Frequency Response Methods – 2 of 3

KOM3712 Control Systems Design

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Spring 2020

- Intro. to Frequency Response Methods
- Bode Diagrams
- **Nyquist Stability Criterion**
- **Gain and Phase Margins**
- Bandwidth, Response Speed, Resonant Peak, etc.
- Constant M Circles and Constant N Circles
- Nichols chart
- Damping Ratio from Phase Margin
- Response Speed from Open-Loop Freq. Response
- Steady-State Error Characteristics from Freq. Resp.
- Systems with Time Delay
- Obtaining Transfer Functions Experimentally

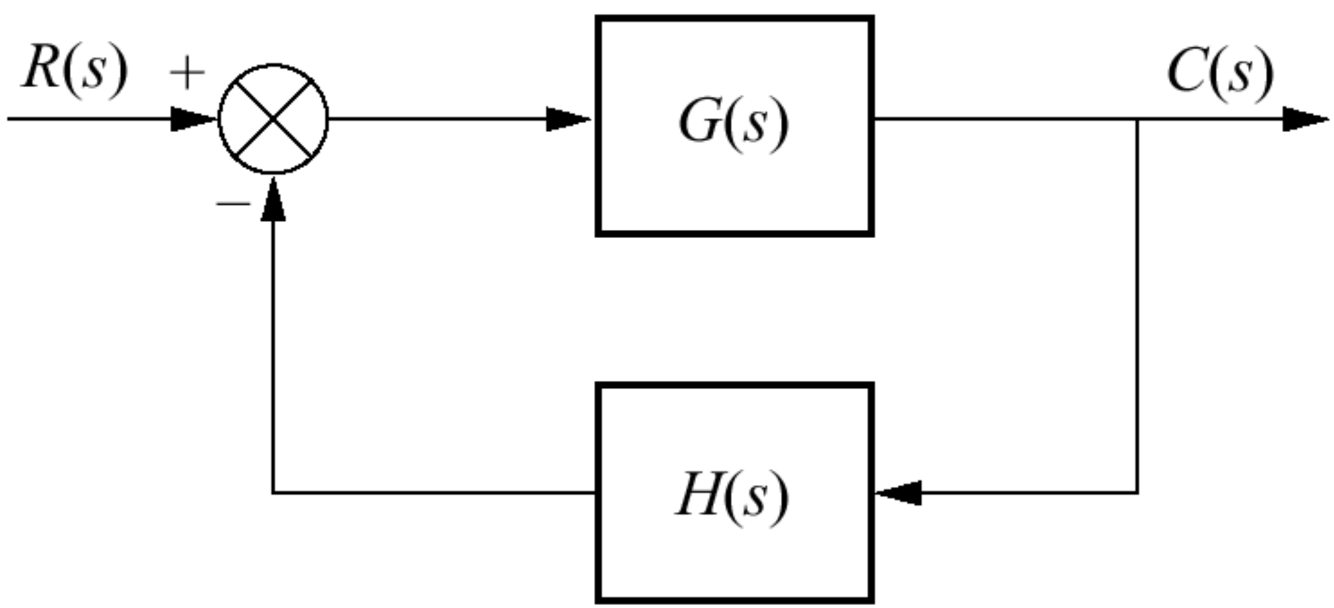
Nyquist Stability Criterion-1

- Nyquist stability criterion is based on the complex analysis result known as ***Cauchy's principle of argument***.
- By applying Cauchy's principle of argument to the *open-loop system* transfer function, we will get information about stability of the closed-loop system transfer function and arrive at the **Nyquist stability criterion** (Nyquist, 1932).
- Nyquist stability can also be used to determine the *relative degree* of system stability by producing the so-called **phase** and **gain stability margins**.
- These stability margins are needed for frequency domain controller design techniques.

Nyquist Stability Criterion-2

- For a SISO feedback system the **closed-loop transfer function** is given by

$$\boldsymbol{T(s)} = \frac{\boldsymbol{G(s)}}{\boldsymbol{1 + G(s)H(s)}}$$



Nyquist Stability Criterion-3

- $G(s)$ and $H(s)$ are usually in the fractional form as shown,

$$G(s) = \frac{N_G}{D_G} \text{ and } H(s) = \frac{N_H}{D_H}$$

- Then the OL TF, the Char. Eqn. and the CL TF can be found respectively as,

$$G(s)H(s) = \frac{N_G N_H}{D_G D_H}$$

$$1 + G(s)H(s) = 1 + \frac{N_G N_H}{D_G D_H} = \frac{D_G D_H + N_G N_H}{D_G D_H}$$

$$T(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{N_G D_H}{D_G D_H + N_G N_H}$$

1) The poles of characteristic equation, $\mathbf{1 + G(s)H(s)}$ are the same as the poles of $\boldsymbol{G(s)H(s)}$, the open-loop system, and
2) The zeros of characteristic equation $\mathbf{1 + G(s)H(s)}$ are the same as the poles of $\boldsymbol{T(s)}$, the closed-loop system.

Nyquist Stability Criterion-4

In summary, we need to establish

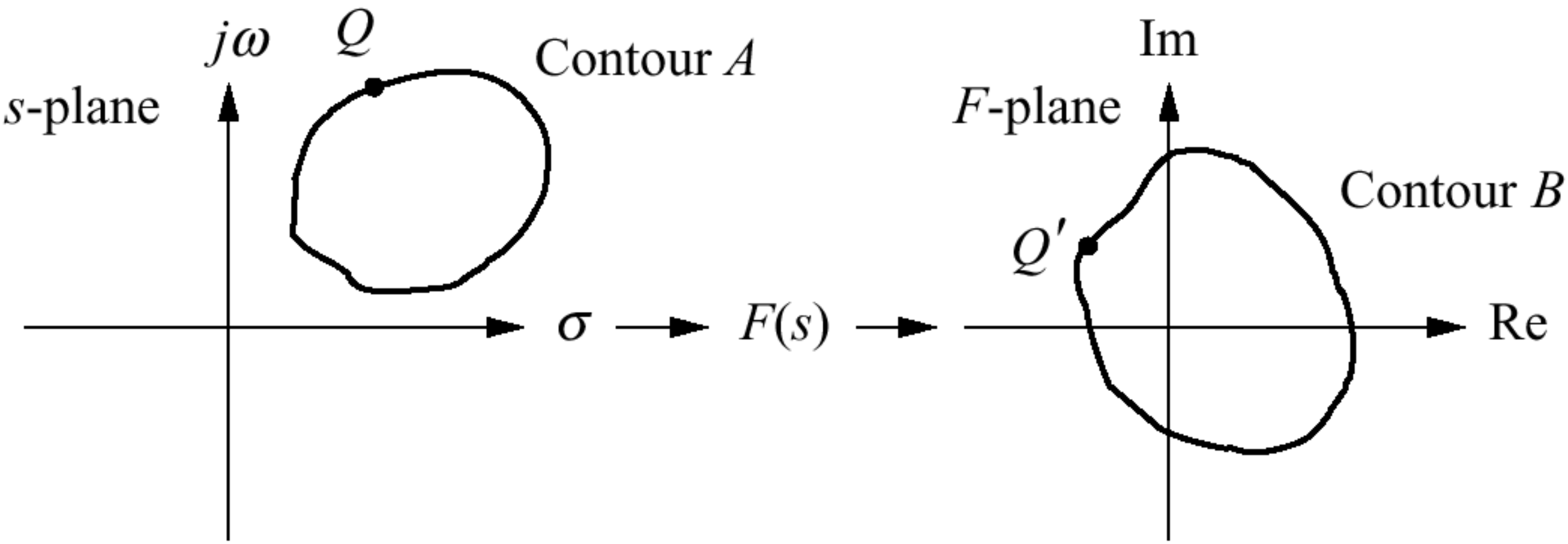
- ✓ The relationship between the poles of $\mathbf{1}+\boldsymbol{G(s)H(s)}$ and the poles of $\boldsymbol{G(s)H(s)}$;
- ✓ The relationship between the zeros of $\mathbf{1}+\boldsymbol{G(s)H(s)}$ and the poles of the closed-loop transfer function, $\boldsymbol{T(s)}$;
- ✓ The concept of mapping points; and
- ✓ The concept of mapping contours .

Nyquist Stability Criterion-5

- Poles of closed-loop transfer function in RHP → the system is unstable.
- Nyquist found way to count, n, closed-loop poles in RHP.
- If count is greater than zero, $n > 0$, system is unstable.
- To do this: **First**, find a way to count closed-loop poles inside a contour.
- **Second**, make the contour equal to the RHP.
- Counting is related to complex functional mapping

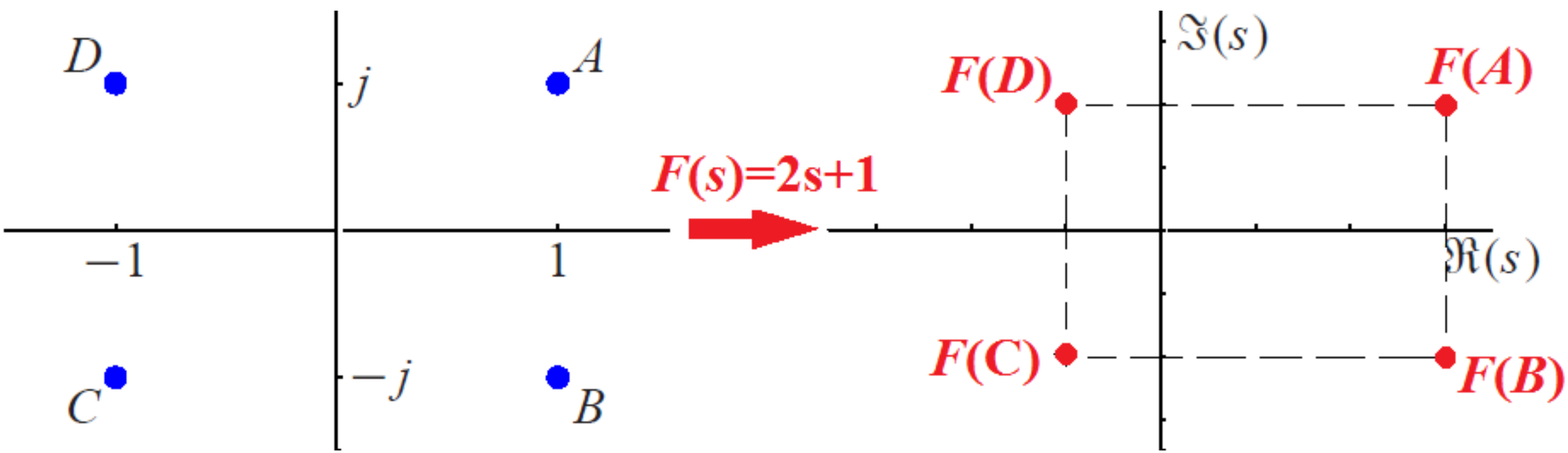
Cauchy principle of argument

- The Nyquist stability test is obtained by applying the **Cauchy principle of argument** to the complex function $\boldsymbol{D(s)}$
- Mapping ***contour A*** through function $\boldsymbol{F(s)}$ to ***contour B***,



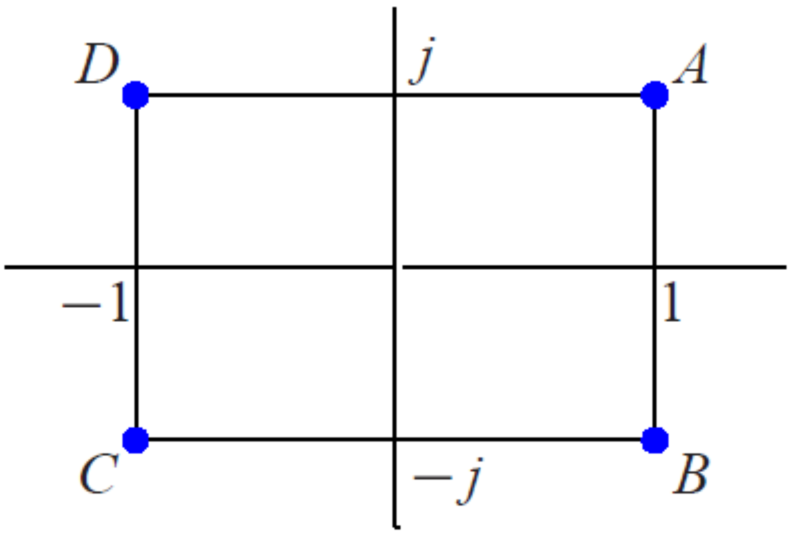
Complex Functional Mapping-1

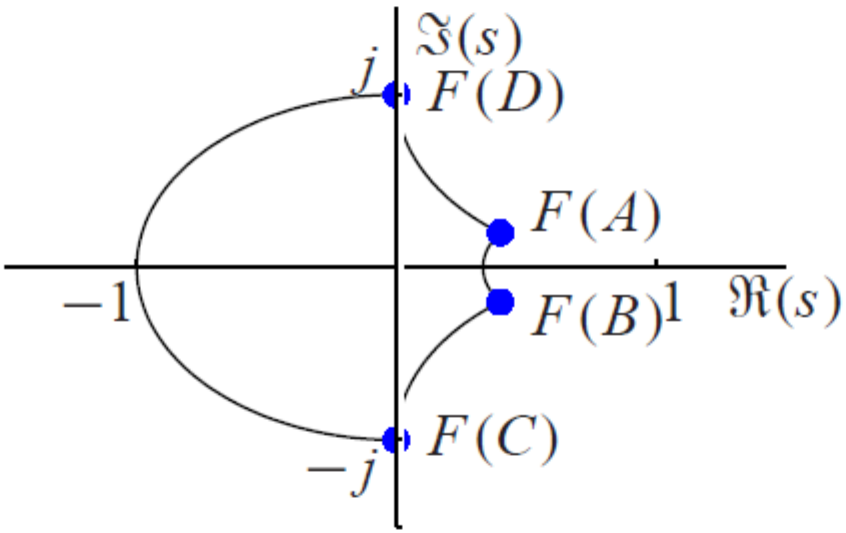
- ***Definition:*** If we take a complex number on the s-plane and substitute it into a function, $F(s)$, another complex number results. This process is called *mapping*.
- **For example,** substituting $s = 4 + j3$ into the function $F(s) = s^2 + 2s + 1$ yields $16 + j30$.
- We say that $4 + j3$ maps into $16 + j30$ through the function $F(s)$, $s^2 + 2s + 1$
- **Example-1:** Map the four points: *A*, *B*, *C*, *D* into $F(s) = 2s + 1$.



Complex Functional Mapping-2

- **Example-2:** Map a square contour (closed path) by $F(s) = \frac{s}{s+2}$.





By drawing maps of a specific contour, using a mapping function related to the plant's open-loop frequency-response, we will be able to determine closed-loop stability of systems.

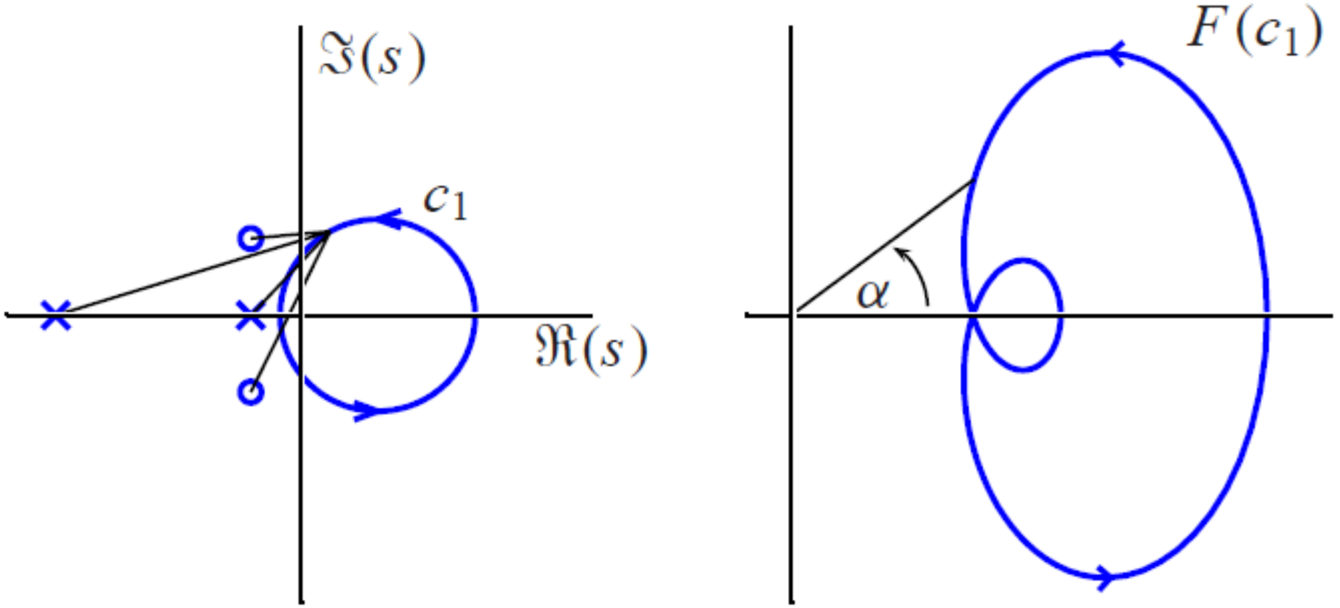
Mapping Function: Poles of the Function

- When we map a contour containing (encircling) poles and zeros of the mapping function, this map will give us information about how many poles and zeros are encircled by the contour.
- We will practice drawing maps when we know poles and zeros.
- Evaluate $G(s)|_{s=s_0}$

$$G(s_o) = |\vec{v}|e^{j\alpha}$$

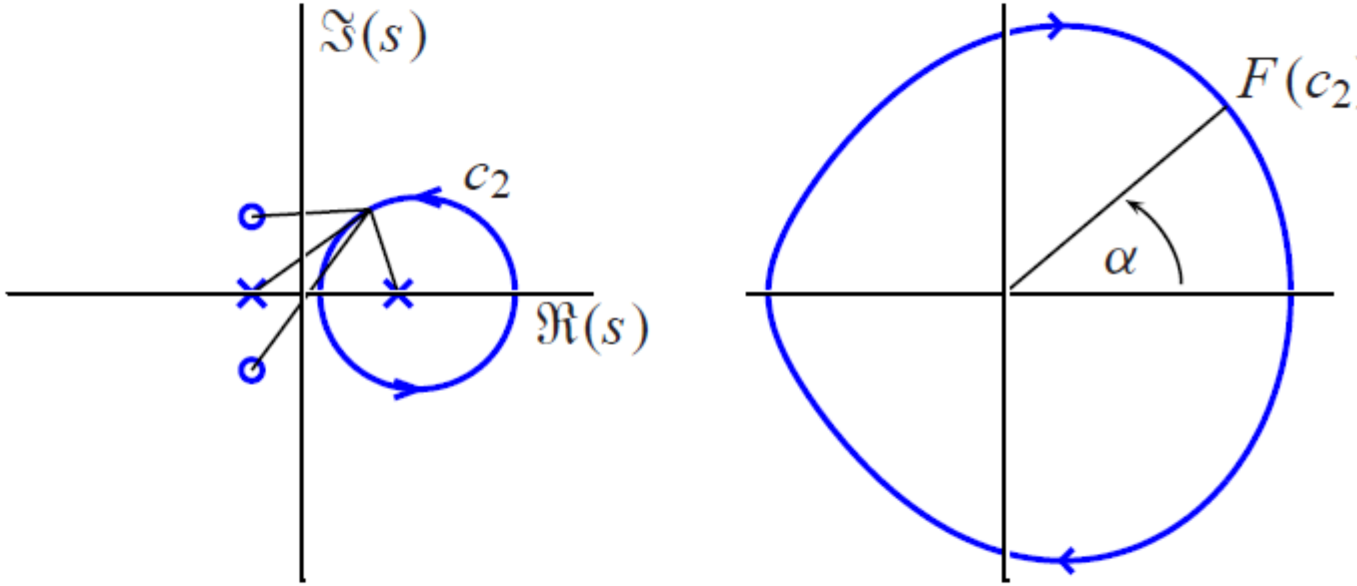
$$\alpha = \sum \angle(\text{zeros}) - \sum \angle(\text{poles}).$$

Example-1



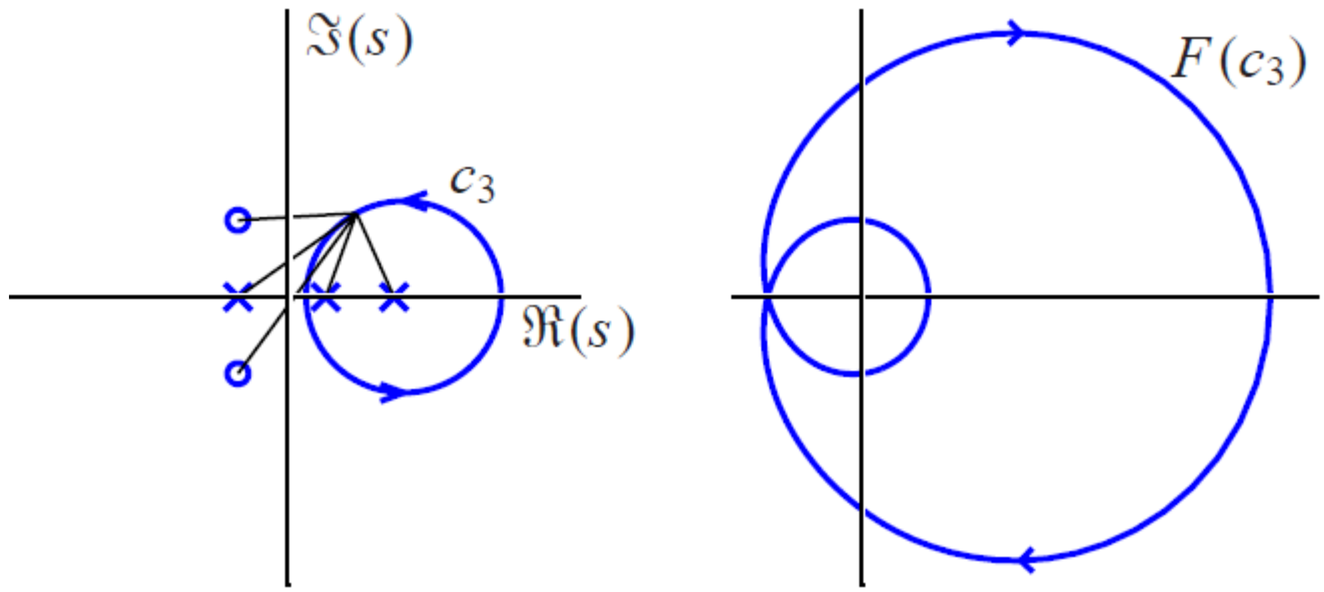
In this example, there are no zeros or poles inside the contour. The phase angle α increases and decreases, but never undergoes a net change of 360^o (does not encircle the origin).

Example-2



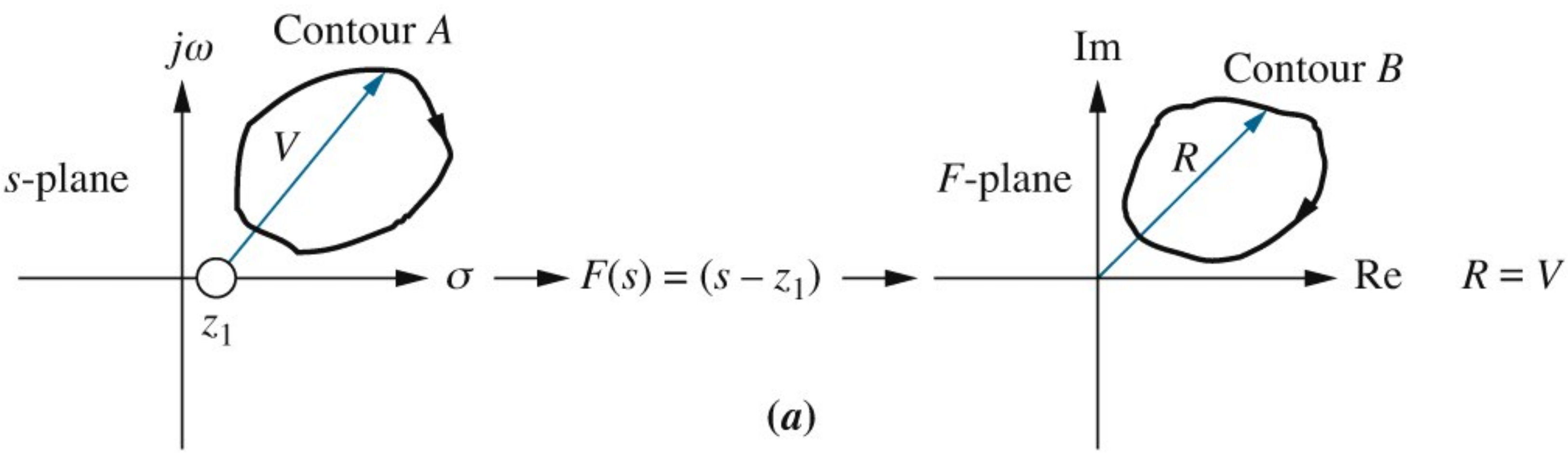
- One pole inside contour.
- Resulting map undergoes 360^{o} net phase change. (Encircles the origin).

Example-3

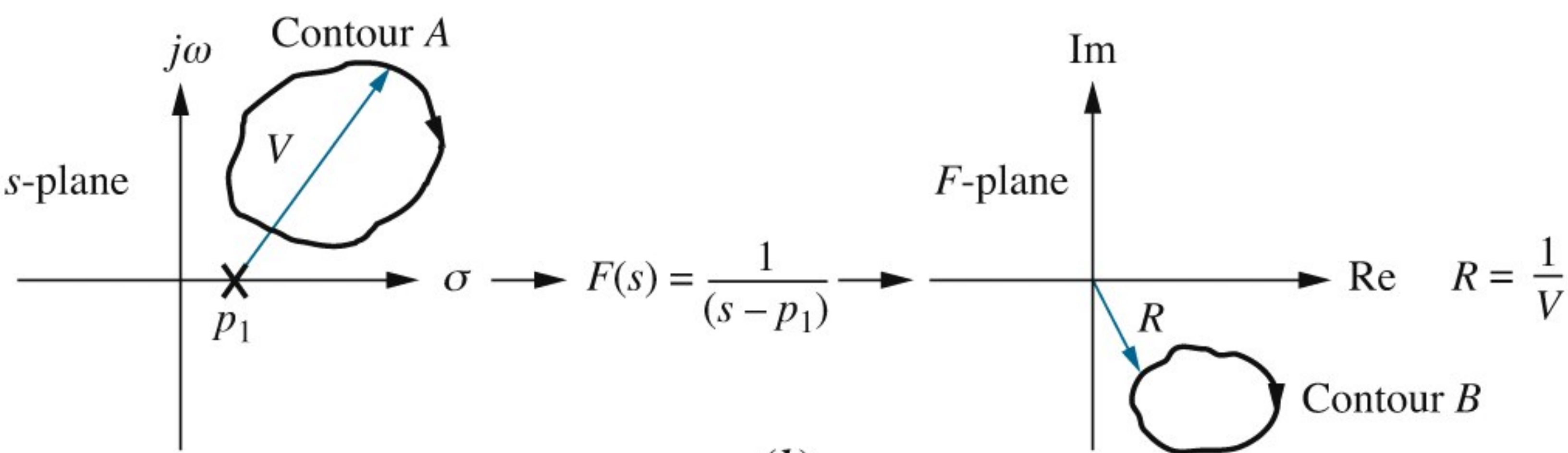


- In this example, there are **two poles inside the contour**, and the map encircles the **origin twice**.
- Resulting map undergoes 360º (x2) net phase change (encircles the origin).

Examples of contour mapping-1

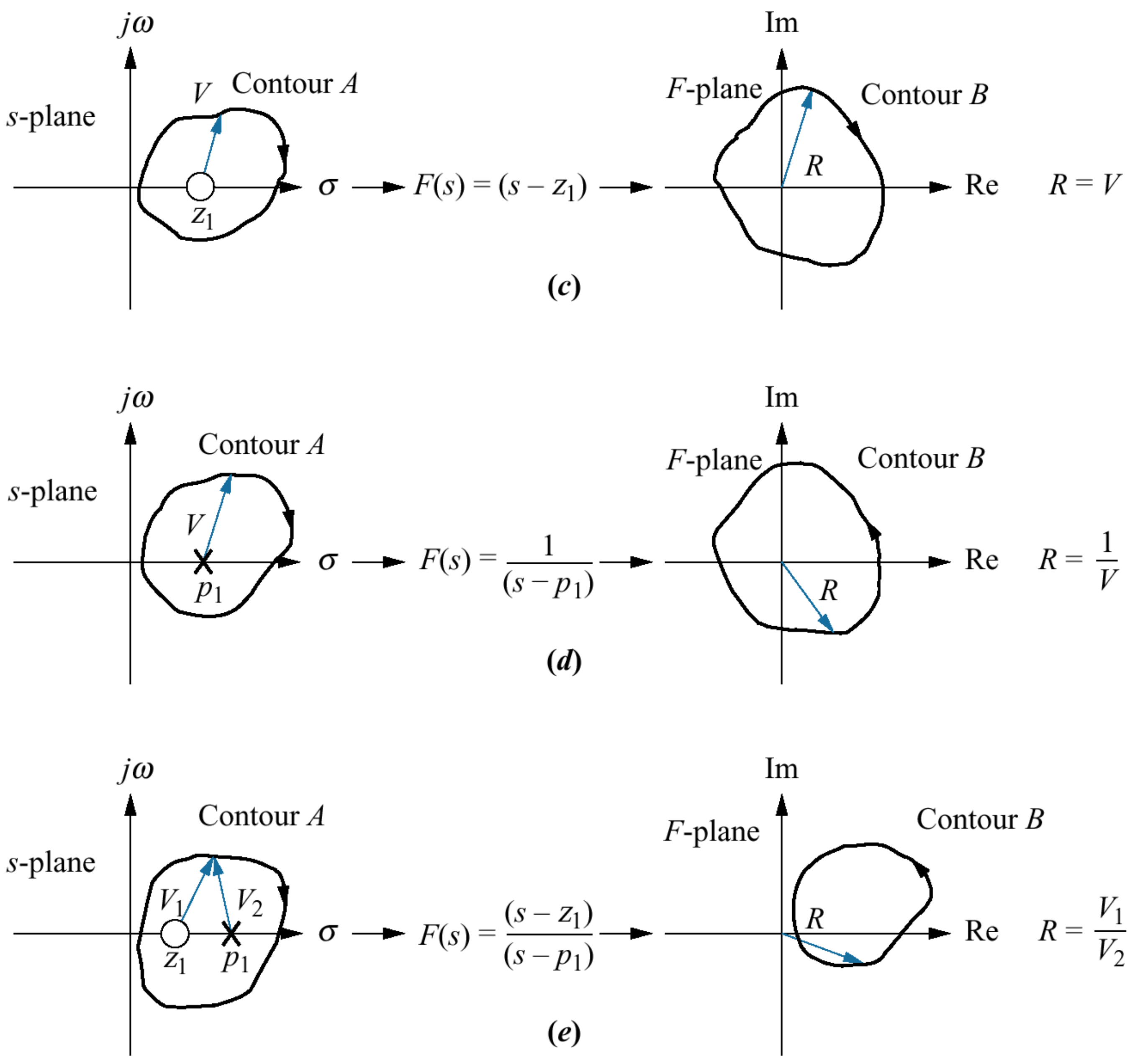


(*a*)



(*b*)

Examples of contour mapping-2

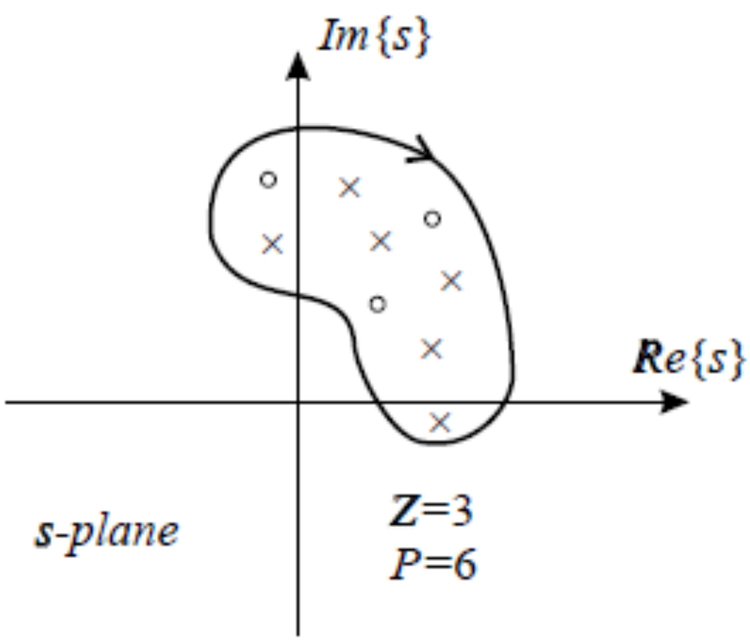


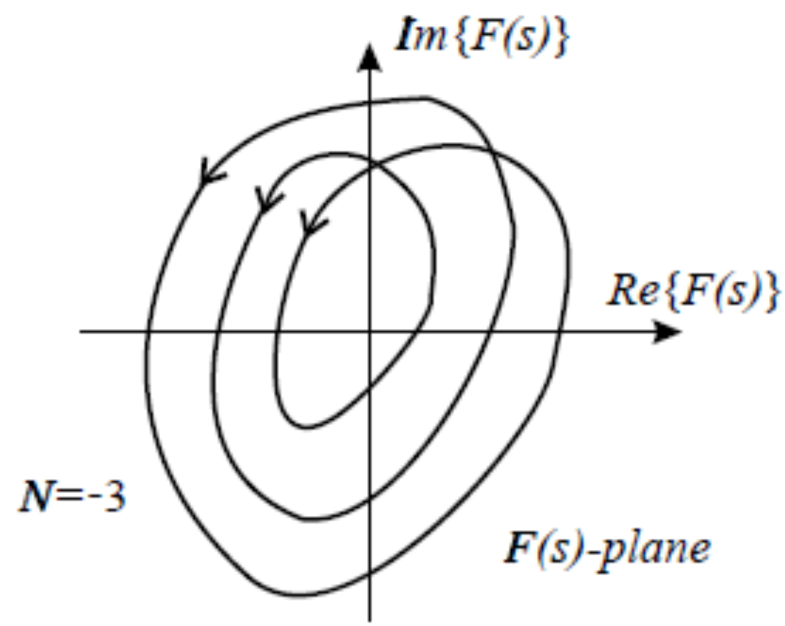
Notes on contour mapping

- You should verify that if we assume a **clockwise** direction for mapping the points on **contour** $\boldsymbol{A}$, then **contour** $\boldsymbol{B}$ maps in a **clockwise** direction if $\boldsymbol{F(s)}$ has just zeros or has just poles that are **not encircled** by the contour.
- The **contour** $\boldsymbol{B}$ maps in a **counterclockwise** direction if $\boldsymbol{F(s)}$ has just poles that are encircled by the contour.
- Also, you should verify that if the pole or zero of $\boldsymbol{F(s)}$ is enclosed by **contour** $\boldsymbol{A}$, the mapping encircles the origin.
- In the last case, the pole and zero rotation cancel, and the mapping does not encircle the origin.

Cauchy's Principle of Argument-1

- Let $\boldsymbol{F}(\boldsymbol{s})$ be an analytic function in a closed region of the complex plane as given in the figure below except at a finite number of points (namely, the poles of $\boldsymbol{F}(\boldsymbol{s})$).
- It is also assumed that $\boldsymbol{F}(\boldsymbol{s})$ is analytic at every point on the contour.



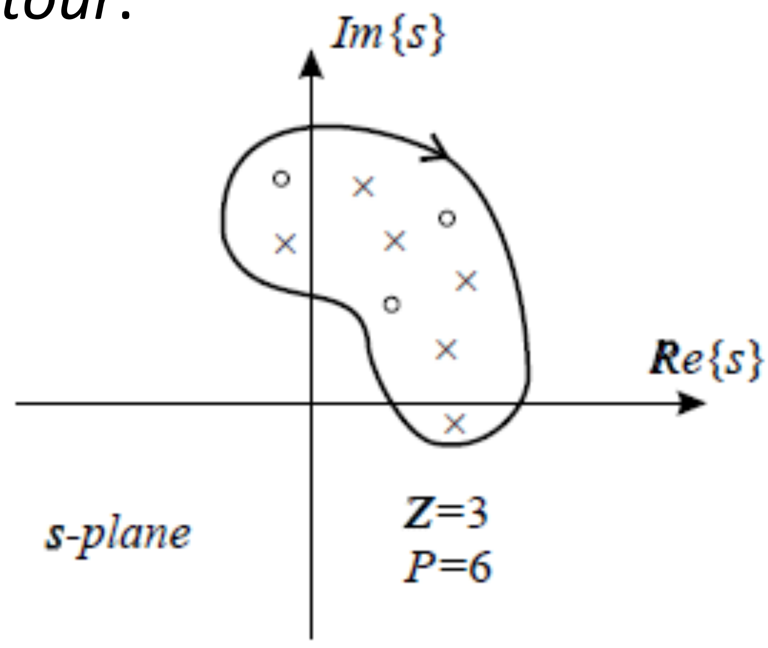


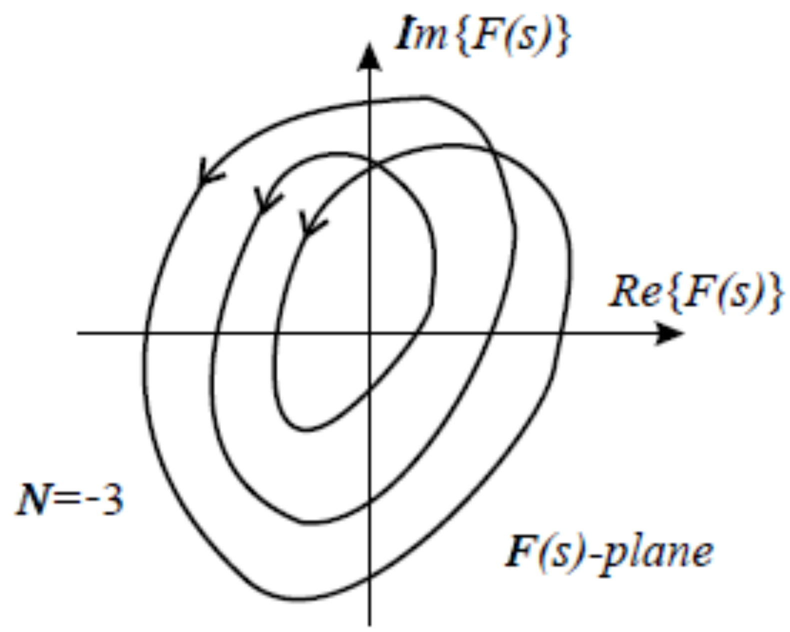
Cauchy's Principle of Argument-2

As $\boldsymbol{s}$ travels around the contour in the $\boldsymbol{s}$-*plane* in the clockwise direction, the function $\boldsymbol{F}(\boldsymbol{s})$ encircles the origin in the $(\mathbf{Re}\{\boldsymbol{F}(\boldsymbol{s})\}, \mathbf{Im}\{\boldsymbol{F}(\boldsymbol{s})\})$-*plane* in the same direction $\boldsymbol{N}$ times,

$$\boldsymbol{N} = \boldsymbol{Z} - \boldsymbol{P}$$

where $\boldsymbol{Z}$ and $\boldsymbol{P}$ stand for the number of zeros and poles (including their multiplicities) of the function $\boldsymbol{F}(\boldsymbol{s})$ *inside the contour*.





Cauchy's Principle of Argument-3

- The above result can be also written as

$$\arg\{F(s)\} = (Z - P)2\pi = 2\pi N$$

which justifies the terminology used, "the principle of argument".

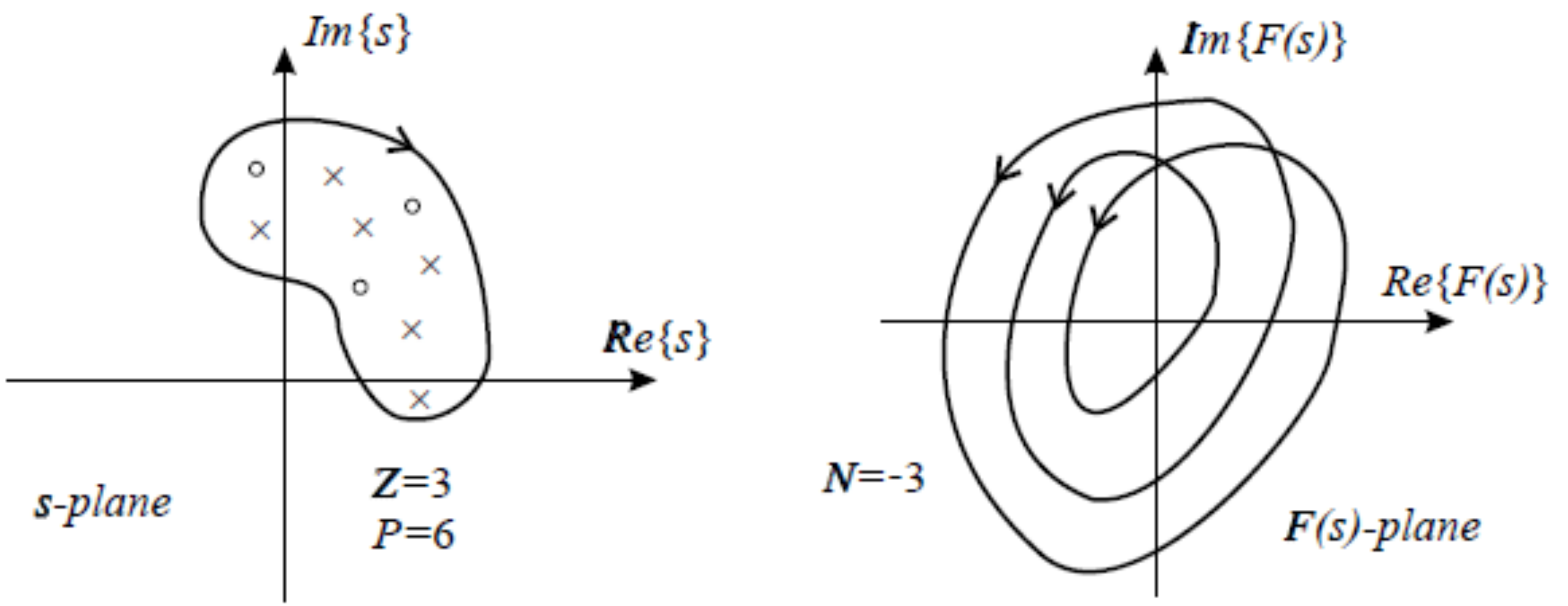


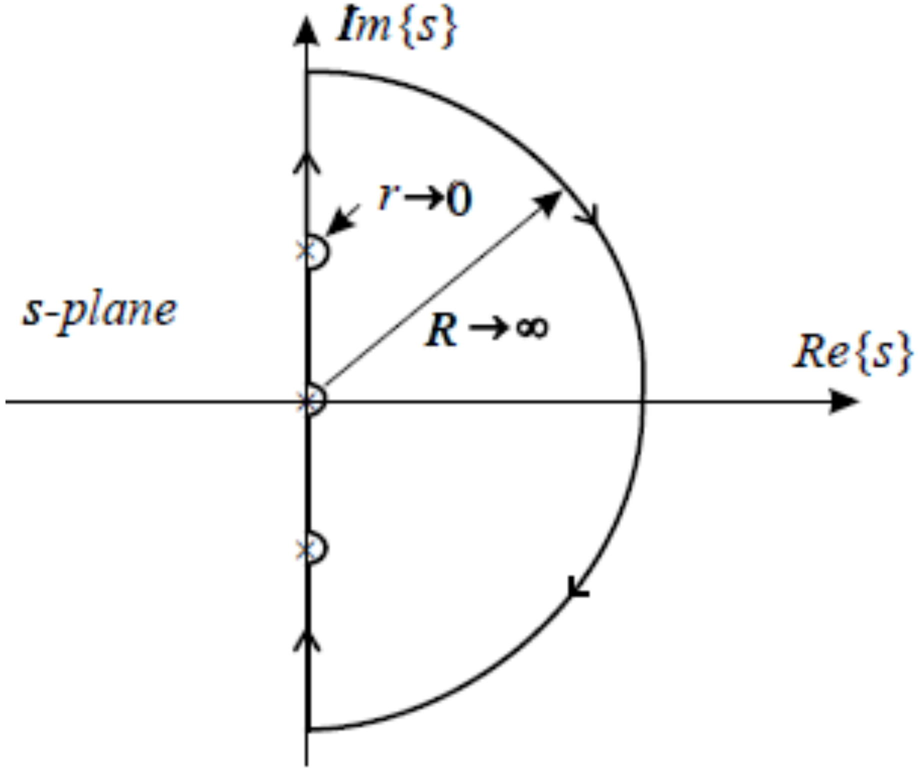
Figure: Cauchy's principle of argument

Nyquist Plot

- The Nyquist plot is a polar plot of the function

$$\boldsymbol{D}(\boldsymbol{s}) = \mathbf{1} + \boldsymbol{G}(\boldsymbol{s})\boldsymbol{H}(\boldsymbol{s})$$

when $\boldsymbol{s}$ travels around the contour as given below,



- The contour in this figure covers the whole unstable half complex plane $\boldsymbol{s}$ (i.e., RHP, $R \to \infty$).
- Since the function $\boldsymbol{D}(\boldsymbol{s})$, according to Cauchy's principle of argument, must be analytic at every point on the contour, the poles of $\boldsymbol{D}(\boldsymbol{s})$ on the imaginary axis must be encircled by infinitesimally small semicircles.

Nyquist Stability Criterion-1

- **Nyquist Stability Criterion** states that the number of unstable closed-loop poles is equal to the number of unstable open-loop poles plus the number of encirclements of the origin of the Nyquist plot for the complex function $\boldsymbol{D(s)}$, i.e. the characteristic equation.

$$\boldsymbol{N = Z - P} \Rightarrow \boldsymbol{Z = P + N}$$

- This can be easily justified by applying Cauchy's principle of argument.
- Note that $\boldsymbol{Z}$ and $\boldsymbol{P}$ represent the numbers of zeros and poles, respectively, of $\boldsymbol{D(s)}$ in the unstable part of the complex plane.
- At the same time,
  - the zeros of $\boldsymbol{D(s)}$ are the closed-loop system poles,
  - and the poles of $\boldsymbol{D(s)}$ are the open-loop system poles.

Nyquist Stability Criterion-2

- The above criterion can be *slightly simplified* if instead of plotting the function $\boldsymbol{D(s) = 1 + G(s)H(s)}$
- We plot only the function $\boldsymbol{G(s)H(s)}$ and count encirclement of the Nyquist plot of $\boldsymbol{G(s)H(s)}$ around the point $(\boldsymbol{-1 + j0})$.
- So that the modified Nyquist criterion has the following form,

The number of unstable closed-loop poles $(\boldsymbol{Z})$ is equal to the number of unstable open-loop poles $(\boldsymbol{P})$ plus the number of encirclements $(\boldsymbol{N})$ of the point $(\boldsymbol{-1 + j0})$ of the Nyquist plot of $\boldsymbol{G(s)H(s)}$, that is

$$\boldsymbol{Z = P + N}$$

Nyquist Stability Criterion-3

- Alternatively, as stated in Nise's book, which is the main textbook of this course, if the definitions are,
- $\boldsymbol{N}$ equals the number of **counterclockwise** rotations of contour *B* (the Nyquist map) around – $\mathbf{1}$.
- $\boldsymbol{P}$ equals the number of poles of $\mathbf{1}+\boldsymbol{G}(\boldsymbol{s})\boldsymbol{H}(\boldsymbol{s})$ inside contour *A* (i.e. the # unstable open-loop poles and they are known).
- $\boldsymbol{Z}$ equals the number of zeros of $\mathbf{1}+\boldsymbol{G}(\boldsymbol{s})\boldsymbol{H}(\boldsymbol{s})$ inside contour *A* (that is the number of unstable closed-loop poles *to be determined*).
- Then $\boldsymbol{N}=\boldsymbol{P}-\boldsymbol{Z}$, hence $\boldsymbol{Z}=\boldsymbol{P}-\boldsymbol{N}$
- Since $\boldsymbol{Z}$ is the number of closed-loop poles inside contour *A*, which encircles the entire right half-plane, for a system to be stable $\boldsymbol{Z}$ must be zero (that means no right–half-plane poles).

In short, the rule is $\boldsymbol{Z}=\mathbf{0}$ for a stable system.

Applying the Nyquist Criterion to Determine Stability

In (a), $P = 0, N = 0, Z = P - N = 0$: **Stable**

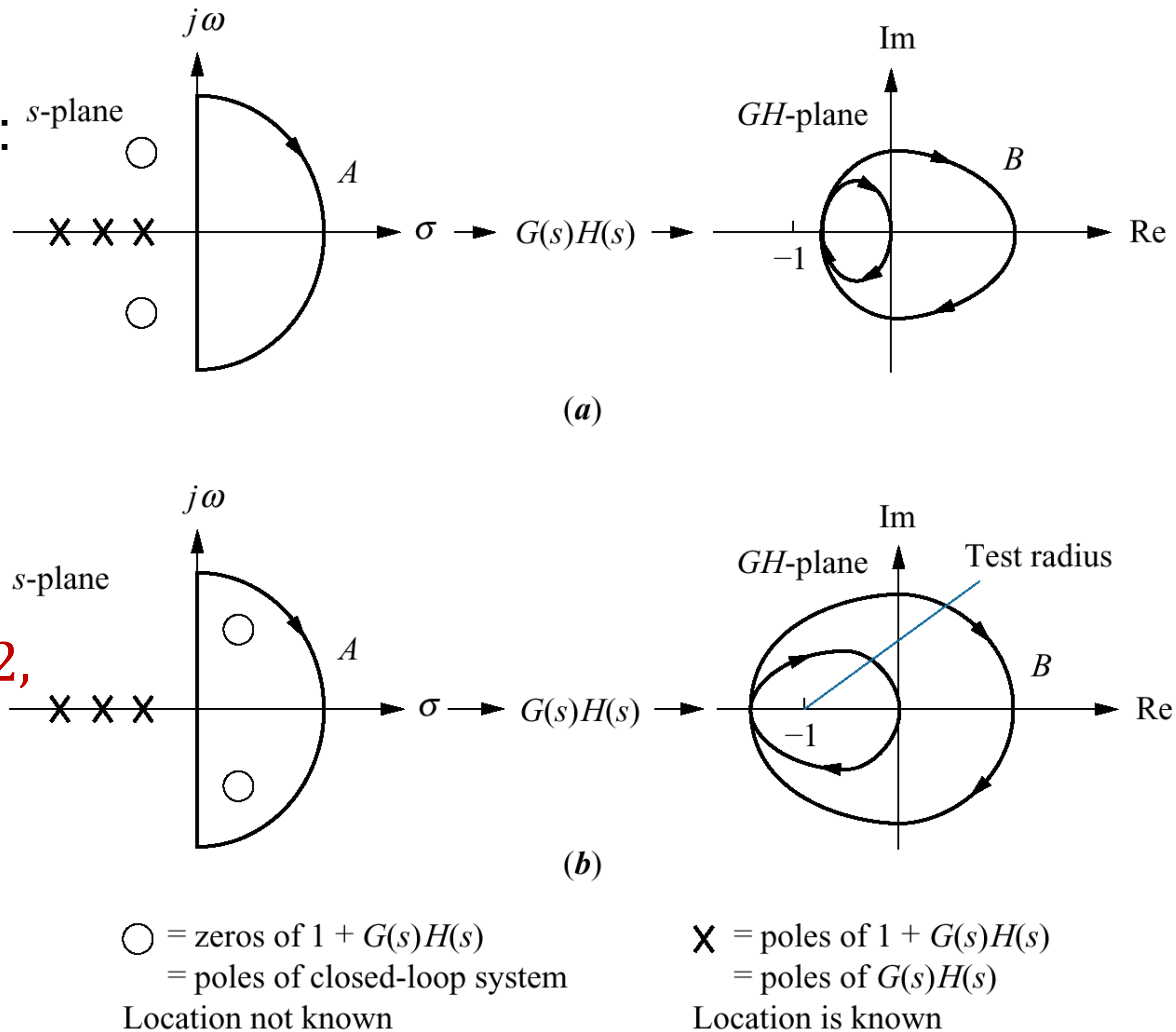
Figure 10.25 Mapping examples: **a.** contour does not enclose closed-loop poles; **b.** contour does enclose closed-loop poles

In (b), $P = 0, N = -2,$

$Z = P - N =$

$0 - (-2) = 2 > 0$:

Unstable

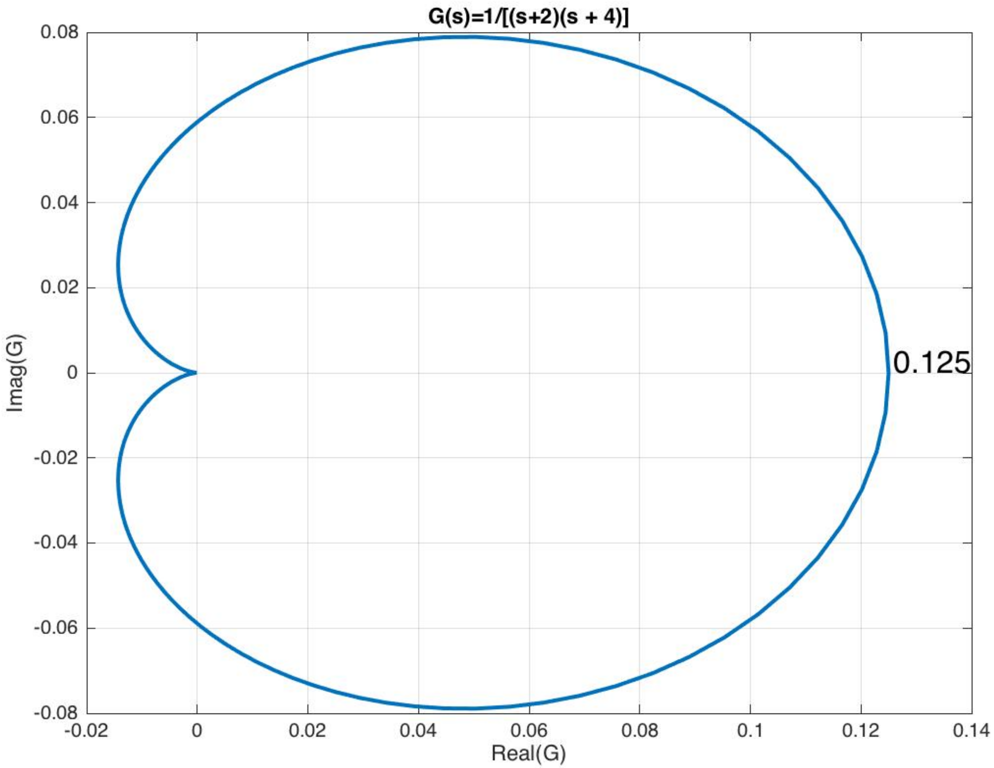


Numerical Example-1

Sketch the Nyquist diagram for a unity feedback system where

$$G(s) = \frac{1}{(s+2)(s+4)}$$

Then determine its stability.



Stability Analysis:

N = P – Z ➔ Z = P – N

P = 0 (no RHP o-l poles)

N = 0 (no encirclements of –1)

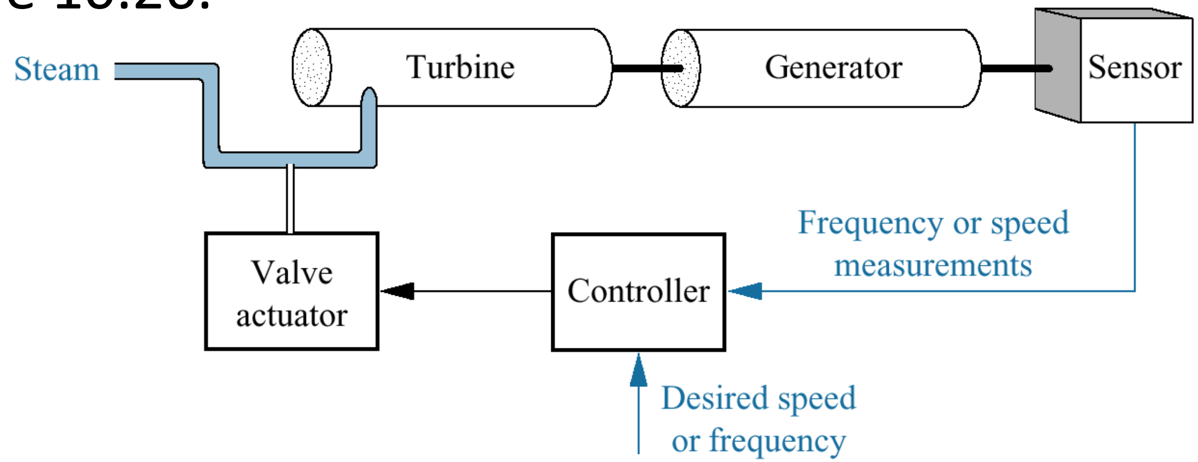
$Z = 0 - 0 = 0$ ➔ Stable

(no RHP c-l poles)

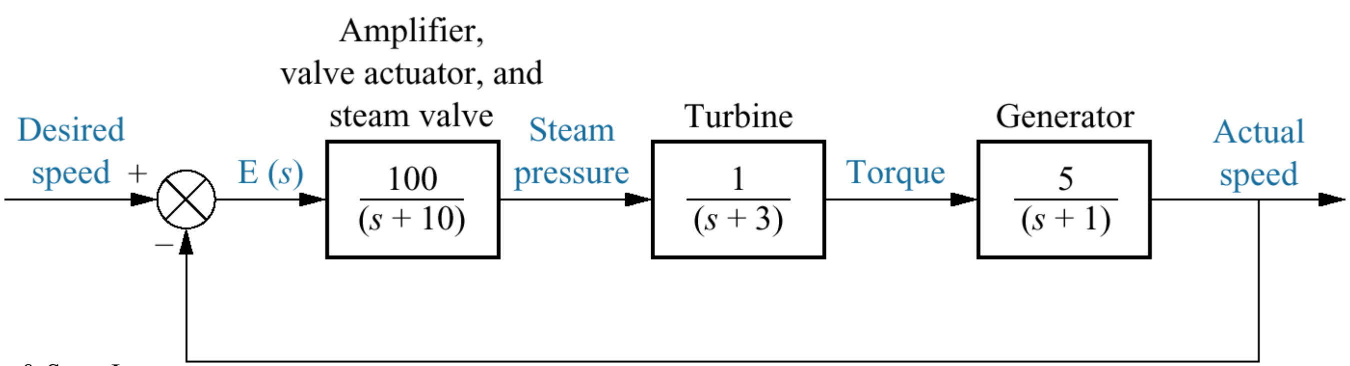
Stability range: $0 < K < \infty$

Example-2 (Example 10.4 from the textbook CSE)

Speed controls find wide application throughout industry and the home. Figure 10.26(a) shows one application. The system block diagram is shown in Figure 10.26(b). Sketch the Nyquist diagram for the system of Figure 10.26.



(*a*)



(*b*)

Figure 10.26
a. Turbine and generator;
b. block diagram of speed control system for Example 10.4

Example-2: Sketching Nyquist plot (Example 10.4 *cont.'s*)

- The open-loop transfer function:

$$\boldsymbol{G(s)} = \frac{\mathbf{500}}{(\boldsymbol{s}+\mathbf{1})(\boldsymbol{s}+\mathbf{3})(\boldsymbol{s}+\mathbf{10})} = \frac{500}{s^3+14s^2+43s+30}$$

- Substituting $s \to j\omega$

$$G(j\omega) = \frac{500}{(j\omega)^3+14(j\omega)^2+43(j\omega)+30} = \frac{500}{(30-14\omega^2)+j\omega(43-\omega^2)}$$

- $G(j\omega) = \frac{500}{30} = 16.67\angle 0°$ for $\omega = 0$ rad/s.
- $G(j\omega)$ will make a total of $3\times(-90°) = -270°$ for $\omega = 0$ to ∞ rad/s.
- Multiplying numerator and denominator by complex conjugate of denominator,

$$G(j\omega) = \frac{500\left[(30-14\omega^2)-j\omega(43-\omega^2)\right]}{(30-14\omega^2)^2+\omega^2(43-\omega^2)^2}$$

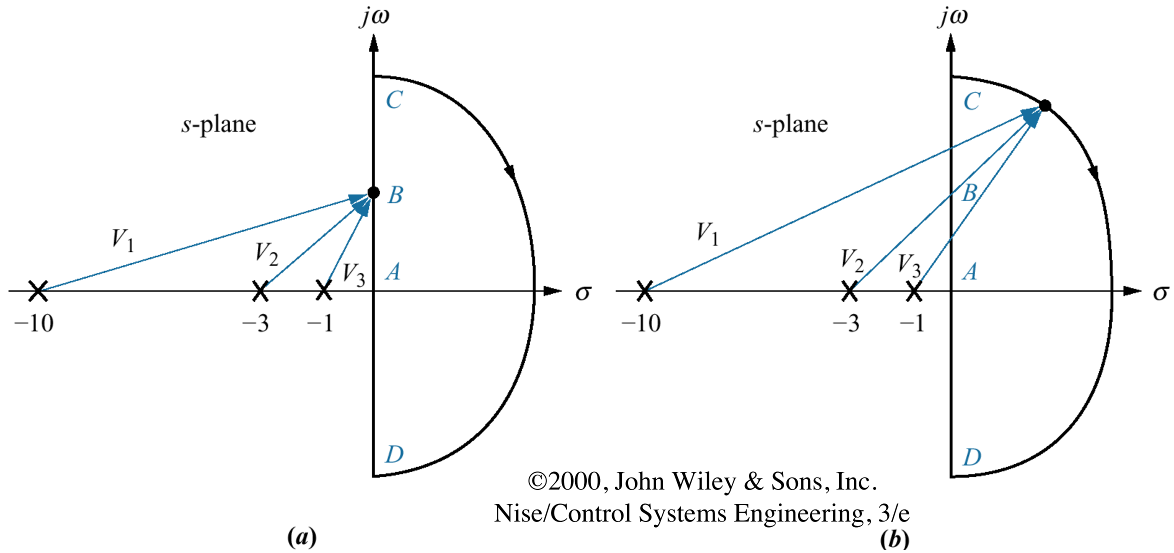
- Imaginary axis interception is found for $\omega = \pm\sqrt{30/14}$:

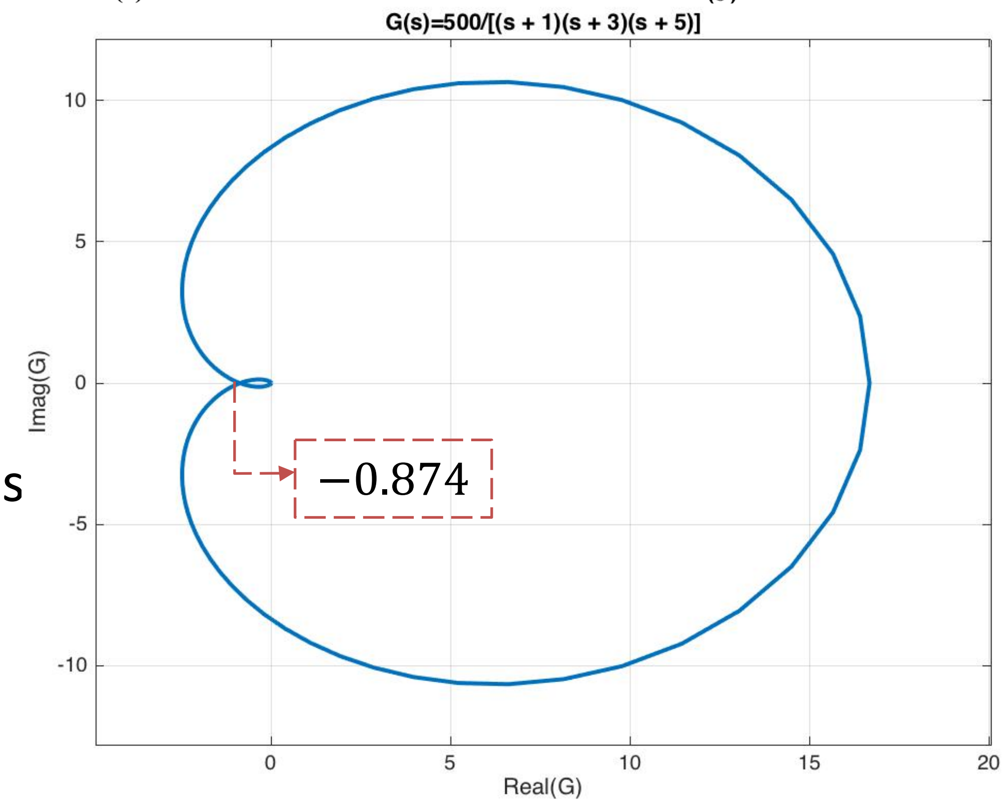
$$G(j\omega) = \frac{-j500\omega(43-\omega^2)}{\omega^2(43-\omega^2)^2} = -j\frac{500}{\omega(43-\omega^2)} = \mp j8.36$$

- Real axis interception is for $\omega = 0$ and $\omega = \pm\sqrt{43}$ (it was $16.67\angle 0°$ for $\omega = 0$):

$$G(j\omega) = \frac{500(30-14\omega^2)}{(30-14\omega^2)^2} = \frac{500}{(30-14\omega^2)} = -0.874$$

Figure 10.27
Vector evaluation of the Nyquist diagram for Example 10.4, where $G(s) = \dfrac{500}{(s+1)(s+3)(s+10)}$:
a. vectors on contour at low frequency; **b.** vectors on contour around infinity; **c.** Nyquist diagram





Stability Analysis:

N = P – Z ➔ Z = P – N

P = 0 (no RHP o-l poles)

N = 0 (no encirclements of –1)

Z = 0 – 0 = 0 Stable (no RHP c-l poles

Stable for:

$$0 < K < 500 * (1/|-0.874|) \rightarrow$$

$$\mathbf{0 < K < 572}$$

Example-2. Summary

$$G(s) = \frac{500}{(s+1)(s+3)(s+10)}$$

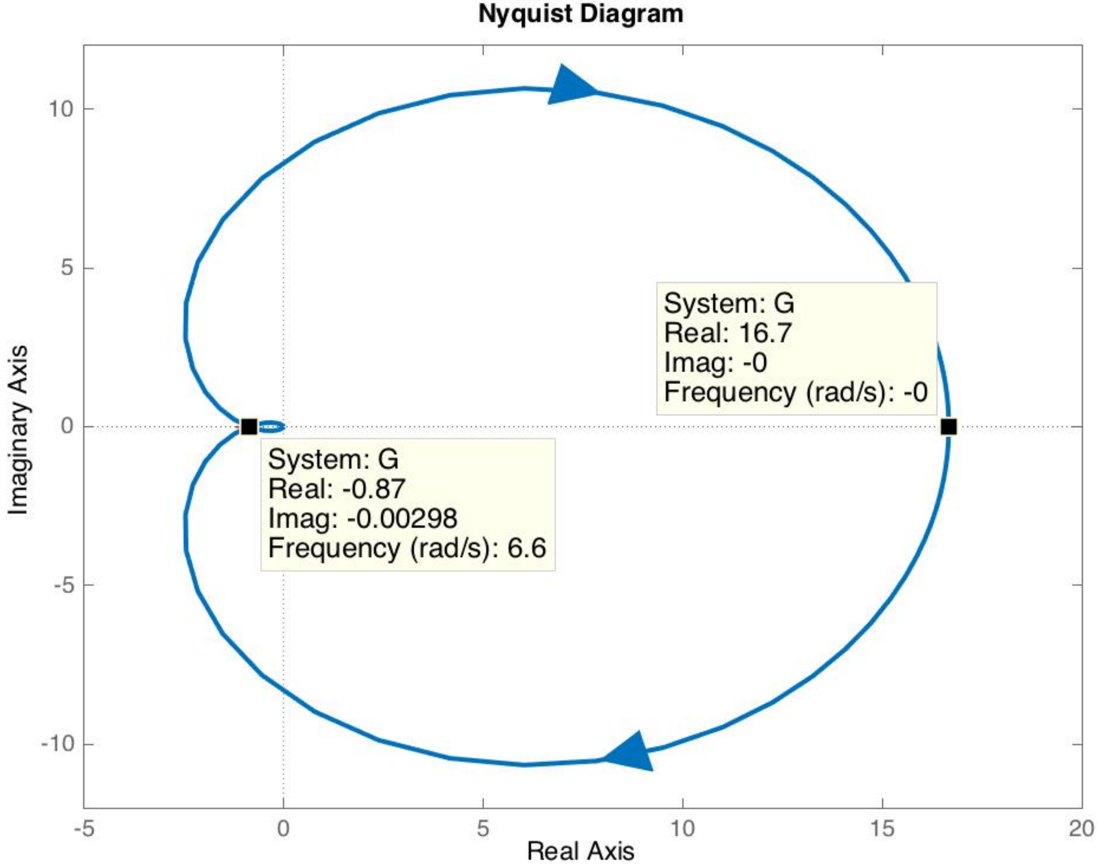
Stability Analysis for $K = 500$:

N = P – Z ➔ Z = P – N;
P = 0 (no RHP o-l poles)
N = 0 (no encirclements of –1)

$Z = 0 - 0 = 0$ ➔ Stable (no RHP c-l poles)

Stability range for gain, K:

$0 < K < 500 * (1/|-0.874|)$ ➔ $\mathbf{0 < K < 572}$



Close look at the real-axis crossing:

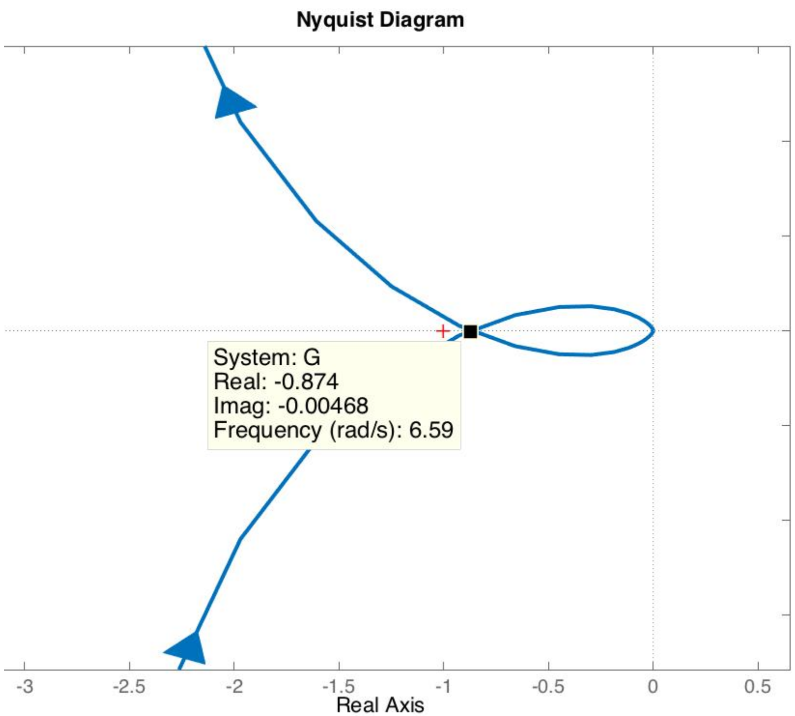
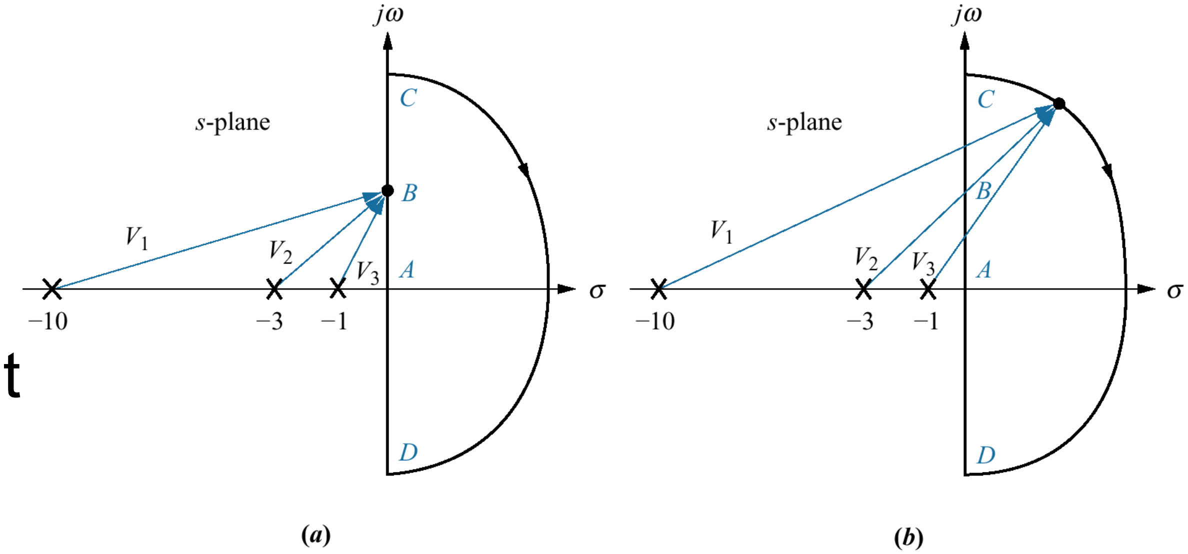
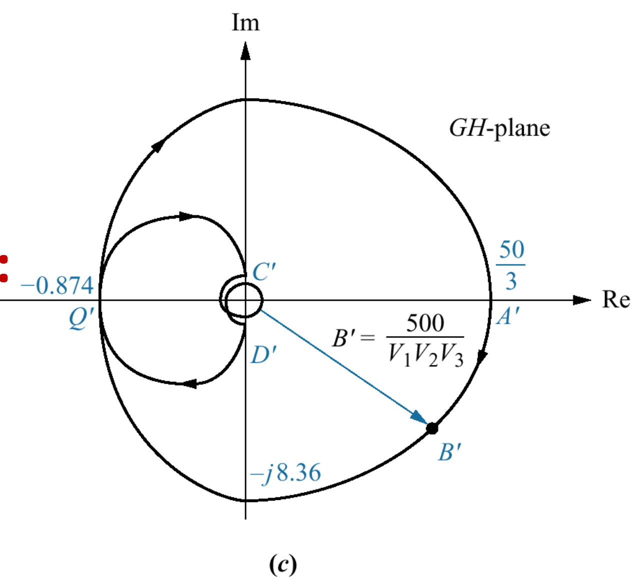


Figure 10.27
Vector evaluation of the Nyquist diagram for Example 10.4:
a. vectors on contour at low frequency;
b. vectors on contour around infinity;
c. Nyquist diagram



Stability Analysis (**N = P – Z** ➔ Z = P – N**):**
P = 0 (no RHP o-l poles)
N = 0 (no encirclements of –1)
Z = 0 – 0 = 0 Stable (no RHP c-l poles)
Stable for:
0 < *K* < 500*(1/0.874) ➔ 0 < *K* < 572



If there is an open-loop pole on the imaginary axis

Figure 10.28 Detouring around open-loop poles:
a. poles on contour;
b. detour right;
c. detour left

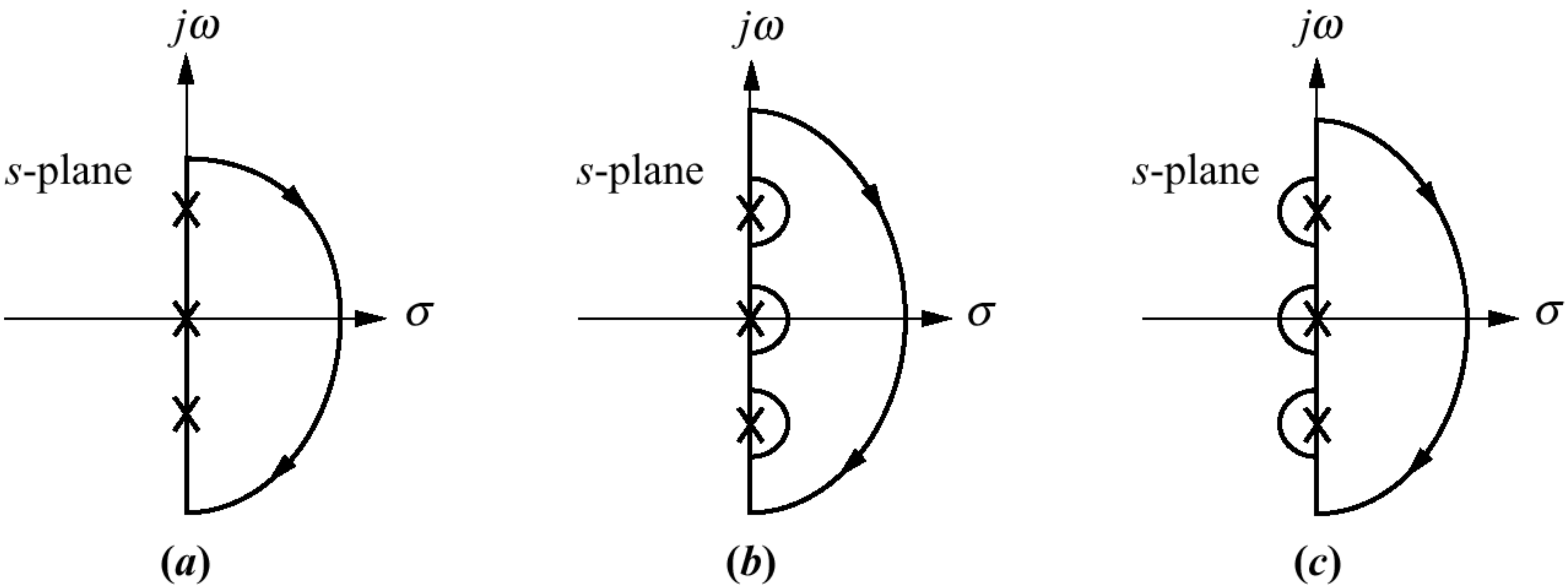
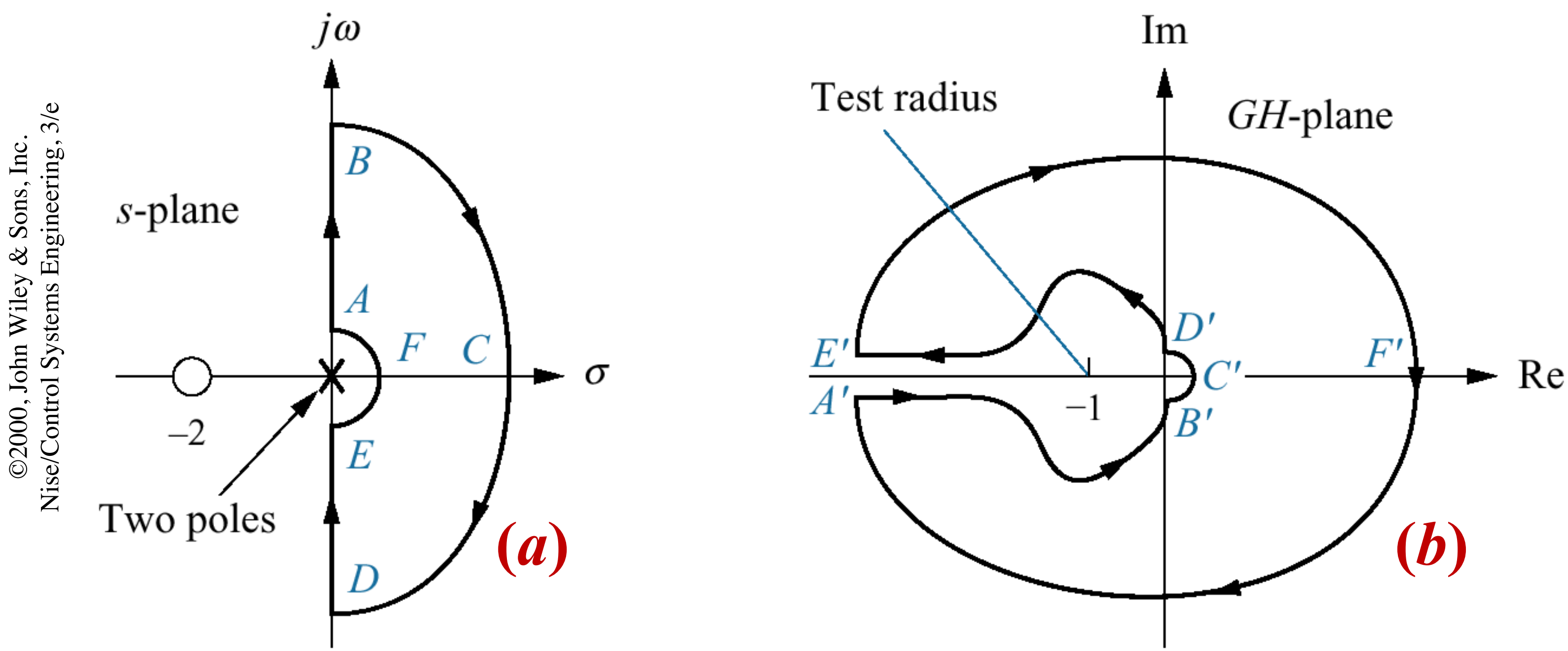


Figure 10.29 a. Contour, where $G(s) = \frac{s+2}{s^2}$; and

b. Nyquist diagram for Example 10.5



@A: $M = \infty$,
$\phi = 0° - (90° + 90°) = -180°$
@B: $M = 0$,
$\phi = 90° - (90° + 90°) = -90°$
@C: $M = 0$,
$\phi = 0° - (0° - 0°) = 0°$
@D: $M = 0$,
$\phi = 0° - (90° + 90°) = 90°$

@E: $M = \infty$,
$\phi = 0° - (-90° - 90°) = 180°$
@F: $M = \infty$,
$\phi = 0° - (0° - 0°) = 0°$

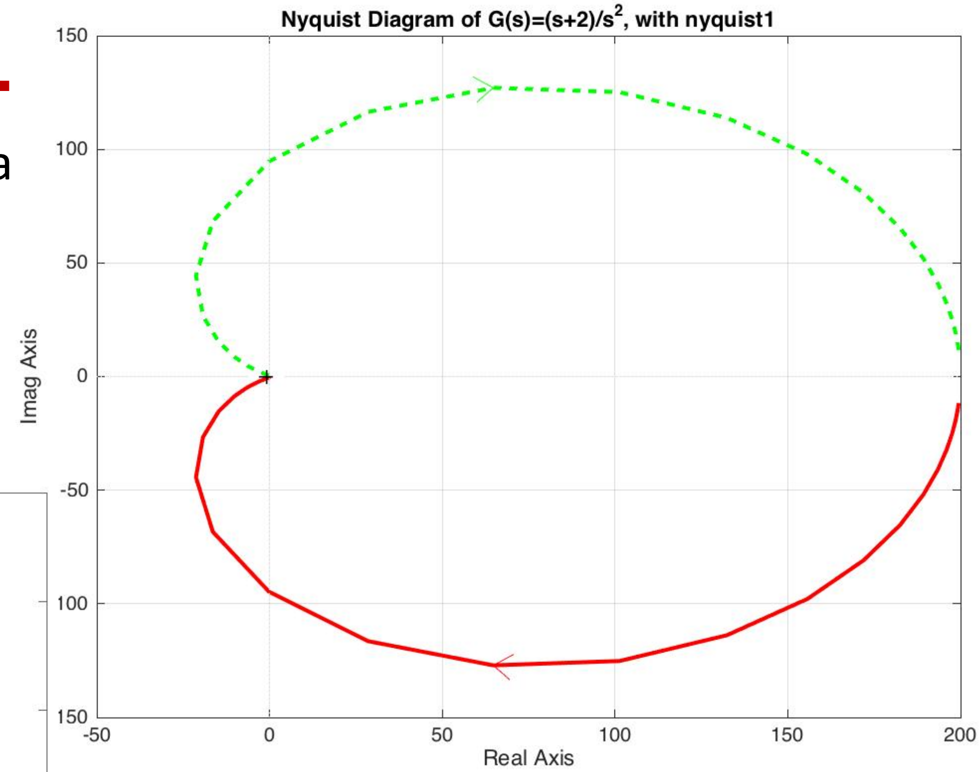
One cw, one ccw (see the test radius).
Hence, N=0. P=0; Z = 0 – 0 = 0:
Stability range: $0 < K < \infty$

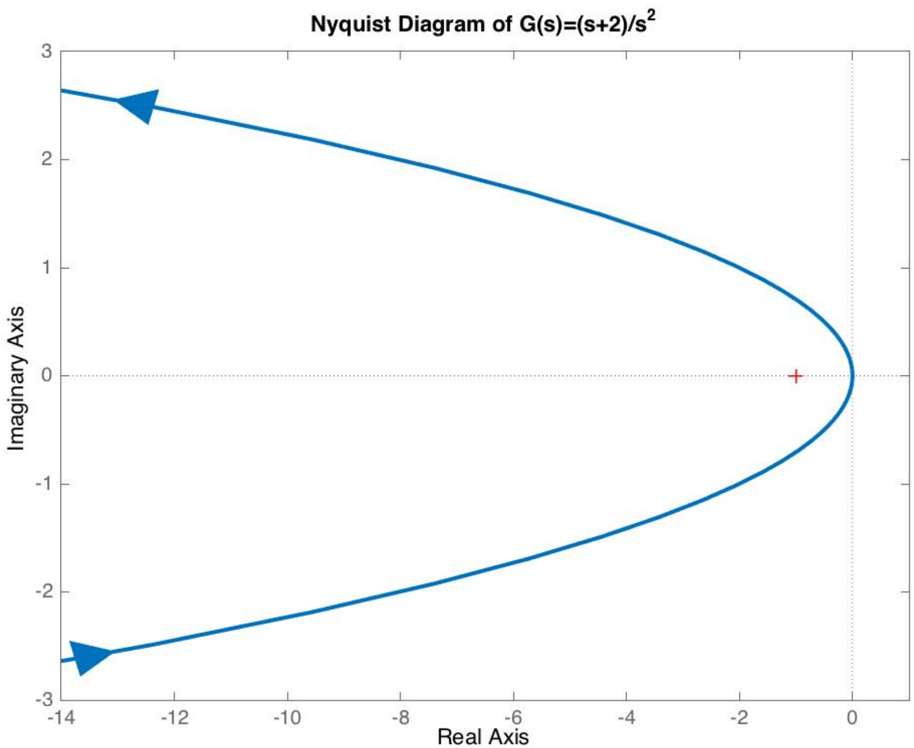
Example 10.5, *cont.'s*..

If we plot it in Matlab by the comma

```
>> nyquist(G)
>> nyquist1(G)
```

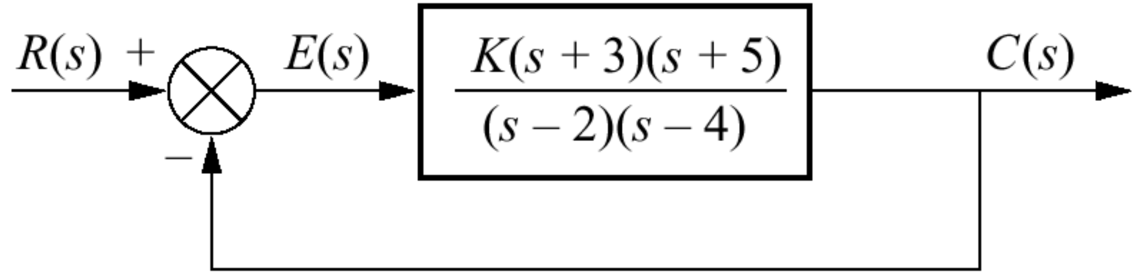
We get the following two plots.





The one at the top, which is obtained by the command of "nyquist1(G)" is found more useful for stability analysis.

Figure 10.30-1 Demonstrating Nyquist stability for the system given.



- $\boldsymbol{G(s) = \frac{K(s+3)(s+5)}{(s-2)(s-4)}} = \frac{K(s^2+8s+15)}{s^2-6s+8}$
- Substituting $s \to j\omega$

$$G(j\omega) = \frac{K[(15-\omega^2) + j8\omega]}{(8-\omega^2) - j6\omega} = \frac{K[(15-\omega^2) + j8\omega][(8-\omega^2) + j6\omega]}{(8-\omega^2)^2 + 36\omega^2}$$

$$= \frac{K[(\omega^4 - 71\omega^2 + 120) + j\omega(154 - 14\omega^2)]}{(8-\omega^2)^2 + 36\omega^2}$$

- $G(j\omega) = \frac{15}{8} = 1.875\angle 0°$ for $\omega = 0$ rad/s.
- Imaginary-axis crossings occur for $\omega = \pm 1.316,\ \pm 8.323$r/s at the following points: $\pm j1.68,\ \mp j1.087$,
- Real-axis interceptions occur for $\omega = 0$ and $\pm\sqrt{154/14}$ ($\pm\sqrt{11} = \pm 3.316$) at the following points: $1.875,\ -1.33$

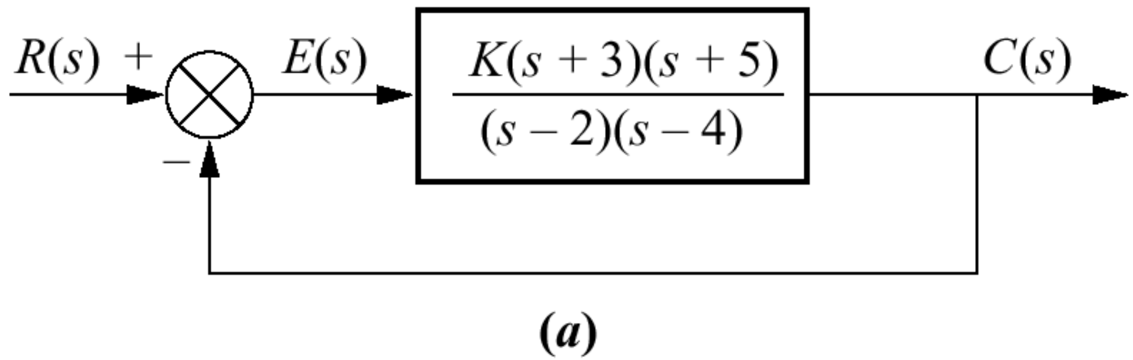
Figure 10.30-1

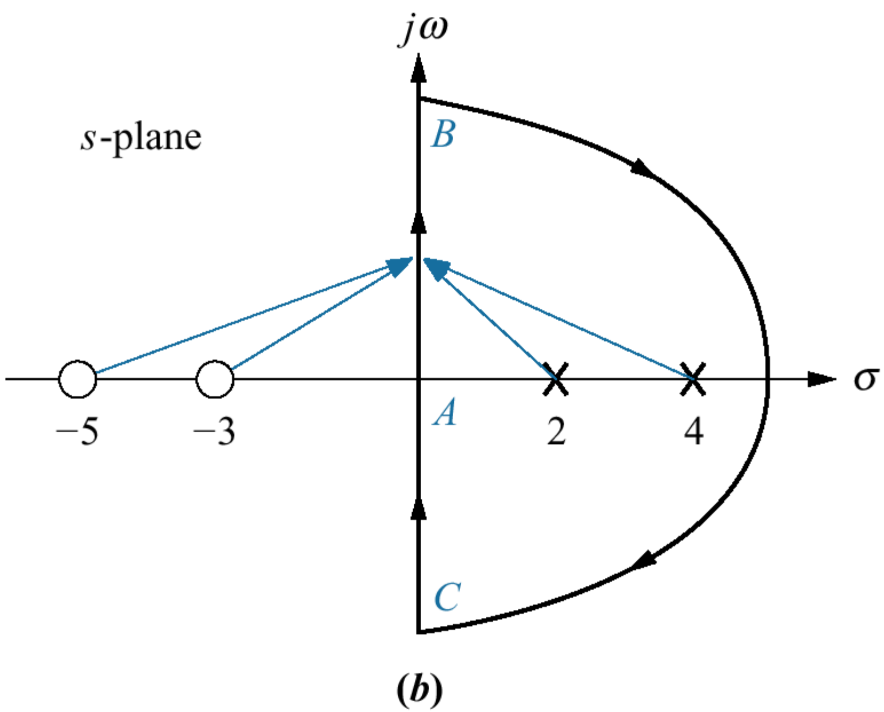
Demonstrating Nyquist stability:

a. system;

b. contour;

c. Nyquist diagram





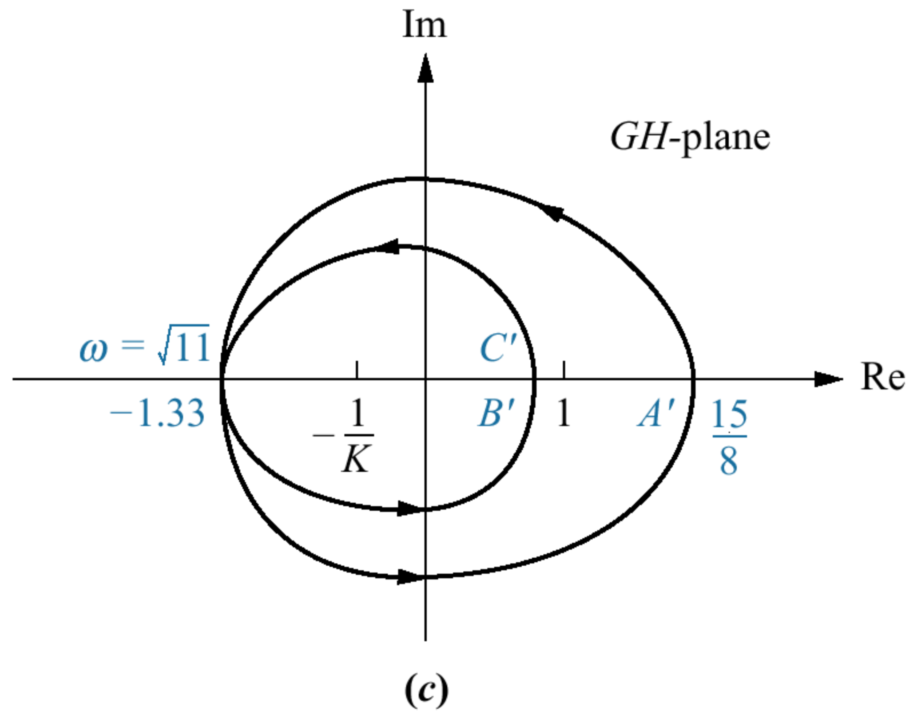
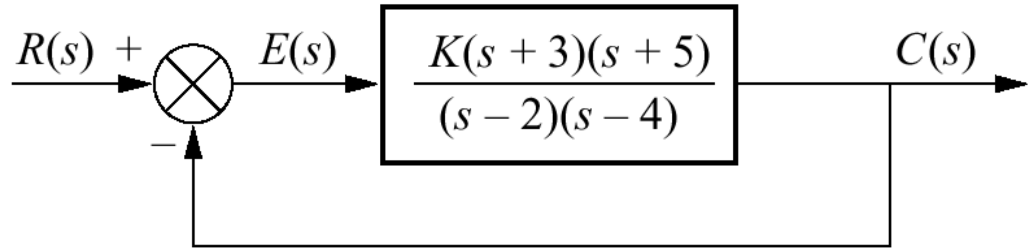


Figure 10.30-2

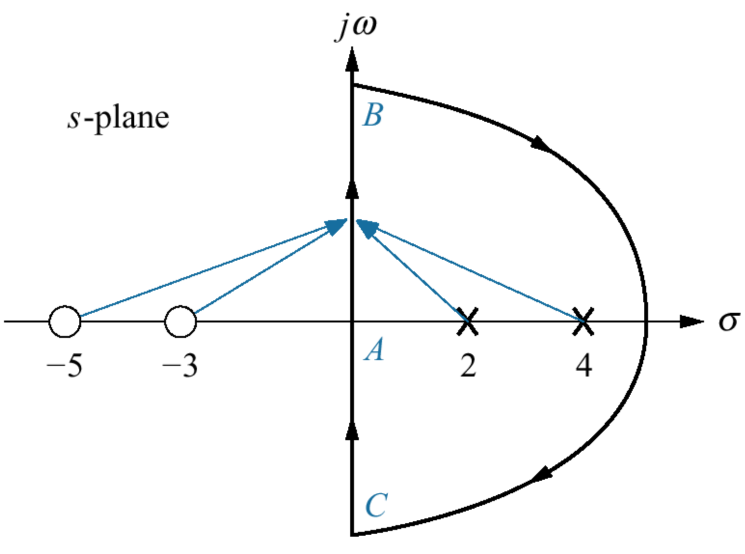
The same example:

a. system; **b.** contour;

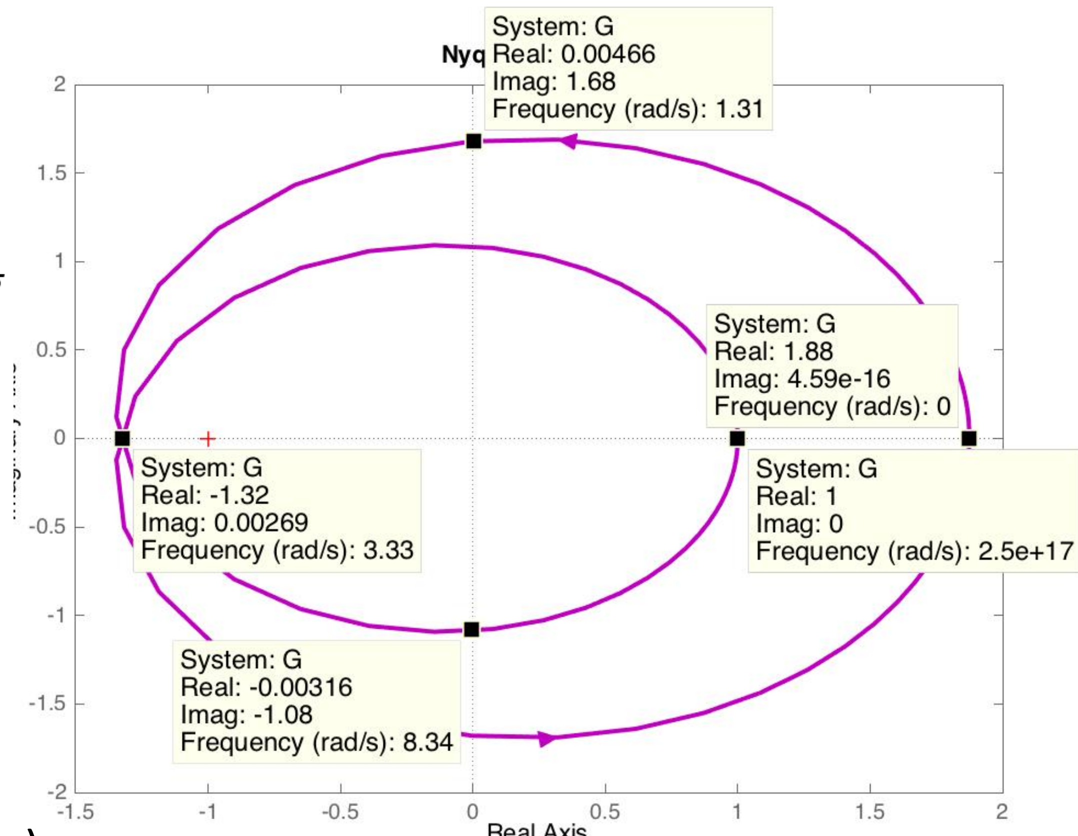
c. Nyquist diagram via Nyquist fn.



(a)



(b)



Stability Analysis for $K = 1$:

N = P – Z ➔ Z = P – N

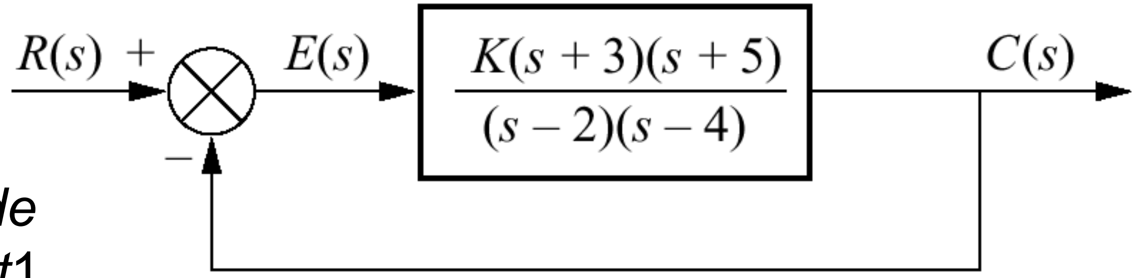
P = 2 (2 RHP o-l poles)

N = 2 (2 encirclements of –1)

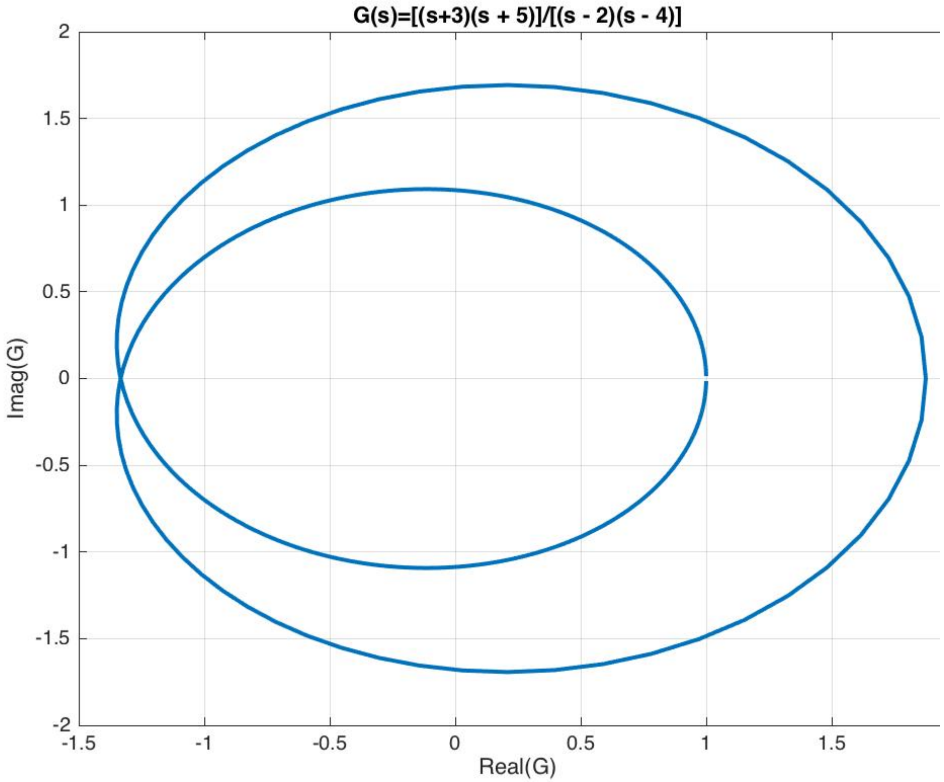
Z = 2 – 2 = 0 Stable (no RHP c-l poles)

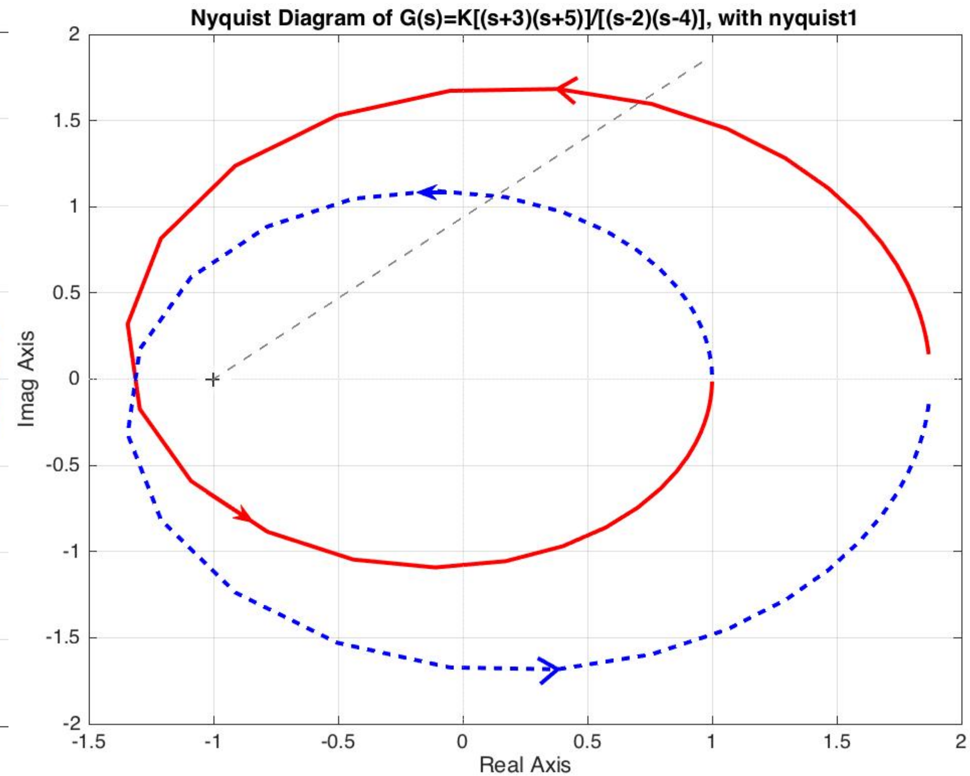
Stability range: $K > 1/1.33$ ➔ $K > 0.75$

Figure 10.30-3
The same example:
a. system;
b. Nyquist diagram via *my code*
c. Nyquist diagram via *Nyquist*1



(a)

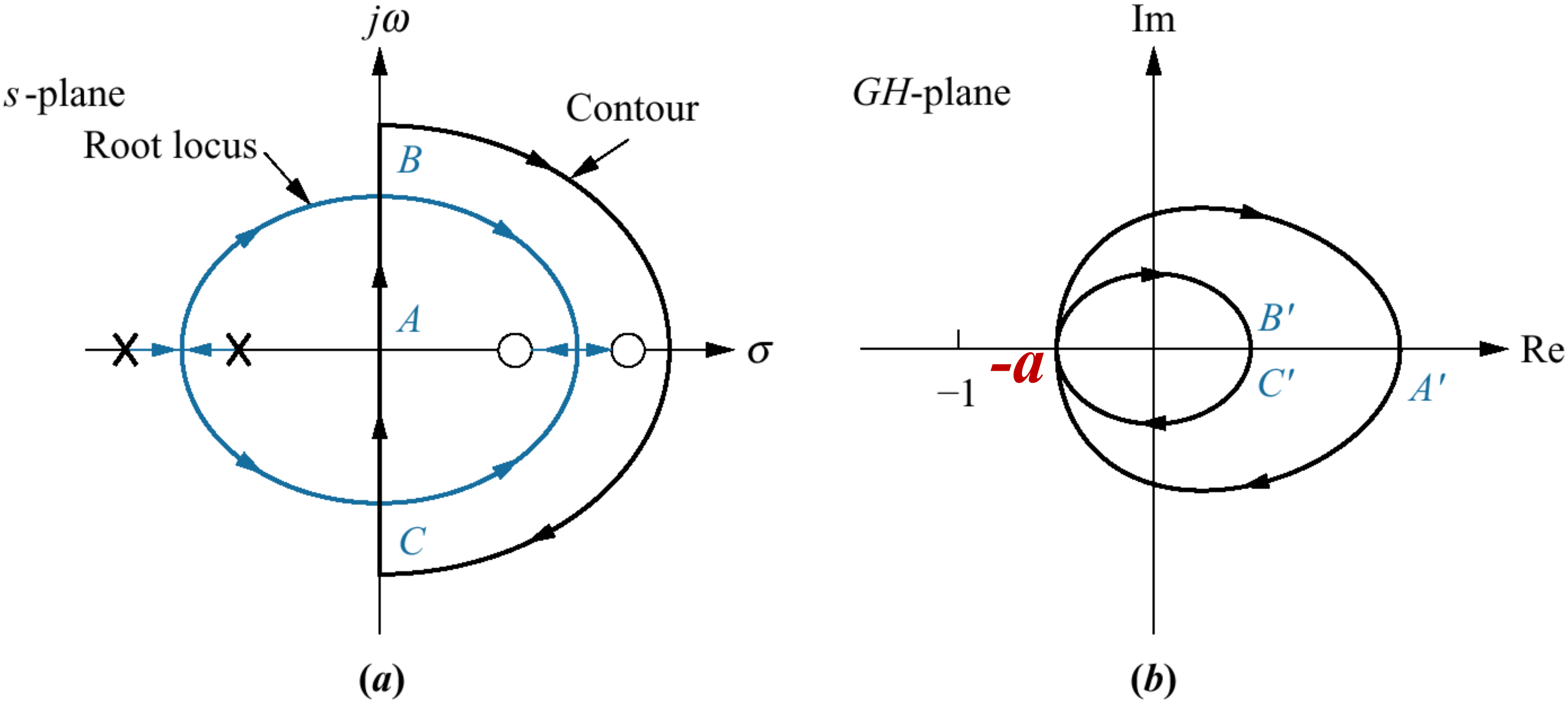




Stability Analysis for $K = 1$: Z = P – N;
P = 2 (2 RHP o-l poles), N = 2 (2 encirclements of –1) ➔
Z = 2 – 2 = 0 **Stable** (no RHP c-l poles)
Stability range: $K > 1/1.33$ ➔ $K > 0.75$

Figure 10.32

a. Contour and root locus of system that is **stable for small** gain and **unstable for large gain**; **b.** Nyquist diagram

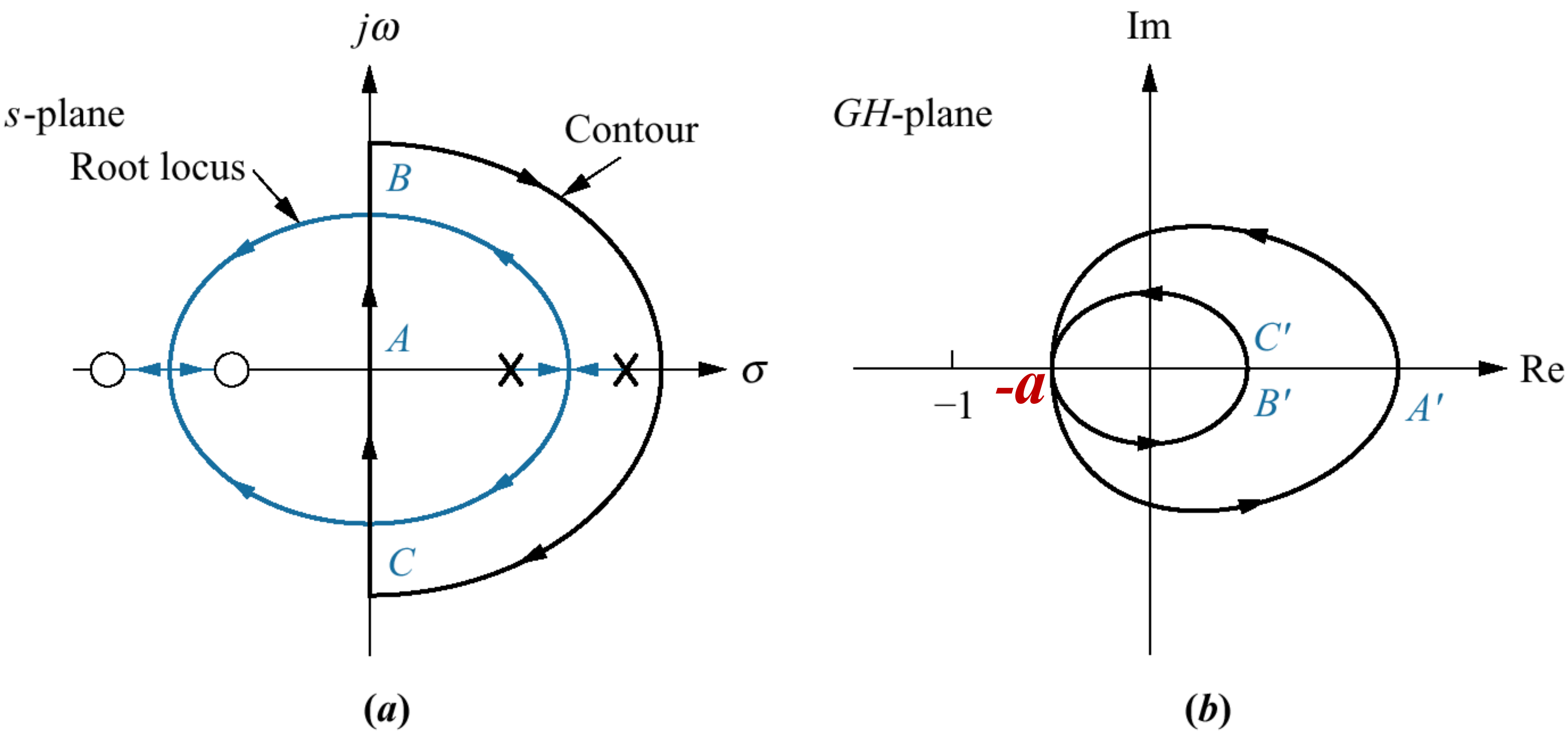


Stability range: $0 < K < 1/|a|$

Considering the Nyquist diagram is plotted for K=1.

Figure 10.33

a. Contour and root locus of system that is **unstable for small** gain and **stable for large gain**; **b.** Nyquist diagram



Stability range: $K > 1/|a|$

Considering the Nyquist diagram is plotted for K=1.

Figure 10.31-1
a. Contour for Example 10.6, where $G(s)=K/[s(s+3)(s+5)]$
b. Nyquist diagram

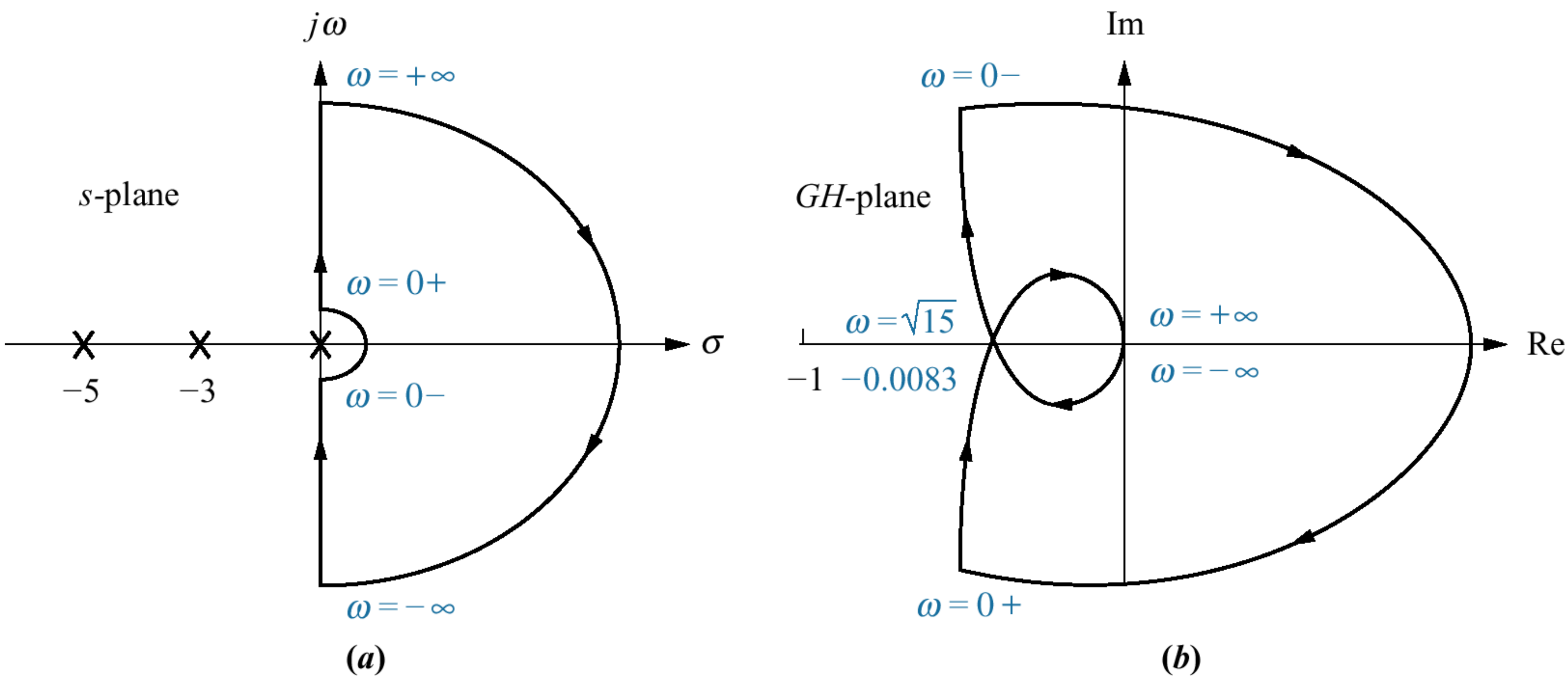
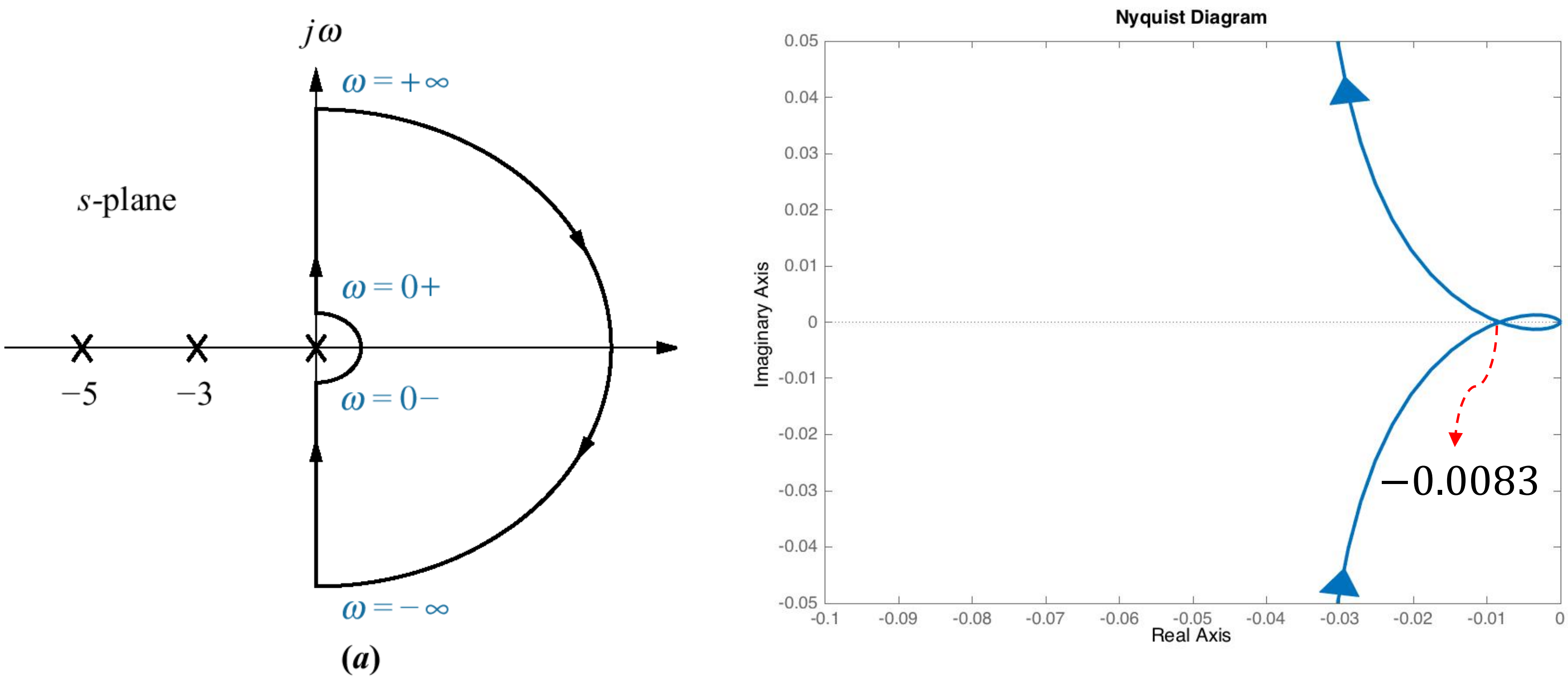


Figure 10.31-2

a. Contour for Example 10.6, where $G(s)=K/[s(s+3)(s+5)]$;

b. Nyquist diagram (via Matlab with close look)



Stability Analysis:

Real-axis crossing occurs for $\omega = \sqrt{15}$ @ $\sigma = -0.0083$

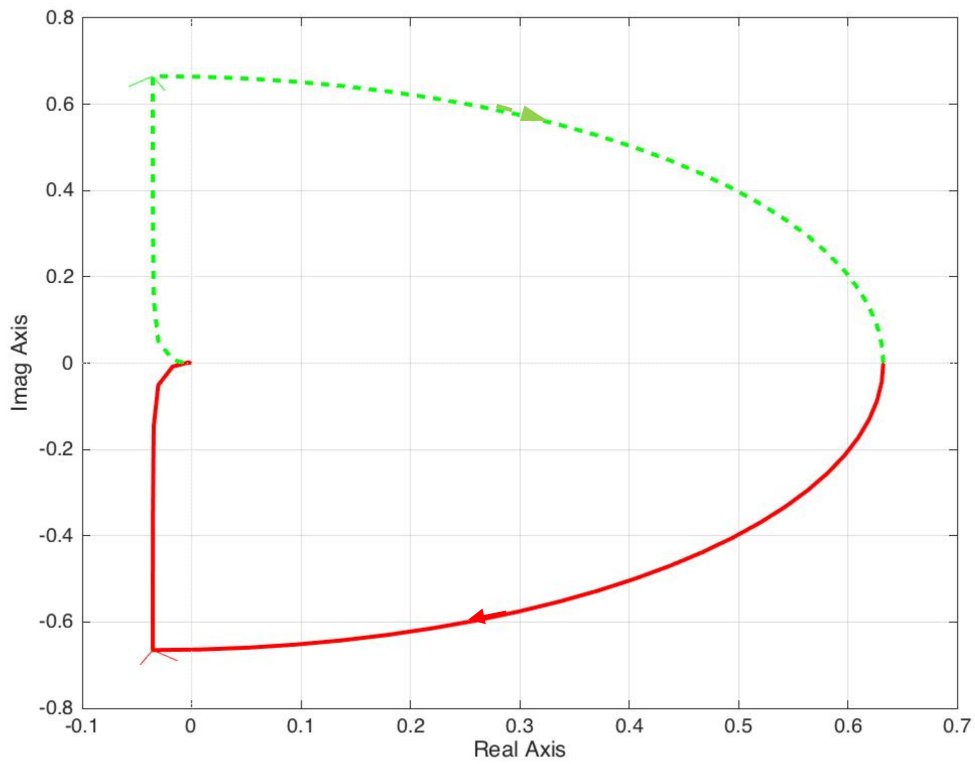
Stability range: $K < 1/0.0083 \rightarrow 0 < K < 120$

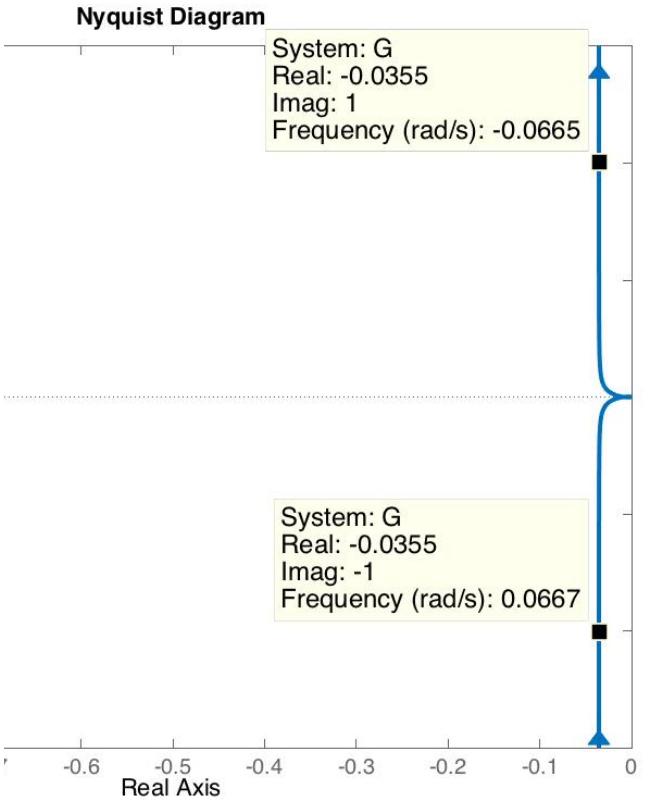
Figure 10.31-2

a. Nyquist diagram for Example 10.6 – Matlab, special Nyquist1;

b. Nyquist diagram for Example 10.6 – Matlab, standard Nyquist (not zoomed);

$G(s)=K/[s(s+3)(s+5)]$





Stability Design via Mapping Positive $j\omega$-Axis

PROBLEM: Find the range of gain for stability and instability, and the gain for marginal stability, for the unity feedback system shown in Figure 10.10, where $G(s) = K/[(s^2+2s+2)(s+2)]$. For marginal stability find the radian frequency of oscillation. Use the Nyquist criterion and the mapping of only the positive imaginary axis.

Solution: Since the open-loop poles are only in the left–half-plane, the Nyquist criterion tells us that we want no encirclements of -1 for stability. **Hence, a gain less than unity at plus and minus180 deg. is required**. Begin by letting $K = 1$ and draw the portion of the contour along the positive imaginary axis as shown in Figure 10.34(a).

$$G(j\omega)H(j\omega) = \left.\frac{1}{(s^2+2s+2)(s+2)}\right|_{s\to j\omega}$$

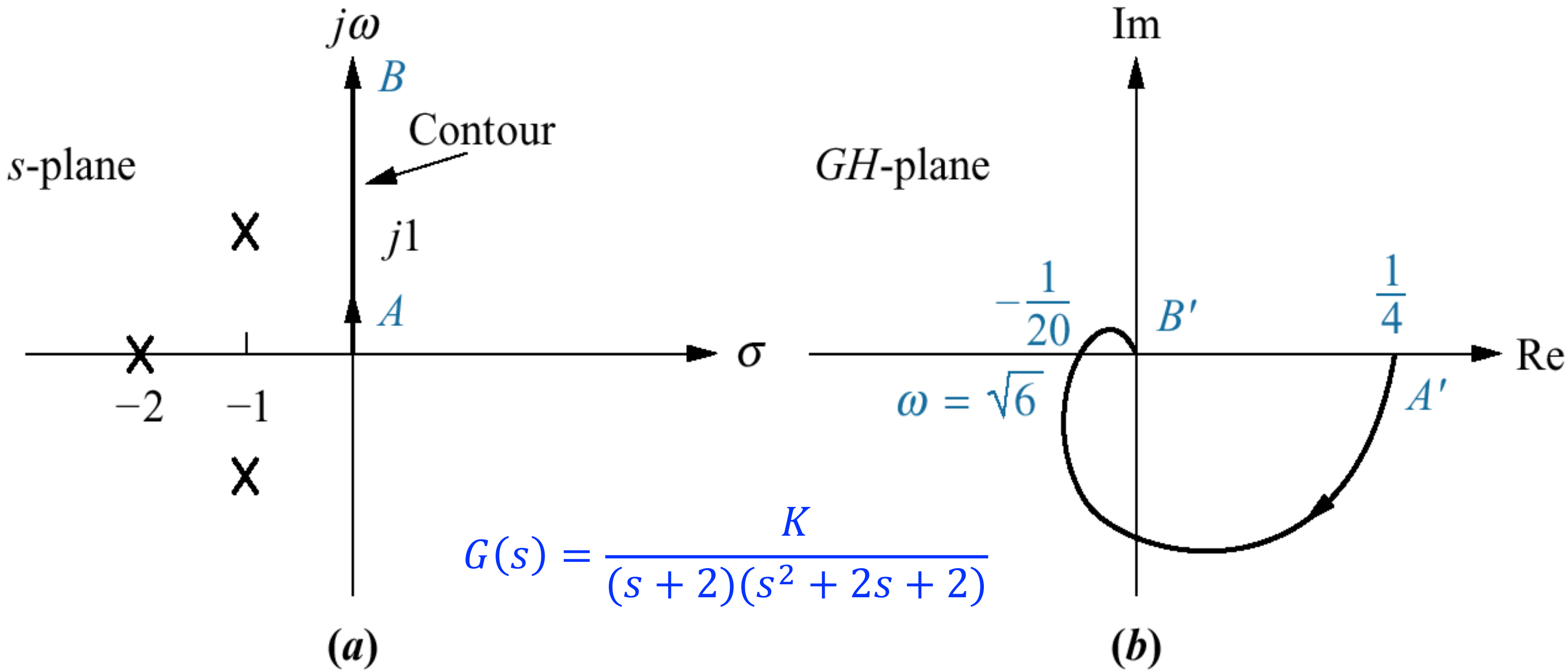
$$= \frac{4(1-\omega^2) - j\omega(6-\omega^2)}{16(1-\omega^2)^2 + \omega^2(6-\omega^2)^2}$$

- $G(j\omega) = \frac{1}{4} = 0.25\angle 0°$ for $\omega = 0$ rad/s
- Imag.-axis crossings occur for $\omega = \pm 1$ at $\mp j0.2$
- Real-axis crossings occur for $\omega = \pm\sqrt{6}$ at -0.05 $(1/20)$

Figure 10.34

a. Portion of contour to be mapped for Example10.7

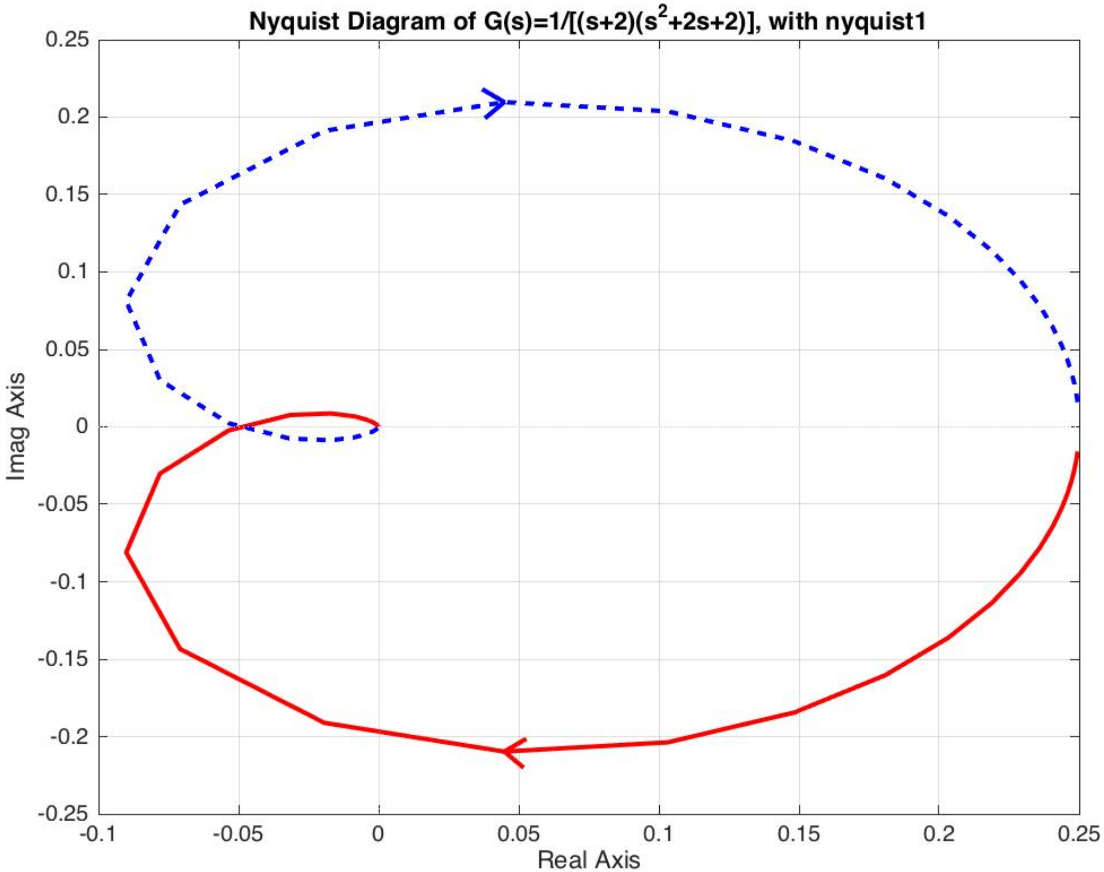
b. Nyquist diagram of mapping of positive imaginary axis

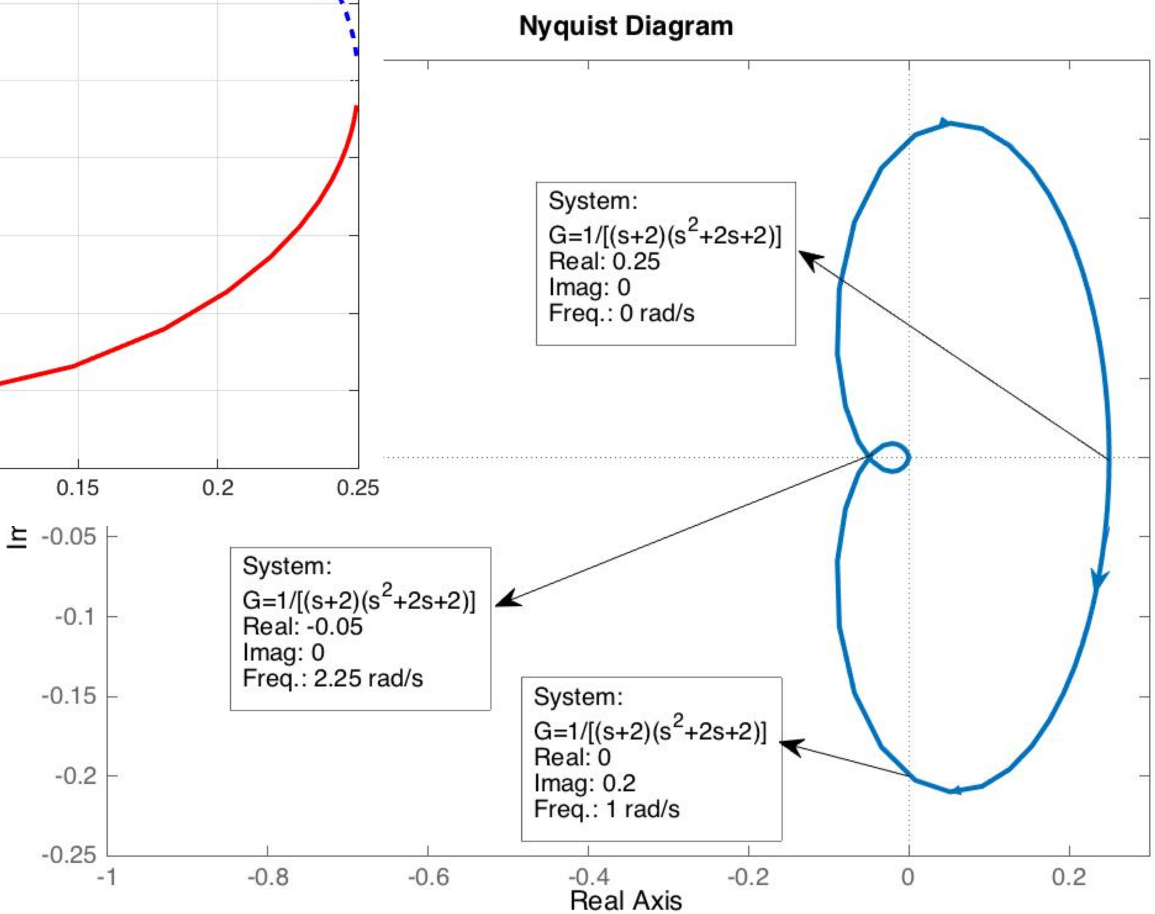


$$G(s) = \frac{K}{(s+2)(s^2+2s+2)}$$

- This closed-loop system is stable if the magnitude of the frequency response is less than unity at 180°.
- **Stability Region for Gain:** $0 < K < 20$
- The frequency of oscillation when the system is marginally stable is $\omega = \sqrt{6} = 2.45$ rad/s

Nyquist plot of $G(s) = \frac{K}{(s+2)(s^2+2s+2)}$ in Matlab





Nyquist diagram for Gain and Phase Margins **(Figure 10.35)**

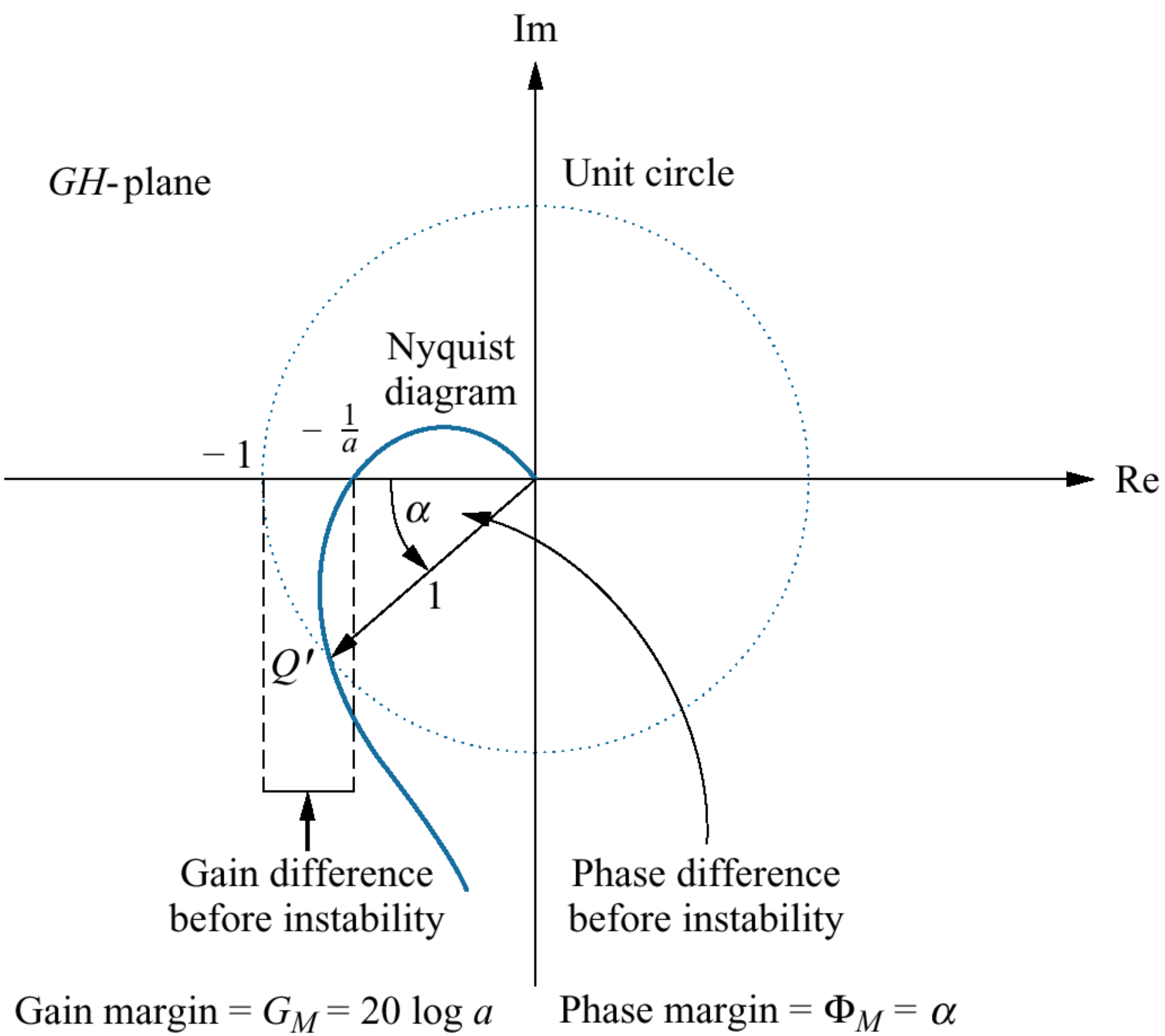
Gain margin, GM:
The gain margin is the change in open-loop gain, expressed in decibels (dB), required at 180° of phase shift to make the closed-loop system unstable.

$$\mathbf{GM = 20.\log \mathit{a}}$$

Phase margin, PM:
The phase margin is the change in open-loop phase shift required at unity gain to make the closed-loop system unstable.

PM = α

(PM and ζ are related)



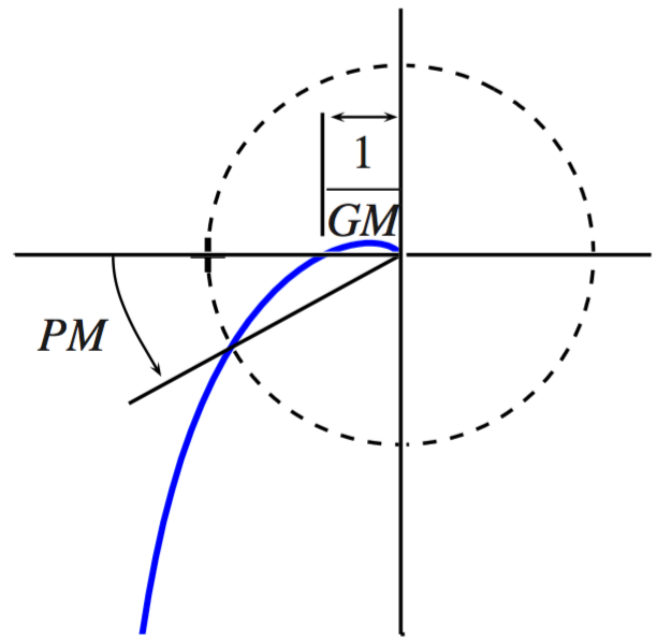
GAIN MARGIN: Factor by which the gain is less than the neutral stability value.

- Gain margin measures "How much can we increase the gain of the loop transfer function $L(s) = D(s)G(s)H(s)$ and still have a stable system?"

- Many Nyquist plots are like this one. Increasing loop gain magnifies the plot.

- $GM = 1/($distance between origin and place where Nyquist map crosses real axis).

- If we increase gain, Nyquist map "stretches" and we may encircle -1.



$$GM(\text{dB}) = 20 \log GM$$

- For a stable system, $GM > 1$ (linear units) or $GM > 0$ dB.

PHASE MARGIN: Phase factor by which phase is greater than neutral stability value.

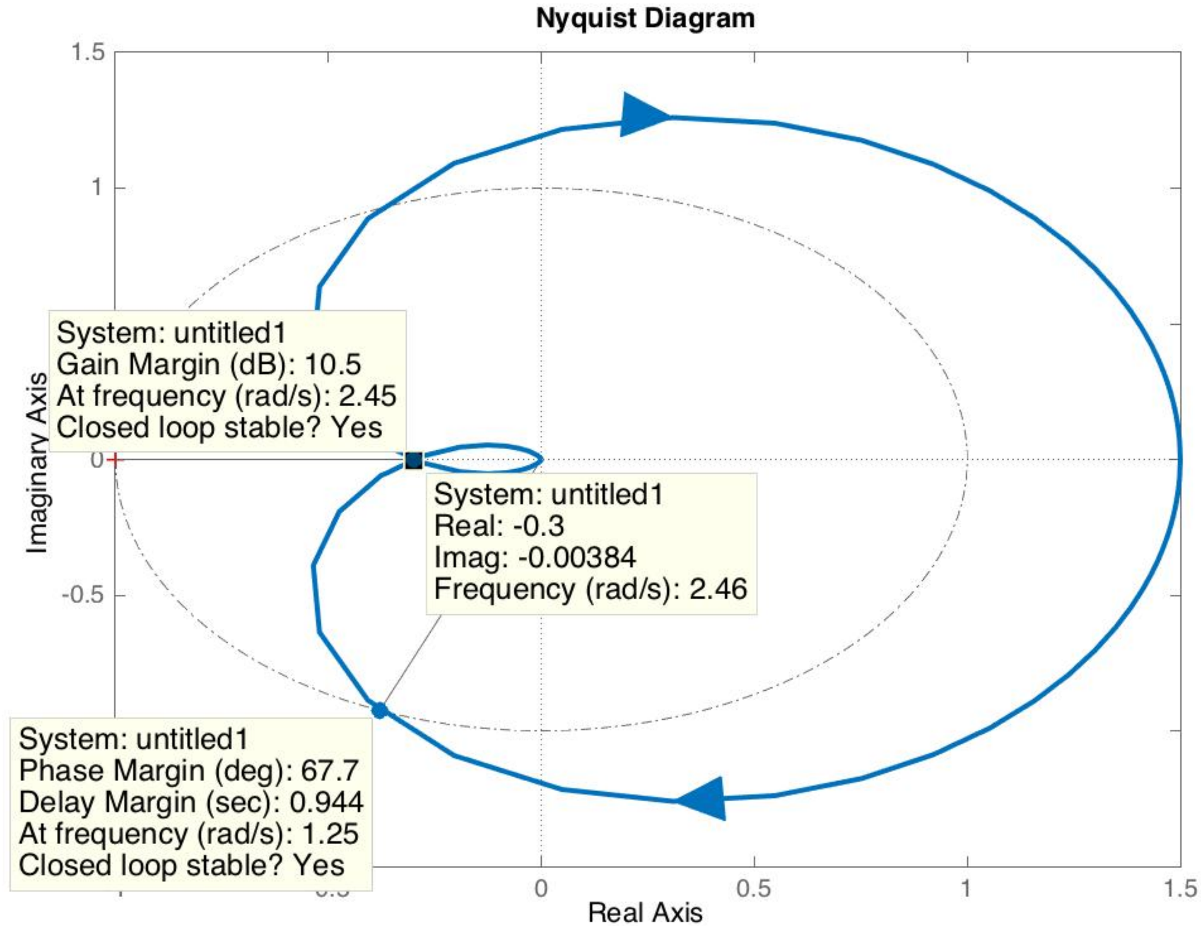
- Phase margin measures "How much delay can we add to the loop transfer function and still have a stable system?"
- $PM =$ Angle to rotate Nyquist plot to achieve neutral stability = intersection of Nyquist with circle of radius 1.
- If we increase open-loop delay, Nyquist map "rotates" and we may encircle -1.
- For a stable system, $PM > 0°$.

IRONY: This is usually easiest to check on Bode plot, even though derived on Nyquist plot!

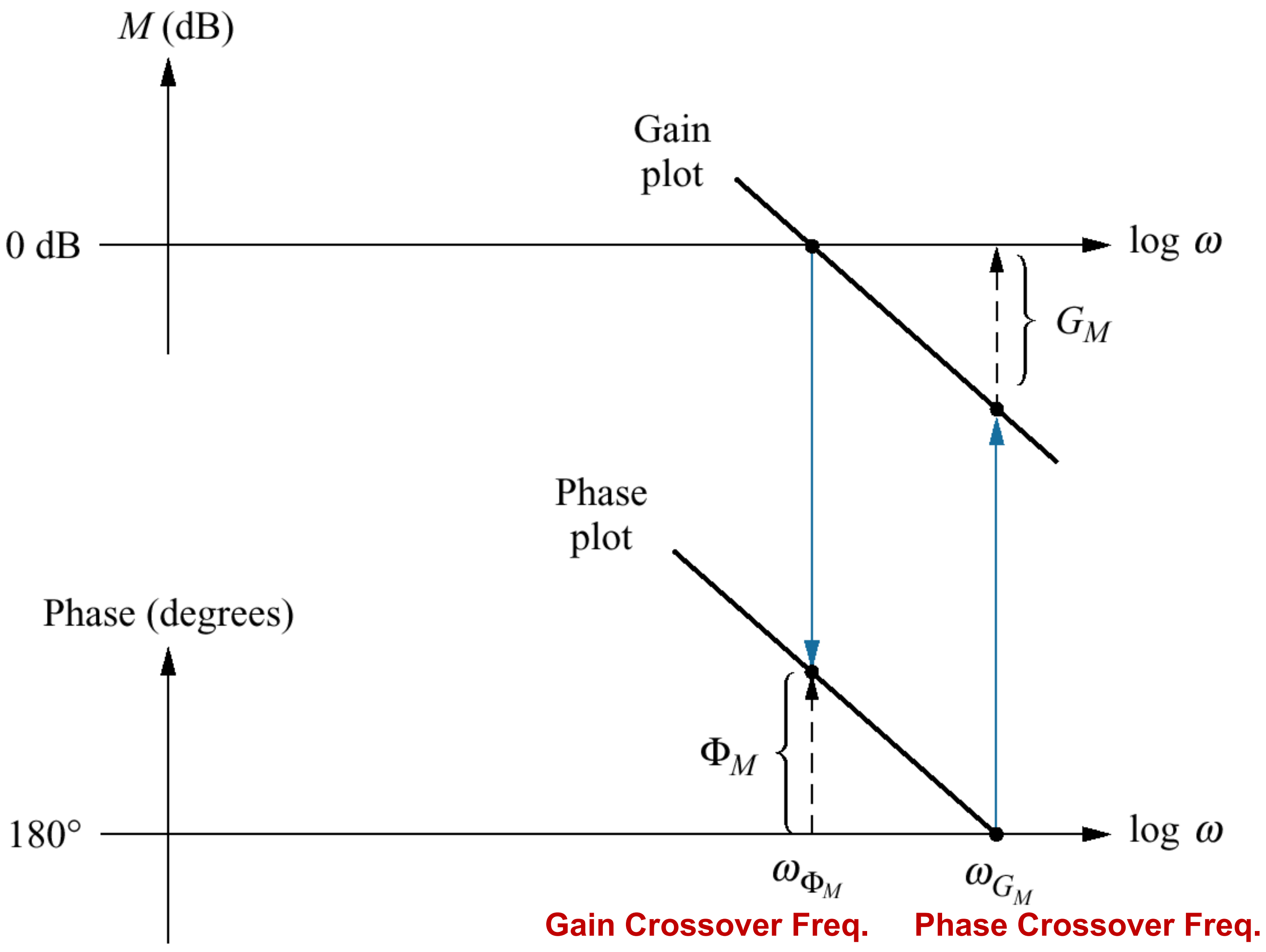
Finding GM & PM for $G(s) = \frac{6}{(s+2)(s^2+2s+2)}$

- The Stability Region was found as $0 < K < 20$.
- Hence, the system is stable for $K = 6$.
- The critical point for stability is $\left|\frac{1}{a}\right| = \left|\frac{20}{6}\right| = 3.33$.
- This is called GM in linear units.
- It is $20\log(3.33) = \mathbf{10.45dB}$.
- How about the PM?
- $|G(j\omega)H(j\omega)| = 1 \rightarrow \omega = 1.253$.
- For $\omega = 1.253$,

  $G(j\omega) = -0.378 - j0.925 = 1\measuredangle - 112.33°$.
- Therefore, $\mathbf{PM} = -112.33° - (-180°) = \mathbf{67.7}°$.

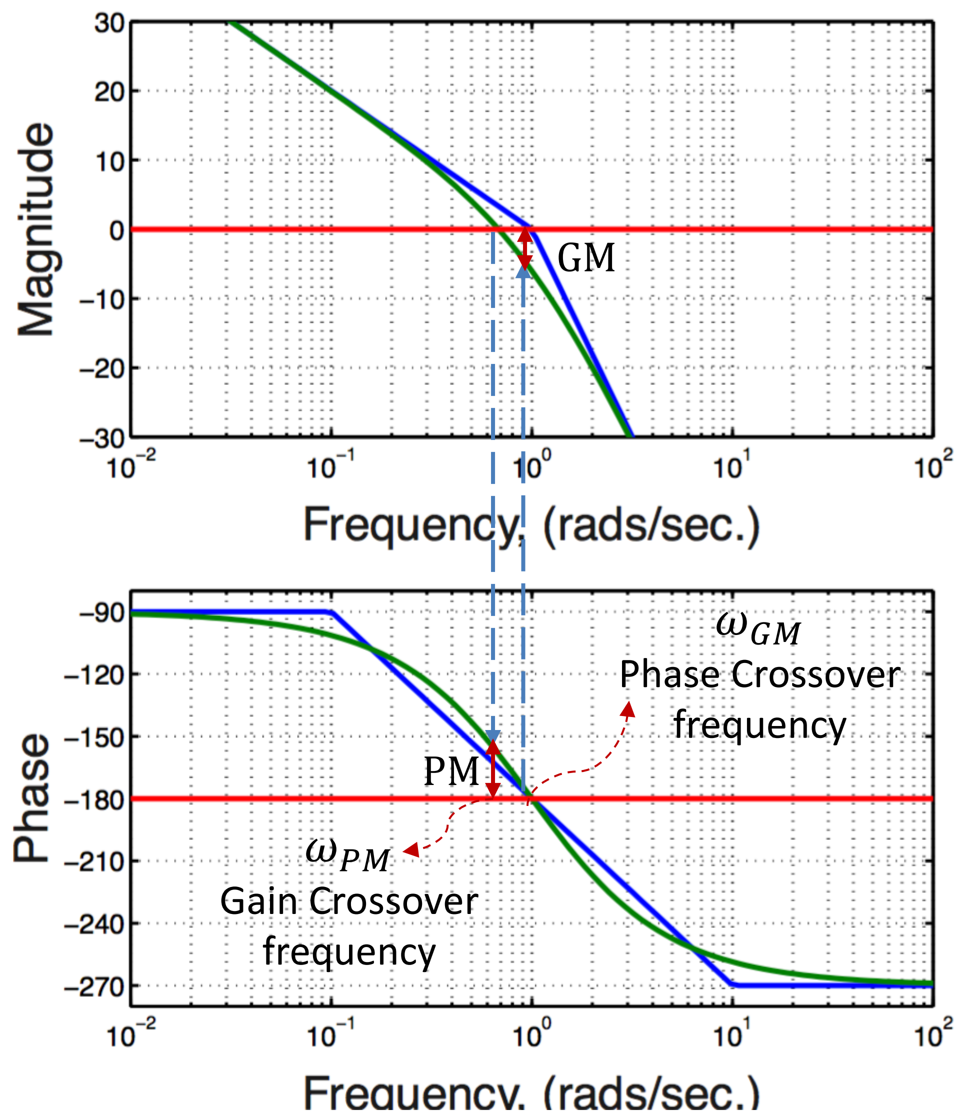


Bode diagrams for Gain and Phase Margins (Fig. 10.37)



- Define gain crossover as frequency where Bode magnitude is 0 dB.
- Define phase crossover as frequency where Bode phase is $-180°$.

- $GM = 1/($Bode gain at phase-crossover frequency) if Bode gain is measured in linear units.
- $GM = (-$ Bode gain at phase-crossover frequency) [dB] if Bode gain measured in dB.
- $PM =$ Bode phase at gain-crossover $-(-180°)$.



Example 10.9

$$G(s) = \frac{K}{(s+2)(s+4)(s+5)}$$

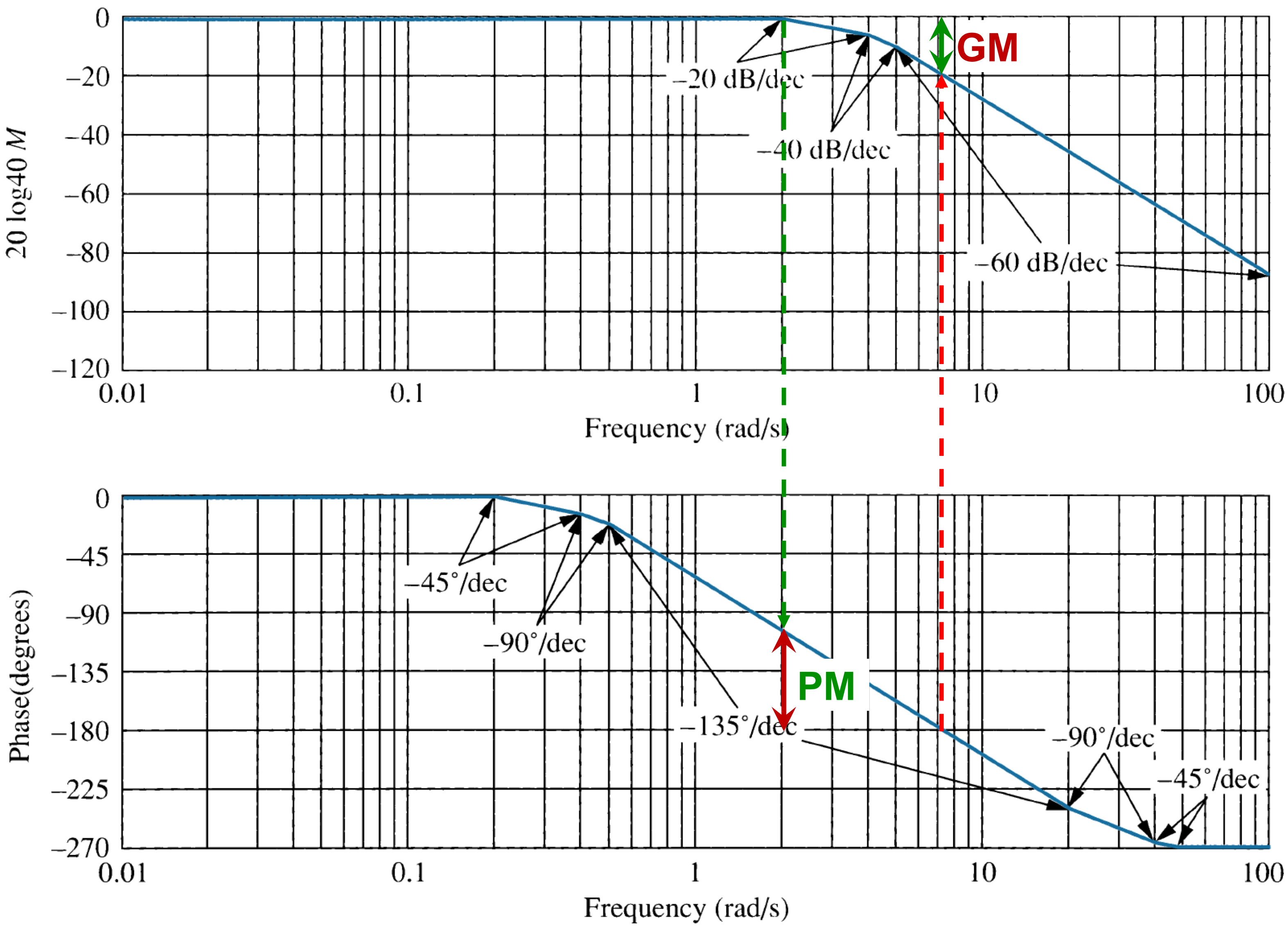
In terms of DC frequencies, $G(s) = \frac{K}{40} \frac{1}{(s/2+1)(s/4+1)(s/5+1)}$.

- Let's plot Bode diagrams for $K = 40$. As seen from the next slide,
- $\mathbf{GM} = 0 - (-20\text{dB}) = 20\text{dB}$ and $\mathbf{PM} = -100 - (-180) = 80°$.
- Since GM ≈ 20 dB, $20 \log x = 20$, $x = 10$. This means we have 10 times 40 before the instability.
- More accurately, $\text{GM} = 19.5\text{dB}$ corresponding to $x = 9.45$.
- Then the stability range will be $0 < K < 40 \cdot 9.45$ ➔ $\mathbf{0 < K < 378}$

Now, let's take *K* = 200 and find GM and PM. $G(s) = \frac{200}{(s+2)(s+4)(s+5)}$

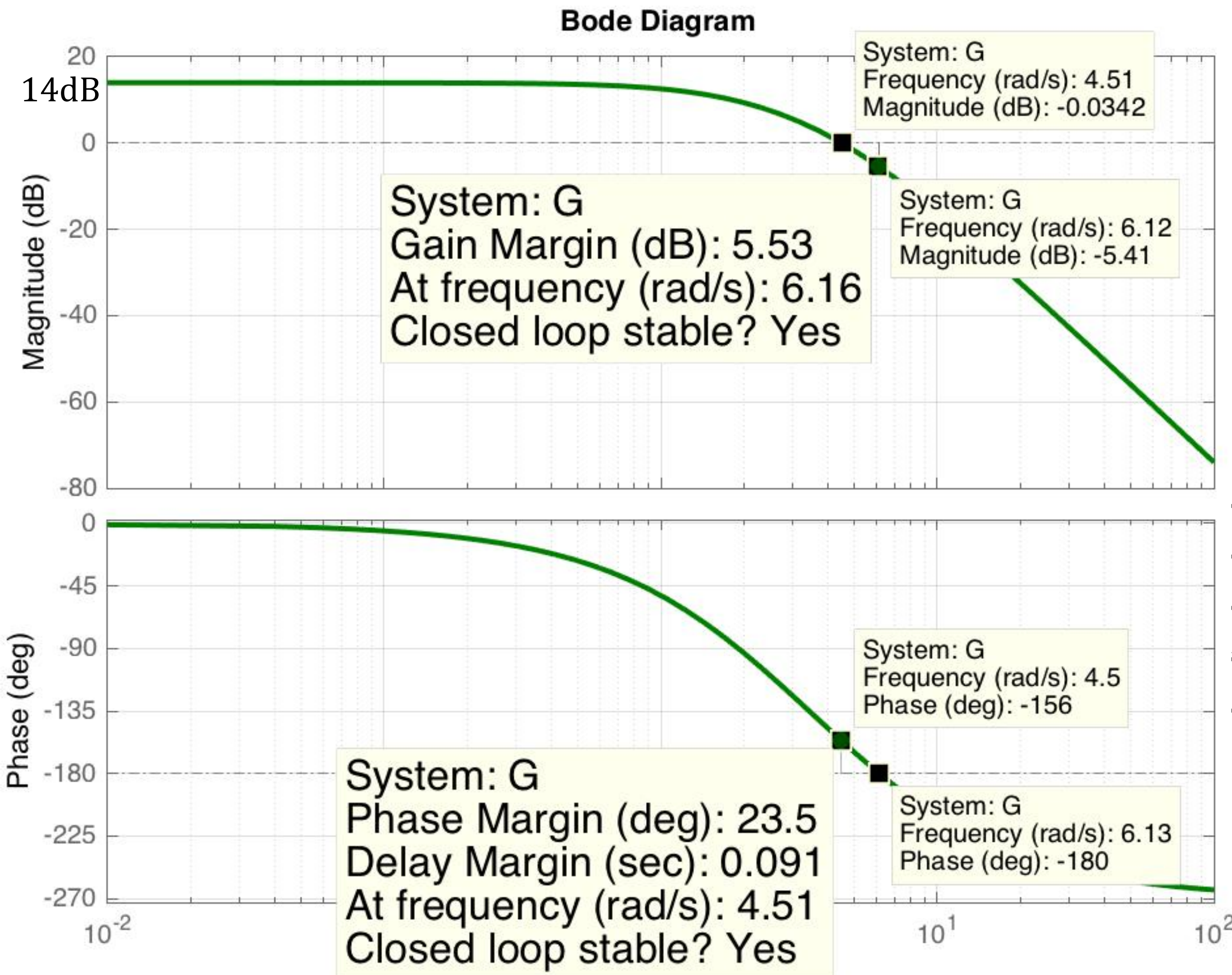
- As seen from the Bode plots, for $K = 200$, $\mathbf{GM} = \mathbf{5.53dB}$ and $\mathbf{PM} = \mathbf{23.5^o}$ are measured.
- Since $\text{GM} = 5.53\text{dB}$, $20 \log x = 5.53$ and $x = 10^{\frac{5.53}{20}} = 1.89$. Therefore the stability range will be $0 < K < 200 \cdot 1.89$ ➔ $\mathbf{0 < K < 378}$

Fig. 10.36 Bode magnitude and phase plots for the sys in Ex.10.9 (the Bode Mag. Plot is drawn for $K = 40$)



$\mathbf{PM} = -100° - (-180°) \approx 80°$ $\qquad$ $\mathbf{GM} = 0 - (-19.5\text{dB}) \approx 20\text{dB}$

Bode plots for $G(s) = \frac{K}{(s+2)(s+4)(s+5)}$, $K = 200$



<u>In Matlab Bode plots using Characteristics → All Stability Margins:</u>

For K = 200, GM=5.53 dB and PM=23.5° found.

$$\text{PM} = -156° - (-180°) = 24° \quad \text{GM} = 0 - (-5.41\text{dB}) = 5.41\text{dB}$$