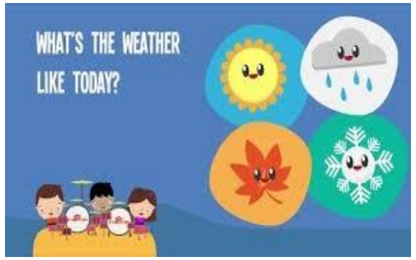
IST1990 Probability And Statistics

- **Lecture 2**

- Introduction to the Theory of Probability
- Random Experiment & Sample Space
- Set Operations for Events
- Axioms of Probability

What is probability?

Most people use terms such as chance, likelihood, or probability to reflect the level of uncertainty about some issues or events. Examples in which these terms may be used are as follows:



• As you watch the news every day, you hear forecasters saying that there is a 70% chance of rain tomorrow.

• As you plan to enter a new business, an expert in the field tells you that the probability of making a first-year profit in this business is only 0.4, or there is a 40% chance that you will make a profit.

•As you take a new course, you may be wondering about the likelihood of passing or failing the course.

Introduction to the Theory of Probability

- Probability is a mathematical tool dealing with the mathematical models of random experiments and measure of belief of the random events (possibilities of occurence of events).

- Random Experiment is an action whose results can not be predicted with certainty before hand.

- The result or realization of the random experiment is called Outcome.

To explain all this context, we will give a simple example:

Assume that a random experiment consist of tossing a coin. We can not predict with certainty which outcome will come, Head or Tail.

- However, we can describe the set of all possible outcomes. It is Sample Space!

- The set of all possible outcomes of the random experiment is called the *sample space* and is denoted by the symbol S.

- Each outcome of the random experiment or each element of the sample space is called an *elementary event.*

- Any subset of sample space is called an *event.*

There are random experiments with countable set and continuum set of outcomes. So, sample space S may have finite or infinite elements.

Sample Space

Countable

- Finite
- Countably Infinite

Uncountable

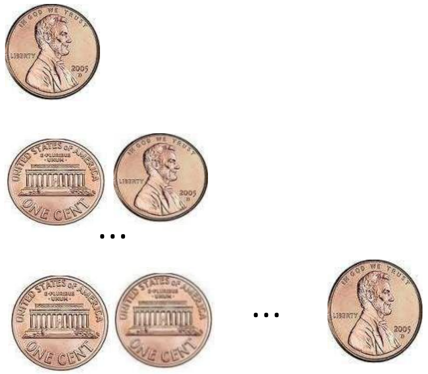
Example (finite S): The sample space S, of possible outcomes when a coin is flipped, may be written

$$S = \{H, T\}$$

where H and T correspond to head and tail, respectively.

Example (countably infinite S): Consider the experiment of tossing a coin until head occurs. What is the sample space?

$$S = \{H, TH, TTH, \ldots, T\ldots TH, \ldots.\}$$



Example : In rolling a six-sided die,

- What is the random experiment?
- What is an outcome?
- What is an event?
- What is a sample space?

Solution:

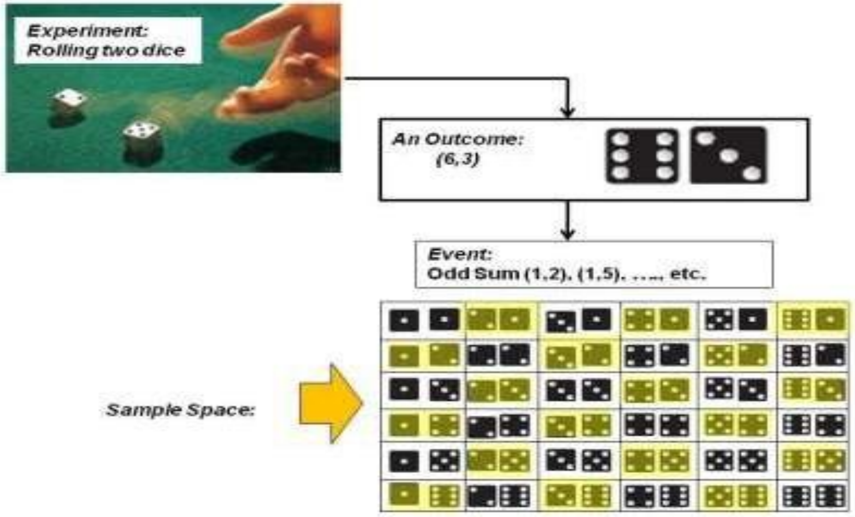
Rolling a six-sided die is the random experiment. A number such as 1, 2, 3, 4, 5, or 6 is the outcome. Observing a certain number, such as odd or even number will be the event. So, an event may be A={the number on the die is even}={2,4,6}

The sample space for this experiment is S = {1, 2, 3, 4, 5, 6}.

Example : In the process of rolling a pair of fair dice,

- What is the random experiment?
- What is an outcome?
- What is an event?
- What is a sample space?

Solution:

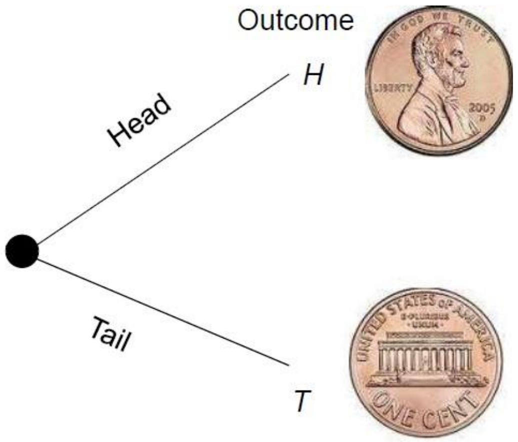


The Venn diagram and the tree diagram

The Venn diagram:

A picture representing events as circles enclosed in an rectangle. The rectangle represents the sample space and each circle represents an event

(a) Venn Diagram

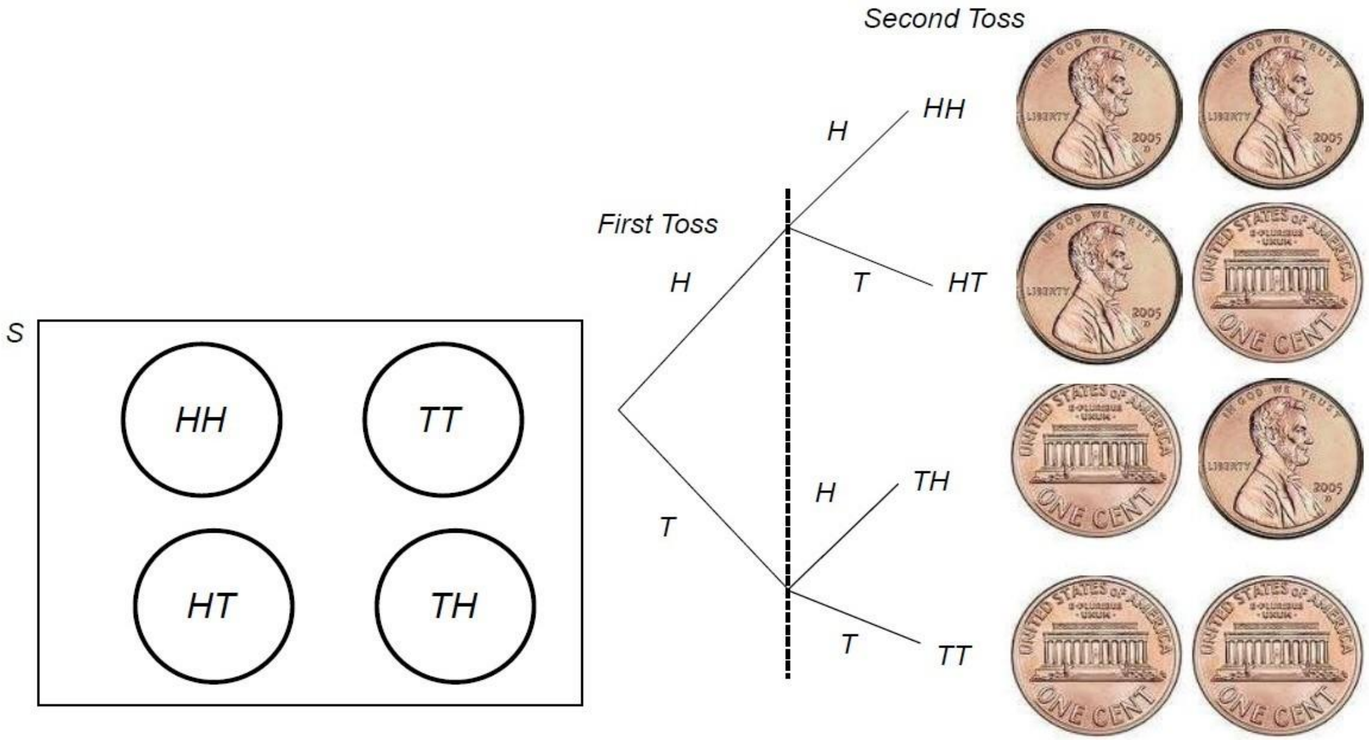


(b) Tree Diagram

The Venn Diagram and the Tree Diagram for the Experiment of Tossing a Coin

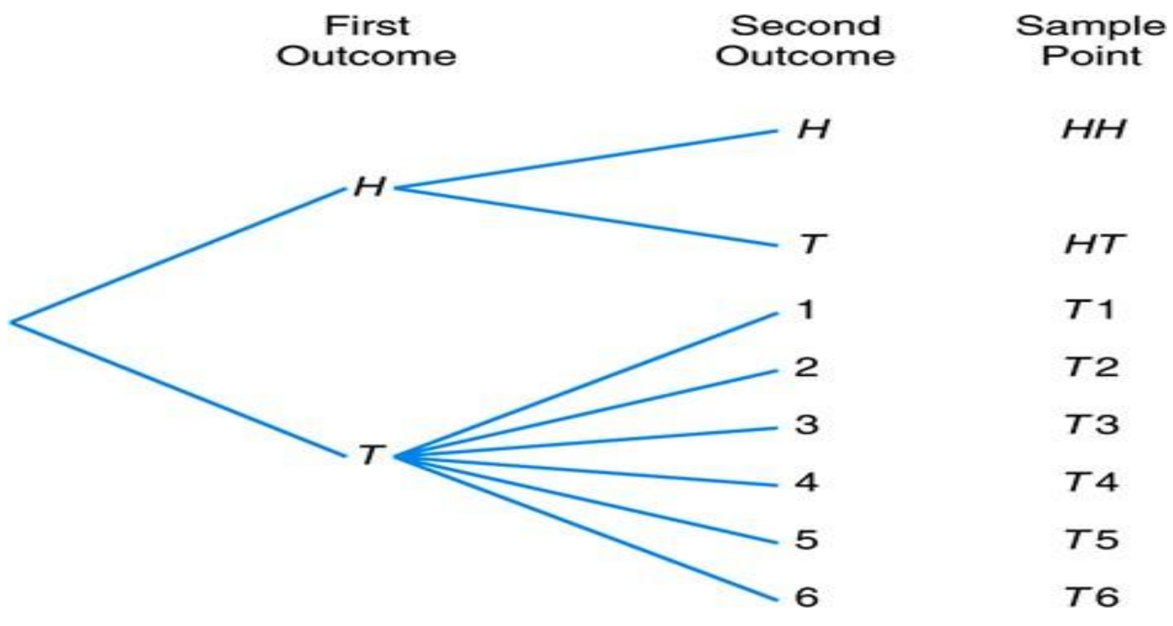
Example: Display the Venn diagram and the tree diagram of the experiment of tossing a coin twice.

Solution:
In this experiment, the following sample space represents all possible outcomes:
$S = \{HH, TT, HT, TH\}$.
The Venn and tree diagrams of this experiment are shown below



(a) Venn Diagram (b) Tree Diagram

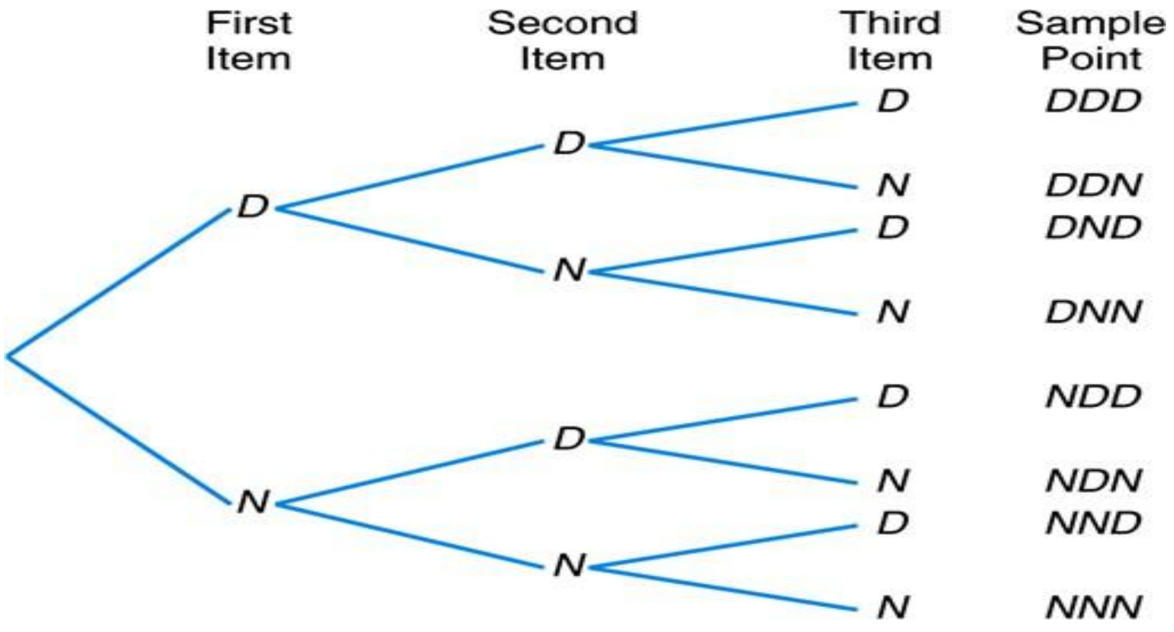
Example: An experiment consists of flipping a coin and then flipping it a second time if a head occurs. If a tail occurs on the first flip, then a die is tossed once. To list the elements of the sample space providing the most information, we construct the *tree diagram*



By proceeding along all paths, we see that the sample space is

$$S = \{HH, HT, T1, T2, T3, T4, T5, T6\}.$$

Example: Suppose that three items are selected at random from a manufacturing process. Each item is inspected and classified defective, D, or non-defective, N. We construct the tree diagram



As we proceed along the other paths, we see that the sample space is

$$S = \{DDD, DDN, DND, DNN, NDD, NDN, NND, NNN\}.$$

Remember the definition: An *event* is a subset of the sample space. Any event is represented by capital letters such as A, B, C...

Consider the sample space of tossing a die

$$S = \{1, 2, 3, 4, 5, 6\}.$$

For example A = {2, 4, 6} is an event defined on the sample space S. One can define 2^6 events on the sample space S.

In the event, we observe only some of elementary events.

If the outcome of the experiment belongs to A (for this example the outcome is 6 or 4 or it is 2), then we say that event A occurs in the experiment.

Empty set Ø is an *impossible event* and S is a *certain event.*

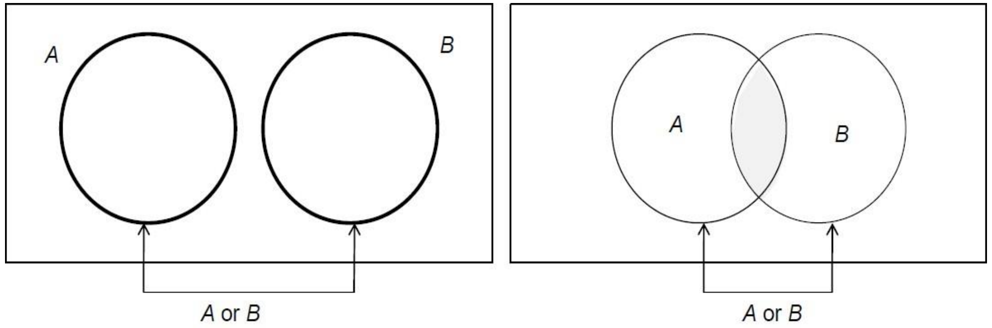
Set Operations for Events

Definition: The *intersection* of two events A and B, denoted by the symbol $A \cap B$, is the event containing all elements that are common to A and B.

Definition: Two events A and B are called *mutually exclusive* or *disjoint*, if

$$A \cap B = \emptyset,$$

that is, if A and B have no elements in common.



(a) mutually exclusive events (b) non-mutually exclusive

In another words, events *A* and *B* are disjoint (or mutually exclusive) if they cannot occur at the same time.

Definition: The *union* of the two events A and B, denoted by the symbol $A \cup B$, is the event containing all the elements that belong to A or B or both.

Set Operations for Events

Definition: The *complement* of an event A with respect to S is the subset of all elements of S that are not in A. We denote the complement of A by the symbol A' or it is defined by

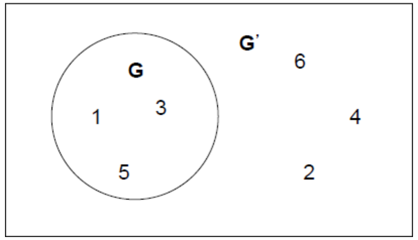
$$A' = S \setminus A,$$

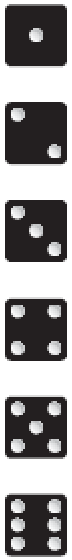
where \ is difference operator.

For two events, $A \setminus B = A \cap B'$ occurs in the experiment, if A occurs and B does not occur.

Example: If G is the event of odd number coming up in rolling a six-sided die once, then the events G and G' are as follows:

S





$G=\{1,3,5\}$ and $G'=\{2,4,6\}$

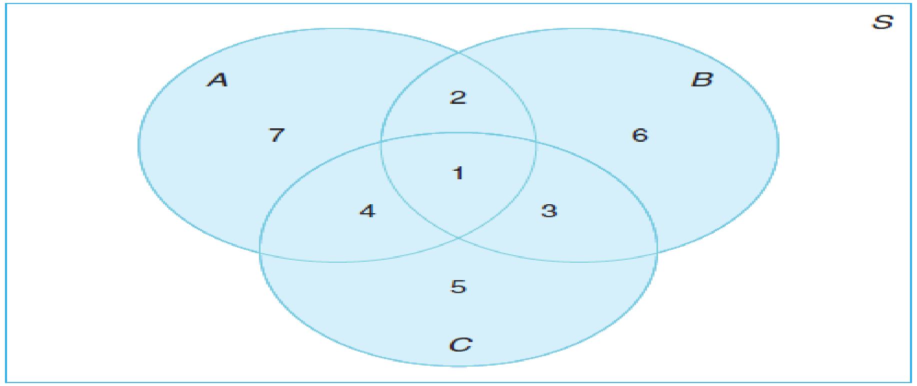
Example: Consider the following sample space

S = {book, cell phone, mp3, paper, stationery, laptop}.

Let A = {book, stationery, laptop, paper}. Then the complement of A is

A' = {cell phone, mp3}.

The relationship between events and the corresponding sample space can be illustrated graphically by means of *Venn diagrams*. In a Venn diagram we let the sample space be a rectangle and represent events by circles drawn inside the rectangle.



we see that

$A \cap B$ = regions 1 and 2,

$B \cap C$ = regions 1 and 3,

$A \cup C$ = regions 1, 2, 3, 4, 5, and 7,

$A \backslash B = A \cap B'$ = regions 4 and 7,

$A \cap B \cap C$ = region 1,

$(A \cup B) \cap C'$ = regions 2, 6, and 7, Notes: The following results may easily be verified by means of Venn diagrams.

(1) $A \cap \varnothing = \varnothing$

(2) $A \cup \varnothing = A$

(3) $A \cap A' = \varnothing$

(4) $A \cup A' = S$

(5) $S' = \varnothing$

(6) $\varnothing' = S$

(7) $(A')' = A$

(8) $(A \cap B)' = (A' \cup B')$ 1^{st} De Morgan Rule

(9) $(A \cup B)' = A' \cap B'$ 2^{nd} De Morgan Rule

Basic Principle of Counting

Multiplication rule: If an operation can be performed in n_1 ways, and if for each of these ways a second operation can be performed in n_2 ways, then the two operations can be performed together in $n_1 n_2$ ways.

Example: How many sample points are there in the sample space when a pair of dice is thrown once?

Solution: The first die can land face-up in any one of $n_1 = 6$ ways. For each of these 6 ways, the second die can also land face-up in $n_2 = 6$ ways. Therefore, the pair of dice can land in

$$n_1 n_2 = (6)(6) = 36$$

possible ways.

$$S = \begin{Bmatrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{Bmatrix}$$

The generalized multiplication rule: If an operation can be performed in n_1 ways, and if for each of these a second operation can be performed in n_2 ways, and for each of the first two a third operation can be performed in n_3 ways, and so forth, then the sequence of *k* operations can be performed in

$$n_1 n_2 \dots n_k$$

ways.

Example: Sam is going to assemble a computer by himself. He has the choice of chips from two brands, a hard drive from four, memory from three, and an accessory bundle from five local stores. How many different ways can Sam order the parts?

Solution: Since

there are

$$n_1 = 2, \qquad n_2 = 4,\ n_3 = 3, \qquad \text{and } n_4 = 5,$$

different ways to order the parts.

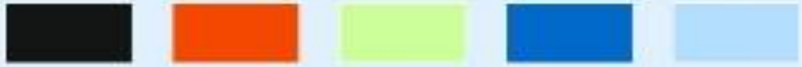
$$n_1 \times n_2 \times n_3 \times n_4 = 2 \times 4 \times 3 \times 5 = 120$$

Example: You are buying a new car.

There are **2** body styles:

sedan or hatchback

There are **5** colors available:



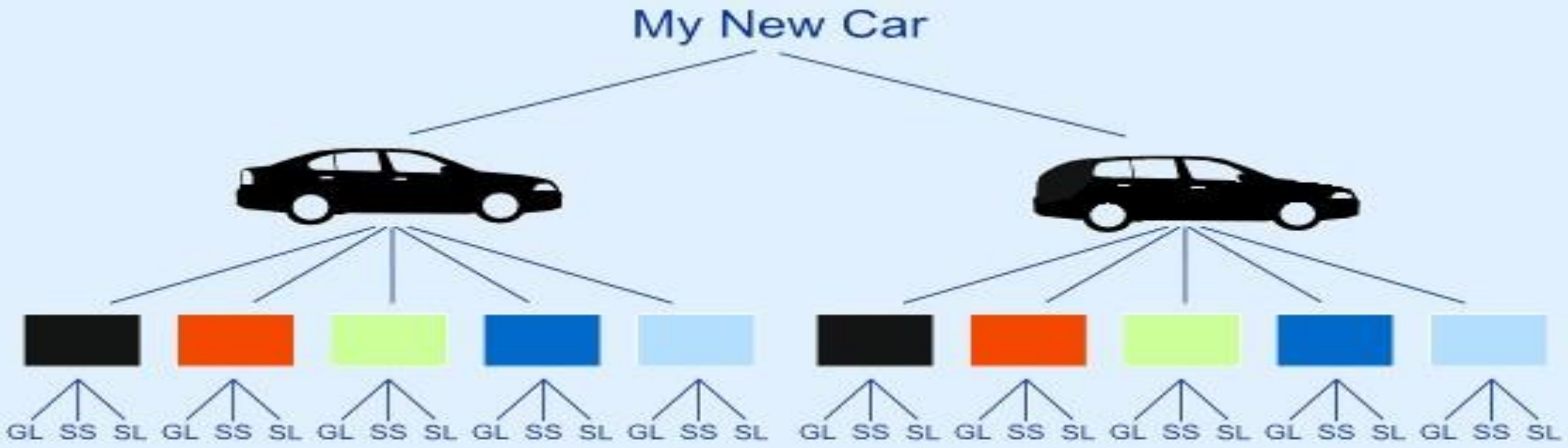
There are **3** models:

- GL (standard model),
- SS (sports model with bigger engine)
- SL (luxury model with leather seats)

How many total choices?

How many total choices?

You can see in this "tree" diagram:



You can count the choices, or just do the simple calculation:

Total Choices = $2 \times 5 \times 3 = 30$

Classical Probability Model. (Sample space having equally likely outcomes)

Let sample space S be a finite set, having N different elements. If we assign equal probabilities to all outcomes of S, then such a model is called classical probability model.

Theorem. If an experiment can result in any one of N different equally likely outcomes, and if exactly n of these outcomes correspond to event A , then the probability of event A is

$$P(A) = \frac{n}{N}$$

n: total # of outcomes of the event A

N: total # of outcomes of the sample space S.

Example: A coin is tossed twice. What is the probability that at least one head occurs?

The sample space for this experiment is $S = \{HH, HT, TH, TT\}$.

A represents the event of at least one head occurring is $A = \{HH, HT, TH\}$, *then*

$$P(A) = \frac{n}{N} = \frac{3}{4}.$$

Example: A die is tossed twice. What is the probability that sum of numbers on die will be less than or equal to 3?

S = {(1,1),(1,2), …,(1,6),…,(6,1),(6,2),…(6,6)} has 36 elementary events.

A= {sum of numbers on die less or equal than 3} = {(1,1),(1,2),(2,1)}. We have 3 outcomes for event A, then

$$P(A) = \frac{n}{N} = \frac{3}{36} = \frac{1}{12}.$$

Interpretation of Probability

The concept of probability is a very abstract one:

- Idea: quantify the "likelihood" or "chance" of the outcome of an experiment…

- One can have a **subjective** interpretation of probability:

The probability of event E, is denoted by $P(E)$, and expressed the degree of belief we assign for the occurence of event E.

It takes a number between 0 and 1, where 0 indicates the event does not occur, and 1 indicates the event certainly occurs (one can also consider percentages).

This might not be very satisfying, as different people will assign different values for the probability of an event…

Interpretation of Probability

- A **frequentist** interpretation of probability:

Suppose we can repeat a random experiment n times. The possible outcome is either in E or not in E. Now let $N(n, E)$ be the number of times the outcome of the experiment is in E. The frequentist interpretation of probability defines $P(E)$ as

$$P(E) \equiv \lim_{n \to \infty} \frac{N(n, E)}{n} .$$

This is simply the fraction of times E happens when we repeat the experiment a large number of times !

Example:

- Take the first card out of a shuffled deck
- Consider the event {the card selected is an ace}

What do we exactly mean by the repetition of an experiment ???

Axioms of Probability

Definition: A **probability** is a numerically valued function that assigns a number $P(A)$ to every event A *of* a sample space S, such that the following axioms hold:

(1) $P(A) \geq 0$ (nonnegativity)

(2) $P(S) = 1$ (normality)

(3) If A_1, A_2, ... is a sequence of mutually exclusive events (*i.e.* $A_i \cap A_j = \emptyset$ for any *i and j*), then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

(sigma additivity)

Axioms of Probability

Example: Random experiment – Flipping a coin $S = \{\text{heads}, \text{tails}\}$

$$H = \{\text{heads}\} \qquad T = \{\text{tails}\}$$

$$P(S) = P(\{\text{heads}, \text{tails}\}) = 1$$
$$P(S') = P(\emptyset) = P(\text{neither heads or tails}) = 0$$
$$P(S) = P(H \cup T) = P(H) + P(T) \ (\text{as } H \cap T = \emptyset)$$

If we have a **fair** coin then $P(H) = P(T) = 1/2$

In general, if you have N possible outcomes:

> If N outcomes are **equally likely** then the probability of each outcome is $1/N$

Basic Properties

Just from the axioms we can deduce a number of simple, but useful properties:

Lemma:

(i) $P(A') = 1 - P(A)$

(ii) $P(\emptyset) = 0$

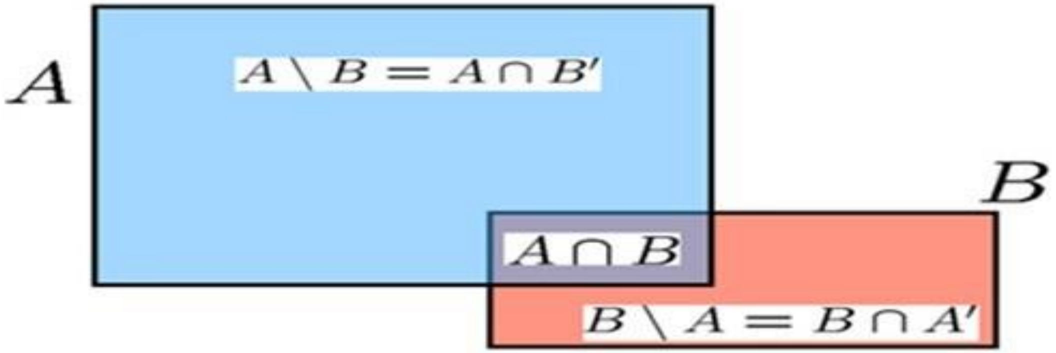
Proof: Note that $S = A \cup A'$, and that A and A' are mutually exclusive. Therefore

$$P(S) = P(A) + P(A') \Leftrightarrow 1 = P(A) + P(A') \ .$$

Applying (i) to the event S immedially shows part (ii).

Addition Rules

Actually, the probability of an event is just a way to **measure** it !!
- you can think of it as the **generalized volume** of the set.

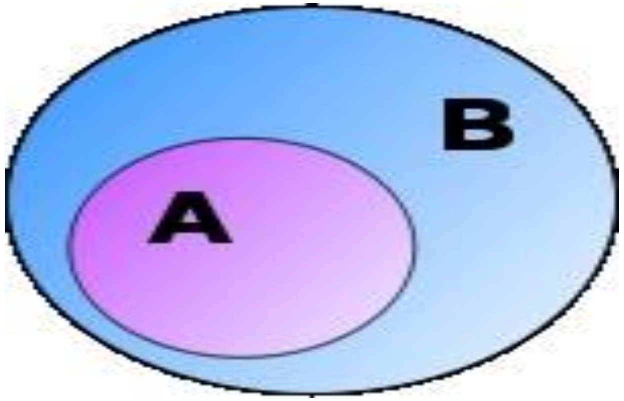


Lemma: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

If the events are mutually exclusive then (and only then!):

$$P(A \cup B) = P(A) + P(B)$$

Lemma: If $A \subset B$, then $P(A) \leq P(B)$.



Example: If a pair of fair dice is rolled,

- What is the probability of rolling a sum of two?
- What is the probability of rolling a sum of seven?
- What is the probability of rolling a sum of four?

Solution:

$$P(B) = P(roll\ a\ seven) = \frac{6}{36} = \frac{1}{6}$$

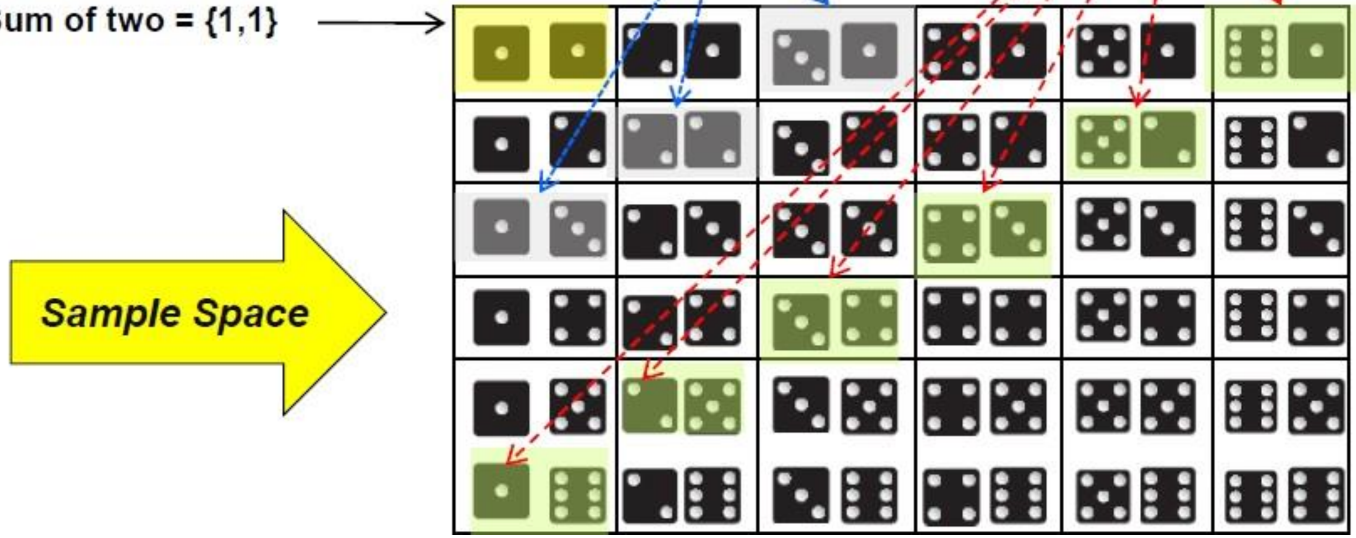
Sum of seven = {(6,1), (1,6), (5,2), (2,5), (4,3), (3,4)

$$P(C) = P(roll\ a\ four) = \frac{3}{36} = \frac{1}{12}$$

Sum of four = {(1,3), (3,1), (2,2)}

$$P(A) = P(roll\ a\ two) = \frac{1}{36}$$

Sum of two = {1,1}



Example. **What is the probability of getting a total either 7 or 11 when a pair of dice are tossed?**

$$S = \begin{Bmatrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{Bmatrix}$$

A : *event that* 7 *occurs* = { (*1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)}*

B : *event that* 11 *occurs* = { (*5, 6), (6, 5)}*

$P(A) = \frac{6}{36}$

$P(B) = \frac{2}{36}$

$P(A \cap B) = 0$

$P(A \cup B) = ?$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{6}{36} + \frac{2}{36} - 0 = \frac{2}{9}$$