KOM3712 Control Systems Design

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Design via State Space Methods – 1 of 4: *Canonical Forms & Pole Placement*

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Textbooks followed mostly for the Design in the State Space are,

- **Modern Control Engineering** (5th ed.), Katsuhiko Ogata, Chap. 9
- **Control Systems Engineering** (7th ed.), Norman S. Nise, Chap. 12

Introduction

- A modern complex system may have many inputs and many outputs, and these may be interrelated in a complicated manner.
- To analyze such a system, it is essential to reduce the complexity of the mathematical expressions, as well as to make use of computers for most of the tedious computations necessary in the analysis.
- The state-space approach to system analysis is best suited from this viewpoint.

State-Space Rep.'s of Transfer-Function Systems

Many techniques are available for obtaining state-space representations of systems with transfer-function.

State-Space Representation in Canonical Forms

- Consider a system defined by the following equations in two forms, namely in t- and s-domains, where u is the input and y is the output.

$$\boxed{\begin{aligned} y^{(n)} + a_1 y^{(n-1)} + \ldots + a_{n-1}\dot{y} + a_n y = \\ b_0 u^{(n)} + b_1 u^{(n-1)} + \ldots + b_{n-1}\dot{u} + b_n u \end{aligned}}$$

- Taking the Laplace transform of both sides of the equation above for zero initial conditions gives the following transfer function,

$$\boxed{\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n}}$$

Controllable Canonical Form

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n}$$

The following state-space representation is called a **controllable canonical form**:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \left[b_n - a_n b_0 \;\vdots\; b_{n-1} - a_{n-1} b_0 \;\vdots\; \cdots \;\vdots\; b_1 - a_1 b_0 \right] \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + b_0 u$$

- The controllable canonical form is important in the **pole-placement** approach to control systems design.
- The poles can be placed precisely using just one (either bottom or top) row of $\mathbf{A}$ without changing any other coefficients.

Alternative Representation of the Controllable Canonical Form

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n}$$

For $n=3$

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^3 + b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3}$$

- There are many (infinitely many) possible state-space representations for this system. One possible representation is CCF, where $b_0 = 0$ is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u; \; y = \begin{bmatrix} b_3 & b_2 & b_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0]u$$

- Alternative for CCF representation of the system is as follows,

CCF for

$\hat{x}_1 = x_3$

$\hat{x}_2 = x_2$

$\hat{x}_3 = x_1$

$$\begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \\ \dot{\hat{x}}_3 \end{bmatrix} = \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} + [0]u$$

Observable Canonical Form:

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n}$$

This state-space representation is called an **observable canonical form**:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + \begin{bmatrix} b_n - a_n b_0 \\ b_{n-1} - a_{n-1} b_0 \\ \cdot \\ \cdot \\ \cdot \\ b_1 - a_1 b_0 \end{bmatrix} u$$

- Note that the state matrix **A** of the state equations given in OCF is the transpose of that of the state matrix of CCF.
- **B** matrix of OCF is the transpose of **C** of CCF,
- **C** matrix of OCF is the transpose of **B** of CCF.

$$y = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix} + b_0 u$$

$$\mathbf{A}_{OCF} = \mathbf{A}_{CCF}^T, \; \mathbf{B}_{OCF} = \mathbf{C}_{CCF}^T, \mathbf{C}_{OCF} = \mathbf{B}_{CCF}^T$$

Alternative Representation of the Observable Canonical Form

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n}$$

For $n=3$

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^3 + b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3}$$

- There are many (infinitely many) possible state-space representations for this system. One possible representation is OCF, where $b_0 = 0$ is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} b_3 \\ b_2 \\ b_1 \end{bmatrix} u; \; y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0]u$$

- Alternative for OCF representation of the system is as follows,

OCF for

$\hat{x}_1 = x_3$

$\hat{x}_2 = x_2$

$\hat{x}_3 = x_1$

$$\begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \\ \dot{\hat{x}}_3 \end{bmatrix} = \begin{bmatrix} -a_1 & 1 & 0 \\ -a_2 & 0 & 1 \\ -a_3 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} + [0]u$$

Diagonal Canonical Form: Here we consider the case where the denominator polynomial involves *only distinct* roots.

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n}$$

$$= b_0 + \frac{c_1}{s + p_1} + \frac{c_2}{s + p_2} + \cdots + \frac{c_n}{s + p_n}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -p_1 & & & & & 0 \\ & -p_2 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ 0 & & & & & -p_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + b_0 u$$

Jordan Canonical Form: Now we consider the case where the denominator polynomial involves *multiple roots* as seen below.

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{(s+p_1)^3(s+p_4)(s+p_5)\cdots(s+p_n)}$$

The partial-fraction expansion of this last equation becomes,

$$\frac{Y(s)}{U(s)} = b_0 + \frac{c_1}{(s+p_1)^3} + \frac{c_2}{(s+p_1)^2} + \frac{c_3}{s+p_1} + \frac{c_4}{s+p_4} + \cdots + \frac{c_n}{s+p_n}$$

A state-space representation of this system in the **Jordan canonical form**:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_n \end{bmatrix} = \left[\begin{array}{ccc:cccc} -p_1 & 1 & 0 & 0 & \cdots & & 0 \\ 0 & -p_1 & 1 & \vdots & & & \vdots \\ 0 & 0 & -p_1 & 0 & \cdots & & 0 \\ \hdashline 0 & \cdots & 0 & -p_4 & & & 0 \\ \cdot & & \cdot & & \cdot & & \\ \cdot & & \cdot & & & \cdot & \\ \cdot & & \cdot & & & & \cdot \\ 0 & \cdots & 0 & 0 & & & -p_n \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + b_0 u$$

Example-1: Consider the system given by

$$\frac{Y(s)}{U(s)} = \frac{s+3}{s^2+3s+2}$$

Obtain state-space representations in the controllable canonical form, observable canonical form, and diagonal canonical form.

Controllable Canonical Form:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Observable Canonical Form:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Diagonal Canonical Form:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 2 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Example-2: State-Space Formulation of Transfer-Function Systems in MATLAB - Consider the transfer-function system,

$$\frac{Y(s)}{U(s)} = \frac{s+10}{s^3+6s^2+5s+10}$$

- There are infinitely many possible state-space representations for this system.
- One possible representation is

$$\left.\begin{aligned}
\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} &= \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -10 & -5 & -6\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 1\end{bmatrix}u\\
y &= \begin{bmatrix}10 & 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + [0]u
\end{aligned}\right\}\ \textbf{CCF}$$

- Another representation in CCF is

$$\left.\begin{aligned}
\begin{bmatrix}\dot{\hat{x}}_1\\ \dot{\hat{x}}_2\\ \dot{\hat{x}}_3\end{bmatrix} &= \begin{bmatrix}-6 & -5 & -10\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}\hat{x}_1\\ \hat{x}_2\\ \hat{x}_3\end{bmatrix} + \begin{bmatrix}1\\ 0\\ 0\end{bmatrix}u\\
y &= \begin{bmatrix}0 & 1 & 10\end{bmatrix}\begin{bmatrix}\hat{x}_1\\ \hat{x}_2\\ \hat{x}_3\end{bmatrix} + [0]u
\end{aligned}\right\}$$

CCF for
$\hat{x}_1 = x_3$
$\hat{x}_2 = x_2$
$\hat{x}_3 = x_1$

MATLAB Program 9–1

```
num = [1    10];
den = [1  6  5  10];
[A,B,C,D] = tf2ss(num,den)

A =

     -6    -5   -10
      1     0     0
      0     1     0

B =

      1
      0
      0

C =

      0    1    10

D =

      0
```

Example-3 Obtain the transfer function of the system defined by the following state-space equations:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -5 & -25 & -8\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}u$$

$$y=\begin{bmatrix}87 & 12 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}; D=0 \rightarrow \qquad \frac{Y(s)}{U(s)}=\frac{12s+87}{s^3+8s^2+25s+5}$$

Example-3 – Matlab Program

```
>> A=[0 1 0; 0 0 1; -5 -25 -8], B=[0; 0; 1], C=[87 12 0], D=0
%or  [num, den] = ss2tf(A,B,C,D,1)
>> [num, den] = ss2tf(A,B,C,D)

num =     0      0     12    87
den =    1.0000    8.0000   25.0000    5.0000

>> G=tf(num,den)
G =
            12 s + 87
      --------------------------
      s^3 + 8 s^2 + 25 s + 5
Continuous-time transfer function.
```

```
>> G=zpk(tf(num,den))
G =
                 12 (s+7.25)
      ----------------------------------------
      (s+0.2143) (s^2 + 7.786s + 23.33)
Continuous-time zero/pole/gain model.
```

Example-4 - Obtain the transfer function of the system defined by the following state-space equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5.008 & -25.1026 & -5.03247 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 25.04 \\ -121.005 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \rightarrow \quad \frac{Y(s)}{U(s)} = \frac{25.04s+5.008}{s^3+5.0325s^2+25.1026s+5.008}$$

MATLAB Program 9–2

```
A = [0  1  0;0  0  1;-5.008  -25.1026  -5.03247];
B = [0;25.04; -121.005];
C = [1  0  0];
D = [0];
[num,den] = ss2tf(A,B,C,D)

num =

        0   -0.0000   25.0400   5.0080

den =

    1.0000   5.0325   25.1026   5.0080
```

$$\boxed{T(s) = \frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}}$$

From Ogata's Book

Example-5 Obtain the transfer function of the system defined by the following state-space equations:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -25 & -5 & -120\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 12\\ -45\end{bmatrix}u$$

$$y = \begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} \rightarrow \frac{Y(s)}{U(s)} = \frac{12s+1395}{s^3+120s^2+5s+25} = \frac{12\,(s+116.25)}{(s+120)(s^2+0.04s+0.21}$$

Example-5 – Matlab Program

```
>> A=[0 1 0; 0 0 1; -25 -5 -120]; B=[0; 12; -45]; C=[1 0 0]; D=0;
>> [num, den] = ss2tf(A,B,C,D)
num =  1.0e+03 *

        0         0    0.0120    1.3950   (0 0 12 1395)

den =
        1.0000  120.0000    5.0000   25.0000
```

$$T(s) = \frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D}$$

```
>> G=tf(num,den)

G =
           12 s + 1395
      --------------------------
      s^3 + 120 s^2 + 5 s + 25
Continuous-time transfer function.
```

Method-2: $\mathrm{G} = \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D}$

```
>> p=sym('p'); I=eye(3); F=p*I-A    % p → s
>> G=C*F^-1*B
G =(12*(p + 120))/(p^3 + 120*p^2 + 5*p + 25) - 45/(p^3 + 120*p^2 + 5*p + 25)
>> G=zpk(tf(num,den))
                  12 (s+116.3)
  -----------------------------------------
       (s+120) (s^2 + 0.03994s + 0.2084)
Continuous-time zero/pole/gain model.
```

Similarity Transformations

- As stated, there are infinitely many possible state-space representations for the system,

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \; ; \mathrm{y} = \mathbf{C}\mathbf{x}$$

- Let us choose an arbitrary invertible $n \times n$ matrix $\mathbf{T}$, and define a new state vector:

$$\mathbf{z} = \mathbf{T}\mathbf{x} \;\rightarrow\; \mathbf{x} = \mathbf{T}^{-1}\mathbf{z} \;\rightarrow\; \dot{\mathbf{x}} = \mathbf{T}^{-1}\dot{\mathbf{z}}$$

- Substitute into the state equation,

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \rightarrow \dot{\mathbf{x}} = \mathbf{T}^{-1}\dot{\mathbf{z}} = \mathbf{A}\mathbf{T}^{-1}\mathbf{z} + \mathbf{B}u$$

- Pre-multiply the whole equation by $\mathbf{T}$,

$$\mathbf{T}\mathbf{T}^{-1}\dot{\mathbf{z}} = \mathbf{T}\mathbf{A}\mathbf{T}^{-1}\mathbf{z} + \mathbf{T}\mathbf{B}u \;\rightarrow$$

$$\dot{\mathbf{z}} = \widehat{\mathbf{A}}\mathbf{z} + \widehat{\mathbf{B}}u, \text{ where } \widehat{\mathbf{A}} = \mathbf{T}\mathbf{A}\mathbf{T}^{-1}, \widehat{\mathbf{B}} = \mathbf{T}\mathbf{B}$$

- The output equation will be, $\mathrm{y} = \mathbf{C}\mathbf{x} = \mathbf{C}\mathbf{T}^{-1}\mathbf{z} \rightarrow$

$$y = \hat{\mathbf{C}}\mathbf{z}, \text{ where } \hat{\mathbf{C}} = \mathbf{C}\mathbf{T}^{-1}$$

- It can be shown that the new state space model has the same eigen values,

$$\dot{\mathbf{z}} = \widehat{\mathbf{A}}\mathbf{z} + \widehat{\mathbf{B}}u; \, y = \hat{\mathbf{C}}\mathbf{z}$$

Similarity Transformations, *cont.'s*…

- The transfer functions corresponding to the original,

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{B}u \,; \mathrm{y} = \mathbf{Cx} \Rightarrow G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$$

and the new model,

$$\dot{\mathbf{z}} = \widehat{\mathbf{A}}\mathbf{z} + \widehat{\mathbf{B}}u;\, y = \widehat{\mathbf{C}}\mathbf{z}, \Rightarrow \hat{G}(s)$$

- will be the same. Now, let's check them out,

$$\begin{aligned}\hat{G}(s) = \widehat{\mathbf{C}}\left(s\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1}\widehat{\mathbf{B}} &= \mathbf{CT}^{-1}\left(s\mathbf{I} - \mathbf{TAT}^{-1}\right)^{-1}\mathbf{TB} \\ &= \mathbf{CT}^{-1}\left(s\mathbf{TT}^{-1} - \mathbf{TAT}^{-1}\right)^{-1}\mathbf{TB} \\ &= \mathbf{CT}^{-1}\mathbf{T}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{T}^{-1}\mathbf{TB} \\ &= \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} = G(s)\end{aligned}$$

- For this reason, the transformation $\mathbf{z} = \mathbf{Tx}$ is referred to as a **similarity transformation**.
- Since $\mathbf{T}$ can be any invertible matrix, and since there are an infinite number of invertible $n \times n$ matrices to choose from, there are an infinite number of realizations for any given transfer function $G(s)$.

Similarity Transformations, *cont.'s*…

- Since both sets $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ and $\left(\widehat{\mathbf{A}}, \widehat{\mathbf{B}}, \hat{\mathbf{C}}\right)$ produce the same transfer function $G(s)$,
- And since the poles of $G(s)$ are the eigenvalues of $\mathbf{A}$ and also $\widehat{\mathbf{A}}$, we see that the following relationship holds:

The eigenvalues of $\mathbf{A}$ are the same as the eigenvalues of $\widehat{\mathbf{A}} = \ \mathbf{TAT^{-1}}$for any $n \times n$ invertible matrix $\mathbf{T}$.

Example - Similarity Transformation

Given the system represented in state space by

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & -7 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathbf{x}$$

transform the system to a new set of state variables, **z**, where the new state variables are related to the original state variables, **x**, as follows:

$$z_1 = 2x_1$$

$$z_2 = 3x_1 + 2x_2$$

$$z_3 = x_1 + 4x_2 + 5x_3$$

Solution:

$$\mathbf{z} = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 2 & 0 \\ 1 & 4 & 5 \end{bmatrix} \mathbf{x} = \mathbf{Tx}$$

The new state space representation will be ,

Example - Similarity Transformation

- System with the original state variables: $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u\ ; y = \mathbf{C}\mathbf{x}$
- System with the new state variables: $\dot{\mathbf{z}} = \hat{\mathbf{A}}\mathbf{z} + \hat{\mathbf{B}}u; y = \hat{\mathbf{C}}\mathbf{z}$

  where, $\hat{\mathbf{A}} = \mathbf{T}\mathbf{A}\mathbf{T}^{-1}$, $\hat{\mathbf{B}} = \mathbf{T}\mathbf{B}$, $\hat{\mathbf{C}} = \mathbf{C}\mathbf{T}^{-1}$
- **Solution in MATLAB:**

```
>> A=[0 1 0; 0 0 1; -2 -5 -7]; B=[0;0;1]; C=[1 0 0]
>> T=[2 0 0; 3 2 0; 1 4 5]
>> A_hat=T*A*T^-1
>> B_hat=T*B
>> C_hat=C*T^-1
```

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & -7 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u$$

$$y = [1 \quad 0 \quad 0]\mathbf{x}$$

$$z_1 = 2x_1$$
$$z_2 = 3x_1 + 2x_2$$
$$z_3 = x_1 + 4x_2 + 5x_3$$

$$\dot{\mathbf{z}} = \begin{bmatrix} -1.5 & 1 & 0 \\ -1.25 & 0.7 & 0.4 \\ -2.55 & 0.4 & -6.2 \end{bmatrix}\mathbf{z} + \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}u$$

$$y = [0.5 \quad 0 \quad 0]\mathbf{z}$$

- Checking the eigen values of $\hat{\mathbf{A}}$:

```
>> eig(A_hat)
   -6.2514 + 0.0000i
   -0.3743 + 0.4240i
   -0.3743 - 0.4240i
```

which are the same as the roots of char. eqn. from A: $s^3 + 7s^2 + 5s + 2 = 0$

Controllability and Observability

- The concepts of controllability and observability were introduced by R. E. Kalman in 1960s and play an important role in the design of control systems in state space.
- In fact, the conditions of controllability and observability may govern the existence of a complete solution to the control system design problem.
- The solution to this problem may not exist if the system considered is not controllable.
- Although most physical systems are controllable and observable, corresponding mathematical models may not possess the property of controllability and observability.
- Then it is necessary to know the conditions under which a system is controllable and observable.

Controllability

- Controlling the pole locations of the closed-loop system means implicitly that the control signal, u, can control the behavior of each state variable in $\mathbf{x}$.
- If any one of the state variables cannot be controlled by the control signal u, then we cannot place the poles of the system where we desire.
- *If an input to a system can be found that takes every state variable from a desired initial state to a desired final state, the system is said to be* ***controllable****; otherwise, the system is* ***uncontrollable***.
- ***Pole placement*** is a viable design technique *only* for systems that are *controllable*.

Controllability by Inspection-1

$$\dot{\mathbf{x}} = \begin{bmatrix} -a_1 & 0 & 0 \\ 0 & -a_2 & 0 \\ 0 & 0 & -a_3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u$$

$$\begin{aligned} \dot{x}_1 &= -a_1 x_1 &&&& + u \\ \dot{x}_2 &= && -a_2 x_2 && + u \\ \dot{x}_3 &= &&&& -a_3 x_3 + u \end{aligned}$$

- Since each of the state equation above is *independent* and decoupled from the rest, the control u affects each of the state variables. Hence, the system is ***controllable***.
- This is the *controllability* test from another perspective.

Controllability by Inspection-2

$$\dot{\mathbf{x}} = \begin{bmatrix} -a_4 & 0 & 0 \\ 0 & -a_5 & 0 \\ 0 & 0 & -a_6 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u$$

or

$$\begin{aligned} \dot{x}_1 &= -a_4 x_1 \\ \dot{x}_2 &= \quad -a_5 x_2 \quad + u \\ \dot{x}_3 &= \quad\quad -a_6 x_3 + u \end{aligned}$$

- From the state equations given above we see that state variable x_1 is not controlled by the control signal u.
- Thus, the system is said to be ***uncontrollable***.
- In summary, a system with distinct eigenvalues and a diagonal system matrix is *controllable* if the input coupling matrix $\mathbf{B}$ does not have any rows that are zero.

The Controllability Matrix

- An n^{th}-order plant whose state equation is
$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$
is completely controllable if the matrix
$$\mathbf{C_M} = [\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B} \quad \ldots \quad \mathbf{A}^{\mathrm{n}-1}\mathbf{B}]$$
is of rank n, i.e., full rank, where $\mathbf{C_M}$ is called *controllability matrix*.
- For single input systems instead of specifying rank n, we can say that $\mathbf{C_M}$ must be non-singular, possess inverse, or have linearly independent rows and columns.
- Also, if $\det(\mathbf{C_M}) \neq 0$, the system is completely controllable.
- "Completely" or "fully controllable" means all state variables are controllable.
- When we say just "controllable", we mean "completely controllable".

Controllability via the Controllability Matrix

- **Problem:** Given the system below determine its controllability.

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u$$

- **Solution:** The controllability matrix is

$$\mathbf{C_M} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ 1 & -1 & 1 \\ 1 & -2 & 4 \end{bmatrix}$$

The determinant of $\mathbf{C_M}$ is -1. So the rank is full (=3). Then the system is **controllable**.

Matlab Solution

```
>> A=[-1 1 0; 0 -1 0; 0 0 -2]
A =
   -1    1    0
    0   -1    0
    0    0   -2

>> B=[0; 1; 1]
B =
    0
    1
    1
```

```
>> CM=[B  A*B  A^2*B]
CM =
    0    1   -2
    1   -1    1
    1   -2    4

>> det(CM)   % or
>> rank(CM)
ans =
   -1
ans =
    3
```

The same result:

The system is **controllable**.

Example-2

- **Problem**: Determine whether the system below is controllable.

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u = \begin{bmatrix} -1 & 1 & 2 \\ 0 & -1 & 5 \\ 0 & 3 & -4 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}u$$

- **Solution:**

```
>> A=[-1 1 2; 0 -1 5; 0 3 -4];
>> B=[2; 1; 1];
>> CM=[B  A*B  A^2*B]
CM =
     2     1     1
     1     4    -9
     1    -1    16
```

>> det(CM)=80 → rank=3, **controllable.**

% or

>> rank(CM)=3 → rank=3, **controllable.**

Control Systems Design in State Space

- The pole-placement method is somewhat similar to the root-locus method in that we place closed-loop poles at desired locations.
- The basic difference is that in the root-locus design we place only the dominant closed-loop poles at the desired locations, while in the pole-placement design we place <u>all closed-loop poles at desired locations</u>.

Introduction

- State-space methods, like transform methods (e.g. transfer functions), are simply tools for **representing, analyzing and designing** feedback control systems.
- However, state-space techniques can be applied to a wider class of systems than transform methods, such as systems with **nonlinearities** and **multiple-input**, **multiple-output**.
- Here in this course, we apply the approach only to linear systems.

Introduction, *cont'd…*

- One of the drawbacks of design methods in the frequency domain, using either root locus or frequency response techniques, is that after designing the location of the dominant 2^{nd}-order pair of poles, we hope that the higher-order poles do not affect the second-order approximation.
- We would like to be able to specify *all closed-loop poles* of the higher-order system.
- Design methods in the frequency domain do not allow us to specify all poles in systems of order higher than 2 because they do not allow for a sufficient number of unknown parameters to place all of the closed-loop poles uniquely:

  One ***gain*** *to adjust*, or ***compensator pole*** *and* ***zero*** *to select*, does not yield a sufficient number of parameters to place all the closed-loop poles at desired locations.

Introduction, *cont'd…*

- Remember, to place n unknown quantities, we need n adjustable parameters.
- State-space methods solve this problem by introducing into the system
  1. other adjustable parameters and
  2. the technique for finding these parameter values, so that we can properly place **all poles** of the closed-loop system.

- On the other hand, state-space methods ***do not allow the specification of closed-loop zero locations*****,** whereas frequency domain methods do allow through placement of the lead compensator zero.
- Also, a state-space design may prove to be very sensitive to parameter changes.

Controller Design

- An n^{th}-order feedback control system has an n^{th}-order closed-loop characteristic equation of the form,

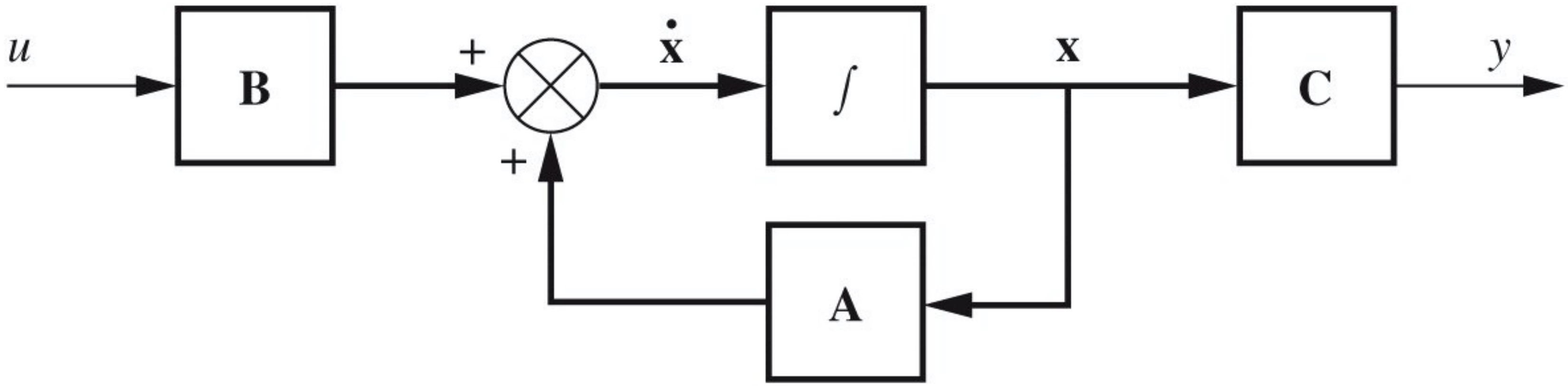
$$s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 = 0 \quad (1)$$

- Since the coefficient of the highest power of s is unity, there are n coefficients whose values determine the system's closed-loop pole locations.
- Thus, if we can introduce n adjustable parameters into the system and relate them to the coefficients in Eq. (1), all of the poles of the closed-loop system can be set to any desired location.

Topology for Pole Placement

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$
$$y = \mathbf{C}\mathbf{x}$$

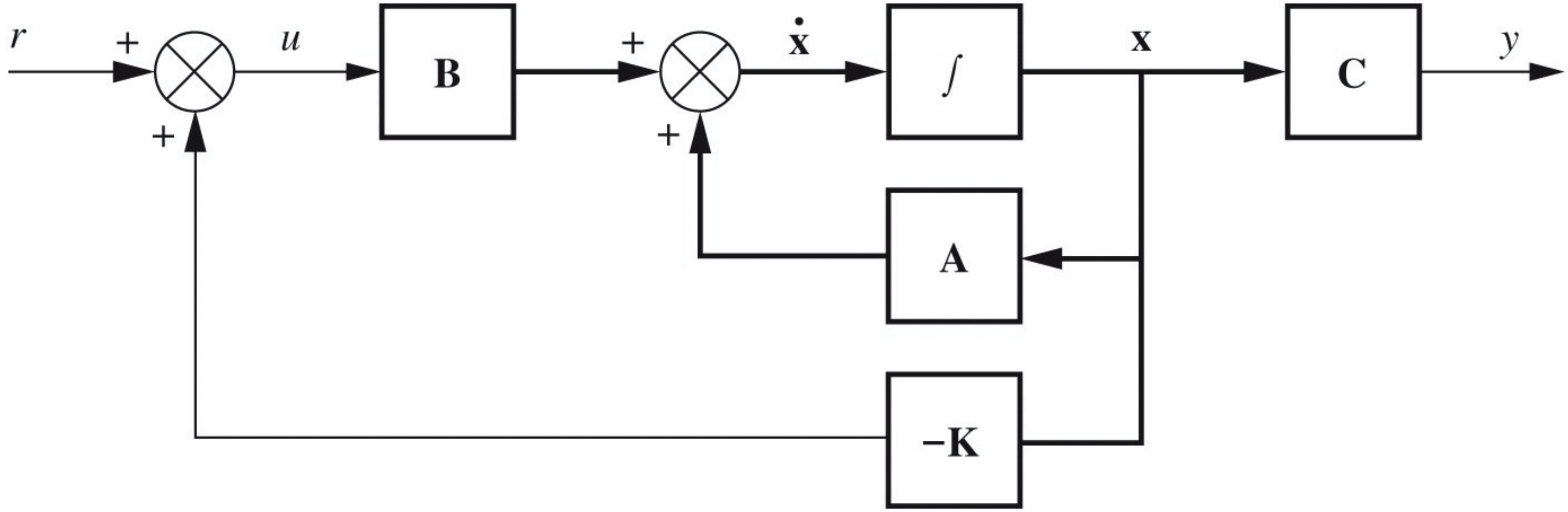
$\dot{\mathbf{x}}$ = state vector (n-vector)
y = output signal (scalar)
u = control signal (scalar)
$\mathbf{A}$ = $n \times n$ constant matrix
$\mathbf{B}$ = $n \times 1$ constant matrix
$\mathbf{C}$ = $1 \times n$ constant matrix



- In the figure the light lines are scalars and the thick lines are vectors.
- In a typical feedback control system, the output, y, is fed back to the summing junction. The topology of the design is different now!

Pole Placement

- Instead of feeding y back, let's feed back all of the *state variables*. That's why it is called **full state** or just **state feedback**.
- If each state variable is fed back to the control signal, u, through a gain, k_i, there would be n gains, k_i, that could be adjusted to yield the required closed-loop pole values.
- The feedback through the gains, k_i, is represented in the figure below by the feedback vector $-\mathbf{K}$.
- Now, $u = -\mathbf{Kx}$ or if there is a reference input $u = -\mathbf{Kx} + r$.
- $\mathbf{K}$ can be chosen properly to place the poles at desired locations.



Pole Placement, *cont'd…*

If the control law is defined as $u = -\mathbf{Kx} + r$, then

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{B}u \quad \mathbf{y} = \mathbf{Cx}$$

$$= \mathbf{Ax} + \mathbf{B}(-\mathbf{Kx} + r)$$

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{BK})\mathbf{x} + \mathbf{B}r$$

- If a plant is of high order and *not* represented in *phase-variable* or *controller canonical form*, the solution for the k_i's can be intricate.
- Thus, it is advisable to transform the system to either of these forms, design the k_i's, and then transform the system back to its original representation.
- We will perform this conversion later, where we develop a method for performing the transformations.
- Until then, let us direct our attention to plants represented in phase-variable form.

Pole Placement for Plants in Phase-Variable Form

To apply pole-placement methodology to plants represented in phase-variable form, take the following steps:

1. Represent the plant in phase-variable form.
2. Feed back each phase variable to the input of the plant through a gain, k_i.
3. Find the characteristic equation for the closed-loop system represented in Step 2.
4. Decide upon all closed-loop pole locations and determine an equivalent (desired) characteristic equation.
5. Equate like coefficients of the characteristic equations from Steps 3 and 4, and
6. Solve for k_i.

Pole Placement for Plants in Phase-Variable Form, *cont.'s*

- Following these steps, the phase-variable representation of the plant is given by $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$ and $y = \mathbf{C}\mathbf{x}$ where,

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix};$$

$$\mathbf{C} = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}$$

- The characteristic equation of the plant is obvious:

$$\boldsymbol{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 = 0}$$

- Now form the closed-loop system by feeding back each state variable to u, forming

$$u = -\mathbf{K}\mathbf{x} \text{ where, } \mathbf{K} = [k_1 \quad k_2 \quad \ldots \quad k_n]$$

Pole Placement for Plants in Phase-Variable Form

- As we found before,

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u, \; u = -\mathbf{K}\mathbf{x} + r$$

$$= \mathbf{A}\mathbf{x} + \mathbf{B}(-\mathbf{K}\mathbf{x} + r)$$

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x} + \mathbf{B}r$$

$$\mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix}, \mathbf{K} = [k_1 \;\; k_2 \;\; \ldots \;\; k_n]$$

$$\mathbf{BK} = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ & & \cdot & & \\ & & \cdot & & \\ 0 & 0 & 0 & \ldots & 0 \\ k_1 & k_2 & k_3 & \ldots & k_n \end{bmatrix}$$

- Subtracting matrix $\mathbf{BK}$ from matrix $\mathbf{A}$,

$$\mathbf{A} - \mathbf{BK} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -(a_0 + k_1) & -(a_1 + k_2) & -(a_2 + k_3) & \cdots & -(a_{n-1} + k_n) \end{bmatrix}$$

- Since the eqn. above is in phase-variable form, the characteristic eqn. of the closed-loop system can be written by inspection as

$$\det(s\mathbf{I} - (\mathbf{A} - \mathbf{BK})) = s^n + (a_{n-1} + k_n)s^{n-1} + (a_{n-2} + k_{n-1})s^{n-2} + \cdots (a_1 + k_2)s + (a_0 + k_1) = 0$$

Pole Placement for Plants in Phase-Variable Form

- The characteristic equation of the plant was

$$s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_1 s + a_0 = 0$$

- The characteristic equation of the closed-loop system was just found as,

$$\det(s\mathbf{I} - (\mathbf{A} - \mathbf{BK})) = s^n + (a_{n-1} + k_n)s^{n-1} + (a_{n-2} + k_{n-1})s^{n-2} + \cdots (a_1 + k_2)s + (a_0 + k_1) = 0$$

- Therefore, for plants represented in phase-variable form, we can write by inspection the closed-loop characteristic equation from the open-loop characteristic equation by adding the appropriate k_i to each coefficient:
- Comparing the coefficients of two equations presented above,

$$a_{n-1} \Longrightarrow a_{n-1} + k_n$$

Pole Placement for Plants in Phase-Variable Form

- Now assume that the *desired characteristic equation* for a proper pole placement is

$$\boldsymbol{s^n + d_{n-1}s^{n-1} + d_{n-2}s^{n-2} + \cdots + d_2 s^2 + d_1 s + d_0 = 0}$$

where the d_i's are the desired coefficients. Equating the desired characteristic equation above to the characteristic equation of the closed-loop system we obtain

$$\det(s\mathbf{I} - (\mathbf{A} - \mathbf{BK})) = s^n + (a_{n-1} + k_n)s^{n-1} + (a_{n-2} + k_{n-1})s^{n-2} + \cdots (a_1 + k_2)s + (a_0 + k_1) = 0$$

$d_i = a_i + k_{i+1};\ i = 0, 1, 2, \dots, n-1$ ➔ $\boxed{k_{i+1} = d_i - a_i}$

- We have found the denominator of the closed-loop transfer function, let us find the numerator.
- For systems represented in phase-variable form, we learned that the numerator polynomial is formed from the coefficients of the output coupling matrix, $\mathbf{C}$.
- We see that the numerators of their transfer functions are the same.

Example-4 (Example 12.1 of Nise's book)

Controller Design for Phase-Variable Form (→CCF)

- **Problem:** Given the plant, $G(s) = \frac{20(s+5)}{s(s+1)(s+4)}$ design the phase-variable feedback gains to yield 9.5% overshoot and a settling time of 0.74 second.
- **Solution:** First, determine the desired closed-loop characteristic equation using the transient response requirements.
- The closed-loop poles can be found as, $9.5\% \rightarrow \zeta = 0.6, T_s = 0.74$ $\rightarrow \omega_n = 9$ rad/s ➔ $s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = \mathbf{-5.4 \pm j7.2}$
- Since the system is third-order, we must select a 3rd closed-loop pole.
- The closed-loop system will have a zero at -5, the same as the open-loop system. We could select the third closed-loop pole to cancel the closed-loop zero. However, to demonstrate the effect of the third pole and the design process, including the need for simulation, let us choose $\mathbf{-5.1}$ as the location of the third closed-loop pole.

Example-4, *cont.'s*

$$G(s) = \frac{20(s+5)}{s(s+1)(s+4)} = \frac{20s+100}{s^3+5s^2+4s}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -4 & -5 \end{bmatrix},\ \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ \mathbf{C} = [100\ \ 20\ \ 0],\ \mathbf{K} = [k_1\ \ k_2\ \ k_3]$$

$$\mathbf{BK} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ k_1 & k_2 & k_3 \end{bmatrix}, \mathbf{A} - \mathbf{BK} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -k_1 & -(4+k_2) & -(5+k_3) \end{bmatrix}$$

After finding closed-loop system's system matrix $\mathbf{A} - \mathbf{BK}$, its characteristic equation can be directly written from its determinant as,

$$\det(s\mathbf{I} - \mathbf{A} + \mathbf{BK}) = s^3 + (5+k_3)s^2 + (4+k_2)s + k_1$$

This equation must match the desired characteristic equation, which is obtained from the desired poles of $s_{1,2} = -5.4 \pm j7.2$ and $s_3 = -5.1$

>> P= poly([s1 s2 s3]) → [413.1 132.08 10.9]

$$\Delta(s) = s^3 + 15.9s^2 + 136.08s + 413.1 = 0$$

Equating the coefficients of last two polynomials,

$$k_1 = 413.1, k_2 = 132.08, k_3 = 10.9$$

Example-4, *cont.'s*

$$G(s) = \frac{20(s+5)}{s(s+1)(s+4)} = \frac{20s+100}{s^3+5s^2+4s}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -4 & -5 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \ \mathbf{C} = [100 \ \ 20 \ \ 0], \ \mathbf{K} = [k_1 \ \ k_2 \ \ k_3]$$

$$\mathbf{BK} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ k_1 & k_2 & k_3 \end{bmatrix}, \mathbf{A} - \mathbf{BK} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -k_1 & -(4+k_2) & -(5+k_3) \end{bmatrix}$$

After finding closed-loop system's system matrix $\mathbf{A} - \mathbf{BK}$, its characteristic equation can be directly written from its determinant as,

$$\det(s\mathbf{I} - \mathbf{A} + \mathbf{BK}) = s^3 + (5+k_3)s^2 + (4+k_2)s + k_1$$

This equation must match the desired characteristic equation, which is obtained from the desired poles of $s_{1,2} = -5.4 \pm j7.2$ and $s_3 = -5.1$

>> P= poly([s1 s2 s3]) → $[1 \ \ 15.91 \ \ 136.4 \ \ 414.45]$

$$\Delta(s) = s^3 + 15.91s^2 + 136.4s + 414.45 = 0$$

Equating the coefficients of last two polynomials,

$$k_1 = 414.45, k_2 = 132.4, k_3 = 10.91$$

Finally, the zero term of the closed-loop transfer function is the same as the zero term of the open-loop system, or $(s+5)$.

Using Eqs. (12.14), we obtain the following state-space representation of the closed-loop system:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -413.1 & -136.08 & -15.9 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r \tag{12.19a}$$

$$y = \begin{bmatrix} 100 & 20 & 0 \end{bmatrix}\mathbf{x} \tag{12.19b}$$

The transfer function is

$$T(s) = \frac{20(s+5)}{s^3 + 15.9s^2 + 136.08s + 413.1} \tag{12.20}$$

Figure 12.5, a simulation of the closed-loop system, shows 11.5% overshoot and a settling time of 0.8 second. A redesign with the third pole canceling the zero at -5 will yield performance equal to the requirements.

Since the steady-state response approaches 0.24 instead of unity, there is a large steady-state error. Design techniques to reduce this error are discussed in Section 12.8.

FIGURE 12.5
Simulation of closed-loop system of Example 12.1

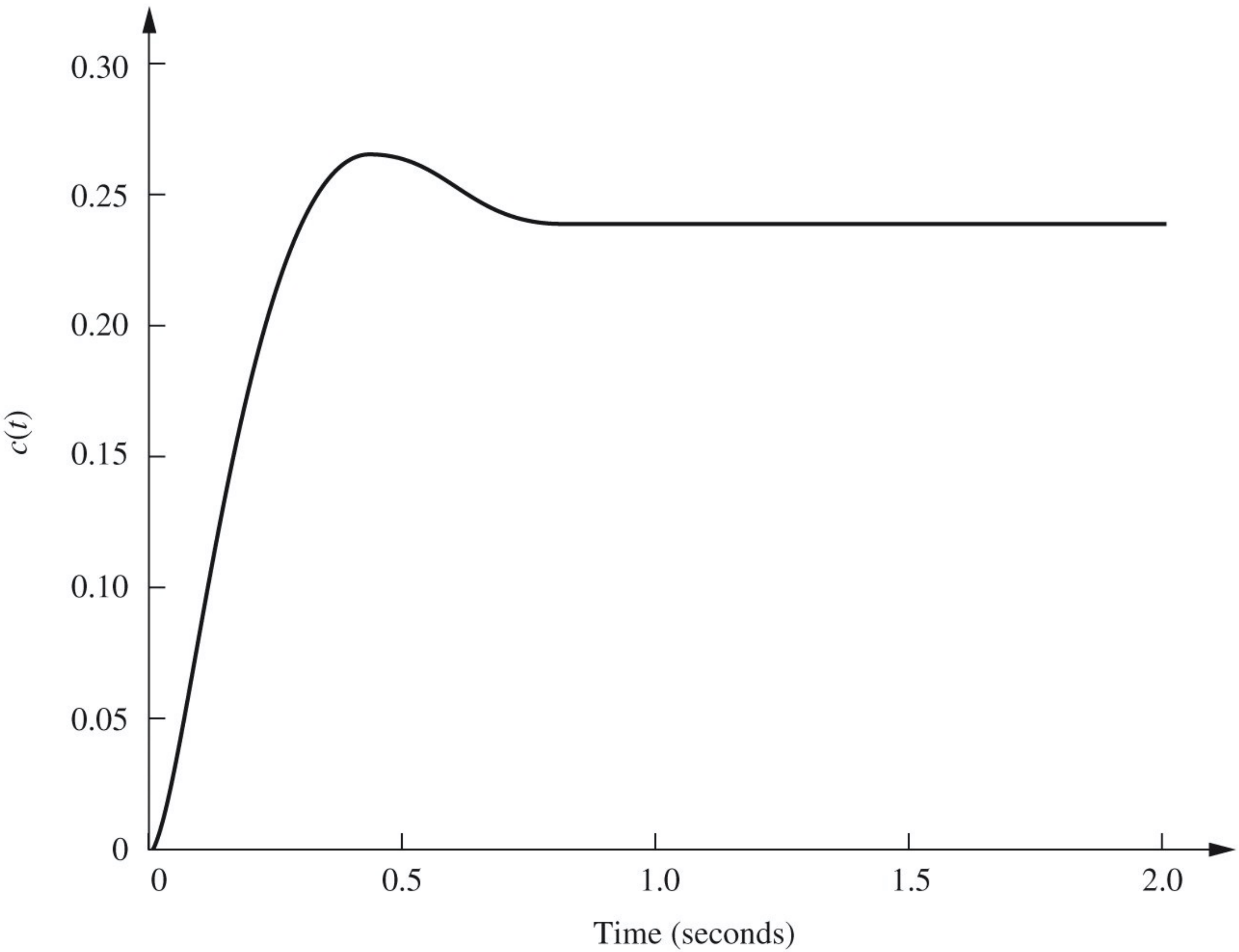
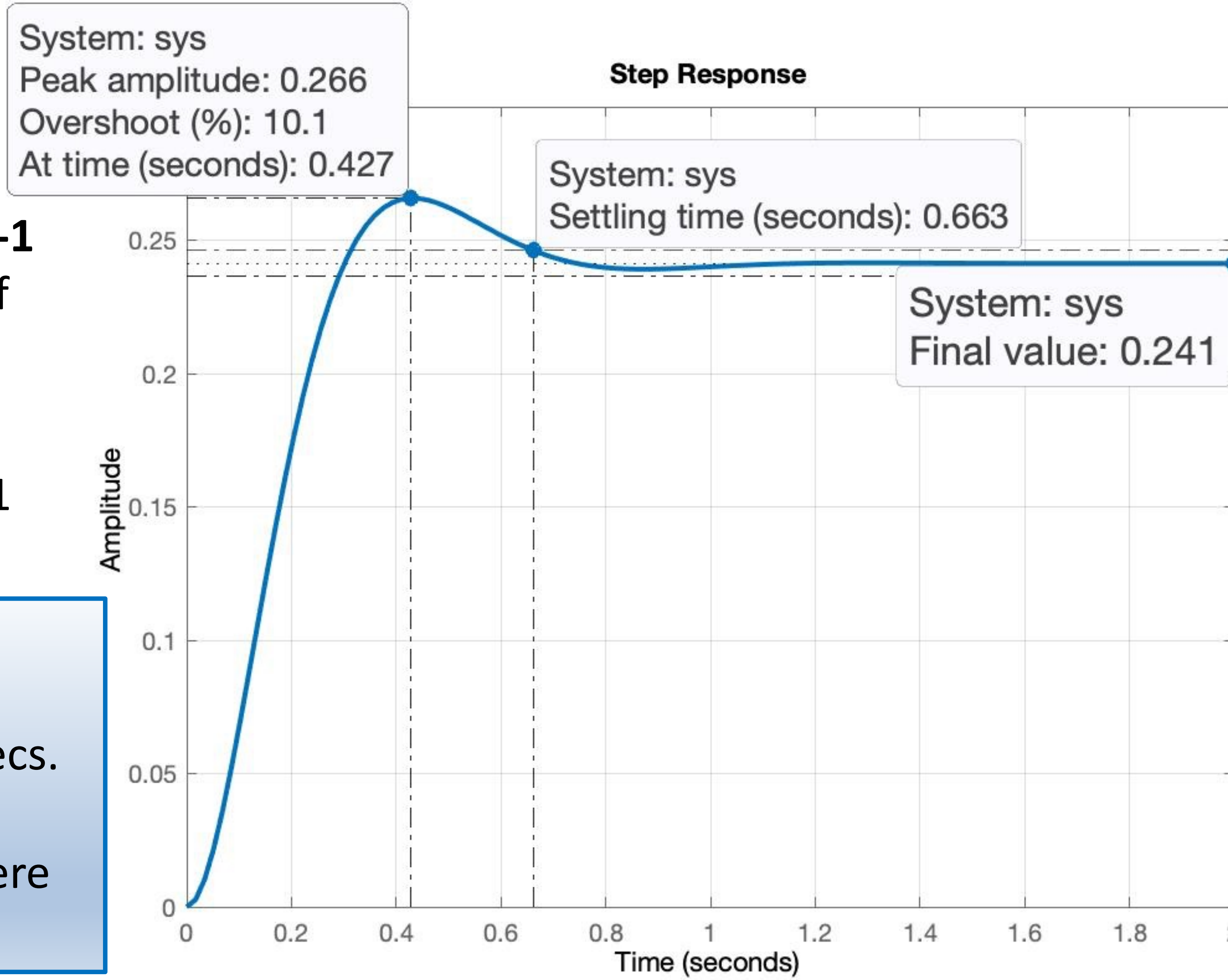


Figure 12.5

Matlab solution of the same problem (Example 12.1)

```
>> A=[0 1 0; 0 0 1; 0 -4 -5], B=[0; 0; 1], C=[100 20 0], D=0;
>> pOS=9.5; zeta=-log(pOS/100)/sqrt(((log(pOS/100))^2 + pi^2))
          zeta =   0.5996
>> Ts=0.74;  wn=4/zeta/Ts    = 9.0147
>> p1=-5.1; p2=-zeta*wn+j*wn*sqrt(1-zeta^2); p3=-zeta*wn-j*wn*sqrt(1-zeta^ 2);
>> P=[p1 p2 p3]
     P =  -5.1000 + 0.0000i  -5.4054 + 7.2143i  -5.4054 - 7.2143i
>> K=place(A,B,P)
     K =  414.4490  132.3996   10.9108
>> [num_d, den_d] = ss2tf(A-B*K,B,C,D)
     num_d =  0     0    20   100;  den_d =  1.0000   15.9108  136.3996  414.4490
>> step(num_d, den_d)
Or
>> Gd=tf(num_d,den_d); step(Gd)
```

FIGURE 12.5-1 Simulation of closed-loop system of Example 12.1 with Matlab



The required transient response specs. are satisfied. However, there is a large e_{ss}.

SOME USEFUL RESULTS IN VECTOR-MATRIX ANALYSIS

Cayley–Hamilton Theorem

- The Cayley–Hamilton theorem is very useful in proving theorems involving matrix equations or solving problems involving matrix equations.
- Consider an $n \times n$ matrix $\mathbf{A}$ and **its characteristic equation**:
  $$|\lambda\mathbf{I} - \mathbf{A}| = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0$$
- The Cayley–Hamilton theorem states that

The matrix $\mathbf{A}$ satisfies its own characteristic equation:

$$\mathbf{A}^n + a_1\mathbf{A}^{n-1} + \cdots + a_{n-1}\mathbf{A} + a_n\mathbf{I} = \mathbf{0}$$

SOME USEFUL RESULTS IN VECTOR-MATRIX ANALYSIS

Cayley–Hamilton Theorem, *cont.'d*...

- Consider an $n \times n$ matrix $\mathbf{A}$ and its characteristic equation:
$$|\lambda\mathbf{I} - \mathbf{A}| = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0$$
- The Cayley–Hamilton theorem states that the matrix $\mathbf{A}$ satisfies its own characteristic equation:
$$\mathbf{A}^n + a_1\mathbf{A}^{n-1} + \cdots + a_{n-1}\mathbf{A} + a_n\mathbf{I} = 0$$
- To prove this theorem, note that $\mathrm{adj}(\lambda\mathbf{I} - \mathbf{A})$ is a polynomial in λ of degree $(n-1)$. That is,
$$\mathrm{adj}(\lambda\mathbf{I} - \mathbf{A}) = \mathbf{B}_1\lambda^{n-1} + \mathbf{B}_2\lambda^{n-2} + \cdots + \mathbf{B}_{n-1}\lambda + \mathbf{B}_n$$

  where $\mathbf{B}_1 = \mathbf{I}$.

  Considering $\mathbf{A}^{-1} = \dfrac{\mathrm{adj}(\mathbf{A})}{\det \mathbf{A}} \rightarrow \det \mathbf{A} = |\lambda\mathbf{I} - \mathbf{A}| = \mathbf{A} \cdot \mathrm{adj}(\mathbf{A})$

$$(\boldsymbol{\lambda}\mathbf{I} - \mathbf{A})\mathbf{adj}(\boldsymbol{\lambda}\mathbf{I} - \mathbf{A}) = [\mathbf{adj}(\boldsymbol{\lambda}\mathbf{I} - \mathbf{A})](\boldsymbol{\lambda}\mathbf{I} - \mathbf{A}) = |\boldsymbol{\lambda}\mathbf{I} - \mathbf{A}|\mathbf{I}$$

We obtain

$$
\begin{aligned}
|\boldsymbol{\lambda}\mathbf{I}-\mathbf{A}| &= \mathbf{I}\boldsymbol{\lambda}^{n}+\boldsymbol{a}_{\mathbf{1}}\mathbf{I}\boldsymbol{\lambda}^{n-1}+\cdots+\boldsymbol{a}_{n-1}\mathbf{I}\boldsymbol{\lambda}+\boldsymbol{a}_{n}\mathbf{I} \\
&= (\boldsymbol{\lambda}\mathbf{I}-\mathbf{A})(\mathbf{B}_{1}\boldsymbol{\lambda}^{n-1}+\mathbf{B}_{2}\boldsymbol{\lambda}^{n-2}+\cdots+\mathbf{B}_{n-\mathbf{1}}\boldsymbol{\lambda}+\mathbf{B}_{n}) \\
&= (\mathbf{B}_{1}\boldsymbol{\lambda}^{n-1}+\mathbf{B}_{2}\boldsymbol{\lambda}^{n-2}+\cdots+\mathbf{B}_{n-1}\boldsymbol{\lambda}+\mathbf{B}_{n})(\boldsymbol{\lambda}\mathbf{I}-\mathbf{A})
\end{aligned}
$$

From this equation, we see that $\mathbf{A}$ and $\boldsymbol{B_i}(i = 1,2,\ldots,n)$ commute. Hence, the product of $(\lambda\mathbf{I}-\mathbf{A})$ becomes zero if either of these is zero. If $\mathbf{A}$ is substituted for λ in this last equation, then clearly $\lambda I-\mathbf{A}$ becomes zero. Hence, we obtain

$$
\mathbf{A}^{n}+a_{1}\mathbf{A}^{n-1}+\cdots+a_{n-1}\mathbf{A}+a_{n}\mathbf{I}=0
$$

This proves the Cayley–Hamilton theorem.

Determination of Matrix $\mathbf{K}$ Using Ackermann's Formula

- There is a well-known formula, known as Ackermann's formula, for the determination of the state feedback gain matrix $\mathbf{K}$. We shall present this formula as follows.

  Consider the system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

  where we use the state feedback control $u = -\mathbf{K}\mathbf{x}$.
- We assume that the system is completely state controllable.
- We also assume that the *desired closed-loop poles* are at

$$s = \mu_1, s = \mu_2, \dots, s = \mu_n.$$

- Use of the state feedback control

$$u = -\mathbf{K}\mathbf{x}$$

modifies the system equation to

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x} \qquad (10\text{-}14)$$

- Let us define

$$\widetilde{\mathbf{A}} = \mathbf{A} - \mathbf{BK}$$

- The desired characteristic equation for this new system is

$$\left|s\mathbf{I} - \widetilde{\mathbf{A}}\right| = |s\mathbf{I} - \mathbf{A} + \mathbf{BK}| = (s - \mu_1)(s - \mu_2) \ldots (s - \mu_n)$$
$$= s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1} s + \alpha_n = 0$$

- Since the Cayley–Hamilton theorem states that $\widetilde{\mathbf{A}}$ satisfies its own characteristic equation, we have

$$\phi\left(\widetilde{\mathbf{A}}\right) = \widetilde{\mathbf{A}}^n + \alpha_1 \widetilde{\mathbf{A}}^{n-1} + \cdots + \alpha_{n-1} \widetilde{\mathbf{A}} + \alpha_n \mathbf{I} = 0 \quad \text{(10-15)}$$

- We shall utilize Equation (10–15) to derive Ackermann's formula. To simplify the derivation, we consider the case where $n = 3$. (For any other positive integer n, the following derivation can be easily extended.)
- Consider the following,

$$\mathbf{A}^2 - \mathbf{ABK} - \mathbf{BKA} - (\mathbf{BK})^2 = \mathbf{A}^2 - \mathbf{ABK} - \mathbf{BK}\underbrace{(\mathbf{A} - \mathbf{BK})}_{\widetilde{\mathbf{A}}}$$

Using the following identities:

$$\mathbf{I} = \mathbf{I}$$

$$\widetilde{\mathbf{A}} = \mathbf{A} - \mathbf{BK}$$

$$\widetilde{\mathbf{A}}^2 = (\mathbf{A} - \mathbf{BK})^2 = \mathbf{A}^2 - \mathbf{ABK} - \mathbf{BK}\widetilde{\mathbf{A}}$$

$$\widetilde{\mathbf{A}}^3 = (\mathbf{A} - \mathbf{BK})^3 = \mathbf{A}^3 - \mathbf{A}^2\mathbf{BK} - \mathbf{ABK}\widetilde{\mathbf{A}} - \mathbf{BK}\widetilde{\mathbf{A}}^2$$

Multiplying the preceding equations in order by α_3 , α_2 , α_1 , and α_0 (where α_0 =1), respectively, and adding the results, we obtain

$$\alpha_3\mathbf{I} + \alpha_2\widetilde{\mathbf{A}} + \alpha_1\widetilde{\mathbf{A}}^2 + \widetilde{\mathbf{A}}^3$$
$$= \alpha_3\mathbf{I} + \alpha_2(\mathbf{A} - \mathbf{BK}) + \alpha_1\left(\mathbf{A}^2 - \mathbf{ABK} - \mathbf{BK}\widetilde{\mathbf{A}}\right) + \mathbf{A}^3 - \mathbf{A}^2\mathbf{BK}$$
$$- \mathbf{ABK}\widetilde{\mathbf{A}} - \mathbf{BK}\widetilde{\mathbf{A}}^2$$

The right hand side of the equation becomes,

$$= \alpha_3\mathbf{I} + \alpha_2\mathbf{A} + \alpha_1\mathbf{A}^2 + \mathbf{A}^3 - \alpha_2\mathbf{BK} - \alpha_1\mathbf{ABK} - \alpha_1\mathbf{BK}\widetilde{\mathbf{A}} - \mathbf{A}^2\mathbf{BK}$$
$$- \quad \mathbf{ABK}\widetilde{\mathbf{A}} - \mathbf{BK}\widetilde{\mathbf{A}}^2 \qquad (10\text{-}16)$$

Referring to Equation (10–15),we have

$$\alpha_3\mathbf{I} + \alpha_2\widetilde{\mathbf{A}} + \alpha_1\widetilde{\mathbf{A}}^2 + \widetilde{\mathbf{A}}^3 = \phi(\widetilde{\mathbf{A}}) = \mathbf{0}$$

Also, we have

$$\alpha_3\mathbf{I} + \alpha_2\mathbf{A} + \alpha_1\mathbf{A}^2 + \mathbf{A}^3 = \phi(\mathbf{A}) \neq \mathbf{0}$$

Substituting the last two equations into Equation (10–16),we have

$$\phi(\widetilde{\mathbf{A}}) = \phi(\mathbf{A}) - \boldsymbol{\alpha}_2\mathbf{BK} - \boldsymbol{\alpha}_1\mathbf{BK}\widetilde{\mathbf{A}} - \mathbf{BK}\widetilde{\mathbf{A}}^2 - \boldsymbol{\alpha}_1\mathbf{ABK} - \mathbf{ABK}\widetilde{\mathbf{A}} - \mathbf{A}^2\mathbf{BK}$$

Since $\phi(\widetilde{\mathbf{A}}) = 0$, we obtain

$$\phi(\mathbf{A}) = \mathbf{B}(\boldsymbol{\alpha}_2\mathbf{K} + \boldsymbol{\alpha}_1\mathbf{K}\widetilde{\mathbf{A}} + \mathbf{K}\widetilde{\mathbf{A}}^2) + \mathbf{AB}(\boldsymbol{\alpha}_1\mathbf{K} + \mathbf{K}\widetilde{\mathbf{A}}) + \mathbf{A}^2\mathbf{BK}$$

$$= [\mathbf{B}\ \ \mathbf{AB}\ \ \mathbf{A}^2\mathbf{B}]\begin{bmatrix} \boldsymbol{\alpha}_2\mathbf{K} + \boldsymbol{\alpha}_1\mathbf{K}\widetilde{\mathbf{A}} + \mathbf{K}\widetilde{\mathbf{A}}^2 \\ \boldsymbol{\alpha}_1\mathbf{K} + \mathbf{K}\widetilde{\mathbf{A}} \\ \mathbf{K} \end{bmatrix} \tag{10-17}$$

Since the system is completely state controllable, the inverse of the controllability matrix, $[\mathbf{B}\ \ \mathbf{AB}\ \ \mathbf{A}^2\mathbf{B}]$

exists. Pre-multiplying both sides of Equation (10–17) by the inverse of the controllability matrix, we obtain

$$[\mathbf{B}\ \ \mathbf{AB}\ \ \mathbf{A}^2\mathbf{B}]^{-1}\phi(\mathbf{A}) = \begin{bmatrix} \boldsymbol{\alpha}_2\mathbf{K} + \boldsymbol{\alpha}_1\mathbf{K}\widetilde{\mathbf{A}} + \mathbf{K}\widetilde{\mathbf{A}}^2 \\ \boldsymbol{\alpha}_1\mathbf{K} + \mathbf{K}\widetilde{\mathbf{A}} \\ \mathbf{K} \end{bmatrix}$$

Pre-multiplying both sides of this last equation by [0 0 1], we obtain

$$[0 \quad 0 \quad 1][\mathbf{B} \;\; \mathbf{AB} \;\; \mathbf{A}^2\mathbf{B}]^{-1}\phi(\mathbf{A})$$

$$= [0 \quad 0 \quad 1]\begin{bmatrix} \alpha_2\mathbf{K} + \alpha_1\mathbf{K}\widetilde{\mathbf{A}} + \mathbf{K}\widetilde{\mathbf{A}}^2 \\ \alpha_1\mathbf{K} + \mathbf{K}\widetilde{\mathbf{A}} \\ \mathbf{K} \end{bmatrix} = \mathbf{K}$$

which can be rewritten as

$$\mathbf{K} = [0 \quad 0 \quad 1][\mathbf{B} \;\; \mathbf{AB} \;\; \mathbf{A}^2\mathbf{B}]^{-1}\phi(\mathbf{A})$$

This last equation gives the required state feedback gain matrix $\mathbf{K}$. For an arbitrary positive integer *n*, we have

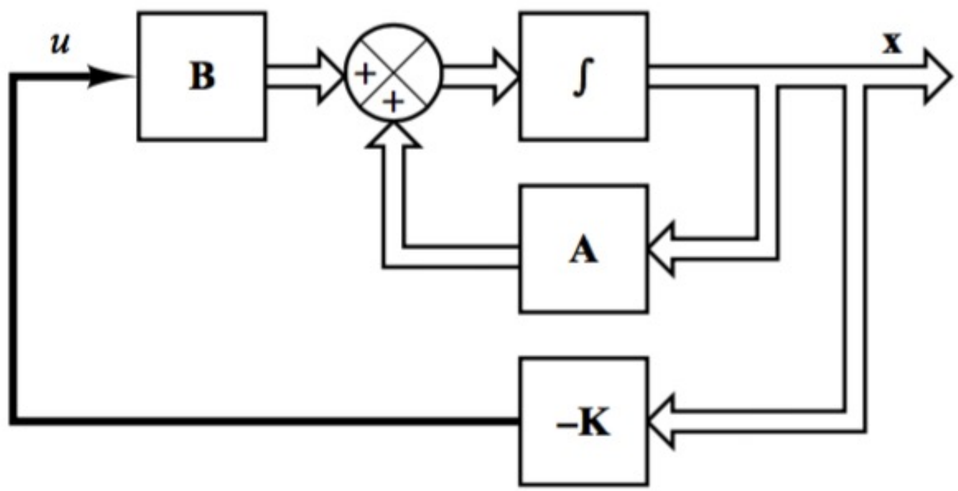
$$\boxed{\mathbf{K} = [0 \quad 0 \quad \ldots \quad 0 \;\; 1][\mathbf{B} \;\; \mathbf{AB} \;\; \ldots \;\; \mathbf{A}^{n-1}\mathbf{B}]^{-1}\phi(\mathbf{A})} \qquad (10\text{-}18)$$

Equation (10–18) is known as Ackermann’s formula for the determination of the state feedback gain matrix $\mathbf{K}$.

Example-1 Regulator Systems and Control Systems. Systems that include controllers can be divided into two categories: **regulator systems** (where the reference input is constant, including zero) and **control systems** (where the reference input is time varying). In what follows we shall consider regulator systems as shown in figure. The plant is given by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$



The system uses the state feedback control $u = -\mathbf{K}\mathbf{x}$. Let us choose the desired closed-loop poles at

$$s_1 = -2 + j4 \quad s_2 = -2 - j4 \quad s_3 = -10$$

Determine the state feedback gain matrix K.

First, we need to check the controllability matrix of the system. Since the controllability matrix M is given by

$$\mathbf{M} = [\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B}] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 31 \end{bmatrix}$$

we find that $|\mathbf{M}| =- 1$, and therefore, rank $\mathbf{M} = 3$. Thus, the system is completely state controllable and arbitrary pole placement is possible. Next, we shall solve this problem. The method is to use Ackermann's formula. Referring to Equation (10–18), we have

$$\mathbf{K} = [\mathbf{0} \quad \mathbf{0} \quad \ldots \quad \mathbf{0} \;\; \mathbf{1}][\mathbf{B} \;\; \mathbf{AB} \quad \ldots \quad \mathbf{A}^{n-1}\mathbf{B}]^{-\mathbf{1}}\phi(\mathbf{A})$$

Since, the new characteristic equation will have the poles of

$$s_1 = -2 + j4 \quad s_2 = -2 - j4 \quad s_3 = -10$$

This characteristic equations coefficients will be 1, 14, 60, 200.

$$\phi(\mathbf{A}) = \mathbf{A}^3 + 14\mathbf{A}^2 + 60\mathbf{A} + 200\mathbf{I}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}^3 + 14\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}^2 + 60\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} + 200\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 199 & 55 & 8 \\ -8 & 159 & 7 \\ -7 & -43 & 117 \end{bmatrix}$$

$$[\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B}] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 31 \end{bmatrix}$$

We can now obtain the gain vector as,

$$\mathbf{K} = [0 \quad 0 \quad 1] \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 31 \end{bmatrix}^{-1} \begin{bmatrix} 199 & 55 & 8 \\ -8 & 159 & 7 \\ -7 & -43 & 117 \end{bmatrix}$$

$$= [0 \quad 0 \quad 1] \begin{bmatrix} 5 & 6 & 1 \\ 6 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 199 & 55 & 8 \\ -8 & 159 & 7 \\ -7 & -43 & 117 \end{bmatrix}$$

$$= [199 \quad 55 \quad 8]$$

With this state feedback, the closed-loop poles are placed at

$$s_{1,2} = -2 \pm j4 \text{ and } s_3 = -10$$

as it is desired.

Controller Design by Matching Coefficients

- **Problem:** Given a plant, $G(s) = \frac{Y(s)}{U(s)} = \frac{10}{(s+1)(s+2)}$
  design a state feedback for the plant represented in cascade form to yield a 15% overshoot with a settling time of 0.5 second. Check if the system is controllable.

  Answer: $\mathbf{K} = [211.5 \quad 13]$
- Solve the same problem with Ackermann formula

- **Problem:** Given a plant, $G(s) = \frac{s+4}{(s+1)(s+2)(s+5)}$
  design a state-variable feedback controller to yield a 20.8% overshoot and a settling time of 4 seconds. Check if the system is controllable.

  Answer: $\mathbf{K} = [-20 \quad 10 \quad -2]$ (*Any method can be used*!)

- **Problem:** Given a plant, $G(s) = \frac{s+4}{(s+1)(s+2)(s+5)}$; design a state-variable feedback controller to yield a 20.8% overshoot and a settling time of 4 seconds. Check if the system is controllable.
- **Solution**

```
>> num=[1 4], den=[conv(conv([1 1], [1 2]),[1 5])]
```

num = 1 4; den = 1 8 17 10 ➔ $G(s) = \frac{s+4}{s^3+8s^2+17s+10}$

```
>> [A,B,C,D] = tf2ss(num, den)
A =  -8   -17   -10
      1     0     0
      0     1     0
B =   1
      0
      0
C =   0     1     4;  D =   0
>> CMM=[B   A*B   A^2*B]
CMM =    1    -8    47
         0     1    -8
         0     0     1
```

➔ det(CMM)=1 or rank(CMM)=3, so it's controllable.

Solution, *continues…*

A = -8 -17 -10
1 0 0
0 1 0

B = 1
0
0

$\mathbf{K} = [k_1 \quad k_2 \quad k_3]$

$$\mathbf{A} - \mathbf{BK} = \begin{bmatrix} -8 - k_1 & -17 - k_2 & -10 - k_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

- The desired system matrix from the desired characteristic equation D=poly([s1 s2 s3]) ➔ $\boldsymbol{D(s) = s^3 + 6s^2 + 13s + 20}$
- $\mathbf{A_D} = \begin{bmatrix} -6 & -13 & -20 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$
- By equating the last two matrices, $\mathbf{K} = [-2 \quad -4 \quad 10]$

Solution continues in Matlab

- **Ackermann's formula:**

$$\mathbf{K} = [0 \quad 0 \quad \dots \quad 0 \quad 1][\mathbf{B} \quad \mathbf{AB} \quad \dots \quad \mathbf{A}^{n-1}\mathbf{B}]^{-1}\phi(\mathbf{A})$$

  >> A=[-8 -17 -10; 1 0 0; 0 1 0]; B=[1; 0; 0]; ➔

  Desired Char. Eqn. for $s_{1,2,3} = -1 \pm 2j, -4$: $s^3 + 6s^2 + 13s + 20 = 0$

  >> CM= [B A*B A^2*B]; rn = rank(A), rn = 3 ➔ full rank, the system is controllable.

$$\phi(\mathbf{A})=\mathbf{A}^3 + 6\mathbf{A}^2 + 13\mathbf{A} + 20\mathbf{I} = \mathbf{Phi}$$

  $\mathbf{K} = [0 \quad 0 \quad 1][\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B}]^{-1}\phi(\mathbf{A})$ ➔

  >> K= [0 0 1]*CM^-1*Phi

  K = -2 -4 10 ➔ $\mathbf{K} = [-2 \quad -4 \quad 10]$

- **Alternatively, using the following two functions (commands) in Matlab:**

  >> K=acker(A, B, P) or

  >> K=place(A, B, P)

  K = -2 -4 10 ➔ $\mathbf{K} = [-2 \quad -4 \quad 10]$

  Both Matlab commands return the same result as the ones obtained via coefficient matching method and Ackermann's formula !

Solution, *from scratch…*

- 20.8% overshoot and a settling time of 4 seconds.

```
>> pOS= 20.8; >> zeta=-log(pOS/100)/sqrt(pi^2 + (log(pOS/100))^2)
```

zeta = 0.4471 ➔ $\zeta = 0.4471;\ T_s = \frac{4}{\zeta\omega_n} = 4$

>> wn=4/zeta/4; wn = 2.2367 ➔ $\omega_n = 2.2367 \text{ rad/s}$

- The desired characteristic equation:

  s1=-zeta*wn-j*wn*sqrt(1-zeta^2); → $s_1 = -1 - 2j$

  s2=-zeta*wn+j*wn*sqrt(1-zeta^2); → $s_2 = -1 + 2j$

>> D=poly([s1 s2])

D = 1.0000 2.0000 5.0030 ➔ $D(s) = s^2 + 2s + 5$

- Let's take the third pole $s_3 = -4$ to cancel the zero.
- Hence the desired characteristic equation will become:

>> D=poly([s1 s2 s3]) ➔ $\boldsymbol{D(s) = s^3 + 6s^2 + 13s + 20}$

As was given, A = [-8 -17 -10; 1 0 0; 0 1 0]; B=[1; 0; 0];

A-B*K = [-(8+k1) -(17+k2) -(10+k3); 1 0 0; 0 1 0];

$\det(s\mathbf{I} - \mathbf{A} + \mathbf{BK}) = s^3 + (8 + k_1)s^2 + (17 + k_2)s + (10 + k_3)$

Comparing the polynomials of $\boldsymbol{D(s)}$ and $\det\,(\boldsymbol{s}\mathbf{I} - \mathbf{A} + \mathbf{BK})$, we get

$k_1 = -2; k_2 = -4; k_3 = 10$ ➔ $\mathbf{K} = [-2 \quad -4 \quad 10]$

Alternative Approaches to Controller Design

- As we proved by Similarity Transformation, there are infinitely many possible state-space representations while there is only one possible transfer function for the system.
- Therefore, different $\mathbf{K}$ matrices can be found for different representations.
- The method consists of transforming the system to phase variables such as CCF, designing the feedback gains, and transforming the designed system back to its original state-variable representation.
- This method requires that we first develop the transformation between a system and its representation in phase-variable form.
- Assume a plant not represented in phase-variable form
- The controllability matrix, $\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{B}u;\ y = \mathbf{C}\mathbf{z}$

$$\mathbf{C}_{\mathbf{Mz}} = [\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B} \quad \dots. \quad \mathbf{A}^{\mathrm{n}-1}\mathbf{B}]$$

Alternative Approaches to Controller Design, *cont.'s*…

- Now, let's make use of the Similarity Transformation.
- Assume that the system can be transformed into the phase-variable ($\mathbf{x}$) representation with the transformation

$$\mathbf{z} = \mathbf{P}\mathbf{x} \rightarrow \dot{\mathbf{z}} = \mathbf{P}\dot{\mathbf{x}}$$

- Substitute into the state equation,

$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{B}u \rightarrow \mathbf{P}\dot{\mathbf{x}} = \mathbf{A}\mathbf{P}\mathbf{x} + \mathbf{B}u$, Pre-multiply the whole equation by $\mathbf{P}^{-1}$:

  - ✓ The state equation: $\dot{\mathbf{x}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{x} + \mathbf{P}^{-1}\mathbf{B}u$,
  - ✓ And the output equation: $y = \mathbf{C}\mathbf{z} \rightarrow y = \mathbf{C}\mathbf{P}\mathbf{x}$

- The controllability matrix of this transformed (to $\mathbf{x}$) system:

$$\mathbf{C}_{\mathbf{Mx}} = \left[\mathbf{P}^{-1}\mathbf{B} \quad (\mathbf{P}^{-1}\mathbf{A}\mathbf{P})(\mathbf{P}^{-1}\mathbf{B}) \quad (\mathbf{P}^{-1}\mathbf{A}\mathbf{P})^{2}(\mathbf{P}^{-1}\mathbf{B}) \quad \ldots. \quad (\mathbf{P}^{-1}\mathbf{A}\mathbf{P})^{n-1}(\mathbf{P}^{-1}\mathbf{B})\right]$$

$$= \left[\mathbf{P}^{-1}\mathbf{B} \quad (\mathbf{P}^{-1}\mathbf{A}\mathbf{P})(\mathbf{P}^{-1}\mathbf{B}) \quad (\mathbf{P}^{-1}\mathbf{A}\mathbf{P})(\mathbf{P}^{-1}\mathbf{A}\mathbf{P})(\mathbf{P}^{-1}\mathbf{B}) \quad \ldots \right.$$

$$\left. \ldots. \quad (\mathbf{P}^{-1}\mathbf{A}\mathbf{P})(\mathbf{P}^{-1}\mathbf{A}\mathbf{P})\ldots(\mathbf{P}^{-1}\mathbf{A}\mathbf{P})(\mathbf{P}^{-1}\mathbf{B})\right]$$

$$\mathbf{C}_{\mathbf{Mx}} = \mathbf{P}^{-1}[\mathbf{B} \quad \mathbf{A}\mathbf{B} \quad \mathbf{A}^{2}\mathbf{B} \quad \ldots. \quad \mathbf{A}^{n-1}\mathbf{B}] = \mathbf{P}^{-1}\mathbf{C}_{\mathbf{Mz}} \rightarrow \boxed{\mathbf{P} = \mathbf{C}_{\mathbf{Mz}}\mathbf{C}_{\mathbf{Mx}}^{-1}}$$

- Thus, the transformation matrix, $\mathbf{P}$, can be found from the two controllability matrices.

Alternative Approaches to Controller Design, *cont.'s*…

- After transforming the system to phase variables, we design the feedback gains as we did before.
- Hence, including both feedback and input, $u = -\mathbf{K_x x} + r$, the transformed system, $\dot{\mathbf{x}} = \mathbf{P^{-1}APx} + \mathbf{P^{-1}B}u$ and $y = \mathbf{CPx}$ becomes,

$$\dot{\mathbf{x}} = \mathbf{P^{-1}APx} - \mathbf{P^{-1}BK_x x} + \mathbf{P^{-1}B}r$$

$$= \left(\mathbf{P^{-1}AP} - \mathbf{P^{-1}BK_x}\right)\mathbf{x} + \mathbf{P^{-1}B}r, \quad y = \mathbf{CPx}$$

- Since this equation is in phase-variable form, the zeros of this closed-loop system are determined from the polynomial formed from the elements of $\mathbf{CP}$.
- Using $\mathbf{z} = \mathbf{Px}$, then $\mathbf{x} = \mathbf{P^{-1}z}$, we transform the transformed system (above) from phase variables back to the original representation and get,

$$\mathbf{P^{-1}\dot{z}} = \mathbf{P^{-1}APP^{-1}z} - \mathbf{P^{-1}BK_x P^{-1}z} + \mathbf{P^{-1}B}r$$

$$\dot{\mathbf{z}} = \mathbf{Az} - \mathbf{BK_x P^{-1}z} + \mathbf{B}r = (\mathbf{A} - \mathbf{BK_x P^{-1}})\mathbf{z} + \mathbf{B}r, \quad y = \mathbf{Cz}$$

Alternative Approaches to Controller Design, *cont.'s*…

- If we compare the last equations, we just obtained,

$$\dot{\mathbf{z}} = \left(\mathbf{A} - \mathbf{B}\mathbf{K}_{\mathbf{x}}\mathbf{P}^{-1}\right)\mathbf{z} + \mathbf{B}r, \quad \mathrm{y} = \mathbf{C}\mathbf{z}$$

- With the ones we obtained for pole placement,

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x} + \mathbf{B}r, \text{ and } \mathrm{y} = \mathbf{C}\mathbf{x}$$

we get the state feedback gain matrix, $\mathbf{K}_{\mathbf{z}}$, as

$$\boxed{\mathbf{K}_{\mathbf{z}} = \mathbf{K}_{\mathbf{x}}\mathbf{P}^{-1}}$$

- As known, the transfer function of this closed-loop system ($\mathbf{G}_{\mathbf{x}}$) is the same as the transfer function for the transformed system ($\mathbf{G}_{\mathbf{z}}$) representing the same system.

- Thus, the zeros of the closed-loop transfer function are the same as the zeros of the uncompensated plant.

Example

$$G(s) = \frac{4(s+6)}{(s+2)(s+3)(s+4)} = \frac{4s+24}{s^3+9s^2+26s+24}$$

```
>> num=[4 24], den=[conv(conv([1 2], [1 3]),[1 4])]
>> A=[0 1 0; 0 0 1; -24   -26   -9], B=[0; 0; 1], C=[24 4 0], D=0;
```

Existing Poles= $s_1 = -2, -3, -4$; New Poles: $s_1 = -5,\ \ s_2 = -8,\ s_3 = -9$

```
>>  num_d=60*[1  6],  den_d=[conv(conv([1  5],  [1  8]),  [1  9])]
>>  Gd=tf(num_d,den_d)
          num_d  =       60      360
          den_d  =        1       22      157      360
>>  Gd=zpk(tf(num_d,den_d))
Gd  =
   60  (s+6)
-----------------
(s+9)  (s+8)  (s+5)
>>  P=[s1  s2  s3],
P  =  -5       -8       -9
>>  Kx=acker(A,  B,  P)   ➔   Kx  =     336     131      13
```

Example, *cont.'s*

New system – system-$\mathbf{z}$:

$$\left.\begin{aligned} z_1 &= x_2 - x_1 \\ z_2 &= x_3 - x_2 \\ z_3 &= 2x_3 - x_1 \end{aligned}\right\} \qquad \mathbf{z} = \mathbf{Tx}$$

>> T=[-1 1 0; 0 -1 1; -1 0 2]

$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{B}u\ ; \mathrm{y} = \mathbf{Cx}$

- $\mathbf{z} = \mathbf{Tx} \rightarrow \mathbf{x} = \mathbf{T}^{-1}\mathbf{z} \rightarrow \dot{\mathbf{x}} = \mathbf{T}^{-1}\dot{\mathbf{z}}$
- $\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{B}u \rightarrow \dot{\mathbf{x}} = \mathbf{T}^{-1}\dot{\mathbf{z}} = \mathbf{AT}^{-1}\mathbf{z} + \mathbf{B}u$ (Pre-multiply by $\mathbf{T}$)
- $\dot{\mathbf{z}} = \mathbf{TAT}^{-1}\mathbf{z} + \mathbf{TB}u \rightarrow \quad \mathbf{A_z} = \mathbf{TAT}^{-1}, \mathbf{B_z} = \mathbf{TB}$
- $\mathrm{y} = \mathbf{Cx} \rightarrow \mathrm{y} = \mathbf{CT}^{-1}\mathbf{z} \rightarrow \quad \mathbf{C_z} = \mathbf{CT}^{-1}$

>> Az=T*A*T^-1, Bz=T*B, Cz=C*T^-1, Dz=0, eig(Az) = $-2, -3, -4$

>> Kz=acker(Az, Bz, P) ➔ **Kz = -816 -947 480**

We found previously that, $\mathbf{K_z} = \mathbf{K_x}\mathbf{P}^{-1}$

>> Kz=Kx*T^-1 ➔ **Kz = -816 -947 480**