Heading numberings follow that in the supplementary material.

S1. White-noise environments

S1.1 Binary model

In[1]:=
```
(* Setup model *)
μ[ϵ_] := If[ϵ === 1, μ₁, μ₂]
s₁[ϵ_] := If[ϵ === 1, σ₁, ρ σ₂]
s₂[ϵ_] := If[ϵ === 1, 0, Sqrt[1 - ρ^2] σ₂]
p[ϵ_] := If[ϵ === 1, 1 - q, q]
```

In[5]:=
```
(* Fitness *)
ϕ[q_] := Evaluate[Sum[p[ϵ] * μ[ϵ], {ϵ, 1, 2}] -
    (Sum[p[ϵ] * s₁[ϵ], {ϵ, 1, 2}]^2 + Sum[p[ϵ] * s₂[ϵ], {ϵ, 1, 2}]^2)/2]
```

In[6]:=
```
Collect[FullSimplify[ϕ[q]], q]
```

Out[6]= $\mu_1 - \frac{\sigma_1^2}{2} + q\left(-\mu_1 + \mu_2 + \sigma_1^2 - \rho\,\sigma_1\,\sigma_2\right) + q^2\left(-\frac{\sigma_1^2}{2} + \rho\,\sigma_1\,\sigma_2 - \frac{\sigma_2^2}{2}\right)$

In[7]:=
```
(* q-star *)
```

In[8]:=
```
qstar = q /. Solve[D[ϕ[q], {q, 1}] == 0, q][[1]]
```

Out[8]= $\frac{-\mu_1 + \mu_2 + \sigma_1^2 - \rho\,\sigma_1\,\sigma_2}{\sigma_1^2 - 2\,\rho\,\sigma_1\,\sigma_2 + \sigma_2^2}$

In[9]:=
```
Expand[D[ϕ[q], {q, 2}]]
```

Out[9]= $-\sigma_1^2 + 2\,\rho\,\sigma_1\,\sigma_2 - \sigma_2^2$

S1.2 Binary model (finite switching speed)

In[10]:=
```
(* Full system *)
m[r1_, r2_] := {{μ1 r1 - b r1 + a r2}, {μ2 r2 + b r1 - a r2}}
g[r1_, r2_] := {{σ1 r1, 0}, {ρ σ2 r2, Sqrt[1 - ρ^2] σ2 r2}}
```

In[12]:=
```
(* x(t) *)
func[r1_, r2_] := r2/(r1 + r2)
```

In[13]:=
```
(* Transformed drift and diffusion for x(t) *)
fx[x_] := Evaluate[FullSimplify[
    (D[func[r1, r2], {{r1, r2}}].m[r1, r2] +
        1/2 Tr[g[r1, r2]ᵀ.D[func[r1, r2], {{r1, r2}, 2}].g[r1, r2]])[[1]]
    /. {r1 → (1 - x) n, r2 → x n}]];
fx[x]
```

Out[13]=

$$b - a\,x - b\,x + (-1 + x)\,x\,\left(\mu 1 - \mu 2 + (-1 + x)\,\sigma 1^2 + (1 - 2\,x)\,\rho\,\sigma 1\,\sigma 2 + x\,\sigma 2^2\right)$$

In[14]:=
```
diffRaw =
  FullSimplify[D[func[r1, r2], {{r1, r2}}]ᵀ.g[r1, r2] /. {r1 → (1 - x) n, r2 → x n}];
gx[x_] := Evaluate[
  FullSimplify[Sqrt[Total[diffRaw²]], Assumptions → x > 0 && σ2 > 0 && x < 1]];
gx[x]
```

Out[15]=

$$-\left((-1 + x)\,x\,\sqrt{\sigma 1^2 - 2\,\rho\,\sigma 1\,\sigma 2 + \sigma 2^2}\right)$$

In[16]:=
```
(* Analytical expression for the exponent *)
Integrate[fx[x] / gx[x], x]
```

Out[16]=

$$-\frac{x\,\left(\mu 1 - \mu 2 - \sigma 1^2 + \rho\,\sigma 1\,\sigma 2\right) + \frac{1}{2}\,x^2\,\left(\sigma 1^2 - 2\,\rho\,\sigma 1\,\sigma 2 + \sigma 2^2\right) - a\,\mathrm{Log}[1 - x] - b\,\mathrm{Log}[x]}{\sqrt{\sigma 1^2 - 2\,\rho\,\sigma 1\,\sigma 2 + \sigma 2^2}}$$

S1.3 Unimodal continuous heterogeneity

ρ differentiable; optimum at $\epsilon = 1$

In[17]:=
```
(* Assumptions *)
ClearAll["Global`*"]
f[x_] := 2 PDF[NormalDistribution[1, η], x]
$Assumptions = η > 0 && ρ[0] == 1 && ρ'[0] == 0;
```

In[20]:=
```
(* Series and expectation for s₁ *)
s₁[x_] := ρ[x - 1] × σ[x];
E₁ = Integrate[f[x] * Normal[Series[s₁[x], {x, 1, 2}]], {x, -∞, 1}];
```

In[22]:=
```
(* Series and expectation for s₂ *)
s₂[x_] := σ[x] * Sqrt[1 - ρ[x - 1]²];
E₂ = Integrate[f[x] * Normal[Series[s₂[x], {x, 1, 2}]], {x, -∞, 1}];
```

In[24]:=
```
(* Fitness *)
ϕ = FullSimplify[Series[Integrate[x * f[x], {x, -∞, 1}] - 1/2 (E1^2 + E2^2), {η, 0, 1}]]
```

Out[24]=

$$\left(1 - \frac{\sigma[1]^2}{2}\right) + \sqrt{\frac{2}{\pi}}\ (-1 + \sigma[1]\ \sigma'[1])\ \eta + O[\eta]^2$$

ρ differentiable; optimum at $0 < \epsilon < 1$

In[25]:=
```
(* Assumptions *)
ClearAll["Global`*"]
f[x_] := PDF[NormalDistribution[ϵ, η], x]
$Assumptions = η > 0 && ρ[0] == 1 && ρ'[0] == 0;
```

In[28]:=
```
(* Series and expectation for s1 *)
s1[x_] := ρ[x - ϵ] × σ[x];
E1 = Integrate[f[x] * Normal[Series[s1[x], {x, ϵ, 2}]], {x, -∞, ∞}]
```

Out[29]=

$$\frac{1}{2}\left(\sigma[\epsilon]\ \left(2 + \eta^2\ \rho''[0]\right) + \eta^2\ \sigma''[\epsilon]\right)$$

In[30]:=
```
(* Series and expectation for s2 *)
s2[x_] := σ[x] * Sqrt[1 - ρ[x - ϵ]^2];
E2 = Integrate[f[x] * FullSimplify[
        Normal[Series[s2[x], {x, ϵ, 2}]], Assumptions → x > ϵ], {x, ϵ, ∞}] +
    Integrate[f[x] * FullSimplify[
        Normal[Series[s2[x], {x, ϵ, 2}]], Assumptions → x < ϵ], {x, -∞, ϵ}];
E2 = FullSimplify[E2];
```

In[33]:=
```
(* Fitness *)
ϕ = FullSimplify[Series[Integrate[x * f[x], {x, -∞, ∞}] - 1/2 (E1^2 + E2^2), {η, 0, 2}]]
```

Out[33]=

$$\left(\epsilon - \frac{\sigma[\epsilon]^2}{2}\right) - \frac{(\sigma[\epsilon]\ ((-2 + \pi)\ \sigma[\epsilon]\ \rho''[0] + \pi\ \sigma''[\epsilon]))\ \eta^2}{2\ \pi} + O[\eta]^3$$

ρ *not* differentiable; optimum at $\epsilon = 1$

In[34]:=
```
(* Assumptions *)
ClearAll["Global`*"]
f[x_] := 2 PDF[NormalDistribution[1, η], x]
$Assumptions = η > 0 && ρ[0] == 1;
```

In[37]:=
```
(* Series and expectation for s1 *)
s1[x_] := ρ[x - 1] × σ[x];
E1 = Integrate[f[x] * Normal[Series[s1[x], {x, 1, 2}]], {x, -∞, 1}];
```

In[39]:=
```
(* Series and expectation for s2 *)
s2[x_] := σ[x] * Sqrt[1 - ρ[x - 1]^2];
E2 = Integrate[f[x] * Normal[Series[s2[x], {x, 1, 2}]], {x, -∞, 1}];
```

In[41]:=
```
(* Fitness *)
```
$$\phi = \text{FullSimplify}\left[\text{Series}\left[\text{Integrate}[x*f[x], \{x, -\infty, 1\}] - \frac{1}{2}\left(E_1{}^2 + E_2{}^2\right), \{\eta, 0, 1\}\right]\right]$$

Out[41]=
$$\left(1 - \frac{\sigma[1]^2}{2}\right) + \frac{\sqrt{2}\left(-\text{Gamma}\left[\frac{3}{4}\right]^2 \sigma[1]^2 \rho'[0] + \sqrt{\pi}\,\left(-1 + \sigma[1]\,\left(\sigma[1]\,\rho'[0] + \sigma'[1]\right)\right)\right)\eta}{\pi} + O[\eta]^2$$

ρ *not* differentiable; optimum at $0 < \epsilon < 1$

In[42]:=
```
(* Assumptions *)
ClearAll["Global`*"]
f[x_] := PDF[NormalDistribution[ϵ, η], x]
$Assumptions = η > 0 && ρ[0] == 1;
```

In[45]:=
```
(* Series and expectation for s1 *)
s11[x_] := ρ[x - ϵ] × σ[x];
s12[x_] := ρ[ϵ - x] × σ[x];
E1 = Integrate[f[x] * Normal[Series[s11[x], {x, ϵ, 2}]], {x, -∞, ϵ}] +
  Integrate[f[x] * Normal[Series[s12[x], {x, ϵ, 2}]], {x, ϵ, ∞}]
```

Out[47]=
$$\sigma[\epsilon] + \frac{\eta\,(-\sigma[\epsilon]\,\rho'[0] + \sigma'[\epsilon])}{\sqrt{2\,\pi}} - \frac{\eta\,(\sigma[\epsilon]\,\rho'[0] + \sigma'[\epsilon])}{\sqrt{2\,\pi}} +$$
$$\frac{1}{4}\eta^2\,(-2\,\rho'[0]\,\sigma'[\epsilon] + \sigma[\epsilon]\,\rho''[0] + \sigma''[\epsilon]) + \frac{1}{4}\eta^2\,(2\,\rho'[0]\,\sigma'[\epsilon] + \sigma[\epsilon]\,\rho''[0] + \sigma''[\epsilon])$$

In[48]:=
```
(* Series and expectation for s2 *)
```

In[49]:=
```
s21[x_] := σ[x] * Sqrt[1 - ρ[x - ϵ]^2];
s22[x_] := σ[x] * Sqrt[1 - ρ[ϵ - x]^2];
E2 = Integrate[f[x] * Normal[Series[s21[x], {x, ϵ, 2}]], {x, -∞, ϵ}] +
  Integrate[f[x] * Normal[Series[s22[x], {x, ϵ, 2}]], {x, ϵ, ∞}]
```

Out[51]=
$$\frac{\eta\left(2\,\sqrt{2}\,\text{Gamma}\left[\frac{3}{4}\right]\sigma[\epsilon]\,\rho'[0] - \eta\,\text{Gamma}\left[\frac{5}{4}\right]\left(-4\,\rho'[0]\,\sigma'[\epsilon] + \sigma[\epsilon]\,\left(\rho'[0]^2 + \rho''[0]\right)\right)\right)}{2 \times 2^{3/4}\,\sqrt{\pi}\,\sqrt{\eta\,\rho'[0]}} +$$
$$\frac{\eta\left(2\,\sqrt{2}\,\text{Gamma}\left[\frac{3}{4}\right]\sigma[\epsilon]\,\rho'[0] - \eta\,\text{Gamma}\left[\frac{5}{4}\right]\left(4\,\rho'[0]\,\sigma'[\epsilon] + \sigma[\epsilon]\,\left(\rho'[0]^2 + \rho''[0]\right)\right)\right)}{2 \times 2^{3/4}\,\sqrt{\pi}\,\sqrt{\eta\,\rho'[0]}}$$

In[52]:=
```
(* Fitness *)
```
$$\phi = \text{FullSimplify}\left[\text{Series}\left[\text{Integrate}[x*f[x], \{x, -\infty, \infty\}] - \frac{1}{2}\left(E_1{}^2 + E_2{}^2\right), \{\eta, 0, 1\}\right]\right]$$

Out[52]=
$$\left(\epsilon - \frac{\sigma[\epsilon]^2}{2}\right) + \frac{\sqrt{2}\left(\sqrt{\pi} - \text{Gamma}\left[\frac{3}{4}\right]^2\right)\sigma[\epsilon]^2\,\rho'[0]\,\eta}{\pi} + O[\eta]^2$$

S1.4 Bimodal continuous heterogeneity

In[53]:=
```
(* Assumptions *)
ClearAll["Global`*"]
f[x_] := 2
$Assumptions = η1 > 0 && η2 > 0 && ρ[0] == 1 && ρ'[0] == 0;
```

In[56]:=
```
(* Series *)
s1[x_] := ρ[x - 1] × σ[x];
s2[x_] := σ[x] * Sqrt[1 - ρ[x - 1]^2];
```

In[58]:=
```
(* Distributions *)
f0[x_] := 2 PDF[NormalDistribution[0, η1], x]
f1[x_] := 2 PDF[NormalDistribution[1, η2], x]
f[x_] := q f0[x] + (1 - q) f1[x]
```

In[61]:=
```
(* Expectations near ϵ = 0 *)
E00 = Integrate[f0[x] Normal[Series[μ[x], {x, 0, 2}]], {x, 0, ∞}];
E10 = Integrate[f0[x] Normal[Series[s1[x], {x, 0, 2}]], {x, 0, ∞}];
E20 = Integrate[f0[x] Normal[Series[s2[x], {x, 0, 2}]], {x, 0, ∞}];
```

In[64]:=
```
(* Expectations near ϵ = 1 *)
E01 = Integrate[f1[x] Normal[Series[μ[x], {x, 1, 2}]], {x, -∞, 1}];
E11 = Integrate[f1[x] Normal[Series[s1[x], {x, 1, 2}]], {x, -∞, 1}];
E21 = Integrate[f1[x] Normal[Series[s2[x], {x, 1, 2}]], {x, -∞, 1}];
```

In[67]:=
```
(* Expectations *)
E0 = q E00 + (1 - q) E01;
E1 = q E10 + (1 - q) E11;
E2 = q E20 + (1 - q) E21;
```

In[70]:=
```
(* Fitness and coefficients *)
ϕ = E0 - 1/2 (E1^2 + E2^2);
```

In[71]:=
```
c0 = FullSimplify[ϕ /. {η1 → 0, η2 → 0}]
```

Out[71]=

$$\mu[1] - \frac{1}{2} q \left(-2\,\mu[0] + 2\,\mu[1] + q\,\sigma[0]^2\right) + (-1+q)\, q\, \rho[-1] \times \sigma[0] \times \sigma[1] - \frac{1}{2} (-1+q)^2\, \sigma[1]^2$$

In[72]:=
```
c1 = FullSimplify[D[ϕ, η1] /. {η1 → 0, η2 → 0}]
```

Out[72]=

$$\sqrt{\frac{2}{\pi}}\; q\, \left(\mu'[0] + (-1+q)\, \rho[-1] \times \sigma[1]\, \sigma'[0] + \sigma[0]\, \left((-1+q)\, \sigma[1]\, \rho'[-1] - q\, \sigma'[0]\right)\right)$$

In[73]:=
```
c2 = FullSimplify[D[ϕ, η2] /. {η1 → 0, η2 → 0}]
```

Out[73]=

$$\sqrt{\frac{2}{\pi}}\; (1-q)$$

$$\left(-\mu'[1] + (q\,\rho[-1] \times \sigma[0] + \sigma[1] - q\,\sigma[1])\, \sigma'[1] - q\, \sqrt{1-\rho[-1]^2}\; \sigma[0] \times \sigma[1]\, \sqrt{-\rho''[0]}\right)$$

In[74]:=

$$\frac{c_2}{(1-q)\ \text{Sqrt}[2\ /\ \pi]}$$

Out[74]=

$$-\mu'[1] + (q\,\rho[-1] \times \sigma[0] + \sigma[1] - q\,\sigma[1])\ \sigma'[1] - q\,\sqrt{1-\rho[-1]^2}\ \sigma[0] \times \sigma[1]\ \sqrt{-\rho''[0]}$$

S2. Poisson environments

S2.2 Binary model (QSS)

In[75]:=

```
(* Fx equation *)
fx[x_, λ_] := b - x (a - b - λ (1 - x))
```

In[76]:=

```
(* Solve QSS *)
Solve[fx[x, λ] == 0, x]
```

Out[76]=

$$\left\{\left\{x \to \frac{-a+b+\lambda-\sqrt{4\,b\,\lambda+(-a+b+\lambda)^2}}{2\,\lambda}\right\}, \left\{x \to \frac{-a+b+\lambda+\sqrt{4\,b\,\lambda+(-a+b+\lambda)^2}}{2\,\lambda}\right\}\right\}$$

S3. Continuously fluctuating environments

S3.2 Bimodal continuous fluctuations

In[77]:=

```
ClearAll["Global`*"]; Remove["Global`*"]; $Assumptions = True;
```

Notation:

- $E1 = E_{\tilde{p}_1}\ (\epsilon) \sim O\ (\eta_1)$
- $E2 = E_{\tilde{p}_2}\ ((1-\epsilon))\ = 1\ -\ E_{\tilde{p}_2}\ (\epsilon) \sim O\ (\eta_2)$
- $V1 = E_{\tilde{p}_1}\ \left(\epsilon^2\right) \sim O\ \left(\eta_1{}^2\right)$, and similar for V2.

In[78]:=

```
(* Governing equation and mixture substitution *)
sys0 = p̃ (λ - 𝔼λ - ω) + ω p̂;
sub0 = {p̂ → q p̂₁ + (1 - q) p̂₂, p̃ → γ p̃₁ + (1 - γ) p̃₂};
sys1 = sys0 /. sub0
```

Out[80]=

$$\omega\ \left(q\,\hat{p}_1 + (1-q)\ \hat{p}_2\right) + (-\mathbb{E}\lambda + \lambda - \omega)\ \left(\gamma\,\tilde{p}_1 + (1-\gamma)\ \tilde{p}_2\right)$$

In[81]:=

```
(* Expansions for λ and 𝔼(λ) *)
sub1 = {
   𝔼λ₁ → Normal[Series[λ[ϵ], {ϵ, 0, 1}]] /. ϵ → E1,
   𝔼λ₂ → Normal[Series[λ[ϵ], {ϵ, 1, 1}]] /. ϵ → 1 - E2
  };
sub2 = 𝔼λ → γ 𝔼λ₁ + (1 - γ) 𝔼λ₂ /. sub1;
```

In[83]:=
```
(* Near ϵ = 0 *)
eq1 = sys1 /. {p̂₂ → 0, p̃₂ → 0} /. {λ → Normal[Series[λ[ϵ], {ϵ, 0, 1}]]} /. sub2;
eq1 = Expand[eq1 * ϵ];
eq1 = eq1 /. ϵ^2 p̃₁ → V1 /. ϵ p̃₁ → E1 /. ϵ p̂₁ → Sqrt[2/π] η₁;
```

In[86]:=
```
(* Near ϵ = 1 *)
eq2 = sys1 /. {p̂₁ → 0, p̃₁ → 0} /. {λ → Normal[Series[λ[ϵ], {ϵ, 1, 1}]]} /. sub2;
eq2 = Expand[eq2 * (1 - ϵ)];
eq2 =
  eq2 /. ϵ^2 p̃₂ → V2 + 1 - 2 E2 /. ϵ p̃₂ → 1 - E2 /. ϵ p̂₂ → (1 - Sqrt[2/π] η₂) /. p̃₂ → 1 /. p̂₂ → 1;
```

In[89]:=
```
(* γ (i.e., mixture weightings) as a function of Δλ *)
γqss[Δλ_] :=
  FullSimplify[(a + b + Δλ - Sqrt[(a + b + Δλ)^2 - 4 b Δλ])/(2 Δλ) /. {a → ω (1 - q), b → ω q}];
```

In[90]:=
```
(* Substitute asymptotic expansion for E1 and E2 *)
sub3 = {E1 → a1 η₁, E2 → a2 η₂};
```

In[91]:=
```
(* Full system, up to O(ηᵢ)*)
sys2 =
 FullSimplify[Expand[{eq1, eq2}] /. {E1 E2 → 0, E1^2 → 0, E2^2 → 0, V1 → 0, V2 → 0}]
```

Out[91]=

$$\left\{\sqrt{\frac{2}{\pi}}\, q\, \omega\, \eta_1 - \mathrm{E1}\, \gamma\, (\omega + (-1+\gamma)\, (\lambda[0] - \lambda[1])),\right.$$
$$\left. -\sqrt{\frac{2}{\pi}}\, (-1+q)\, \omega\, \eta_2 + \mathrm{E2}\, (-1+\gamma)\, (\omega + \gamma\, (\lambda[0] - \lambda[1]))\right\}$$

In[92]:=
```
(* System following substitution of γ *)
sys3 =
  FullSimplify[sys2 /. γ → γqss[𝔼λ₂ - 𝔼λ₁ /. sub1] /. {λ[0] → 0, λ[1] → Δλ}] /. sub3;
```

In[93]:=
```
(* Obtain get correction terms... *)
sub4 = Solve[{
    0 == Limit[D[sys3, η₁], {η₁ → 0, η₂ → 0}][[1]],
    0 == Limit[D[sys3, η₂], {η₁ → 0, η₂ → 0}][[2]]}, {a1, a2}][[1]]
```

Out[93]=

$$\left\{\mathrm{a1} \to \sqrt{\frac{2}{\pi}},\ \mathrm{a2} \to \sqrt{\frac{2}{\pi}}\right\}$$

In[94]:=
```
(* Expression for 𝔼(λ) *)
sol =
  𝔼λ /. sub2 /. γ → γqss[𝔼λ₂ - 𝔼λ₁ /. sub1] /. sub3 /. sub4 /. {λ[0] → 0, λ[1] → Δλ};
```

In[95]:=
```
(* Coefficients *)
coef = {
  c₀ → Limit[sol, {η₁ → 0, η₂ → 0}],
  c₁ → Limit[D[sol, η₁], {η₁ → 0, η₂ → 0}],
  c₂ → Limit[D[sol, η₂], {η₁ → 0, η₂ → 0}]}
```

Out[95]=

$$\left\{c_0 \to \frac{1}{2}\left(\Delta\lambda - \omega + \sqrt{\Delta\lambda^2 + (2-4\,q)\,\Delta\lambda\,\omega + \omega^2}\right),\right.$$

$$c_1 \to \frac{\left(-\Delta\lambda - \omega + 2\,q\,\omega + \sqrt{\Delta\lambda^2 + (2-4\,q)\,\Delta\lambda\,\omega + \omega^2}\right)\lambda'[0]}{\sqrt{2\,\pi}\,\sqrt{\Delta\lambda^2 + (2-4\,q)\,\Delta\lambda\,\omega + \omega^2}},$$

$$\left.c_2 \to -\frac{\left(\Delta\lambda + \omega - 2\,q\,\omega + \sqrt{\Delta\lambda^2 + (2-4\,q)\,\Delta\lambda\,\omega + \omega^2}\right)\lambda'[1]}{\sqrt{2\,\pi}\,\sqrt{\Delta\lambda^2 + (2-4\,q)\,\Delta\lambda\,\omega + \omega^2}}\right\}$$

In[96]:=
```
FullSimplify[coef /. Δλ^2 + (2 - 4 q) Δλ ω + ω^2 → Δ]
```

Out[96]=

$$\left\{c_0 \to \frac{1}{2}\left(\sqrt{\Delta} + \Delta\lambda - \omega\right),\ c_1 \to \frac{\left(\sqrt{\Delta} - \Delta\lambda + (-1+2\,q)\,\omega\right)\lambda'[0]}{\sqrt{2\,\pi}\,\sqrt{\Delta}},\ c_2 \to -\frac{\left(\sqrt{\Delta} + \Delta\lambda + \omega - 2\,q\,\omega\right)\lambda'[1]}{\sqrt{2\,\pi}\,\sqrt{\Delta}}\right\}$$

S3.2 Bimodal continuous fluctuations

In[97]:=
```
ClearAll["Global`*"]; Remove["Global`*"];
$Assumptions = η > 0 && ξ > 0 && σ > 0;
```

A. General case

In[99]:=
```
(* Governing equation *)
sys0 = p̃ (λ - 𝔼λ - ω) + ω p̂;
```

In[100]:=
```
(* Target distribution moments *)
Meval[n_] := Integrate[ϵ^n * PDF[NormalDistribution[0, η], ϵ], {ϵ, -∞, ∞}]
```

In[101]:=
```
(* Substitutions for λ *)
sub0 = λ → Normal[Series[λ[ϵ], {ϵ, 0, 2}]];
sub1 = 𝔼λ → λ /. sub0 /. {ϵ^2 → F₂, ϵ → F₁};
sys1 = Expand[sys0 * ϵ^n /. sub0 /. sub1]
```

Out[103]=

$$\hat{p}\,\epsilon^n\,\omega - \tilde{p}\,\epsilon^n\,\omega + \tilde{p}\,\epsilon^{1+n}\,\lambda'[0] - \tilde{p}\,\epsilon^n\,F_1\,\lambda'[0] + \frac{1}{2}\,\tilde{p}\,\epsilon^{2+n}\,\lambda''[0] - \frac{1}{2}\,\tilde{p}\,\epsilon^n\,F_2\,\lambda''[0]$$

In[104]:=

```
(* Integrate (substitute moments) *)
```

$$\text{sys2} = \text{sys1} \,/.\, \left\{\hat{p}\, \epsilon^{n} \to M_n,\ \tilde{p}\, \epsilon^{2+n} \to F_{n+2},\ \tilde{p}\, \epsilon^{1+n} \to F_{n+1},\ \tilde{p}\, \epsilon^{n} \to F_n\right\}$$

Out[104]=

$$-\omega F_n + \omega M_n - F_1 F_n \lambda'[0] + F_{1+n} \lambda'[0] - \frac{1}{2} F_2 F_n \lambda''[0] + \frac{1}{2} F_{2+n} \lambda''[0]$$

In[105]:=

```
(* Order n = 1 *)
```

$$\text{test0} = \text{sys2} \,/.\, n \to 1 \,/.\, M_1 \to \text{Meval}[1] \,/.\, \left\{F_1{}^2 \to 0,\ F_2 \to 0,\ F_3 \to 0\right\}$$

Out[105]=

$$-\omega F_1$$

In[106]:=

```
(* Order n = 2 *)
```

$$\text{sys3} = \text{sys2} \,/.\, \{\{n \to 1\}, \{n \to 2\}\} \,/.\, \left\{F_1{}^2 \to 0,\ F_1 F_2 \to 0,\ F_3 \to 0,\ F_2{}^2 \to 0,\ F_4 \to 0\right\} \,/.\, \{M_2 \to \text{Meval}[2],\ M_1 \to \text{Meval}[1]\}$$

Out[106]=

$$\left\{-\omega F_1 + F_2 \lambda'[0],\ \eta^2 \omega - \omega F_2\right\}$$

In[107]:=

```
(* Solve... *)
```

$$\text{sol0} = \text{Solve}[\text{Table}[0 == \text{expr}, \{\text{expr}, \text{sys3}\}], \{F_1, F_2\}][[1]]$$

Out[107]=

$$\left\{F_1 \to \frac{\eta^2 \lambda'[0]}{\omega},\ F_2 \to \eta^2\right\}$$

In[108]:=

```
(* Expression for Eλ *)
```

$$\text{sol1} = \mathbb{E}\lambda \,/.\, \text{FullSimplify}[\text{sub1} \,/.\, \text{sol0}]$$

Out[108]=

$$\lambda[0] + \frac{1}{2} \eta^2 \left(\frac{2 \lambda'[0]^2}{\omega} + \lambda''[0]\right)$$

B. Specific model

In[109]:=

```
(* Setup form of λ, equilibrium environmental state distribution *)
```

$$\lambda[\epsilon\_, w\_] := k\, \text{Exp}\left[\frac{-(\epsilon - w)^2}{\xi^2}\right];$$

```
fw[w_] := PDF[NormalDistribution[0, σ], w];
```

In[111]:=

```
(* Coefficients as a function of w *)
```

$$c_0[w\_] := \lambda[0, w]$$

$$c_2[w\_] := \left(\frac{D[\lambda[\epsilon, w], \epsilon]^2}{\omega} + \frac{D[\lambda[\epsilon, w], \{\epsilon, 2\}]}{2}\right) /.\ \epsilon \to 0$$

In[113]:=

```
(* Integrate coefficients *)
coef = FullSimplify[{
   C₀ → Integrate[c₀[w] * fw[w], {w, -∞, ∞}],
   C₂ → Integrate[c₂[w] * fw[w], {w, -∞, ∞}]}]
```

Out[113]=

$$\left\{C_0 \to \frac{k\,\xi}{\sqrt{\xi^2 + 2\,\sigma^2}},\ C_2 \to -\frac{k\,\xi}{\left(\xi^2 + 2\,\sigma^2\right)^{3/2}} + \frac{4\,k^2\,\sigma^2}{\xi\,\left(\xi^2 + 4\,\sigma^2\right)^{3/2}\,\omega}\right\}$$