# IST1990 Probability And Statistics

- **Lecture 3**

- Conditional Probability and Independence
- Total Probability Rule
- Bayes Theorem

# Conditional Probability

This is a very important concept. In most cases events "share" information, and so we want to see how can we take this into account

Let $A$ and $B$ be two arbitrary events. Then the **conditional probability** of B given A is denoted by

$$P(B|A)$$

You should read the above expression as "the probability of $B$ given $A$".

**Example:** taking a card out of a shuffled deck

$$A = \{\text{the selected card is a red queen}\}$$

$$B = \{\text{the selected card is of hearts}\}$$

$$P(A) = 2/52 \approx 0.0384$$

$$P(B) = 13/52 = 0.25$$

$$P(B|A) = 1/2$$

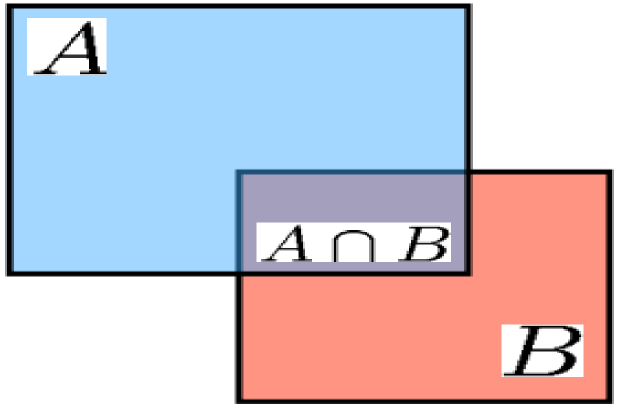
$$P(A|B) = 1/13 \approx 0.0769$$

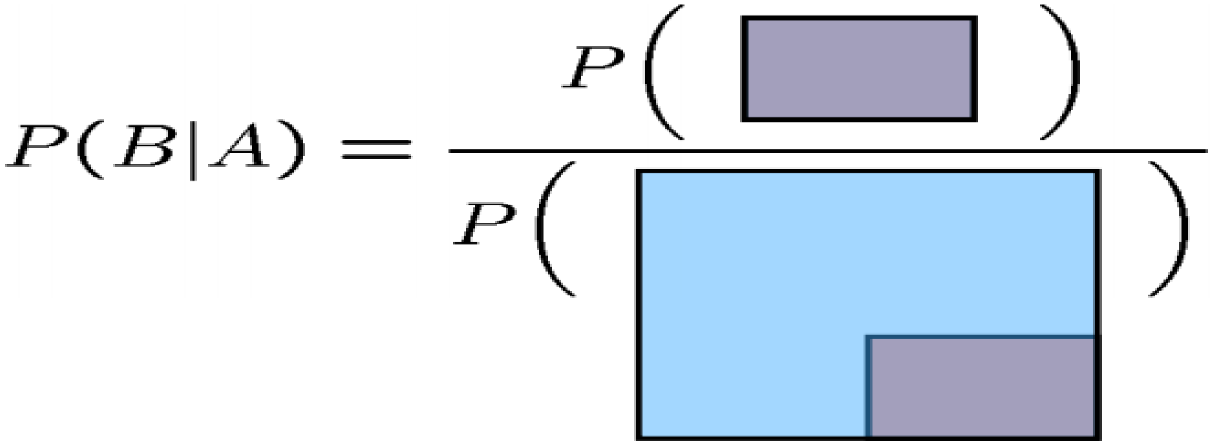
# Conditional Probability

**Definition:** Conditional Probability

Let $A$ and $B$ be two events, and assume $P(A) > 0$. The **conditional probability** of B given A is defined as

$$P(B|A) = \frac{P(A \cap B)}{P(A)} .$$



# Example.

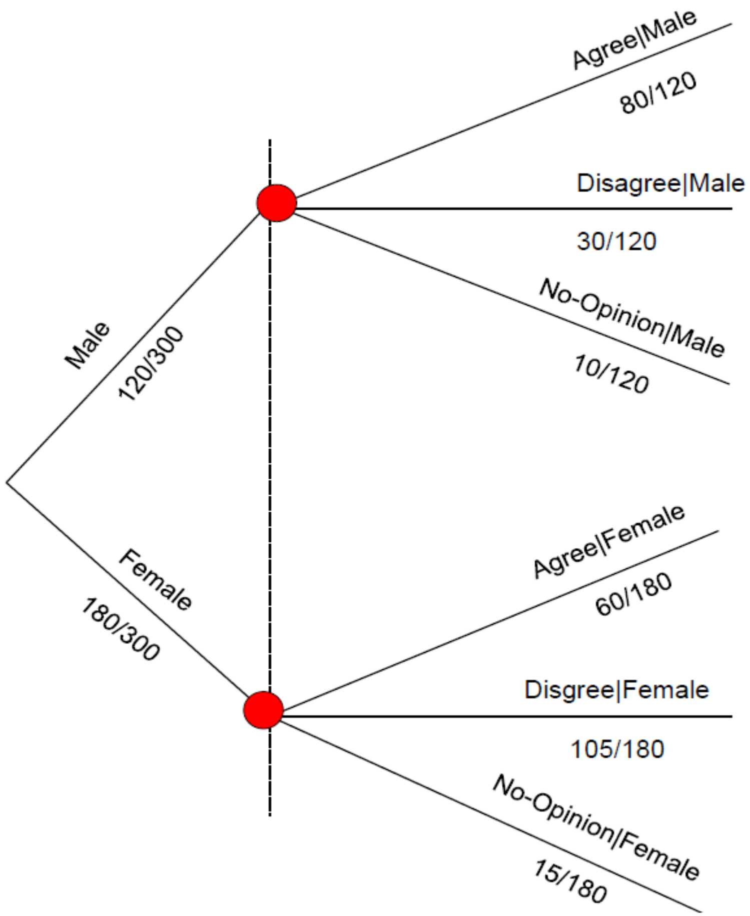
In a public school system, the issue of dress code was brought up as a result of some violations. Three hundred students categorized by gender were asked about their opinion on this issue. The Table below gives a two-way classification (contingency table) of the responses of students.

| | Agree | Disagree | No-opinion | Total |
|---|---|---|---|---|
| Male | 80 | 30 | 10 | 120 |
| Female | 60 | 105 | 15 | 180 |
| Total | 140 | 135 | 25 | 300 |

What is the probability that the student will agree with dress code, given that the student is male?

| | Agree (A) | Disagree (D) | No-opinion (N) |
|---|---|---|---|
| Male (M) | $M \cap A$ | $M \cap D$ | $M \cap O$ |
| Female (F) | $F \cap A$ | $F \cap D$ | $F \cap O$ |

| | Agree ($A$) | Disagree ($D$) | No-opinion ($N$) | Total |
|---|---|---|---|---|
| Male ($M$) | 0.267 | 0.100 | 0.033 | **0.400** |
| Female ($F$) | 0.200 | 0.350 | 0.050 | **0.600** |
| Total | **0.467** | **0.450** | **0.083** | 1 |




| | **Agree** | **Disagree** | **No-opinion** | **Total** |
|---|---|---|---|---|
| **Male** | **80** | **30** | **10** | **120** |
| **Female** | **60** | **105** | **15** | **180** |
| **Total** | **140** | **135** | **25** | **300** |

$$P(Agree \mid Male) = \frac{P(Agree \cap Male)}{P(Male)} = \frac{\frac{80}{300}}{\frac{120}{300}} = \frac{80}{120} = 0.667$$

# Example. Radar Detection

If an aircraft is present in a certain area, a radar detects it and generates an alarm signal with probability 0.99. If an aircraft is not present, the radar generates a (false) alarm with probability 0.10. We assume that an aircraft is present with probability 0.05. What is the probability of no aircraft presence and a false alarm? What is the probability of aircraft presence and no detection?

Solution:

A sequential representation of the experiment is appropriate here. Let A and B be the events

$$A = \{\text{an aircraft is present}\},$$
$$B = \{\text{the radar registers an aircraft presence}\},$$

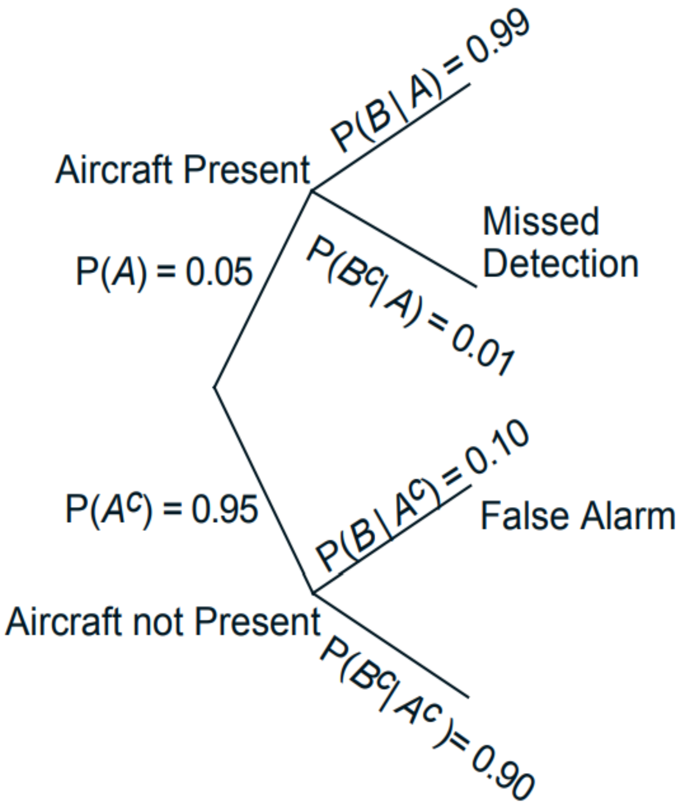
and consider also their complements

$$A^c = \{\text{an aircraft is not present}\},$$
$$B^c = \{\text{the radar does not register an aircraft presence}\}.$$

The given probabilities are recorded along the corresponding branches of the tree describing the sample space, as shown in Figure. The desired probabilities of false alarm and missed detection are:

$$\mathbf{P}(\text{false alarm}) = \mathbf{P}(A^c \cap B) = \mathbf{P}(A^c)\mathbf{P}(B \mid A^c) = 0.95 \cdot 0.10 = 0.095,$$

$$\mathbf{P}(\text{missed detection}) = \mathbf{P}(A \cap B^c) = \mathbf{P}(A)\mathbf{P}(B^c \mid A) = 0.05 \cdot 0.01 = 0.0005.$$



# Partition of Sample Space

**Definition:** If the events $E_1, E_2, \ldots, E_k$ are

i) mutually exclusive events , i.e. $E_i \bigcap E_j = \varnothing \; for \; \forall i \neq j$ , and

ii) exhaustive events, i.e. $\bigcup_{i=1}^{k} E_i = S$ ,

then the collection $\{E_1, E_2, \ldots, E_k\}$ forms a partition of Sample Space S.

You should think of a partition as a way to "cut up" a set into pieces. This colorful diagram is an example of a partition of a sample space.

**Example:** We are tossing a pair of dice. Then the sample space:

S={(1,1),(1,2),...,(1,6),...,(6,1),(6,2),...,(6,6)}.

Assume that

$E_i$ ={the first die is equal to i}, i=1,2,...,6.

Then the collection $\{E_1, E_2, ..., E_k\}$ forms a partition of Sample Space S.

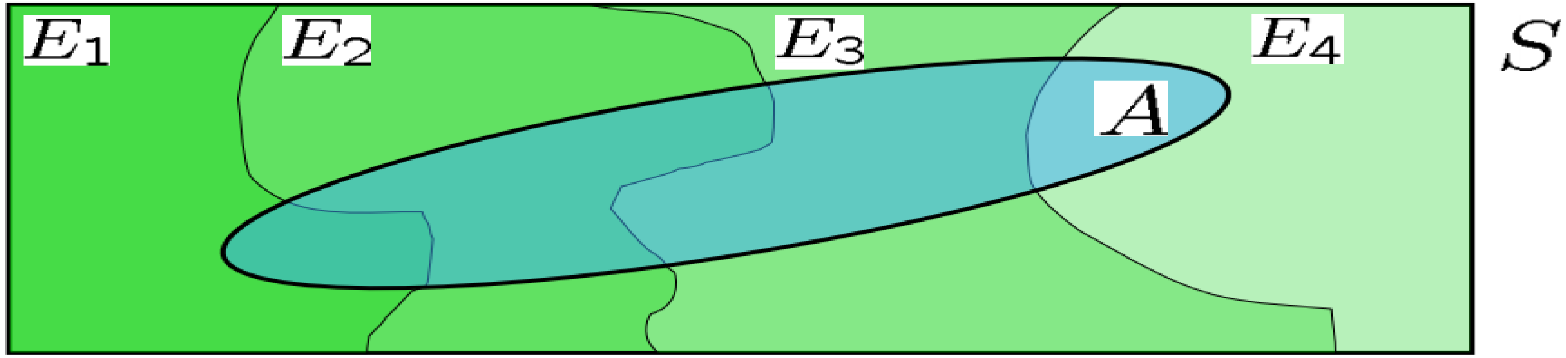
# Total Probability Rule for Multiple Events

Let $A$ be an arbitrary event, and let $E_1, \ldots, E_k$ be $k$ mutually exclusive and exhaustive events. That is,

$$\cup_{i=1}^{k} E_i = S, \quad \text{and} \quad \forall i \neq j \ E_i \cap E_j = \emptyset .$$

Then

$$\begin{aligned} P(A) &= P(A \cap E_1) + P(A \cap E_2) + \cdots + P(A \cap E_k) \\ &= P(A|E_1)P(E_1) + P(A|E_2)P(E_2) + \cdots + P(A|E_k)P(E_k) . \end{aligned}$$




if k=2, then

**P(A) =P(A|E) .P(E) + P(A|E′) .P(E′)**

**Proof:** $A = A \cap S = A \cap (\bigcup_{i=1}^{k} E_i) = A \cap E_1 \cup A \cap E_2 ... \cup A \cap E_k$.

Since $E_i \bigcap E_j = \varnothing \; for \; \forall i \neq j$, then $AE_i \bigcap AE_j = \varnothing \; for \; \forall i \neq j$, as well.

Then just taking probability and using the 3rd axiom:

$$
\begin{aligned}
P(A) &= P(A \cap E_1 \cup A \cap E_2 ... \cup A \cap E_k) \\
&= P(A \cap E_1) + P(A \cap E_2) + ... + P(A \cap E_k) \\
&= \sum_{i=1}^{k} P(A \cap E_i) = \sum_{i=1}^{k} P(A | E_i) P(E_i).
\end{aligned}
$$

# Example

The probability is 1% that an electrical connector that is kept dry fails during the warranty period of a portable computer. If the connector is ever wet, the probability of a failure during the warranty period is 5%. If 90% of the connectors are kept dry and 10% are wet, what proportion of connectors fail during the warranty period?

# Solution:

Define event notation for selecting a dry connector:

Let D = "the randomly selected connector is kept dry". Then

$D^c$= "the randomly selected connector has been wet".

Similarly, define event notation for selecting a connector that fails:

*F* = "the randomly selected connector fails during warranty".

$F^c$ = "the randomly selected connector does not fail during warranty".

The following probabilities are given:

$P(F \mid D) = 0.01$.
$P(F \mid D^c) = 0.05$.
$P(D) = 0.9$.

By using the Total Probability Rule, we determine the fraction of connectors that fail during the warranty period:

$$P(F) = P(F \mid D)\,P(D) + P(F \mid D^c)\,P(D^c)$$

$$= (0.01)\,(0.9) + (0.05)\,(1-0.9)$$

$$= 0.014$$

That is, 1.4 %.

# Homework.

The edge roughness of slit paper products increases as knife blades wear. Only 1% of products slit with new blades have rough edges, 3% of products slit with blades of average sharpness exhibit roughness, and 5% of products slit with worn blades exhibit roughness.

If 25% of the blades in manufacturing are new, 60% are of average sharpness, and 15% are worn, what is the probability that a randomly selected product has rough edges?

# Independence of Events

Sometimes two events might not be "related":

• Given that the outcome of the experiment is in A might not affect the probability that this same outcome is also in B…

**Example:** Take a card out of a shuffled deck

$$A = \{\text{the selected card is a queen}\}$$
$$B = \{\text{the selected card is of hearts}\}$$

$$P(A) = \frac{4}{52} = \frac{1}{13}, \quad P(B) = \frac{13}{52} = \frac{1}{4}, \quad P(A \cap B) = 1/52$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/52}{13/52} = \frac{1}{13} = P(A)$$

This means that B doesn't give any probabilistic information about A !!!

# Definition of Independence

**Definition:** Independence

Two events $A$ and $B$ are independent are equivalent if any of the following equivalent statements is true:

(i) $P(A \cap B) = P(A)P(B)$

(ii) $P(B|A) = P(B)$

(iii) $P(A|B) = P(A)$

**Independence is a truly probabilistic concept, and not a set relationship !!!**

The events $E_1, E_2, \ldots, E_k$ are **jointly** independent if

$$P(E_1 \cap E_2 \cap \cdots \cap E_k) = P(E_1)P(E_2) \cdots P(E_k)$$

**Challenge:** Can you construct a probability model and three events that are **pairwise** independent but **not jointly independent**?

# Example

One ball is drawn randomly from a bowl containing four balls numbered 1, 2, 3, and 4. Define the following three events:

Let $A$ be the event that a 1 or 2 is drawn. That is, $A = \{1, 2\}$.

Let $B$ be the event that a 1 or 3 is drawn. That is, $B = \{1, 3\}$.

Let $C$ be the event that a 1 or 4 is drawn. That is, $C = \{1, 4\}$.

Are events $A$, $B$, and $C$ pairwise independent? Are they mutually independent?

## Solution:

Left as a class exercise…

# Properties

**Lemma:** The following statements are equivalent:

(i) $A$ and $B$ are independent

(ii) $P(A \cap B) = P(A)P(B)$

(iii) $P(A' \cap B) = P(A')P(B)$

(iv) $P(A \cap B') = P(A)P(B')$

(v) $P(A' \cap B') = P(A')P(B')$

**Proof:** left as exercise...

# Independence is IMPORTANT !!!

Independence is a key assumption in most probability models, e.g.:

- We assume each memory chip from a certain manufacturer fails independently from one another
- Processes arriving to a server from different users are assume to arrive independently from one another

**Example:** There are three elevators in the MetaForum, but one is currently broken. The other two fail independently with probability 0.2 and 0.1 respectively.

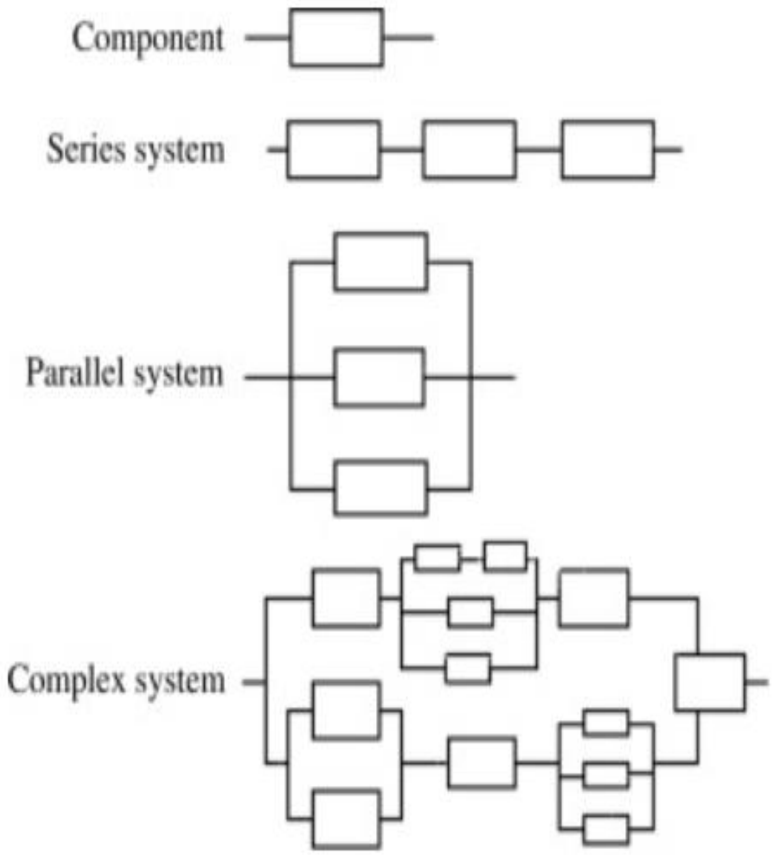
What is the probability I cannot take the elevators to get to the $7^{th}$ floor?

A - 0.1    B - 0.2    C - 0.3    D - 0.02

## Suppose that there are n components in parallel:

- P(Parallel system works)=P(at least one component works)= P( E1 or E2 or…or En works)=P($\bigcup_{i=1}^{n} Ei$)

- P(Parallel system fails)=P(all components fail)=
  P( E1 and E2 and …and En fail)=P($\bigcap_{i=1}^{n} E_i{}'$)



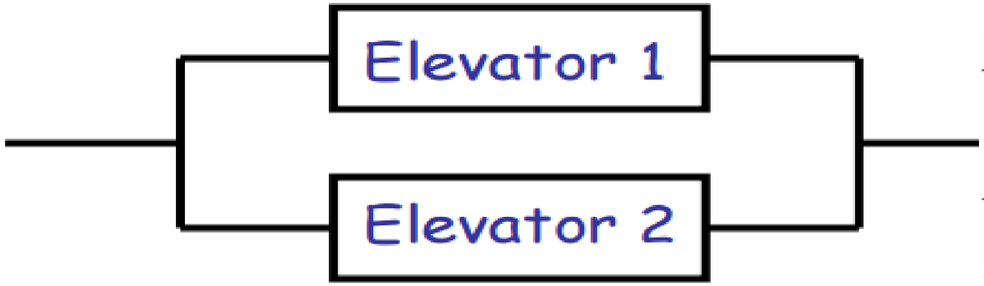

## Suppose that there are n components in series:

- P(Series system works)= P(all components work)=
  P( E1 and E2 and …and En work)=P($\bigcap_{i=1}^{n} E_i$)

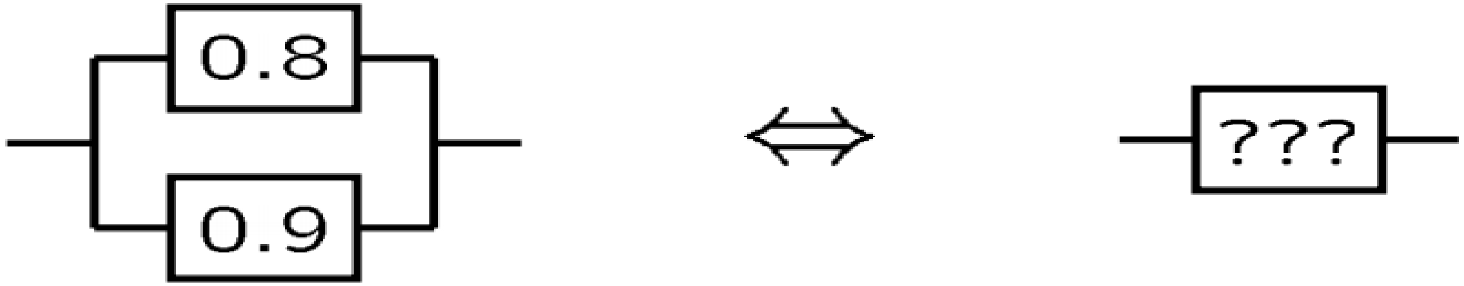
- P(Series system fails)=P(at least one component fails)=
  P( E1 or E2 or…or En fails)=P($\bigcup_{i=1}^{n} E_i{}'$)

# A Simple Example

We can think of this setting as a circuit:




$$\begin{aligned} P(E_1) &= P(\text{Elevator 1 works}) \\ &= 1 - 0.2 = 0.8 \\ P(E_2) &= P(\text{Elevator 2 works}) \\ &= 1 - 0.1 = 0.9 \end{aligned}$$

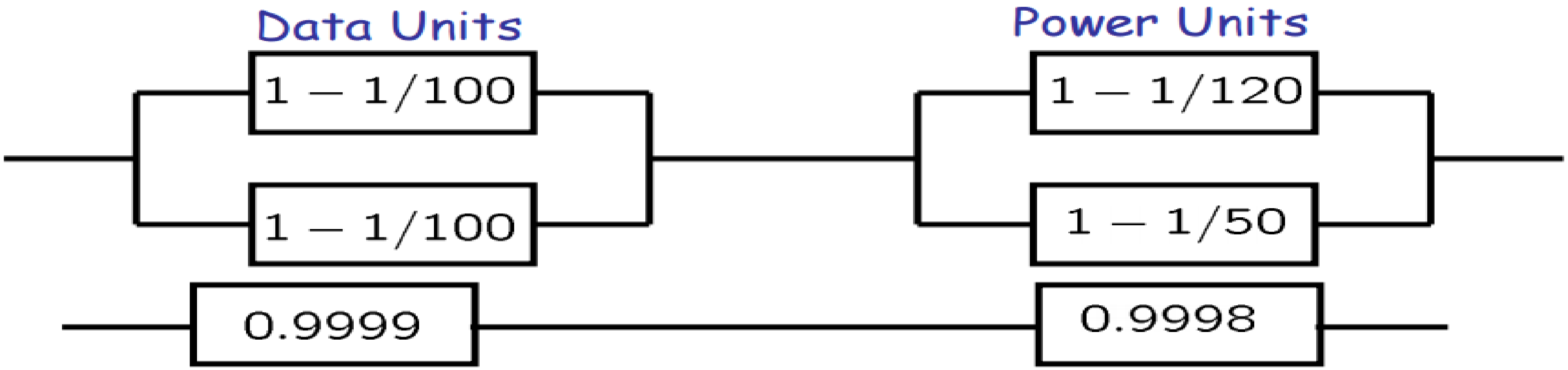



$$\begin{aligned} &P(\text{at least one of the elevators works}) = P(E_1 \cup E_2) \\ &= \quad 1 - P((E_1 \cup E_2)') \\ &= \quad 1 - P(E_1' \cap E_2') \\ &= \quad 1 - P(E_1')P(E_2') = 1 - (1-0.8)(1-0.9) = 0.98 \end{aligned}$$

# Another Example

**Example:** A harddrive backup unit uses redundancy to ensure reliable operation. It consists of two redundant data storage units (each with probability of failure of 1/100) and two independent power supply units each with probabilities of failure respectively 1/50 (electric grid unit) and 1/120 (battery unit).

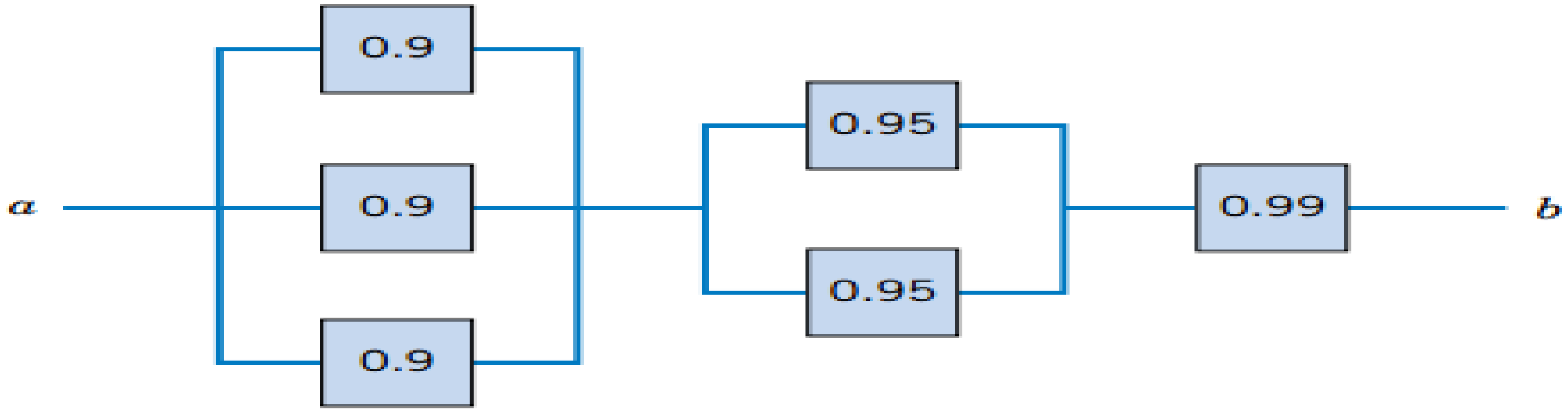
What is the overall probability of failure of such system?




$$\text{Overall probability success} = 0.9999 * 0.9998 = 0.9997$$

# Example

The following circuit operates only if there is a path of functional devices from left to right. The probability that each device functions is shown on the graph. Assume that devices fail independently. What is the probability that the circuit operates?




Solution is left as a class exercise…

# Bayes' s Rule

In many situations in practice what we can measure (estimate) are conditional probabilities. It is rather useful to be able to relate various conditional probabilities to one another.

Reverend Thomas Bayes (1702-1761):

Let $A$ and $B$ be two events, and $P(B) > 0$. Then

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Proof: $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B|A)P(A)}{P(B)}$

Despite it' s simplicity, this is a very powerful result, and forms the basis of many inference procedures used nowadays (e.g. both the GPS system and the communication encoding used in cellphones rely on Bayes' s rule).

REV. T. BAYES

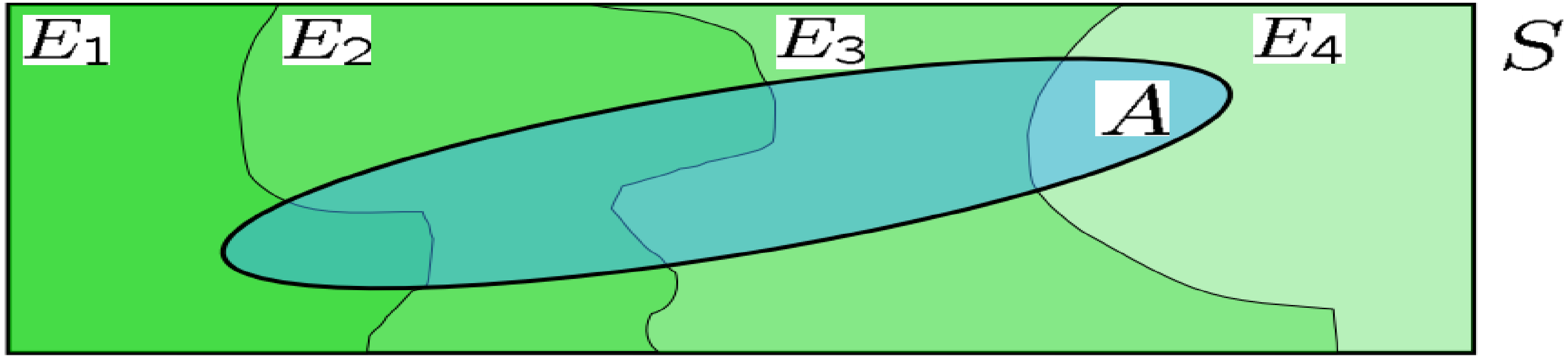
# Bayes' s Rule

Let $A$ be an arbitrary event such that $P(A) > 0$, and let $E_1, \ldots, E_k$ be $k$ mutually exclusive and exhaustive events. That is,

$$\cup_{i=1}^{k} E_k = S, \quad \text{and} \quad \forall i \neq j \; E_i \cap E_j = \emptyset \; .$$

Then

$$P(E_1|A) = \frac{P(A|E_1)P(E_1)}{P(A|E_1)P(E_1) + P(A|E_2)P(E_2) + \cdots + P(A|E_k)P(E_k)}$$



# A Real Life Example

The doctor knows that only 0.8 percent of the people in the country are diseased. A patient goes to see a doctor. If the patient is diseased, the probability is 90 percent that he/she will have a positive test. If the patient is not diseased, then the probability is 7 percent that he/she will stil have a positive test.

Now the question is: if the patient has a positive test, what is the probability that the patient is actually diseased ?

# Solution:

The doctors answer for this question ranged from 1 to 99%.
What is the correct answer ???

**Enter your answer as a percentage**
**(e.g. 47% corresponds to 47)**

T={patient has positive test}

D={patient is diseased}

$$P(D) = 0.008$$
$$P(T|D) = 0.9$$
$$P(T|D^C) = 0.07$$

We desire to know that what $P(D|T)$ is … $P(D|T) = \frac{P(T|D)P(D)}{P(T)} = ???$

Using Total Probability Rule:

$$P(T) = P(T|D)P(D) + P(T|D^c)P(D^c)$$

$$= 0.9 * 0.008 + 0.07 * (1 - 0.008) = 0.07664$$

Therefore:

$$P(D|T) = \frac{P(T|D)}{P(T)} = \frac{0.9 * 0.008}{0.07664} = 0.09395$$

**So a patient with positive test has only about 9.5 % chance of actually having disease.**

# Example

A desk lamp produced by The Luminar Company was found to be defective ($D$) with probability 0.01475. There are three factories ($A$, $B$, $C$) where such desk lamps are manufactured. A Quality Control Manager (QCM) is responsible for investigating the source of found defects. This is what the QCM knows about the company's desk lamp production and the possible source of defects:

| Factory | % of total production | Probability of defective lamps |
|---|---|---|
| A | $0.35 = P(A)$ | $0.015 = P(D \mid A)$ |
| B | $0.35 = P(B)$ | $0.010 = P(D \mid B)$ |
| C | $0.30 = P(C)$ | $0.020 = P(D \mid C)$ |

If a randomly selected lamp is defective, what is the probability that the lamp was manufactured in factory A? And, if a randomly selected lamp is defective, what is the probability that the lamp was manufactured in factory B?

## Solution:

Using Bayes Theorem, the probability that a lamp was manufactured in factory B given that it is defective is:

$$P(B|D) = \frac{P(B \cap D)}{P(D)} = \frac{P(D|B) \times P(B)}{P(D)} = \frac{(0.01)(0.35)}{0.01475} = 0.237$$

# Example

An e-mail message can travel through one of two routes to the server. The probability of transmission error in each of the servers and the proportion of messages that travel each route are shown in the following table. Assume that the servers are independent.

| | | probability of error | | | |
|---|---|---|---|---|---|
| | percentage of messages | server 1 | server 2 | server 3 | server 4 |
| route 1 | 30 | 0.01 | 0.015 | | |
| route 2 | 70 | | | 0.02 | 0.003 |

(a) What is the probability that a message will arrive without error?

(b) If a message arrives in error, what is the probability it was sent through route 1?

# Solution:

## (a) What is the probability that a message will arrive without error?

P(route 1 choosen) = 0.3

P(route 2 choosen) = 0.7

P(Server 1 and Server 2 have no error ) =
P(Server 1 has no error) . P(Server 2 has no error)
(from independence)

P(no error|route 1) = (1- 0.01)x(1 -0.015) = 0.97515

P(error |route 1)= 1- P(no error|route 1) =1-0.97515=0.02485

Similarly,

P(no error|route 2) = (1 -0.02)x(1 -0.003) = 0.97706

P(no error) = P(no error|route 1)xP(route 1) + P(no error|route 2)xP(route 2)

= 0.97515 x 0.3 + 0.97706 x 0.7 = 0.97648

## (b) If a message arrives in error, what is the probability it was sent through route 1?

$$P(\text{route 1}|\text{error}) = \frac{P(\text{error}|\text{route1})P(\text{route1})}{P(\text{error})}$$

$$= \frac{(1-0.97515) \times 0.3}{(1-0.9764)}$$

$$= \frac{0.02485 \times 0.3}{0.0236}$$

$$= 0.3159$$

**Homework.** In a certain assembly plant, three machines, $B_1$, $B_2$ and $B_3$ make 30%, 45% and 25%, respectively, of the products. It is known from past experience that 2%, 3%, and 2% of the product made by each machine, respectively, are defective.

a) Find the probability that a randomly selected product is defective?

b) If a product were chosen randomly and found to be defective, what is the probability that it was made by machine $B_3$?