Simplified Lotka-Volterra model

In[1]:=
```
(* Setup the model *)
$Assumptions = b ≠ 0;
F[x_] := {a x[[1]] + b x[[2]], c x[[2]] + d x[[1]] × x[[2]]}
G[x_] := DiagonalMatrix[{p, s}]
```

In[4]:=
```
(* Moment equations *)
expr =
    D[x[t]^i y[t]^j, {{x[t], y[t]}}].F[{x[t], y[t]}] +
    1/2 Tr[G[{x[t], y[t]}]^T.D[x[t]^i y[t]^j, {{x[t], y[t]}, 2}].G[{x[t], y[t]}]];
mex[i_, j_] := Evaluate[FullSimplify[Expand[expr] /.
      Flatten[Table[x[t]^(i+p) y[t]^(j+q) → m_(i+p,j+q)[t], {p, -2, 4}, {q, -2, 4}]]]];
meq[i_, j_] := m_(i,j)'[t] == mex[i, j]
mzero[i_, j_] := m_(i,j)'[t] - mex[i, j]
msol[i_, j_] := FullSimplify[Solve[meq[i + 1, j - 1], m_(i,j)[t]][[1]]] /. m_(0,0)[t] → 1 /.
  Table[D[m_(0,0)[t], {t, k}] → 0, {l, 1, 5}]
```

In[9]:=
```
(* Look at stencil *)
```

In[10]:=
```
vars = Quiet[Select[Variables[meq[i, j][[2]], #[[0]][[1]] === m &]]
rows = Table[var[[0]][[2]], {var, vars}] /. i → 3;
cols = Table[var[[0]][[3]], {var, vars}] /. j → 3;
SparseArray[Table[{rows[[k]], cols[[k]]} → vars[[k]], {k, 1, Length[vars]}]] //
 MatrixForm
```

Out[10]=
$$\{m_{-2+i,j}[t],\ m_{-1+i,1+j}[t],\ m_{i,-2+j}[t],\ m_{i,j}[t],\ m_{1+i,j}[t]\}$$

Out[13]//MatrixForm=
$$\begin{pmatrix} 0 & 0 & m_{-2+i,j}[t] & 0 \\ 0 & 0 & 0 & m_{-1+i,1+j}[t] \\ m_{i,-2+j}[t] & 0 & m_{i,j}[t] & 0 \\ 0 & 0 & m_{1+i,j}[t] & 0 \end{pmatrix}$$

First order necessarily satisfied equation

In[14]:=
```
expr1 = FullSimplify[2 b mzero[0, 1] /. msol[0, 1] /. D[msol[0, 1], t] /. msol[1, 1]]
```

Out[14]=
$$d\,p^2 + 2\,a\,c\,m_{1,0}[t] + 2\,a\,d\,m_{2,0}[t] - 2\,(a + c)\,m_{1,0}'[t] - d\,m_{2,0}'[t] + 2\,m_{1,0}''[t]$$

Second order necessarily satisfied equation

In[15]:=
```
expr2 = FullSimplify[
  2 b^2 mzero[0, 2] /. msol[1, 2] /. msol[0, 2] /. D[msol[0, 2], t] /. msol[3, 1] /.
          msol[2, 1] /. D[msol[2, 1], t] /. msol[1, 1] /. D[msol[1, 1], t] /.
        D[msol[1, 1], {t, 2}] /. msol[0, 1] /. D[msol[0, 1], t] /. m0,0[t] → 0]
```

Out[15]=

$$-2\,c\,(a+c)\,p^2-6\,(a+c)\,d\,p^2\,m_{1,0}[t]-\left(4\,a\,c\,(a+c)+3\,d^2\,p^2\right)m_{2,0}[t]-4\,a^2\,d\,m_{3,0}[t]-6\,a\,c\,d\,m_{3,0}[t]-2\,a\,d^2\,m_{4,0}[t]+6\,d\,p^2\,m_{1,0}'[t]+2\,a^2\,m_{2,0}'[t]+8\,a\,c\,m_{2,0}'[t]+2\,c^2\,m_{2,0}'[t]+\frac{16}{3}\,a\,d\,m_{3,0}'[t]+2\,c\,d\,m_{3,0}'[t]+\frac{1}{2}\,d^2\,m_{4,0}'[t]-3\,a\,m_{2,0}''[t]-3\,c\,m_{2,0}''[t]-\frac{4}{3}\,d\,m_{3,0}''[t]+m_{2,0}^{(3)}[t]$$

Third order necessarily satisfied equation

In[16]:=
```
sub1 =
  FullSimplify[msol[0, 3] /. msol[1, 2] /. msol[2, 2] /. D[msol[1, 2], t]] /. msol[
            4, 1] /. msol[3, 1] /. D[msol[3, 1], t] /. msol[2, 1] /.
        D[msol[2, 1], t] /. D[msol[2, 1], {t, 2}] /. msol[1, 1] /.
      D[msol[1, 1], t] /. msol[0, 1] /. D[msol[0, 1], t];
sub2 =
  msol[1, 3] /. msol[3, 2] /. msol[2, 2] /. D[msol[2, 2], t] /. msol[0, 2] /. msol[5,
          1] /. msol[4, 1] /. D[msol[4, 1], t] /. msol[3, 1] /. D[msol[3, 1], t] /.
      D[msol[3, 1], {t, 2}] /. msol[2, 1] /. msol[1, 1] /. D[msol[1, 1], t];
```

In[18]:=
```
expr3 = FullSimplify[6 b^3 mzero[0, 3] /. sub1 /. D[sub1, t] /. sub2 /. msol[0, 1]]
```

Out[18]=

$$\frac{9}{2}\,(3\,a+7\,c)\,d\,p^4+9\left(c\,(3\,a+c)\,(a+2\,c)\,p^2+4\,d^2\,p^4+2\,b^2\,(a+c)\,s^2\right)m_{1,0}[t]+\frac{9}{2}\,d\left(\left(12\,a^2+37\,a\,c+12\,c^2\right)p^2+2\,b^2\,s^2\right)m_{2,0}[t]+18\,a^3\,c\,m_{3,0}[t]+45\,a^2\,c^2\,m_{3,0}[t]+18\,a\,c^3\,m_{3,0}[t]+84\,a\,d^2\,p^2\,m_{3,0}[t]+45\,c\,d^2\,p^2\,m_{3,0}[t]+18\,a^3\,d\,m_{4,0}[t]+69\,a^2\,c\,d\,m_{4,0}[t]+36\,a\,c^2\,d\,m_{4,0}[t]+\frac{45}{4}\,d^3\,p^2\,m_{4,0}[t]+24\,a^2\,d^2\,m_{5,0}[t]+\frac{45}{2}\,a\,c\,d^2\,m_{5,0}[t]+\frac{9}{2}\,a\,d^3\,m_{6,0}[t]-9\,a^2\,p^2\,m_{1,0}'[t]-66\,a\,c\,p^2\,m_{1,0}'[t]-51\,c^2\,p^2\,m_{1,0}'[t]-24\,b^2\,s^2\,m_{1,0}'[t]-\frac{177}{2}\,a\,d\,p^2\,m_{2,0}'[t]-\frac{183}{2}\,c\,d\,p^2\,m_{2,0}'[t]-6\,a^3\,m_{3,0}'[t]-48\,a^2\,c\,m_{3,0}'[t]-48\,a\,c^2\,m_{3,0}'[t]-6\,c^3\,m_{3,0}'[t]-35\,d^2\,p^2\,m_{3,0}'[t]-\frac{69}{2}\,a^2\,d\,m_{4,0}'[t]-\frac{229}{4}\,a\,c\,d\,m_{4,0}'[t]-9\,c^2\,d\,m_{4,0}'[t]-\frac{163}{10}\,a\,d^2\,m_{5,0}'[t]-\frac{9}{2}\,c\,d^2\,m_{5,0}'[t]-\frac{3}{4}\,d^3\,m_{6,0}'[t]+15\,a\,p^2\,m_{1,0}''[t]+33\,c\,p^2\,m_{1,0}''[t]+30\,d\,p^2\,m_{2,0}''[t]+11\,a^2\,m_{3,0}''[t]+32\,a\,c\,m_{3,0}''[t]+11\,c^2\,m_{3,0}''[t]+\frac{35}{2}\,a\,d\,m_{4,0}''[t]+10\,c\,d\,m_{4,0}''[t]+\frac{23}{10}\,d^2\,m_{5,0}''[t]-6\,p^2\,m_{1,0}^{(3)}[t]-6\,a\,m_{3,0}^{(3)}[t]-6\,c\,m_{3,0}^{(3)}[t]-\frac{5}{2}\,d\,m_{4,0}^{(3)}[t]+m_{3,0}^{(4)}[t]$$

In[19]:=
```
vars3 = Quiet[Select[Variables[expr2], #[[0]][[1]] === m || #[[0]][[1]][[1]] === m &]]
```

Out[19]=

$$\{m_{1,0}[t],\ m_{2,0}[t],\ m_{3,0}[t],\ m_{4,0}[t],\ m_{1,0}'[t],\ m_{2,0}'[t],\ m_{3,0}'[t],\ m_{4,0}'[t],\ m_{2,0}''[t],\ m_{3,0}''[t],\ m_{2,0}^{(3)}[t]\}$$

In[20]:=
```
Collect[expr3, vars3]
```

Out[20]=

$$\frac{9}{2}(3a+7c)\,d\,p^4 + 9\left(c\,(3a+c)\,(a+2c)\,p^2 + 4\,d^2\,p^4 + 2\,b^2\,(a+c)\,s^2\right) m_{1,0}[t] +$$
$$\frac{9}{2}\,d\left(\left(12\,a^2+37\,a\,c+12\,c^2\right)p^2 + 2\,b^2\,s^2\right) m_{2,0}[t] +$$
$$\left(18\,a^3\,c + 45\,a^2\,c^2 + 18\,a\,c^3 + 84\,a\,d^2\,p^2 + 45\,c\,d^2\,p^2\right) m_{3,0}[t] +$$
$$\left(18\,a^3\,d + 69\,a^2\,c\,d + 36\,a\,c^2\,d + \frac{45\,d^3\,p^2}{4}\right) m_{4,0}[t] + 24\,a^2\,d^2\,m_{5,0}[t] +$$
$$\frac{45}{2}\,a\,c\,d^2\,m_{5,0}[t] + \frac{9}{2}\,a\,d^3\,m_{6,0}[t] + \left(-9\,a^2\,p^2 - 66\,a\,c\,p^2 - 51\,c^2\,p^2 - 24\,b^2\,s^2\right) m_{1,0}'[t] +$$
$$\left(-\frac{177}{2}\,a\,d\,p^2 - \frac{183}{2}\,c\,d\,p^2\right) m_{2,0}'[t] + \left(-6\,a^3 - 48\,a^2\,c - 48\,a\,c^2 - 6\,c^3 - 35\,d^2\,p^2\right) m_{3,0}'[t] +$$
$$\left(-\frac{69\,a^2\,d}{2} - \frac{229\,a\,c\,d}{4} - 9\,c^2\,d\right) m_{4,0}'[t] - \frac{163}{10}\,a\,d^2\,m_{5,0}'[t] - \frac{9}{2}\,c\,d^2\,m_{5,0}'[t] -$$
$$\frac{3}{4}\,d^3\,m_{6,0}'[t] + 15\,a\,p^2\,m_{1,0}''[t] + 33\,c\,p^2\,m_{1,0}''[t] + 30\,d\,p^2\,m_{2,0}''[t] +$$
$$\left(11\,a^2 + 32\,a\,c + 11\,c^2\right) m_{3,0}''[t] + \frac{35}{2}\,a\,d\,m_{4,0}''[t] + 10\,c\,d\,m_{4,0}''[t] + \frac{23}{10}\,d^2\,m_{5,0}''[t] -$$
$$6\,p^2\,m_{1,0}^{(3)}[t] - 6\,a\,m_{3,0}^{(3)}[t] - 6\,c\,m_{3,0}^{(3)}[t] - \frac{5}{2}\,d\,m_{4,0}^{(3)}[t] + m_{3,0}^{(4)}[t]$$