KOM3712 Control Systems Design

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Design via State Space Methods – 2 of 4: *Observers*

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Textbooks followed mostly for the Design in the State Space are,

- **Modern Control Engineering** (5th ed.), Katsuhiko Ogata, Chap. 9
- **Control Systems Engineering** (7th ed.), Norman S. Nise, Chap. 12

Observers: Introduction

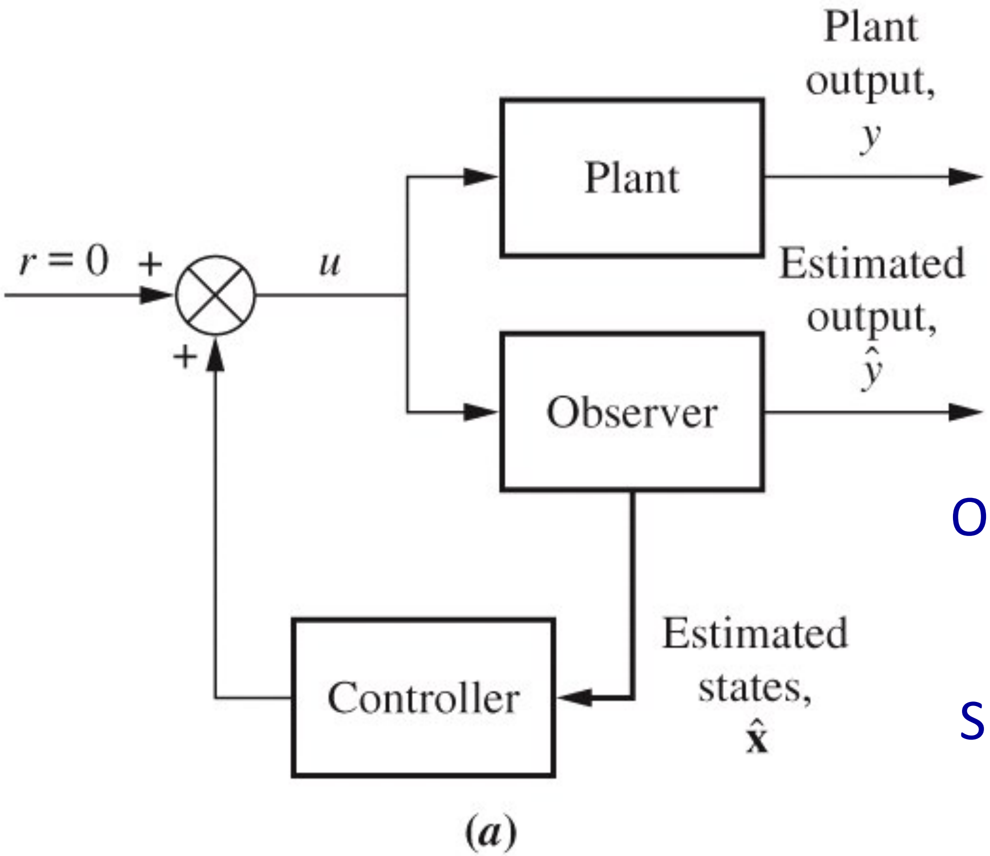
- Requirements for state feedback controller design:
  - ✓ Controllability
  - ✓ Entire state vector availability
- Unavailable state variables should be estimable, otherwise, state feedback control implementation is impossible
- Challenges to state variable sensing/measurement:
  - Sensor accuracy: may deteriorate over time
  - Measurement cost: may be too high for very large systems
  - Availability: some states may be unmeasurable

Introduction, *cont'd…*

- Controller design relies upon access to the state variables for feedback through adjustable gains (remember: $u = -\mathbf{Kx}$).
- This access can be provided by hardware. For example, gyros can measure position and velocity on a space vehicle or UAV.
- Sometimes it is impractical to use this hardware for reasons of cost, accuracy, or availability.
- For example, in powered flight of space vehicles, inertial measuring units (IMU) can be used to calculate the acceleration.
- However, their alignment deteriorates with time; thus, other means of measuring acceleration may be desirable.
- In other applications, some of the state variables may not be available at all, or it is too costly to measure them or send them to the controller.
- If the state variables are not available because of system configuration or cost, it is possible to *estimate the states*.
- Estimated states, rather than actual states, are then fed to the controller.

Introduction, *cont.'s…*

- **Observer/estimator**: a subsystem for estimating the unmeasurable state variables
- The observer model is normally based on the plant model



Open-loop observer

Plant model: $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \quad \ldots(1)$

$y = \mathbf{C}\mathbf{x}$

Observer model: $\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}u \quad \ldots(2)$

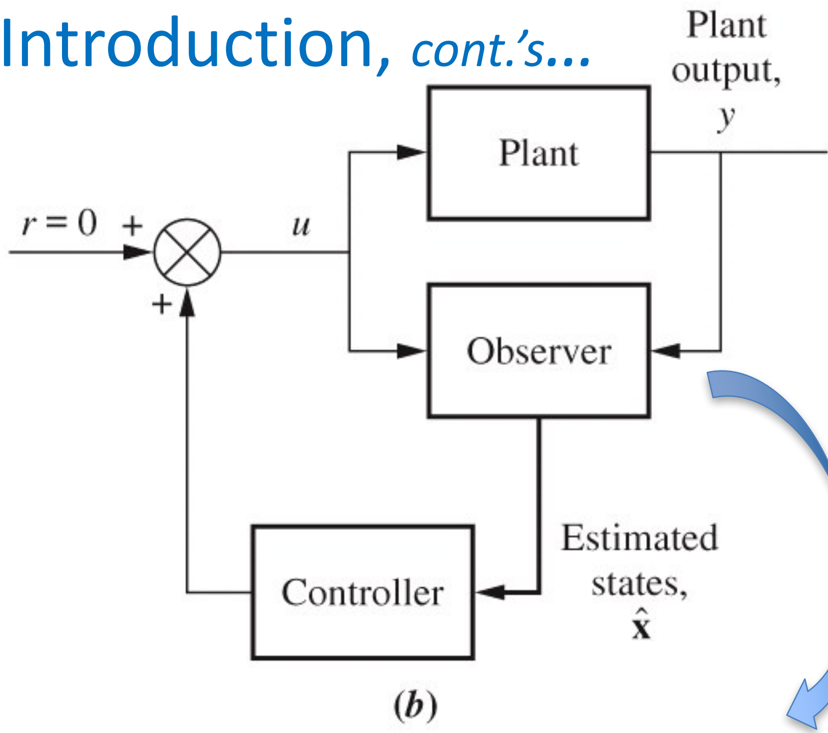
$\hat{y} = \mathbf{C}\hat{\mathbf{x}}$

Subtract (2) from (1) :

$$\dot{\mathbf{x}} - \dot{\hat{\mathbf{x}}} = \mathbf{A}(\mathbf{x} - \hat{\mathbf{x}}) \quad \ldots(3)$$

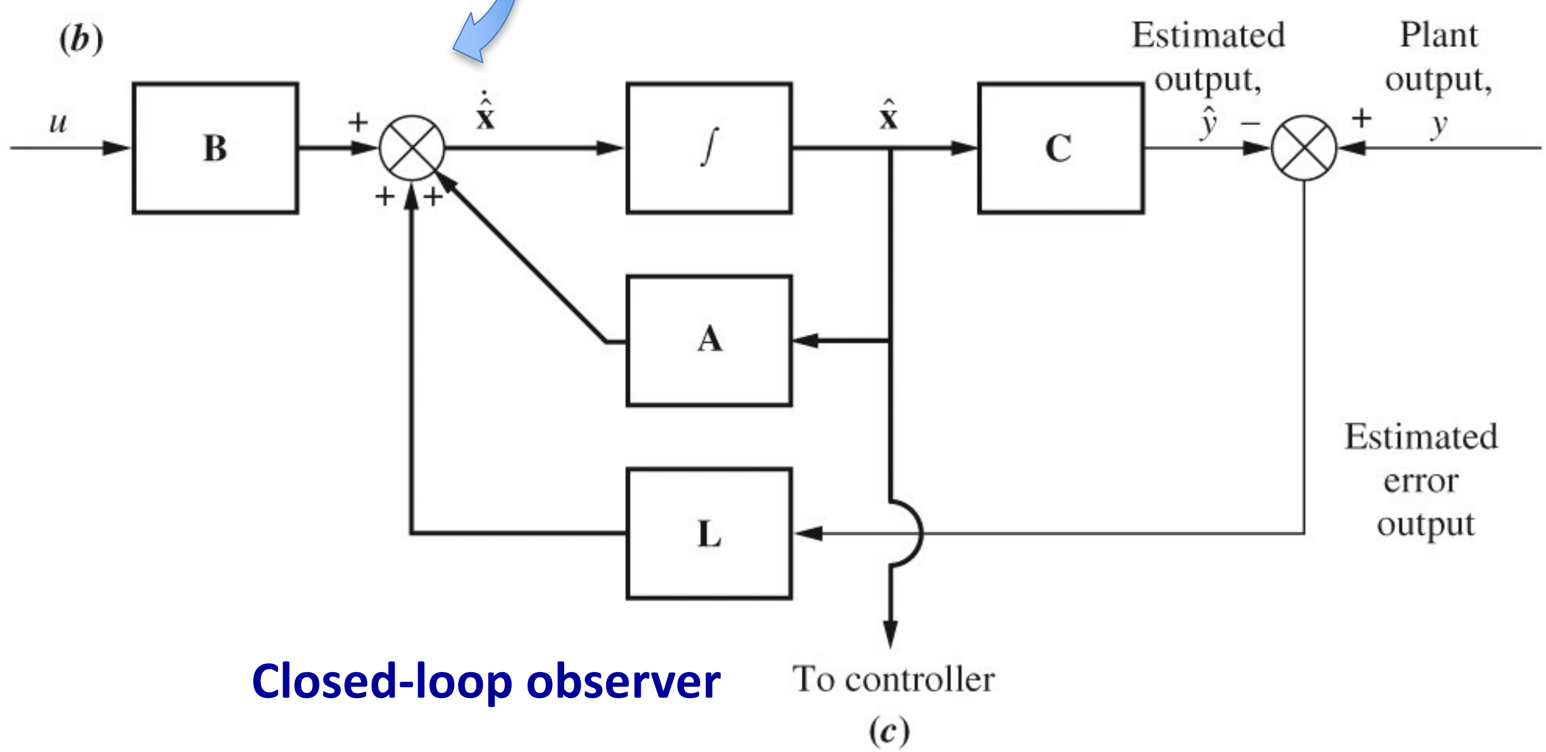
$$y - \hat{y} = \mathbf{C}(\mathbf{x} - \hat{\mathbf{x}})$$

Introduction, *cont.'s...*



- ✓ Inputs to observer:
  - Plant control input, u
  - Plant measured output, y
- ✓ Loop is closed in order to improve the error dynamics

  (i.e. to speed up the convergence of estimated states to actual states)



Closed-loop observer

Observer Canonical Form (OCF) for Observers!

- Observer design (like state feedback controller design) is a pole-placement problem
- Observer design is simplified if system is represented in Observer Canonical Form (OCF):

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + \begin{bmatrix} b_n - a_n b_0 \\ b_{n-1} - a_{n-1} b_0 \\ \cdot \\ \cdot \\ \cdot \\ b_1 - a_1 b_0 \end{bmatrix} u$$

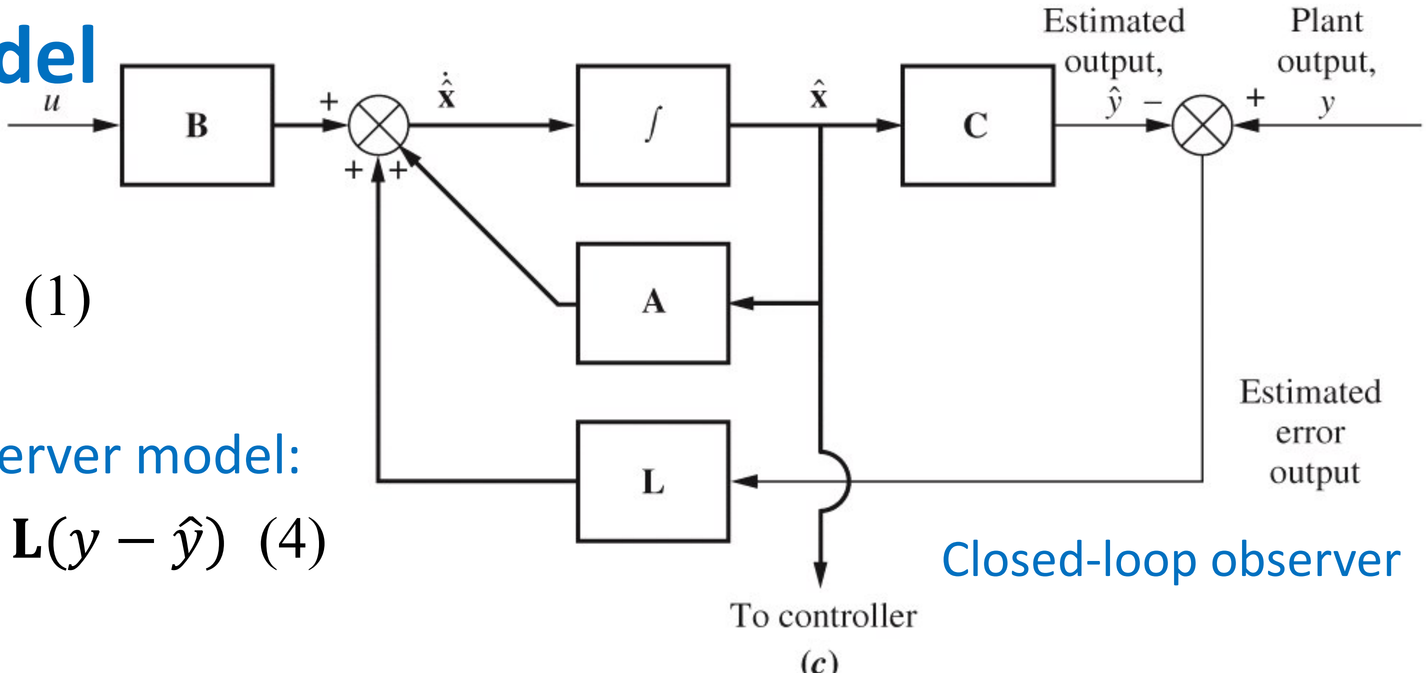
$$y = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix} + b_0 u$$

An alternative OCF is placing the negative coefficients of a's in reverse order with respect to the above OCF in the first column.

Observer – more on its structure

- Observer is designed to have much faster (say, **5 times or more**) transient response than that of controlled loop
- This is achieved by placing observer poles farther to the left of the s-plane than the control loop's closed-loop poles
- Observer design is a trade-off between speed of response and need for noise rejection
- It may be necessary to design several observer gain matrices, then evaluate the system performance for each of them

Observer model



Closed-loop observer

- Plant model:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \qquad (1)$$
$$y = \mathbf{C}\mathbf{x}$$

- Closed-loop observer model:

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}u + \mathbf{L}(y - \hat{y}) \quad (4)$$
$$\hat{y} = \mathbf{C}\hat{\mathbf{x}}$$

- State vector error estimation model (subtract (4) from (1)):

$$(\dot{\mathbf{x}} - \dot{\hat{\mathbf{x}}}) = \mathbf{A}(\mathbf{x} - \hat{\mathbf{x}}) - \mathbf{L}(y - \hat{y}) \qquad (5a)$$
$$y - \hat{y} = \mathbf{C}(\mathbf{x} - \hat{\mathbf{x}}) \qquad (5b)$$

- Substitute (5b) into (5a):

$$(\dot{\mathbf{x}} - \dot{\hat{\mathbf{x}}}) = \mathbf{A}(\mathbf{x} - \hat{\mathbf{x}}) - \mathbf{LC}(\mathbf{x} - \hat{\mathbf{x}}) \rightarrow (\dot{\mathbf{x}} - \dot{\hat{\mathbf{x}}}) = (\mathbf{A} - \mathbf{LC})(\mathbf{x} - \hat{\mathbf{x}})$$

- Let $\mathbf{x} - \hat{\mathbf{x}} = \boldsymbol{e}_x$ $\rightarrow$

$$\boxed{\begin{aligned} \dot{\boldsymbol{e}}_x &= (\mathbf{A} - \mathbf{LC})\boldsymbol{e}_x \\ y - \hat{y} &= \mathbf{C}\boldsymbol{e}_x \end{aligned}} \quad (6)$$

In fact, the estimated model has the same dynamics ((4)):

$$\dot{\hat{\mathbf{x}}} = (\mathbf{A} - \mathbf{LC})\hat{\mathbf{x}} + \mathbf{B}u + \mathbf{L}y$$

Observer model, *cont.'s*

$$\dot{\boldsymbol{e}}_x = (\mathbf{A} - \mathbf{LC})\boldsymbol{e}_x \qquad (6)$$ → State estimation error dynamics

- If eigenvalues of (6) are all stable, estimated state vector converges to actual state vector
- Moreover, the dynamics of (6) can be shaped by means of the proper design of $\mathbf{L}$
- Eigenvalues of *error dynamical system* are determined from:

$$\det[s\mathbf{I} - (\mathbf{A} - \mathbf{LC})] = 0 \qquad (7)$$

Observer design problem:

1. Choose the observer eigenvalues to ensure stable and fast error dynamic response
2. Determine the value of $\mathbf{L}$ accordingly

Observer model - *caution*:

$\dot{\boldsymbol{e}}_{\boldsymbol{x}} = (\mathbf{A} - \mathbf{LC})\boldsymbol{e}_x$ (6) → State estimation error dynamics

Observer design problem – *caution*!

1. The system model was assumed to be exact. Hence the matrices $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$, representing the physical system are used in the derivation of (6). This cannot be true.
2. Also, disturbances on the physical system have been ignored. A more accurate state space model of the plant is,

$$\boxed{\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{Ax}(t) + \mathbf{B}u(t) + \mathbf{B}_w w(t) \\ y(t) &= \mathbf{Cx}(t) + v(t) \end{aligned}} \quad (7)$$

where, $w(t)$ is a vector of the disturbance inputs and $v(t)$ represents the sensor errors. If this model is considered the error dynamics (6) become,

$$\dot{\boldsymbol{e}}_{\boldsymbol{x}} = (\mathbf{A} - \mathbf{LC})\boldsymbol{e}_x + \mathbf{B}_w w(t) - \mathbf{L}v(t)$$

We see that errors will not die out over time, even the modeling inaccuracies are ignored.

Observer design: System in **OCF**

(1)

$$\mathbf{A}-\mathbf{LC}=\begin{bmatrix} -a_{n-1} & 1 & 0 & 0 & \cdots & 0 \\ -a_{n-2} & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_1 & 0 & 0 & 0 & \cdots & 1 \\ -a_0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \\ \vdots \\ l_{n-1} \\ l_n \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -(a_{n-1}+l_1) & 1 & 0 & 0 & \cdots & 0 \\ -(a_{n-2}+l_2) & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -(a_1+l_{n-1}) & 0 & 0 & 0 & \cdots & 1 \\ -(a_0+l_n) & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

(2) Closed-loop observer characteristic equation:

$$\det[s\mathbf{I}-(\mathbf{A}-\mathbf{LC})] = s^n + (a_{n-1}+l_1)s^{n-1} + (a_{n-2}+l_2)s^{n-2} + \cdots + (a_1+l_{n-1})s + (a_0+l_n) = 0$$

(3) Desired closed-loop characteristic equation:

$$\Delta_{CL-\text{desired}} = s^n + d_{n-1}s^{n-1} + d_{n-2}s^{n-2} + \cdots + d_1 s + d_0 = 0$$

(3) Observer Gain Vector $\mathbf{L}$ is found by matching the coefficients of (2) and (3),

$$\boxed{l_i = d_{n-i} - a_{n-i} \quad i = 1, 2, \dots, n}$$

Observer design example: **OCF**

Observer Design for Observer Canonical Form

PROBLEM: Design an observer for the plant

$$G(s) = \frac{(s+4)}{(s+1)(s+2)(s+5)} = \frac{s+4}{s^3+8s^2+17s+10}$$

which is represented in observer canonical form. The observer will respond 10 times faster than the controlled loop designed in Example 12.4.

(1) System in OCF:

$$A = \begin{bmatrix} 0 & 0 & -10 \\ 1 & 0 & -17 \\ 0 & 1 & -8 \end{bmatrix}, \quad B = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \quad D = 0$$

$$\mathbf{A} - \mathbf{LC} = \begin{bmatrix} 0 & 0 & -10 \\ 1 & 0 & -17 \\ 0 & 1 & -8 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & -10 \\ 1 & 0 & -17 \\ 0 & 1 & -8 \end{bmatrix} - \begin{bmatrix} 0 & 0 & l_1 \\ 0 & 0 & l_2 \\ 0 & 0 & l_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -(10+l_1) \\ 1 & 0 & -(17+l_2) \\ 0 & 1 & -(8+l_3) \end{bmatrix}$$

(2) Closed-loop observer characteristic equation:

$$\Delta_{CL} = s^3 + (8+l_3)s^2 + (17+l_2)s + 10 + l_1 = 0 \quad \ldots(8)$$

Observer design example: **OCF**, *cont.'s…*

Observer Design for Observer Canonical Form

PROBLEM: Design an observer for the plant

$$G(s) = \frac{(s+4)}{(s+1)(s+2)(s+5)} = \frac{s+4}{s^3+8s^2+17s+10}$$

which is represented in observer canonical form. The observer will respond 10 times faster than the controlled loop designed in Example 12.4.

Design a state-variable feedback controller to yield a 20.8% overshoot and a settling time of 4 seconds for the plant…

(3) Desired observer closed-loop characteristic equation:

Observer should respond 10 times faster than controlled loop

Closed-loop control system dominant poles: $p_{1,2\_sys} = -1 \pm j2$

Observer dominant poles: $p_{1,2\_obs} = -10 \pm j20$

Third observer pole chosen to be <u>10 times</u> the <u>real component</u> of the <u>dominant</u> closed-loop <u>observer poles</u> $p_{3\_obs} = -100$

$$\Delta_{CL\_des} = s^3 + 120s^2 + 2500s + 50000 \quad \ldots(9)$$

Observer design example: **OCF**, *cont.'s…*

Observer Design for Observer Canonical Form

PROBLEM: Design an observer for the plant

$$G(s) = \frac{(s+4)}{(s+1)(s+2)(s+5)} = \frac{s+4}{s^3+8s^2+17s+10}$$

which is represented in observer canonical form. The observer will respond 10 times faster than the controlled loop designed in Example 12.4.

$$\Delta_{CL} = s^3 + (8+l_3)s^2 + (17+l_2)s + 10 + l_1 = 0 \quad \ldots(8)$$

$$\Delta_{CL\_des} = s^3 + 120s^2 + 2500s + 50000 \quad \ldots(9)$$

(4) Determine L:

$$L = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} = \begin{bmatrix} 49990 \\ 2483 \\ 112 \end{bmatrix}$$

Observability

- **Observability** is a property of a dynamical system that determines whether it is possible to *estimate the system's state vector* or not
- The observer estimates the state vector based on the measured output and the control input
- Thus a state variable can be estimated only if it directly affects the output
- When a system is in Diagonal Form, it is easy to see the direct effect of each state variable on the output

Observability, *cont.'s*…

- **Definition**: a system is **observable** if the initial state, $\boldsymbol{x}(t_o)$, can always be determined, given the input $u(t)$, and the output $y(t)$, measured over a finite interval $[t_0, t_1]$.
- Otherwise, the system is not observable.
- Plant model: $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$

$$y = \mathbf{C}\mathbf{x}$$

- Observability matrix:
- $$\mathbf{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \\ \vdots \\ \mathbf{CA}^{\mathbf{n-1}} \end{bmatrix}$$
- System is observable only if the observability matrix is of full rank

Observer design: system not in **OCF**

- For a system not in Observer Canonical Form, hand-computation of the observer gain matrix tends to be cumbersome and tedious (for higher-order systems)

- Two design methods for systems not in OCF
  1. Transformation of system to **OCF**
  2. Direct coefficient matching of closed-loop characteristic equations

Observer design: system not in **OCF**
Transformation to OCF (*if possible*)

- System not in OCF:

  $$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{B}u;\ y = \mathbf{C}\mathbf{z}$$

- System transformed to OCF

  $$\dot{\mathbf{x}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{x} + \mathbf{P}^{-1}\mathbf{B}u;\ y = \mathbf{C}\mathbf{P}\mathbf{x}$$

- Observability matrices of systems in OCF and non-OCF:

  $\mathbf{P} = \mathbf{O}_{\mathbf{z}}^{-1}\mathbf{O}_{\mathbf{x}}$ ➔ Transformation matrix

  $\mathbf{O}_{\mathbf{z}}$ ➔ Original System's Observability Matrix

  $\mathbf{O}_{\mathbf{x}}$ ➔ Transformed (to OCF) System's Observability Matrix

- Observer gain matrix:

  $$\mathbf{L}_{\mathbf{z}} = \mathbf{P}\mathbf{L}_{\mathbf{x}}$$

Observability Example: Transformation to OCF

PROBLEM: Design an observer for the plant

$$G(s) = \frac{1}{(s+1)(s+2)(s+5)}$$

represented in cascade form. The closed-loop performance of the observer is governed by the characteristic polynomial used in Example 12.5: $s^3 + 120s^2 + 2500s + 50{,}000$.

System in cascade form:

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{B}u = \begin{bmatrix} -5 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \mathbf{z} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \mathbf{C}\mathbf{z} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathbf{z}$$

(1) Determine $\mathbf{O}_z$
(2) Determine **OCF** representation of system
(3) Determine $\mathbf{O}_x$
(4) Determine $\mathbf{L}_x$
(5) Determine $\mathbf{P}$
(6) Determine $\mathbf{L}_z$

Observability Example:
Direct coefficient matching

PROBLEM: A time-scaled model for the body's blood glucose level is shown in Eq. (12.105). The output is the deviation in glucose concentration from its mean value in mg/100 ml, and the input is the intravenous glucose injection rate in g/kg/hr (*Milhorn, 1966*).

$$G(s) = \frac{407(s+0.916)}{(s+1.27)(s+2.69)}$$

Design an observer for the phase variables with a transient response described by $\zeta = 0.7$ and $\omega_n = 100$.

(1) Derive phase-variable state space representation

(2) Derive $\det[s\mathbf{I} - \mathbf{A} + \mathbf{LC}]$

(3) Derive desired observer closed-loop characteristic equation

(4) Compare results of (2) and (3) to determine $\mathbf{L}$