Two-dimensional Ornstein-Uhlenbeck model

In[1]:=
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$Assumptions = a ≠ -d;
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In[2]:=
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(* System *)
A = {{a, b}, {c, d}}; B = {e, f}; S = {{p, 0}, {r, s}};
F[x_] := -A.(x - B)
G[x_] := S
```

In[5]:=
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(* Moment equations *)
expr =
    D[x[t]^i y[t]^j, {{x[t], y[t]}}].F[{x[t], y[t]}] +
    1/2 Tr[G[{x[t], y[t]}]^T.D[x[t]^i y[t]^j, {{x[t], y[t]}, 2}].G[{x[t], y[t]}]];
mex[i_, j_] := Evaluate[Expand[expr] /.
    Flatten[Table[x[t]^(i+p) y[t]^(j+q) → m_(i+p,j+q)[t], {p, -2, 2}, {q, -2, 2}]]]
```

In[7]:=
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(* Stationary distribution *)
KroneckerSum[a_, b_] := KroneckerProduct[a, IdentityMatrix[Length[A]]] +
    KroneckerProduct[IdentityMatrix[Length[a]], b];
M = {e, f};
Σ =
  FullSimplify[ArrayReshape[Inverse[KroneckerSum[A, A]].Flatten[S.S^T], {2, 2}]];
```

In[10]:=
```
VarY = FullSimplify[
    ArrayReshape[Inverse[KroneckerSum[A, A]].Flatten[S.S^T], {2, 2}]][[2, 2]]
```

Out[10]=
$$\frac{c^2 p^2 + a (a+d) \left(r^2+s^2\right) - c \left(2 a p r + b \left(r^2+s^2\right)\right)}{2 (a+d) (-b c + a d)}$$

In[11]:=
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(* Conditional distribution *)
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In[12]:=
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m01 = FullSimplify[f + Σ[[1, 2]]/Σ[[1, 1]] (x0 - e)]
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Out[12]=
$$f - \frac{\left(c d p^2 - 2 a d p r + a b \left(r^2+s^2\right)\right) (-e + x_0)}{d (a+d) p^2 - b p (c p + 2 d r) + b^2 \left(r^2+s^2\right)}$$

In[13]:=
```
m02 = FullSimplify[Σ[[2, 2]] - Σ[[1, 2]]^2/Σ[[1, 1]] + m01^2]
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Out[13]=
$$-\frac{\left(c d p^2 - 2 a d p r + a b \left(r^2+s^2\right)\right)^2}{2 (a+d) (-b c + a d) \left(d (a+d) p^2 - b p (c p + 2 d r) + b^2 \left(r^2+s^2\right)\right)} +$$
$$\frac{c^2 p^2 + a (a+d) \left(r^2+s^2\right) - c \left(2 a p r + b \left(r^2+s^2\right)\right)}{2 (a+d) (-b c + a d)} +$$
$$\left(f - \frac{\left(c d p^2 - 2 a d p r + a b \left(r^2+s^2\right)\right) (-e + x_0)}{d (a+d) p^2 - b p (c p + 2 d r) + b^2 \left(r^2+s^2\right)}\right)^2$$

In[14]:=
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Q3 = FullSimplify[d x0 - b m01 + b f - d e]
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Out[14]=

$$-\frac{(a+d)\left((d\,p-b\,r)^2+b^2\,s^2\right)(e-x_0)}{d\,(a+d)\,p^2-b\,p\,(c\,p+2\,d\,r)+b^2\left(r^2+s^2\right)}$$

In[15]:=
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FullSimplify[Expand[(Q3/((a + d) (e - x0) ((d p - b r)^2 + b^2 s^2)))^-1] + (a d - b c) p^2]
```

Out[15]=

$$-(d\,p-b\,r)^2-b^2\,s^2$$