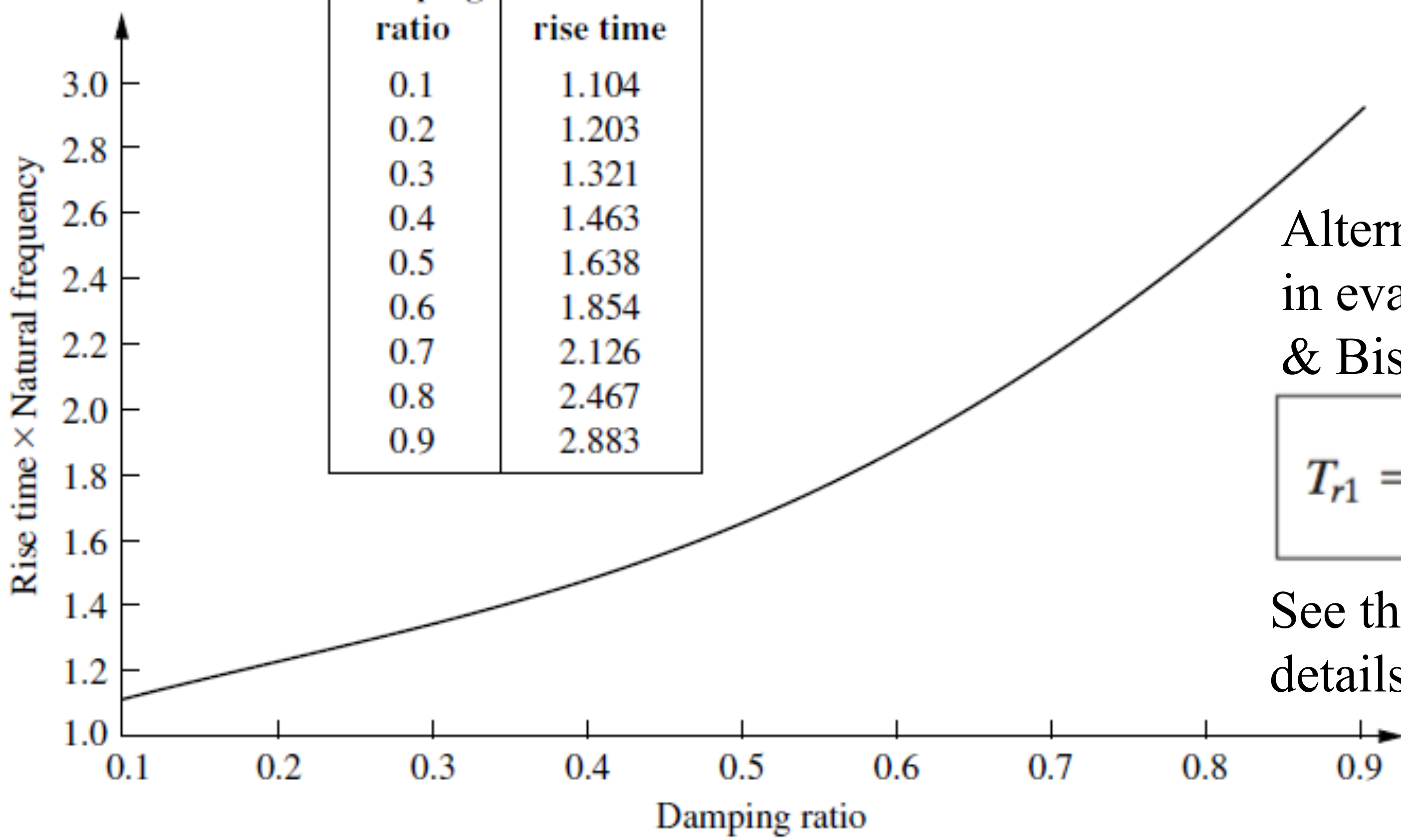
- We can verify that the numerator of this equation varies from 3.91 to 4.74 as ζ varies from 0 to 0.9. Thus, we can approximate it by

$$\boxed{T_s \cong \frac{4}{\zeta\omega_n}}$$

Evaluation of T_r for Underdamped Systems

A precise analytical relationship between rise time and damping ratio, ζ, can not be found. However, using a computer and $c(t)$, the rise time can be found. The following figure gives a relation between natural frequency, damping ratio and rise time.

Damping ratio	Normalized rise time
0.1	1.104
0.2	1.203
0.3	1.321
0.4	1.463
0.5	1.638
0.6	1.854
0.7	2.126
0.8	2.467
0.9	2.883



Alternative approach in evaluating T_r, Dorf & Bishop:

$$\boxed{T_{r1} = \frac{2.16\zeta + 0.60}{\omega_n},}$$

See the next slide for details

Evaluation of T_r for Underdamped Systems, *cont.'s*…

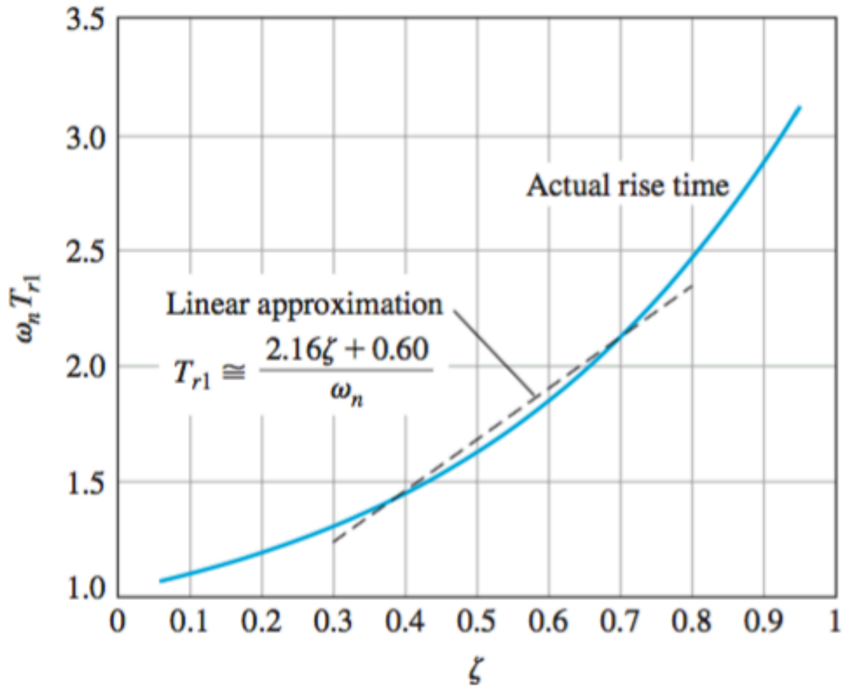


FIGURE 5.8 Normalized rise time, T_{r1}, versus ζ for a second-order system.

Alternative approach in evaluating T_r is based on the linear approximation of the ζ vs. $\omega_n T_r$ curve. *Ref.:* "Modern Control Systems", Pearson, R.C. Dorf and R.H. Bishop

The swiftness of step response can be measured as the time it takes to rise from 10% to 90% of the magnitude of the step input. This is the definition of the rise time, T_{r1}, shown in Figure 5.6. The normalized rise time, $\omega_n T_{r1}$, versus $\zeta(0.05 \leq \zeta \leq 0.95)$ is shown in Figure 5.8. Although it is difficult to obtain exact analytic expressions for T_{r1}, we can utilize the linear approximation

$$\boxed{T_{r1} = \frac{2.16\zeta + 0.60}{\omega_n},} \tag{5.17}$$

which is accurate for $0.3 \leq \zeta \leq 0.8$. This linear approximation is shown in Figure 5.8.

Example-3

Find $T_p, T_s, \%OS$ and T_r for the transfer function given below,

$$G(s) = \frac{50}{s^2 + 15s + 100}$$

What would be final value of the unit step response?

Solution: $G(s) = \frac{50}{s^2+15s+100} = \frac{50}{s^2+2\zeta\omega_n s+\omega_n^2}$

$$\therefore\ \omega_n = \sqrt{100} = 10 \text{ rad/s and } 2\zeta\omega_n = 15 \rightarrow \zeta = \frac{15}{2\omega_n} = \frac{15}{20} = 0.75$$

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{3.14}{10\sqrt{1-0.75^2}} = 0.475 \text{ sec},\ T_s \cong \frac{4}{\zeta\omega_n} = \frac{4}{0.75\cdot 10} = 0.533 \text{ sec}$$

$$\%OS = 100.e^{-\zeta\pi/\sqrt{1-\zeta^2}} = 100.e^{-0.75\pi/\sqrt{1-0.75^2}} = 2.84$$

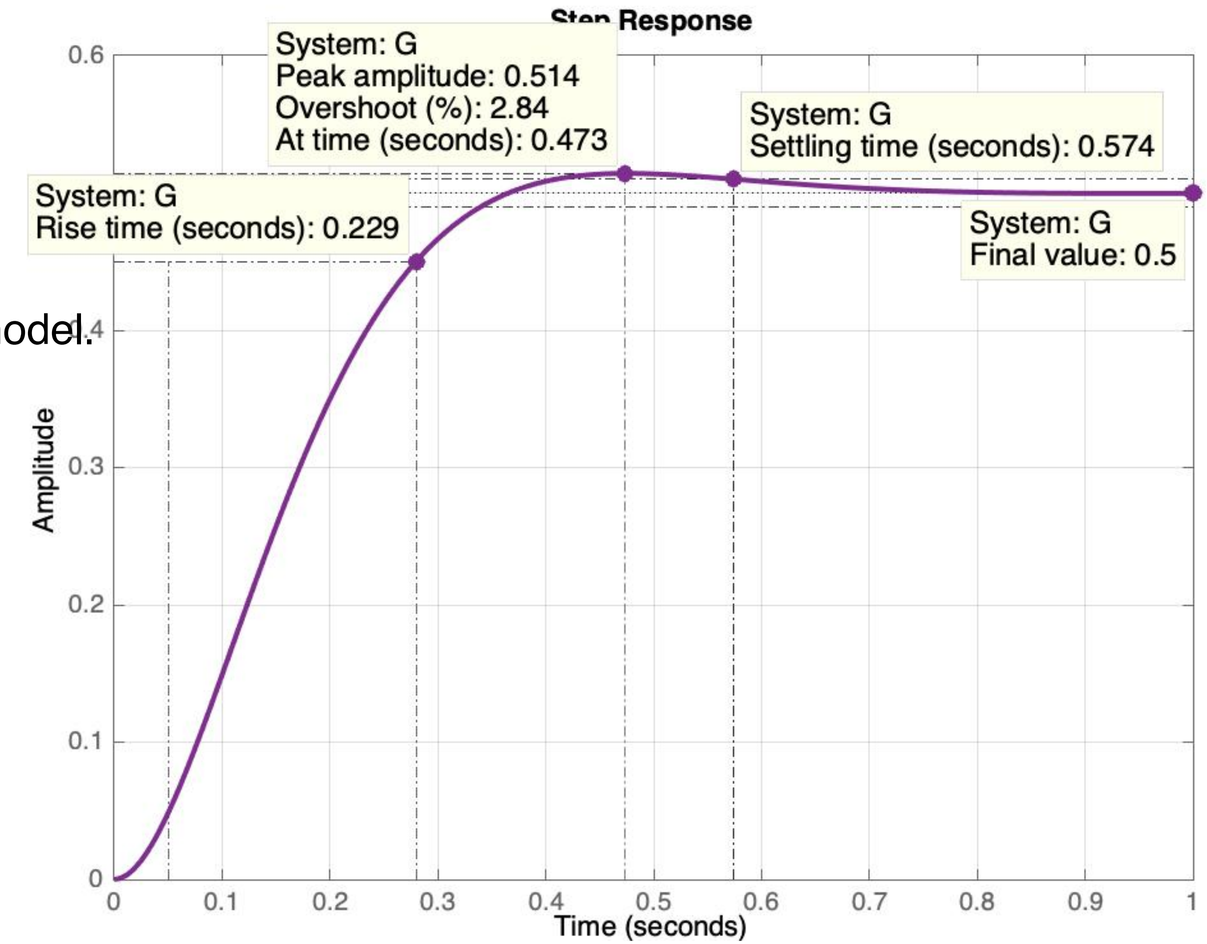
- Using the table for $\zeta = 0.75$, which corresponds to the Normalized rise time of $\omega_n T_r = 2.3 \rightarrow T_r = 0.23$ sec,
- Using the equation obtained by linearizing the nonlinear curve of normalized rise time vs damping ratio,

$$T_r = \frac{2.16\zeta+0.60}{\omega_n} = 0.222 \text{ sec}$$

Damping ratio	Normalized rise time
0.1	1.104
0.2	1.203
0.3	1.321
0.4	1.463
0.5	1.638
0.6	1.854
0.7	2.126
0.8	2.467
0.9	2.883

Example-3, *cont.'s:* solving in Matlab

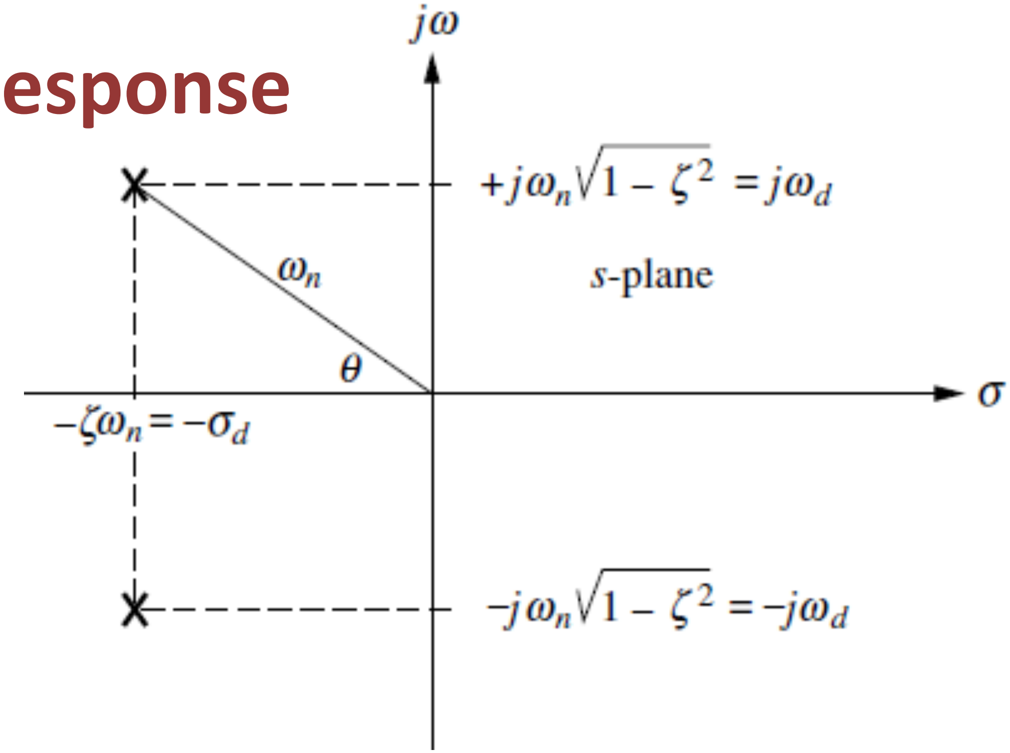
```
>> s=tf('s'), format compact;
>> G=zpk(50/(s^2 + 15*s +100)),
G =
              50
      -------------------------
       (s^2 + 15s + 100)
Continuous-time zero/pole/gain model.

>> step(G)

>> wn=sqrt(100), zeta=15/2/wn
wn =  10
zeta = 0.7500

>> Tp=pi/(wn*sqrt(1-zeta^2))
Tp = 0.4750

>> Ts=4/zeta/wn
Ts =  0.5333

>> pOS=100*exp(-zeta*pi/sqrt(1-zeta^2))
pOS =  2.8375
```



Pole Location and Response

- From the pole plot for a general, underdamped second-order system, the radial distance from the origin to the pole is the natural frequency, ω_n
- Hence, the

$$\cos(\theta) = \zeta$$



- Now, we can evaluate peak time and settling time in terms of the pole location.

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\mathrm{Im(poles)}} = \frac{\pi}{\omega_d}$$

$$T_s \cong \frac{4}{\zeta\omega_n} = \frac{4}{\mathrm{Re(poles)}} = \frac{4}{\sigma_d}$$

- ω_d is called the *damped frequency of oscillation*, and
- σ_d is called the *exponential damping frequency*.

Pole Location and Response *cont.'s*.

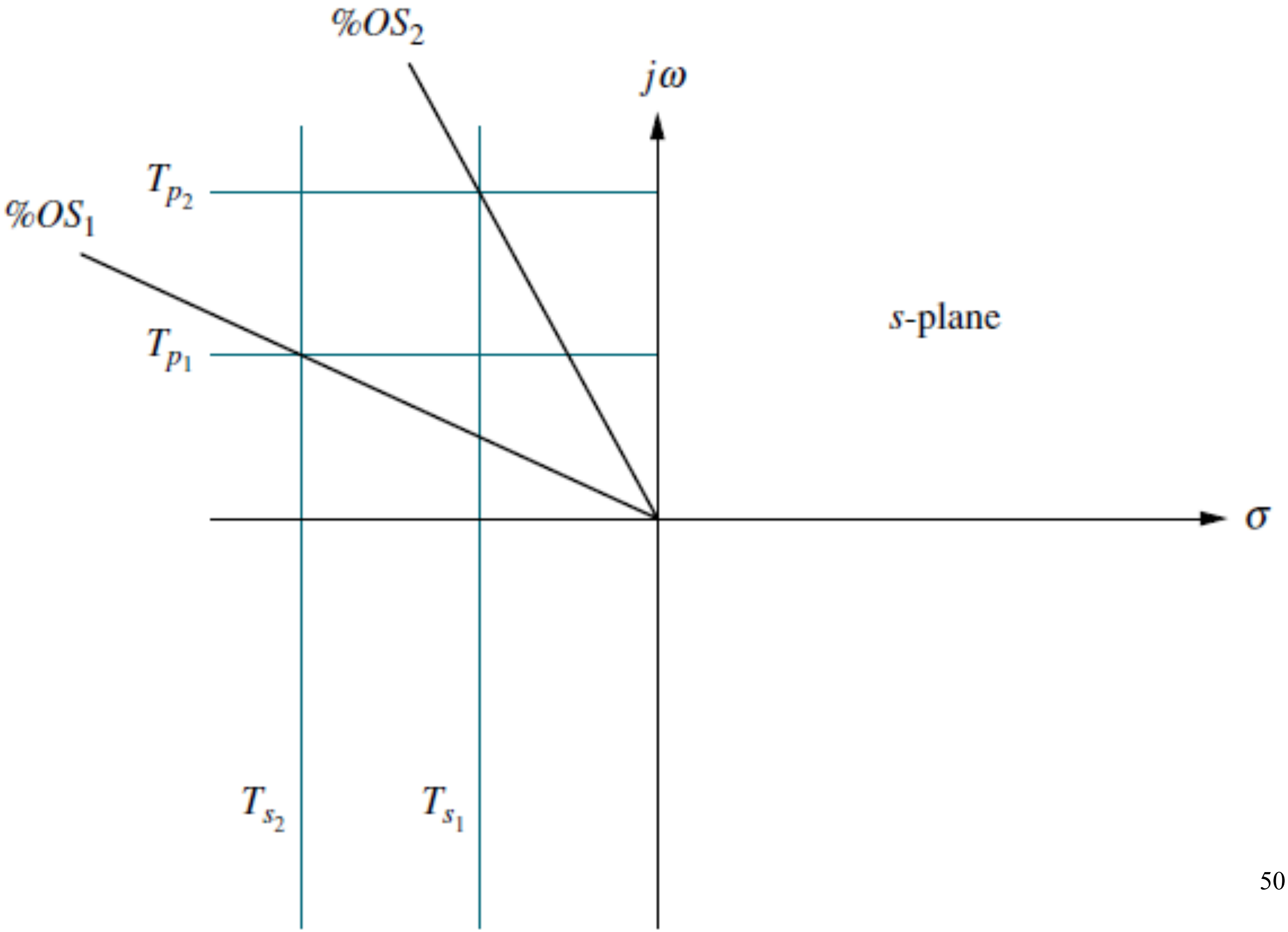
Therefore,

- T_p is inversely proportional to the imaginary part of the pole. Since horizontal lines on the *s*-plane are lines of constant imaginary value, *they are also lines of constant peak time*.

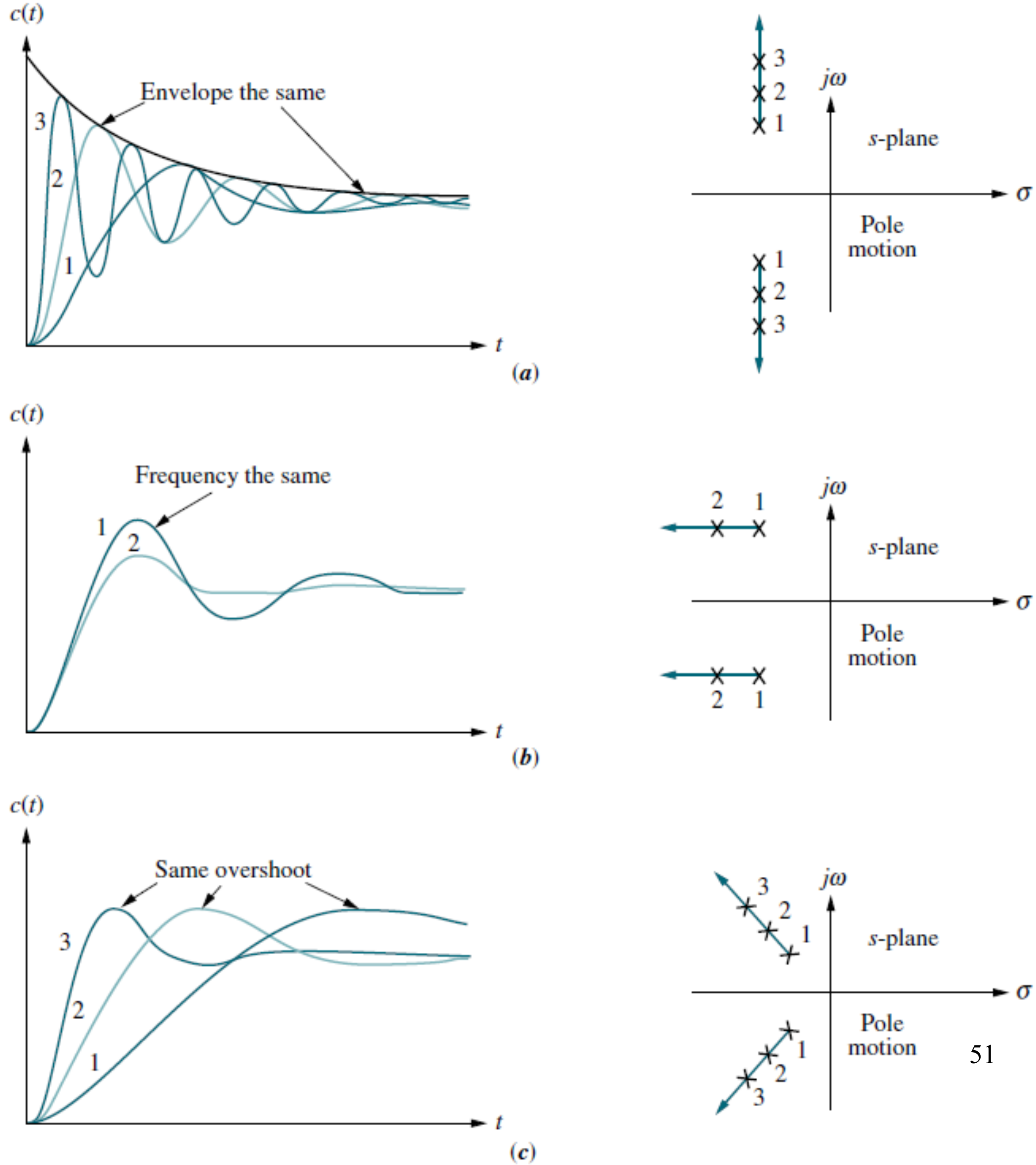
- T_s is inversely proportional to the real part of the pole. Since vertical lines on the *s*-plane are lines of constant real value, *they are also lines of constant settling time*.

- Finally, since $\zeta = \cos(\theta)$, radial lines are lines of constant ζ. Since percent overshoot is only a function of ζ, *radial lines are thus lines of constant percent overshoot,* $\%OS$.

Pole Location and Response *cont.'s*.



Pole Location and Response *cont.'s*.



Example-4

Problem: Given the pole plot, find ζ, ω_n ,T_p, $\%OS$ and T_s

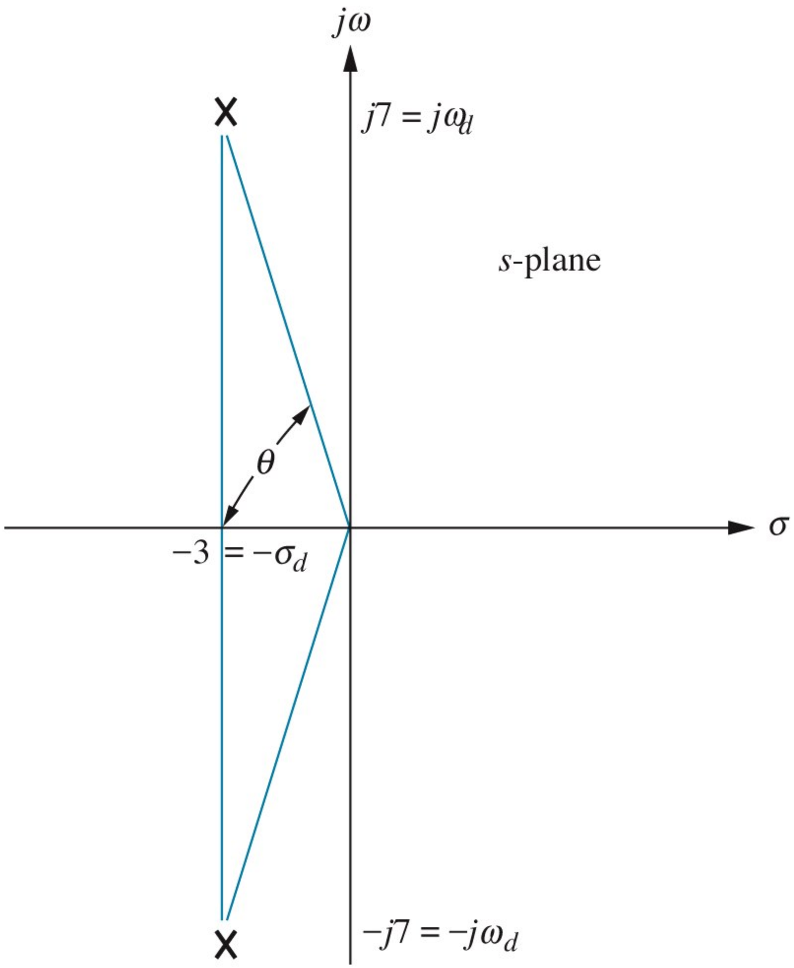


Figure 4.20

Solution:

The poles are at $-3 \pm j7$

$$\omega_n = \sqrt{3^2 + 7^2} = 7.616 \text{ rad/s}$$

$$\zeta = \cos\theta = \frac{3}{7.616} = 0.3939$$

$$\%OS = 100.\, e^{-\zeta\pi/\sqrt{1-\zeta^2}} = \mathbf{26}$$

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\text{Im(poles)}} = \frac{\pi}{7}$$

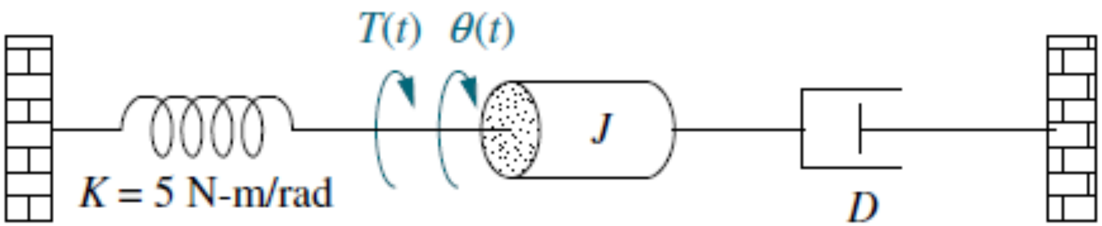
$$\boldsymbol{T_p = 0.4488 \text{ sec}}$$

$$T_s \cong \frac{4}{\zeta\omega_n} = \frac{4}{\text{Re(poles)}} = \frac{4}{3}$$

$$\boldsymbol{T_s = 1.33 \text{ sec}}$$

Example-5

Given the system, find J and D to yield %20 overshoot and a settling time of 2s for a step input of torque $T(t)$,



$$G(s) = \frac{1/J}{s^2 + \frac{D}{J}s + \frac{K}{J}} \quad \Longrightarrow \quad \omega_n = \sqrt{\frac{K}{J}} \qquad 2\zeta\omega_n = \frac{D}{J}$$

$$T_s = 2 = \frac{4}{\zeta\omega_n} \quad \Longrightarrow \quad \zeta\omega_n = 2 \quad \Longrightarrow \quad 2\zeta\omega_n = 4 = \frac{D}{J}$$

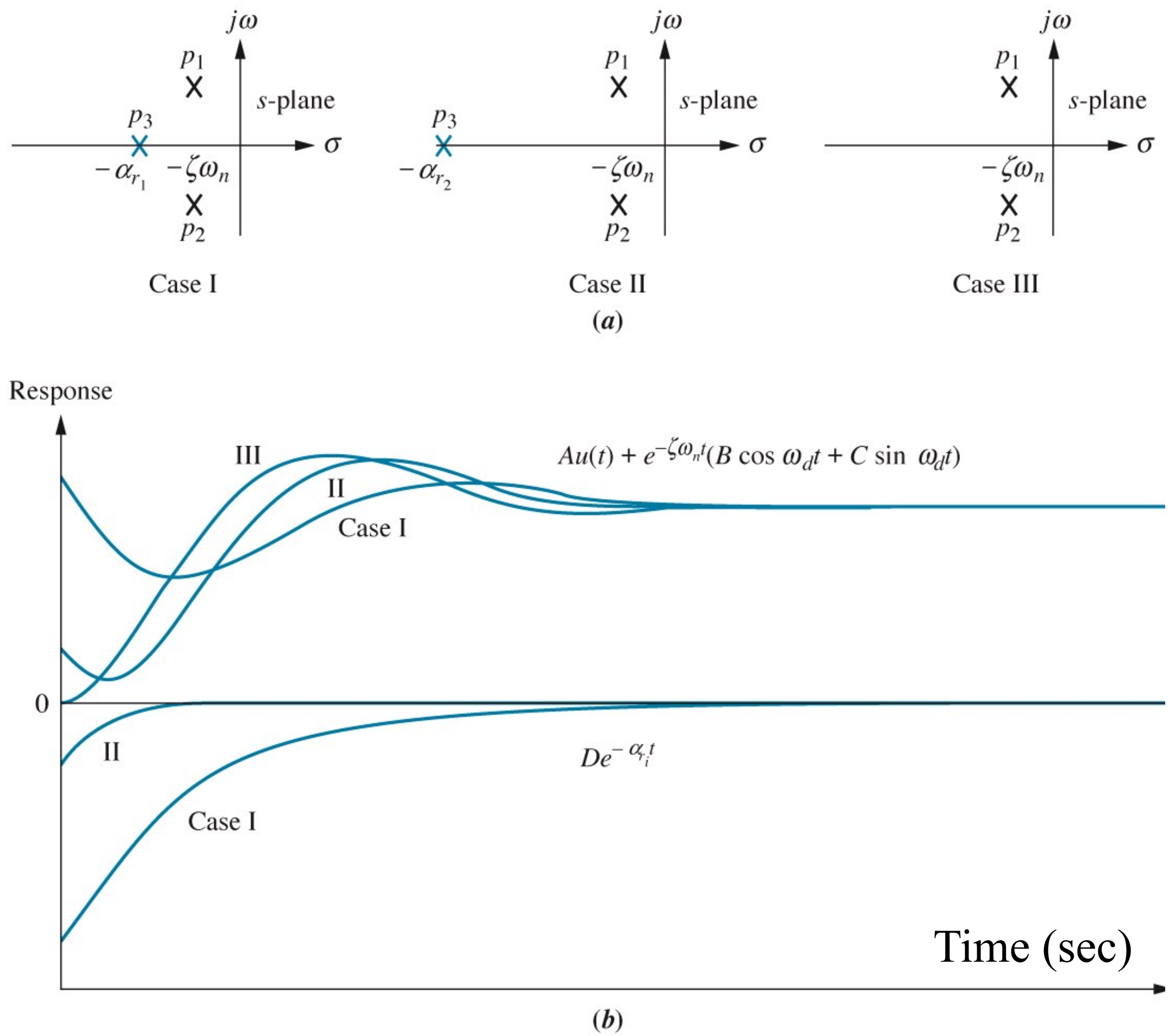
$$\zeta = \frac{4}{2\omega_n} = 2\sqrt{\frac{J}{K}}$$

20% overshoot implies $\zeta = 0.456$ $\Longrightarrow$ $\zeta = 2\sqrt{\frac{J}{K}} = 0.456$ $\Longrightarrow$ $\frac{J}{K} = 0.052$

$K = 5$ N-m/rad $\Longrightarrow$ $D = 1.04$ N-m-s/rad $\Longrightarrow$ $J = 0.26$ kg-m^2

Figure 4.23

Component responses of a three-pole system: **(*a*)** pole plot; **(*b*)** component responses:



Non-dominant pole is

- near dominant second-order pair (Case I),
- far from the pair (Case II), behaves like 2nd-order systems, and
- at infinity (Case III): it is almost 2nd-order.

Conclusion:

2nd-order approximation is valid if,

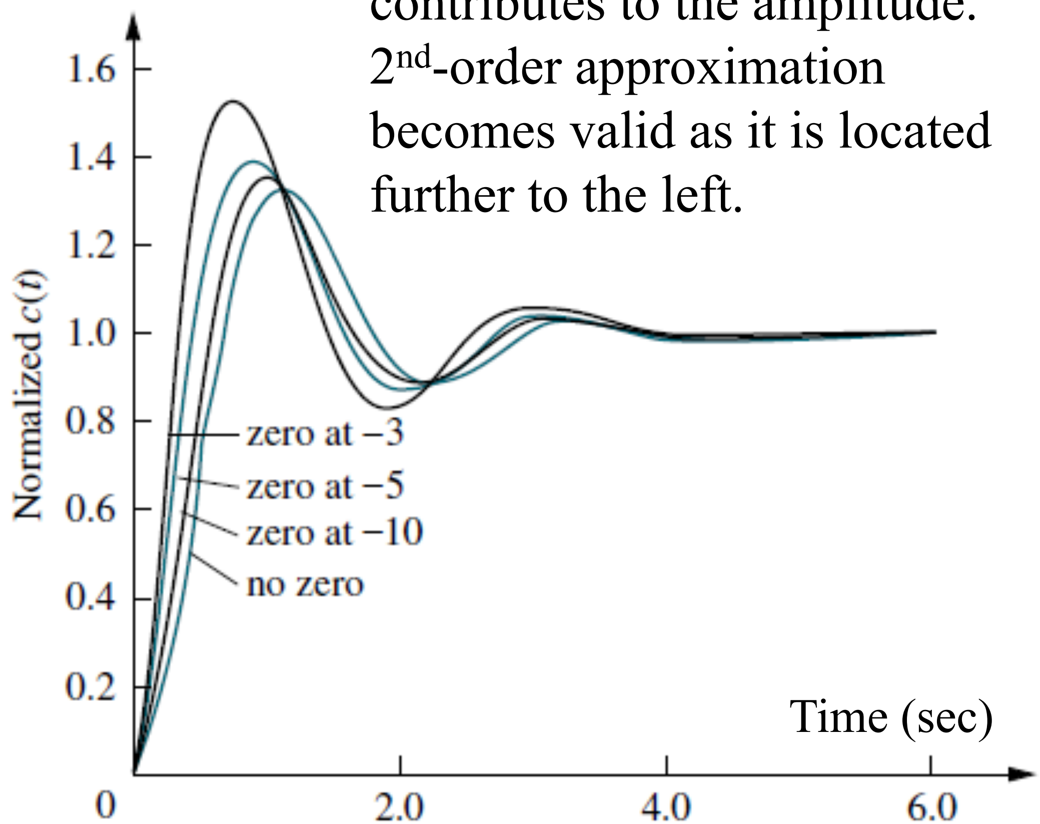
$$|-\alpha_r| > 5|-\zeta\omega_n|$$

Systems with Additional Poles and Zeros

- If a system has more than two poles or has zeros, we cannot use the formulas to calculate the performance specifications that we derived.
- However, under certain conditions, a system with more than two poles can be approximated as a second-order system that has just two complex *dominant poles*.

- We saw that the zeros of a response affect the residue, or amplitude, of a response component but do not affect the nature of the response exponential, damped sinusoid, and so on.
- Starting with a two-pole system with poles at $-1 \pm j2.828$, we consecutively add zeros at -3, -5, and -10. The results, normalized to the steady-state value, are plotted in the figure.

Conclusion: The zero contributes to the amplitude. 2nd-order approximation becomes valid as it is located further to the left.



$$T_1(s) = \frac{24.542}{s^2 + 4s + 24.542}$$

$$T_2(s) = \frac{245.42}{(s + 10)(s^2 + 4s + 24.542)}$$

$$T_3(s) = \frac{73.626}{(s + 3)(s^2 + 4s + 24.542)}$$

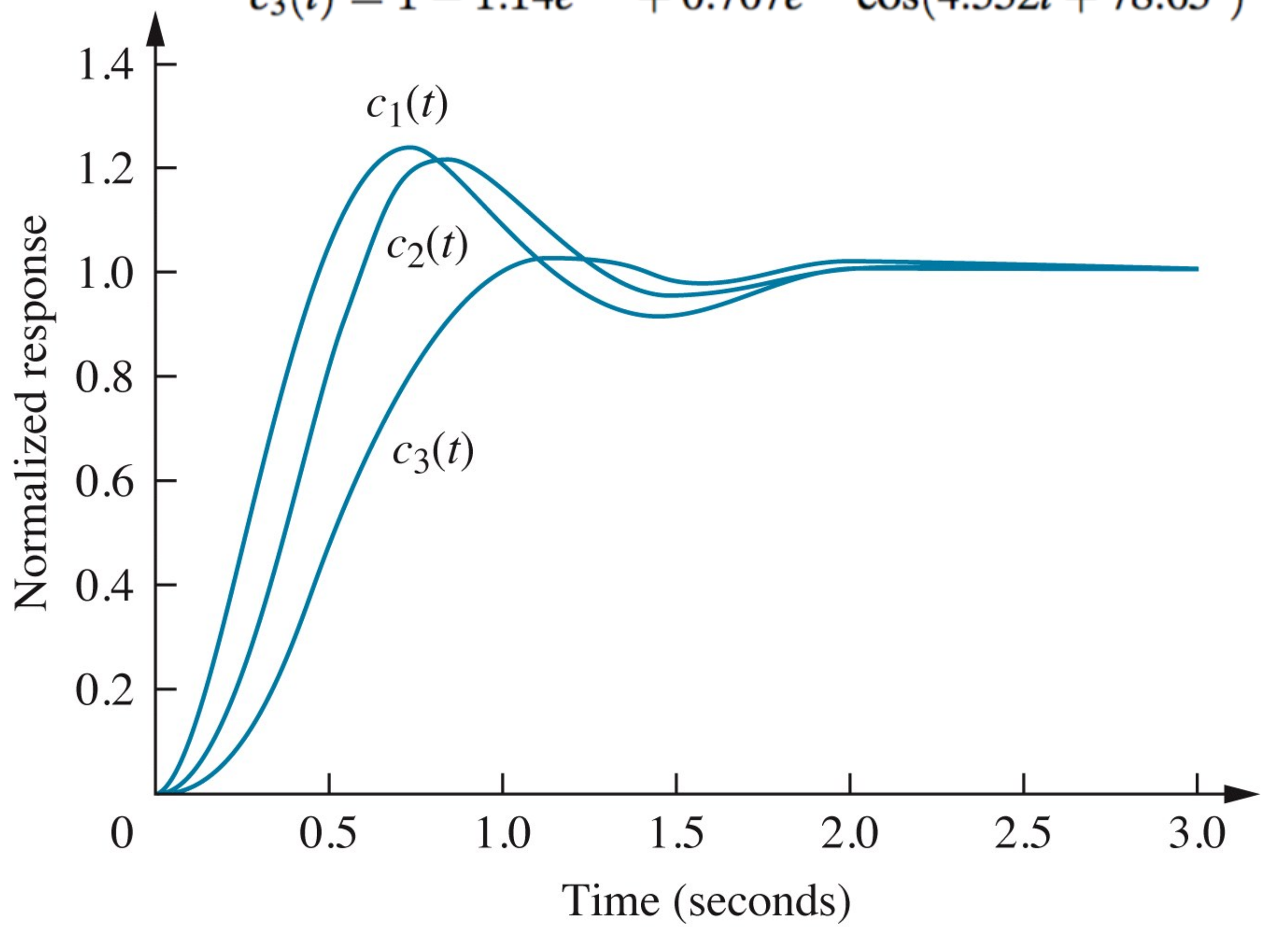
$$c_1(t) = 1 - 1.09e^{-2t}\cos(4.532t - 23.8°)$$

$$c_2(t) = 1 - 0.29e^{-10t} - 1.189e^{-2t}\cos(4.532t - 53.34°)$$

$$c_3(t) = 1 - 1.14e^{-3t} + 0.707e^{-2t}\cos(4.532t + 78.63°)$$

Step responses of system $T_1(s)$, system $T_2(s)$,
and system $T_3(s)$

Conclusion: The poles are at $-2 \pm j4.53$
A pole is added first at -10 (T_2) and then at -3 (T_3).
Since -10 is much further to the left compared to -3 with respect to -2, 2nd-order approximation valid for T_2



Step response of a nonminimum-phase system

An interesting phenomenon occurs if there is a zero in the right half-plane (RHP).

- The response starts evolving from an opposite direction.
- This kind of systems are called nonminimum-phase systems.
- Unlike having a RHP pole, these systems having a RHP zero are not unstable but must be dealt with carefully depending on the application.