## CLE model

In[1]:=
```
(* Setup the model *)
$Assumptions = x[t] > 0 && y[t] > 0;
a[x_] := {α x[[1]]^2, β x[[2]], γ, δ x[[2]], ϵ , ζ x[[1]]};
ν = {{-2, 2, 0, 0, 1, -1}, {1, -1, 1, -1, 0, 0}};
F[x_] := FullSimplify[a[x].ν^T]
G[x_] := FullSimplify[ν.DiagonalMatrix[Sqrt[a[x]]]]
```

In[6]:=
```
Expand[F[{x, y}]]
```

Out[6]=

$$\left\{-2 x^2 \alpha + 2 y \beta + \epsilon - x \zeta,\ x^2 \alpha - y \beta + \gamma - y \delta\right\}$$

In[7]:=
```
G[{x, y}]
```

Out[7]=

$$\left\{\left\{-2 \sqrt{x^2 \alpha},\ 2 \sqrt{y \beta},\ 0,\ 0,\ \sqrt{\epsilon},\ -\sqrt{x \zeta}\right\},\ \left\{\sqrt{x^2 \alpha},\ -\sqrt{y \beta},\ \sqrt{\gamma},\ -\sqrt{y \delta},\ 0,\ 0\right\}\right\}$$

In[8]:=
```
(* Moment equations *)
expr =
   D[x[t]^i y[t]^j, {{x[t], y[t]}}].F[{x[t], y[t]}] +
   1/2 Tr[G[{x[t], y[t]}]^T.D[x[t]^i y[t]^j, {{x[t], y[t]}, 2}].G[{x[t], y[t]}]];
mex[i_, j_] := Evaluate[FullSimplify[Expand[expr] /.
     Flatten[Table[x[t]^(i+p) y[t]^(j+q) → m_{i+p,j+q}[t], {p, -2, 4}, {q, -2, 4}]]]];
meq[i_, j_] := m_{i,j}'[t] == mex[i, j]
mzero[i_, j_] := m_{i,j}'[t] - mex[i, j]
msol[i_, j_] := FullSimplify[Solve[meq[i + 1, j - 1], m_{i,j}[t]][[1]] /. m_{0,0}[t] → 1 /.
  Table[D[m_{0,0}[t], {t, k}] → 0, {l, 1, 5}]]
```

In[13]:=
```
mex[i, j]
```

Out[13]=

$$\begin{aligned}\frac{1}{2}\Big(&(-1+i)\, i\, \epsilon\, m_{-2+i,j}[t] + 4\,(-1+i)\, i\, \beta\, m_{-2+i,1+j}[t] - 4\, i\, j\, \beta\, m_{-1+i,j}[t] + 2\, i\, \epsilon\, m_{-1+i,j}[t] - \\ & i\, \zeta\, m_{-1+i,j}[t] + i^2 \zeta\, m_{-1+i,j}[t] + 4\, i\, \beta\, m_{-1+i,1+j}[t] - j\, \gamma\, m_{i,-2+j}[t] + j^2 \gamma\, m_{i,-2+j}[t] - \\ & j\, \beta\, m_{i,-1+j}[t] + j^2 \beta\, m_{i,-1+j}[t] + 2\, j\, \gamma\, m_{i,-1+j}[t] - j\, \delta\, m_{i,-1+j}[t] + j^2 \delta\, m_{i,-1+j}[t] - \\ & 4\, i\, \alpha\, m_{i,j}[t] + 4\, i^2 \alpha\, m_{i,j}[t] - 2\, j\, \beta\, m_{i,j}[t] - 2\, j\, \delta\, m_{i,j}[t] - 2\, i\, \zeta\, m_{i,j}[t] - \\ & 4\, i\, j\, \alpha\, m_{1+i,-1+j}[t] - 4\, i\, \alpha\, m_{1+i,j}[t] - j\, \alpha\, m_{2+i,-2+j}[t] + j^2 \alpha\, m_{2+i,-2+j}[t] + 2\, j\, \alpha\, m_{2+i,-1+j}[t]\Big)\end{aligned}$$

In[14]:=
```
(* Stencil *)
```

In[15]:=
```
vars = Quiet[Select[Variables[meq[i, j][[2]]], #[[0]][[1]] === m &]]
rows = Table[var[[0]][[2]], {var, vars}] /. i → 3;
cols = Table[var[[0]][[3]], {var, vars}] /. j → 3;
SparseArray[Table[{rows[[k]], cols[[k]]} → vars[[k]], {k, 1, Length[vars]}]] //
 MatrixForm
```

Out[15]=

$$\{m_{-2+i,j}[t], m_{-2+i,1+j}[t], m_{-1+i,j}[t], m_{-1+i,1+j}[t], m_{i,-2+j}[t], m_{i,-1+j}[t], m_{i,j}[t], m_{1+i,-1+j}[t], m_{1+i,j}[t], m_{2+i,-2+j}[t], m_{2+i,-1+j}[t]\}$$

Out[18]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & m_{-2+i,j}[t] & m_{-2+i,1+j}[t] \\ 0 & 0 & m_{-1+i,j}[t] & m_{-1+i,1+j}[t] \\ m_{i,-2+j}[t] & m_{i,-1+j}[t] & m_{i,j}[t] & 0 \\ 0 & m_{1+i,-1+j}[t] & m_{1+i,j}[t] & 0 \\ m_{2+i,-2+j}[t] & m_{2+i,-1+j}[t] & 0 & 0 \end{pmatrix}$$

## First order necessarily satisfied equation

In[19]:=
```
FullSimplify[2 β mzero[0, 1] /. msol[0, 1] /. D[msol[0, 1], t] /. m0,0[t] → 1]
```

Out[19]=

$$-2\beta\gamma - (\beta+\delta)\,\epsilon + (\beta+\delta)\,\zeta\, m_{1,0}[t] + 2\,\alpha\,\delta\, m_{2,0}[t] + (\beta+\delta+\zeta)\, m_{1,0}'[t] + 2\,\alpha\, m_{2,0}'[t] + m_{1,0}''[t]$$

## Second order necessarily satisfied equation

In[20]:=
```
expr2 = FullSimplify[
  8 β^2 mzero[0, 2] /. msol[0, 2] /. D[msol[0, 2], t] /. msol[2, 1] /. D[msol[2, 1],
       t] /. msol[1, 1] /. D[msol[1, 1], t] /. D[msol[1, 1], {t, 2}] /.
    msol[0, 1] /. D[msol[0, 1], t] /. D[msol[0, 1], {t, 2}] /. m0,0[t] → 1]
vars2 = Quiet[Select[Variables[expr2], #[[0]][[1]] === m || #[[0]][[1]][[1]] === m &]]
```

Out[20]=

$$\begin{aligned}
&-2\,\beta^2\,(4\,\gamma+\epsilon)+2\,\beta\,\epsilon\,(4\,\gamma+2\,\epsilon+\zeta)+\\
&2\,\delta\,\epsilon\,(-4\,\alpha+\delta+2\,\epsilon+\zeta)-2\,\Big(4\,\beta\,\gamma\,(\beta+\delta)+2\,\big(-2\,\alpha\,\delta+(\beta+\delta)^2\big)\,\epsilon+\\
&\quad\big(\beta^2+4\,\beta\,(\gamma+\delta+\epsilon)+\delta\,(-12\,\alpha+3\,\delta+4\,\epsilon)\big)\,\zeta+3\,(\beta+\delta)\,\zeta^2\Big)\,m_{1,0}[t]+\\
&4\,\big(16\,\alpha^2\,\delta+(\beta+\delta)\,\zeta\,(\beta+\delta+\zeta)-2\,\alpha\,(2\,\delta\,(\delta+\epsilon)+5\,\delta\,\zeta+\beta\,(2\,\gamma+\delta+\epsilon+2\,\zeta))\big)\,m_{2,0}[t]-\\
&64\,\alpha^2\,\delta\,m_{3,0}[t]+8\,\alpha\,\beta\,\delta\,m_{3,0}[t]+8\,\alpha\,\delta^2\,m_{3,0}[t]+8\,\alpha\,\beta\,\zeta\,m_{3,0}[t]+\\
&16\,\alpha\,\delta\,\zeta\,m_{3,0}[t]+16\,\alpha^2\,\delta\,m_{4,0}[t]-12\,\beta\,\gamma\,m_{1,0}'[t]+16\,\alpha\,\delta\,m_{1,0}'[t]-4\,\beta\,\delta\,m_{1,0}'[t]-\\
&4\,\delta^2\,m_{1,0}'[t]+4\,\alpha\,\epsilon\,m_{1,0}'[t]-10\,\beta\,\epsilon\,m_{1,0}'[t]-10\,\delta\,\epsilon\,m_{1,0}'[t]+12\,\alpha\,\zeta\,m_{1,0}'[t]-\\
&9\,\beta\,\zeta\,m_{1,0}'[t]-13\,\delta\,\zeta\,m_{1,0}'[t]-4\,\epsilon\,\zeta\,m_{1,0}'[t]-3\,\zeta^2\,m_{1,0}'[t]+32\,\alpha^2\,m_{2,0}'[t]-\\
&8\,\alpha\,\beta\,m_{2,0}'[t]+2\,\beta^2\,m_{2,0}'[t]-32\,\alpha\,\delta\,m_{2,0}'[t]+4\,\beta\,\delta\,m_{2,0}'[t]+2\,\delta^2\,m_{2,0}'[t]-\\
&8\,\alpha\,\epsilon\,m_{2,0}'[t]-20\,\alpha\,\zeta\,m_{2,0}'[t]+8\,\beta\,\zeta\,m_{2,0}'[t]+8\,\delta\,\zeta\,m_{2,0}'[t]+2\,\zeta^2\,m_{2,0}'[t]-\\
&32\,\alpha^2\,m_{3,0}'[t]+8\,\alpha\,\beta\,m_{3,0}'[t]+\frac{44}{3}\,\alpha\,\delta\,m_{3,0}'[t]+8\,\alpha\,\zeta\,m_{3,0}'[t]+8\,\alpha^2\,m_{4,0}'[t]+\\
&8\,\alpha\,m_{1,0}''[t]-2\,\beta\,m_{1,0}''[t]-6\,\delta\,m_{1,0}''[t]-4\,\epsilon\,m_{1,0}''[t]-5\,\zeta\,m_{1,0}''[t]-12\,\alpha\,m_{2,0}''[t]+\\
&3\,\beta\,m_{2,0}''[t]+3\,\delta\,m_{2,0}''[t]+3\,\zeta\,m_{2,0}''[t]+\frac{16}{3}\,\alpha\,m_{3,0}''[t]-2\,m_{1,0}^{(3)}[t]+m_{2,0}^{(3)}[t]
\end{aligned}$$

Out[21]=

$$\begin{aligned}
&\{m_{1,0}[t],\ m_{2,0}[t],\ m_{3,0}[t],\ m_{4,0}[t],\ m_{1,0}'[t],\ m_{2,0}'[t],\\
&\ m_{3,0}'[t],\ m_{4,0}'[t],\ m_{1,0}''[t],\ m_{2,0}''[t],\ m_{3,0}''[t],\ m_{1,0}^{(3)}[t],\ m_{2,0}^{(3)}[t]\}
\end{aligned}$$

In[22]:=
```
coef2 = Join[{expr2 /. Table[var → 0, {var, vars2}]}, Coefficient[expr2, vars2]]
```

Out[22]=

$$\begin{aligned}
&\Big\{-2\,\beta^2\,(4\,\gamma+\epsilon)+2\,\beta\,\epsilon\,(4\,\gamma+2\,\epsilon+\zeta)+2\,\delta\,\epsilon\,(-4\,\alpha+\delta+2\,\epsilon+\zeta),\\
&-2\,\Big(4\,\beta\,\gamma\,(\beta+\delta)+2\,\big(-2\,\alpha\,\delta+(\beta+\delta)^2\big)\,\epsilon+\\
&\quad\big(\beta^2+4\,\beta\,(\gamma+\delta+\epsilon)+\delta\,(-12\,\alpha+3\,\delta+4\,\epsilon)\big)\,\zeta+3\,(\beta+\delta)\,\zeta^2\Big),\\
&4\,\big(16\,\alpha^2\,\delta+(\beta+\delta)\,\zeta\,(\beta+\delta+\zeta)-2\,\alpha\,(2\,\delta\,(\delta+\epsilon)+5\,\delta\,\zeta+\beta\,(2\,\gamma+\delta+\epsilon+2\,\zeta))\big),\\
&-64\,\alpha^2\,\delta+8\,\alpha\,\beta\,\delta+8\,\alpha\,\delta^2+8\,\alpha\,\beta\,\zeta+16\,\alpha\,\delta\,\zeta,\ 16\,\alpha^2\,\delta,\\
&-12\,\beta\,\gamma+16\,\alpha\,\delta-4\,\beta\,\delta-4\,\delta^2+4\,\alpha\,\epsilon-10\,\beta\,\epsilon-10\,\delta\,\epsilon+12\,\alpha\,\zeta-9\,\beta\,\zeta-13\,\delta\,\zeta-4\,\epsilon\,\zeta-3\,\zeta^2,\\
&32\,\alpha^2-8\,\alpha\,\beta+2\,\beta^2-32\,\alpha\,\delta+4\,\beta\,\delta+2\,\delta^2-8\,\alpha\,\epsilon-20\,\alpha\,\zeta+8\,\beta\,\zeta+8\,\delta\,\zeta+2\,\zeta^2,\\
&-32\,\alpha^2+8\,\alpha\,\beta+\frac{44\,\alpha\,\delta}{3}+8\,\alpha\,\zeta,\ 8\,\alpha^2,\ 8\,\alpha-2\,\beta-6\,\delta-4\,\epsilon-5\,\zeta,\\
&-12\,\alpha+3\,\beta+3\,\delta+3\,\zeta,\ \frac{16\,\alpha}{3},\ -2,\ 1\Big\}
\end{aligned}$$