CSE 351 Signals and Systems **(Due: 01/04/20)**

Homework #1

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Problem 1: Determine whether below systems are linear or non-linear.

To find out if the systems are linear (or not), we need to test the following conditions on them.

1. Condition (say $\mathbf{F}_1$)

$$\boxed{f_1(t)} \longrightarrow \boxed{\quad\quad} \longrightarrow \boxed{y_1(t)} \qquad \boxed{f_2(t)} \longrightarrow \boxed{\quad\quad} \longrightarrow \boxed{y_2(t)}$$

2. Condition (say $\mathbf{F}_2$)

$$\boxed{k_1 f_1(t) + k_2 f_2(t)} \longrightarrow \boxed{\quad\quad} \longrightarrow \boxed{k_1 y_1(t) + k_2 y_2(t)}$$

Let's test them.

(a) $\dfrac{dy}{dt} + 2y(t) = f^2(t)$

$$\frac{dy_1}{dt} + 2y_1(t) = {f_1}^2(t)$$
$$\frac{dk_1 y_1(t)}{dt} + 2k_1 y_1(t) = k_1 {f_1}^2(t)$$

$$\frac{dy_2}{dt} + 2y_2(t) = {f_2}^2(t)$$
$$\frac{dk_2 y_2(t)}{dt} + 2k_2 y_2(t) = k_2 {f_2}^2(t)$$

Add them up, and substitute the variables, then get the results of the first condition.

$$\frac{d}{dt}[k_1 y_1(t) + k_2 y_2(t)] + 2[k_1 y_1(t) + k_2 y_2(t)] = k_1 f_1^2(t) + k_2 f_2^2(t)$$
$$k_1 f_1^2(t) + k_2 f_2^2(t) = F_1(t)$$

Same thing for the second condition.

$$\frac{d}{dt}[k_1 y_1(t) + k_2 y_2(t)] + 2[k_1 y_1(t) + k_2 y_2(t)] = [k_1 f_1(t) + k_2 f_2]^2$$
$$k_1 f_1^2(t) + k_2 f_2^2(t) = F_2(t)$$

Hence, $F_1(T) \neq F_2(T)$. System is non-linear.

(b) $\dfrac{dy}{dt} + 3ty(t) = t^2 f(t)$

Again, we test the conditions.

$$\frac{dy_1}{dt} + 3ty_1(t) = t^2 f_1(t)$$
$$\frac{dk_1 y_1(t)}{dt} + 3tk_1 y_1(t) = t^2 k_1 f_1(t)$$

$$\frac{dy_2}{dt} + 3ty_2(t) = t^2 f_2(t)$$
$$\frac{dk_2 y_2(t)}{dt} + 3tk_2 y_2(t) = t^2 k_2 f_2(t)$$

Add them up.

$$\frac{d}{dt}[k_1 y_1(t) + k_2 y_2(t)] + 3t[k_1 y_1(t) + k_2 y_2(t)] = t^2[k_1 f_1(t) + k_2 f_2(t)]$$
$$k_1 f_1^2(t) + k_2 f_2^2(t) = F_1(t)$$

$$\frac{d}{dt}[k_1 y_1(t) + k_2 y_2(t)] + 2[k_1 y_1(t) + k_2 y_2(t)] = t^2[k_1 f_1(t) + k_2 f_2]^2$$
$$k_1 f_1^2(t) + k_2 f_2^2(t) = F_1(t)$$

Hence, $F_1(T) = F_2(T)$. System is linear.

Problem 2:

(a) For the LTIC system with the below system equation, find the zero- input response ($y_0(t)$) where the initial conditions are $y_0(0) = 2$ and $\dfrac{dy_0(0)}{dt} = -1$.
$(D^2 + 5D + 6)y(t) = (D + 1)f(t)$.

Let's find the characteristic equation.

$$\lambda^2 + 5\lambda + 6 = 0$$
$$(\lambda + 2)(\lambda + 3) = 0$$

$\lambda_1 = -2$, $\lambda_2 = -3$.

$$y_0(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$
$$= c_1 e^{-2t} + c_2 e^{-3t}$$
$$y_0'(t) = -2c_1 e^{-2t} - 3c_2 e^{-3t}$$

Let's substitute for given, $y_0(0) = 2$, $y_0'(0) = -1$.

$$c_1 + c_2 = 2$$
$$-2c_1 - 3c_2 = 1$$

$c_1 = 5$, $c_2 = -3$. Hence, $y_0(t) = 5e^{-2t} - 3e{-}3t$.

(b) For the LTIC system with the unit impulse response of $h(t) = e^{-t}u(t)$. Find the zero state response of the system $y(t)$ if input is $f(t) = u(t)$.

Let's find the convolution of those functions.

$$\begin{aligned} y(t) &= h(t) * f(t) \\ &= \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} u(\tau)u(t-\tau)e^{\tau-t}d\tau \end{aligned}$$

Let's call, $A = u(\tau)u(t-\tau)e^{\tau-t}d\tau$. Then;

$$\begin{aligned} \int_{-\infty}^{\infty} A &= \int_{-\infty}^{0} A + \int_{0}^{t} A + \int_{t}^{\infty} A \\ u(t)(t-\tau)u &\neq 0 \\ \textbf{only for } 0 \leq \tau \leq t \rightarrow u(t)(t-\tau)u &= 1 \\ y(t) &= \int_0^t A \\ y(t) &= \int_0^t u(\tau)u(t-\tau)e^{\tau-t}d\tau \\ &= \int_0^t e^{\tau-t}d\tau \\ &= e^{-t}\int_0^t e^{\tau}d\tau \\ &= e^{-t}(e^t-1) \\ &= 1-e^t \end{aligned}$$

Hence, $y(t) = 1 - e^t$.

Problem 3:

(a) Find the unit impulse response $h[k]$ of the following system: $y[k+1] + 2y[k] = f[k]$.
We know, $h[k] = \dfrac{b_0}{a_0}\delta[k] + y_0[k]u[k]$. We can write the system as, $E + 2y[k] = f[k]$. Let's find the characteristic equation.

$$\begin{aligned} (\gamma + 2) &= 0 \\ \gamma &= -2 \\ y_0(k) &= c(\gamma)^k \\ &= c(-2)^k \end{aligned}$$

$a_0 = 2$, $b_0 = 1$, hence $h[k] = \frac{1}{2}\delta[k] + c(-2)^k$. Approaching iteratively, to solve c;

$$\begin{aligned} (E+2)h[k] &= \delta[k] \\ h[k+1] + 2h[k] &= \delta[k] \end{aligned}$$

Setting $k = -1$, we get $\rightarrow h[0] = 0$. Setting $k = 0$ and $h[0] = 0$, we get $\rightarrow c = -\frac{1}{2}$.

Hence, we get as a result, $h[k] = \frac{1}{2}\delta[k] - \frac{1}{2}(-2)^k u(k)$

(b) Determine the zero-state response of the LTID system with the unit impulse response of $h[k] = (-2)^k u[k]$

if the input $f[k] = e^{-k}u[k]$.

Let's apply convolution sum. y[t] = f[k] * h[k]

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\begin{aligned}
y[k] &= \sum_{n=-\infty}^{\infty} f[n].h[k-n] \\
&= \sum_{n=-\infty}^{\infty} e^{-n}u[n](-2)^{n-k}u[n-k] \\
&= \sum_{n=-\infty}^{\infty} e-n(-2)^{n-k} \\
&= (-2)^{k}\sum_{n=0}^{k} e^{-n}(-2)^{n} \\
&= (-2)^{-k}\sum_{n=0}^{k}(\frac{-2}{e})^{n} \\
&= (-2)^{-k}\frac{1-(\frac{-2}{e})^{k+1}}{1+\frac{2}{e}} \\
&= (-2)^{-k}e\frac{1-(\frac{-2}{e})^{k+1}}{e+2} \\
&= \frac{(-2)^{-k}e-(-2)e^{-k}}{e+2} \\
&= 0.576(-2)^{-k}+0.423e^{-k}
\end{aligned}
$$

Or without the numbers substituted right in, it can be restated as $[\frac{(\frac{1}{e})^{k+1}-(-2)^{k+1}}{\frac{1}{e}+2}]u(k)$.