Advanced Placement Calculus AB

Preparing for AP Calculus AB

- a short summary -

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Contents

1 Introduction

1.1 About

The following notes aim to provide a discussion and summary of selected pre-requisite topics for students interested in enrolling in a full-year Advanced Placement Calculus AB course. The content spans a number of topics that my students have found challenging in recent years. Please note that this document does not cover all the pre-requisite material for AP Calculus AB, nor does it replace a full year honors precalculus experience.

Please feel free to share any feedback or suggestions on the web at ap-calc.github.com, where updated versions and corrections of Preparing for AP Calculus AB will be published in the future.

1.2 On the web

The site ap-calc.github.io contains miscellaneous resources for students and teachers of Advanced Placement Calculus AB and BC. It was originally created when I first started teaching in 2005, for the purpose of making classroom materials available online for my students. It includes selected tutorials, references, worksheets, multiple choice and free response practice, summer review packets, as well as links to external resources and learning tools.

2 Sets, Real Numbers

2.1 Working with Sets

A set is a collection of things, usually numbers. Sets can be finite or infinite. We describe them by listing the elements explicitly (if finite) using Venn diagrams, word descriptions, or by means of a mathematical statement such as, for example, an inequality or an equation that the elements must satisfy.

Example Here are a few sets.

$$A = \{x \in \mathbb{R} \text{ such that } x \geq 0\}$$

$$B = \{x \text{ such that } x^2 - 3x + 2 = 0\}$$

$$C = \{\text{apple, orange, pear}\}$$

$$D = \{\text{cherry, orange, apple}\}$$

A universal set is used in most branches of mathematics. The universal set in single-variable calculus is the infinite set of real numbers. We describe it using the notation $\mathbb{R}$ or $(-\infty, \infty)$ or in plain English: all real numbers. Visually, we think of the real numbers as one of the axes (the x- or y-axis).

When working with sets, we need to be comfortable with at least two operations: the union and the intersection, defined below, respectively:

$$A \cup B = \{ \text{ any element that belongs to A or B}\}$$

$$A \cap B = \{ \text{ any element that belongs to both A and B}\}$$

Considering the four sets from the example above, the reader can confirm the following:

$$A \cup B = A$$

$$A \cap B = B = \{1, 2\}$$

$$C \cup D = \{\text{apple, cherry, orange, pear}\}$$

$$C \cap D = \{\text{apple, orange}\}$$

We use the symbol ϕ to describe the empty set, a set with no elements in it. For example, $A \cap C = \phi$, since sets A and C do not share any elements.

2.2 Domain and Range

The domain of a function consists of all the real numbers for which the function is defined.We describe domain as a part of (or possibly all of) the x-axis. Graphically, we look for a collection of values for which the graph of the function exists. Visually, the domain is determined by projecting the graph of the function onto the horizontal axis. (See Figure 1.)

We define the range of a function in a similar fashion. It consists of all the possible output values on the y-axis. Visually, the range can be determined by projecting the graph of the function onto the y-axis. When commenting on the domain and range of functions that involve two or more expressions, we try to use terms like: union, intersection, and describe in the form of interval notation, inequalities, or in words.

Example

The domain of $f(x) = \sqrt{x-3} + \sqrt{2-x}$ is the intersection of the domains of the individual square root expressions. Since the individual domains are $x \geq 3$ and $x \leq 2$, respectively, the overall domain of $f(x)$ is $2 \leq x \leq 3$. A function exists when all of the terms and operations in its algebraic expression are meaningful, hence the intersection. Note that we can determine the range of this function with the aid of technology (graphing) or calculus.
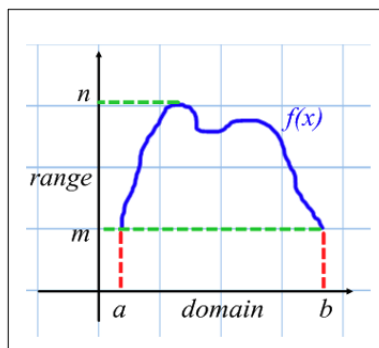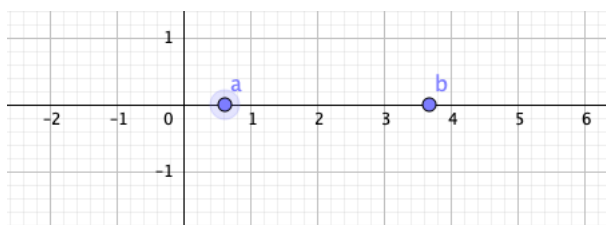

Figure 1: Domain and Range

2.3 Interval notation

Figure 2: Points on the number line

On the real number, suppose there are two distinct points $x = a$ and $x = b$ such that $a < b$. The open interval notation (first row) is easy to confuse with the coordinates of a point on the xy-plane. Fortunately, most of the time we will be able to tell from the context.

Notation	Read as…
$(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$	An open interval from a to b.
$[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}$	A half-open interval, closed at a, open at b.
$(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$	A half-open interval, open at a, closed at b.
$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$	A closed interval from a to b.
$(-\infty, \infty)$	Describes the entire real number line.

Figure 3: Interval Notation

3   Distance Formula

3.1   On the number line

Since distance is a non-negative quantity, on the real number line, we use the operation of absolute value to define the distance between two points $x = a$ and $x = b$:

$$d(a, b) = |b - a| = |a - b|$$

We translate the English sentence "x is within 3 units of 5" into the mathematical statement with the absolute value notation:

$$|x - 5| < 3$$

Rewriting as a double inequality

$$-3 < x - 5 < 3$$

and solving

$$2 < x < 8$$

we find that the solution describes the open interval $(2, 8)$. Excluded end-points are labeled as open circles "∘" on the number line. If the end-points are included, we we make them solid, as shown in the figure.
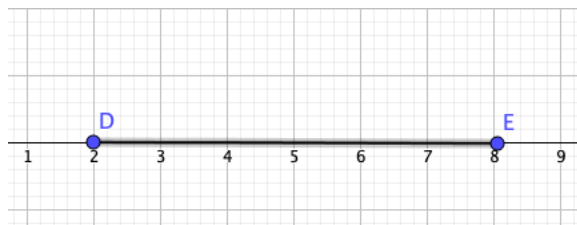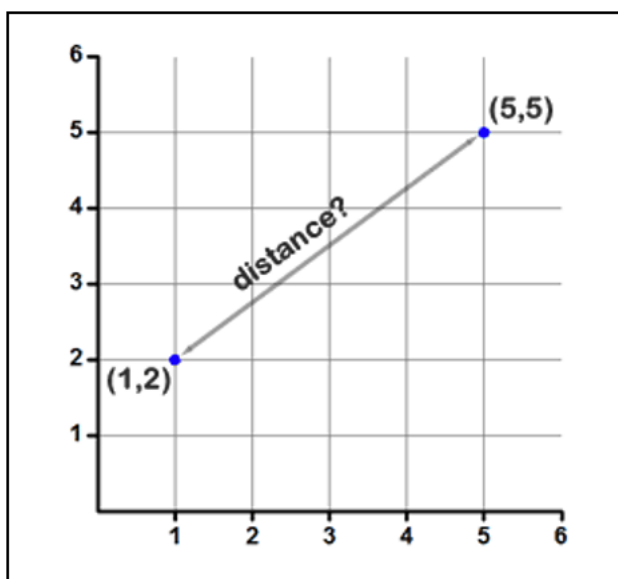
Figure 4: Distance on the number line

3.2   On the plane

On the Cartesian plane, the Pythagorean theorem is used to define and determine the distance between two points $A(x_1, y_1)$ and $B(x_2, y_2)$.

Figure 5: Distance on the xy-plane

The right triangle with legs $(x_2 - x_1)$ and $(y_2 - y_1)$ has hypotenuse with length $d = |AB|$, so we write:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$A(x_1, y_1) = (1, 2) \quad B(x_2, y_2) = (5, 5)$$

$$d = \sqrt{(5 - 1)^2 + (5 - 2)^2} = \sqrt{25} = 5$$

Example

A circle of radius r centered at the origin is defined as the set of all points on the plane whose distance from the origin is r. We confirm the equation of the circle by applying the distance formula to the variable point (x, y) on the circle and the origin $(0, 0)$:

$$r = \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2}$$

$$x^2 + y^2 = r^2$$

4 Factoring

4.1 Common formulas

Difference of Squares	Square of the Sum and Difference
$a^2 - b^2 = (a-b)(a+b)$	$a^2 + 2ab + b^2 = (a+b)^2$
	$a^2 - 2ab + b^2 = (a-b)^2$
Difference of Cubes	Sum of Cubes
$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$	$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
Factoring a Minus Sign	Difference of even powers
$a - b = -(-a+b) = -(b-a)$	$a^{2m} - b^{2m} = (a^m - b^m)(a^m + b^m)$

Figure 6: Some Factoring Formulas

4.2 Splitting the middle term

Assuming the expression $ax^2 + bx + c$ is factorable, we first guess two integers p and q such that $p + q = b$ and $pq = ac$, and then rewrite the quadratic expression as a four-term sum:

$$ax^2 + px + qx + c$$

The middle term bx is now split into the sum $px + qx$. Grouping the terms two by two leads to complete factoring.

Example

Factor $2x^2 - x - 3$ by splitting the middle term.

$$p + q = -1 \text{ and } pq = 2(-3) = -6 \rightarrow p = 2, q = -3$$

$$2x^2 - x - 3 = 2x^2 + (2x - 3x) - 3$$
$$= (2x^2 + 2x) + (-3x - 3)$$
$$= 2x(x + 1) - 3(x + 1)$$
$$= (2x - 3)(x + 1)$$

4.3   Quadratic Expressions

Consider the quadratic equation in its general form: $ax^2 + bx + c = 0$. Assuming $a \neq 0$, complete the square by factoring a and splitting the middle term:

$$a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = 0$$
$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$
$$x^2 + 2\frac{b}{2a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$
$$\left(x + \frac{b}{2a}\right)^2 = \frac{-4ac + b^2}{2a} \Rightarrow {}_1x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{D}}{2a}$$

The Quadratic Formula

The expression $D = b^2 - 4ac$ is called a discriminant. Its value (positive or negative or zero) determines the number of solutions the quadratic equation has. In the case of a quadratic function $f(x) = ax^2 + bx + c$, the discriminant determines the number of zeroes, or graphically, how many times the parabola intersects the horizontal axis (x-intercepts). The chart below summarizes the possibilities.
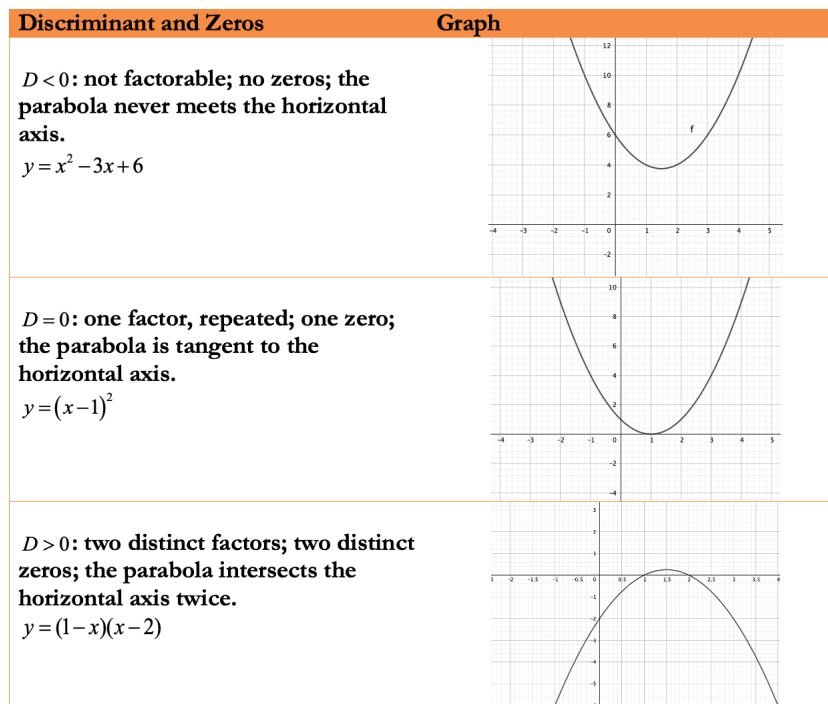
Discriminant and Zeros	Graph
$D < 0$: **not factorable; no zeros; the parabola never meets the horizontal axis.** $y = x^2 - 3x + 6$	
$D = 0$: **one factor, repeated; one zero; the parabola is tangent to the horizontal axis.** $y = (x-1)^2$	
$D > 0$: **two distinct factors; two distinct zeros; the parabola intersects the horizontal axis twice.** $y = (1-x)(x-2)$	

Figure 7: The Quadratic Function

4.4 Expressions with Fractional Exponents

When given an expression with fractional exponents, factoring the term with the least exponent usually does the trick. The 'left-over' expression often has integer exponents, as shown in this example:

$$\frac{x^{\frac{-1}{2}} + 2x^{\frac{3}{2}} - x^{\frac{-3}{2}}}{4x^{\frac{-3}{2}}} = \frac{x^{\frac{-3}{2}}\left[x + 2x^3 - 1\right]}{4x^{\frac{-3}{2}}} = \frac{x + 2x^3 - 1}{4}$$

5 Functions

5.1 Transformations

Common geometric transformations of the graph of a function $y = f(x)$ are summarized here:

New Form	Description
$f(x)+a$	Vertical translation (or shift) by a units up $(a>0)$ or down $(a<0)$
$a \cdot f(x)$	Vertical stretch $(a>1)$ or shrink $(0<a<1)$
$-f(x)$	Reflection over the horizontal axis
$\lvert f(x) \rvert$	Portions of the graph below the x-axis reflect about $y=0$
$f(x-a)$	Horizontal shift by a units to the right $(a>0)$ or to the left $(a<0)$
$f(a \cdot x)$	Horizontal shrink $(a>1)$ or stretch $(0<a<1)$ by factor a
$f(-x)$	Reflection over the vertical axis
$f(\lvert x \rvert)$	Portions of the graph to the left of y-axis reflect about $x=0$

<center>Transformations: Table Summary</center>

Example

Consider $y = f(x) = 1 - x$. Apply the transformations in the given order:

Horizontally stretch by a factor of 2

Reflect over the horizontal axis

Vertically shift by 5 units upward

$$y = 1 - \frac{x}{2}$$
$$y = -(1 - \frac{x}{2}) = \frac{x}{2} - 1$$
$$y = \frac{x}{2} - 1 + 5 = \frac{x}{2} + 4$$

5.2 Symmetry: Even, Odd, Neither

Odd functions have graphs with symmetry about the origin. Algebraically, $f(x)$ is odd if and only if $f(-x) = -f(x)$ for all x in the domain. Examples of odd functions include $f(x) = x^3, g(x) = -x + k, h(x) = \sin(x)$.

Even functions have graphs with symmetry about the y-axis. Algebraically, $f(x)$ is even if and only if $f(x) = f(-x)$ for all x in the domain. Examples of even functions include $f(x) = -x^2, g(x) = \lvert x \rvert, h(x) = \cos(x)$.

Functions like $f(x) = x^2 - 2x$, which does not satisfy either type of symmetry, are labeled as 'neither odd nor even'.

Example

Show that $f(x) = x^5 - x$ is an odd function.

Solution We need to prove that $f(-x) = -f(x)$ for all x. Starting with the left side, we write:

$$f(-x) = (-x)^5 - (-x) = -x^5 + x = -(x^5 - x) = -f(x)$$

Since we transformed the left side into the right, we have shown that the algebraic definition of an odd function holds true for this function.

Technology helps confirm the symmetries of odd (about the origin) and even (about the y-axis) functions:
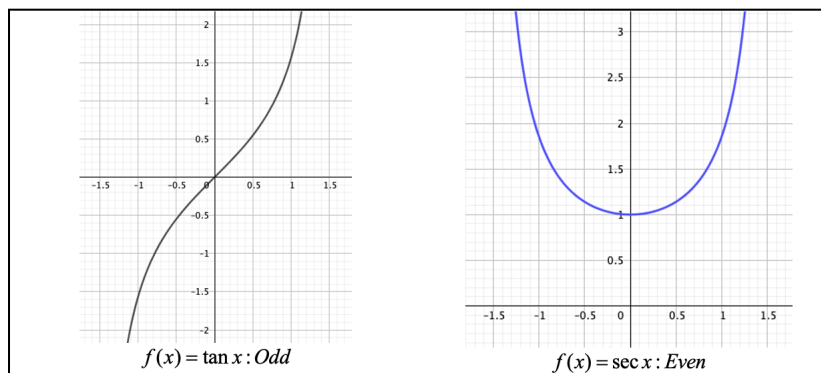

Figure 8: Odd and Even Functions

Example

Is there a function that is both even and odd?

Solution

Assume that there exists a function that is both odd and even. Let (a, b) be a point on the graph of $y = f(x)$, so that $f(a) = b$. By the property of even functions, it follows that point $(-a, b)$ must also be on the graph, since $f(a) = f(-a)$. By the property of odd functions, it follows that the point $(-a, -b)$ must also be on the graph of $y = f(x)$, since $f(-a) = -f(a)$. Hence, we must have a function that passes through points $(-a, b)$ and $(-a, -b)$. It follows that such a function fails the Vertical Line Test whenever b is nonzero. We conclude that the function $f(x) = 0$ is the only function that is both odd and even.
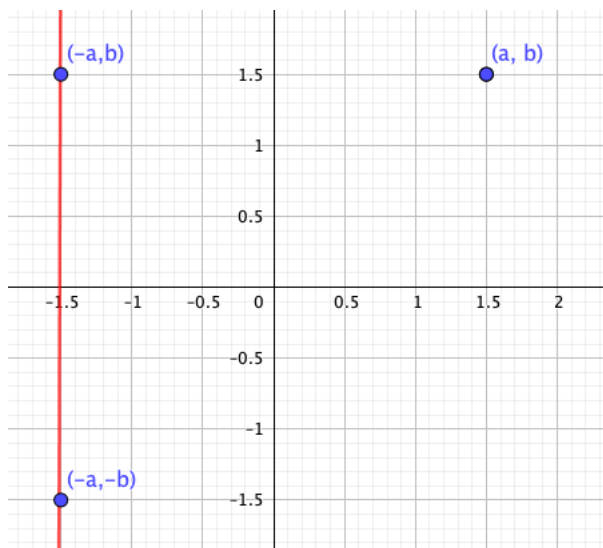
Figure 9: Building a function that is both even and odd

5.3   Algebra of Functions

Let $f(x)$ and $g(x)$ be two functions with domains A and B, respectively. We define four new functions and identify their domains:

New Function	Name	Domain
$f + g = (f + g)(x) = f(x) + g(x)$	Direct Sum	$A \cap B$
$f - g = (f - g)(x) = f(x) - g(x)$	Difference	$A \cap B$
$fg = (fg)(x) = f(x) \cdot g(x)$	Product	$A \cap B$
$f/g = (f/g)(x) = f(x)/g(x)$	Quotient	$A \cap B$ $(g(x) \neq 0)$

Figure 10: Algebra of Functions

Geometrically, the graphs of the new functions above are obtained by performing arithmetic operations on pairs of y-values at each independent value x in the domain $A \cap B$ and plotting the new points. For example, if $h(x) = f(x) - g(x)$, we can compute $h(2) = f(2) - g(2)$ and plot the point $(2, h(2))$ on the graph.

Example

In firm theory models in economics, the total cost function is the direct sum of two 'old' functions: fixed costs (a constant function) and variable costs. We write $C(x) = F(x) + V(x)$, or simply $C(x) = F + V(x)$, since $F(x)$ is constant. Graphically, the total cost function is a vertical translation of the variable cost function by F units upward. (Note that the domain in this example is $x \geq 0$.)
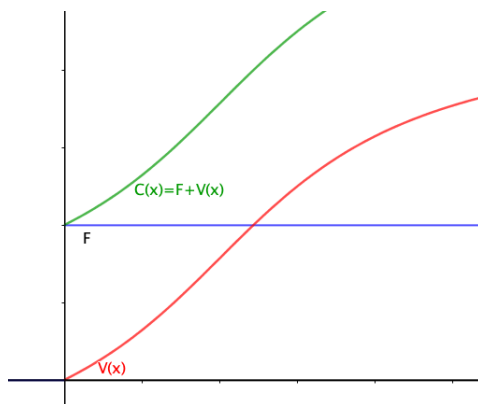
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Figure 11: A direct sum example

$F(x)$ (blue) describes the fixed costs. $V(x)$ (red) describes the variable cost function, which passes through the origin. The variable cost is zero when there is no output. $C(x) = F(x) + V(x)$ (green) describes the total cost and is the direct sum of the fixed and variable cost curves.

Example

Find the domain of $h(x) = \sqrt{2 - x} + \frac{1}{\sqrt{x-1}}$. Express the answer in interval notation.
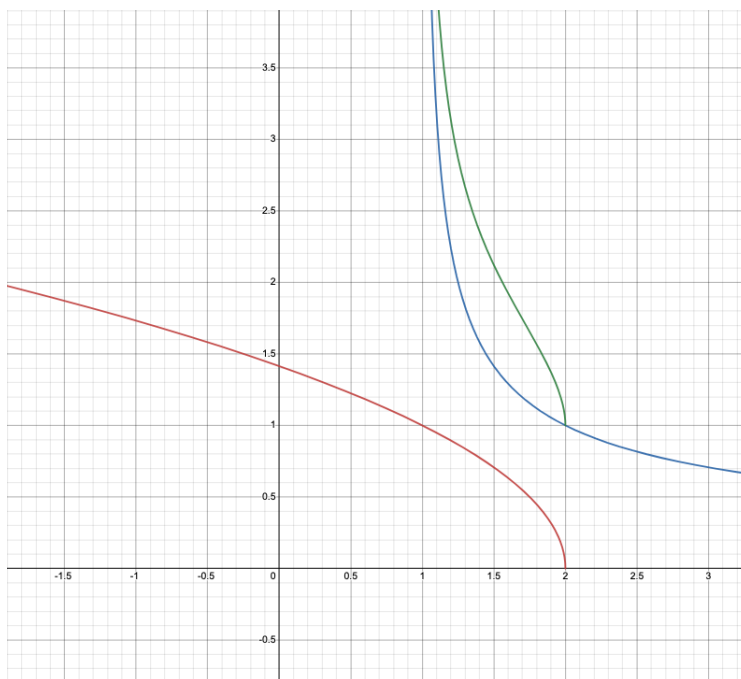
Figure 12: Green $(h(x))$ is the sum of red $(f(x))$ and blue $(g(x))$

If we let $f(x) = \sqrt{2 - x}$ and $g(x) = \frac{1}{\sqrt{x-1}}$, the domain of $h(x)$ will be dom $f \cap$ dom g

$$\text{dom } f : x \leq 2$$

$$\text{dom } g : x > 1$$

$$\text{dom } h = \text{dom } f \cap \text{dom } g : 1 < x \leq 2$$

Using interval notation, the domain of $h(x)$ is $(1, 2]$. The graph confirms that $h(x)$ is only defined for x-values between 1 and 2, excluding 1.

5.4   Compositions

The composition of $f(x)$ with $g(x)$ is a new function $h(x)$, defined as:

$$h(x) = f \circ g = (f \circ g)(x) = f(g(x))$$

$g(x)$ is the inner function, whereas $f(x)$ is the outer function. We think of the inner function as the rule that is applied to an input x first, and the outer function as the algebraic rule that is applied before the final output.
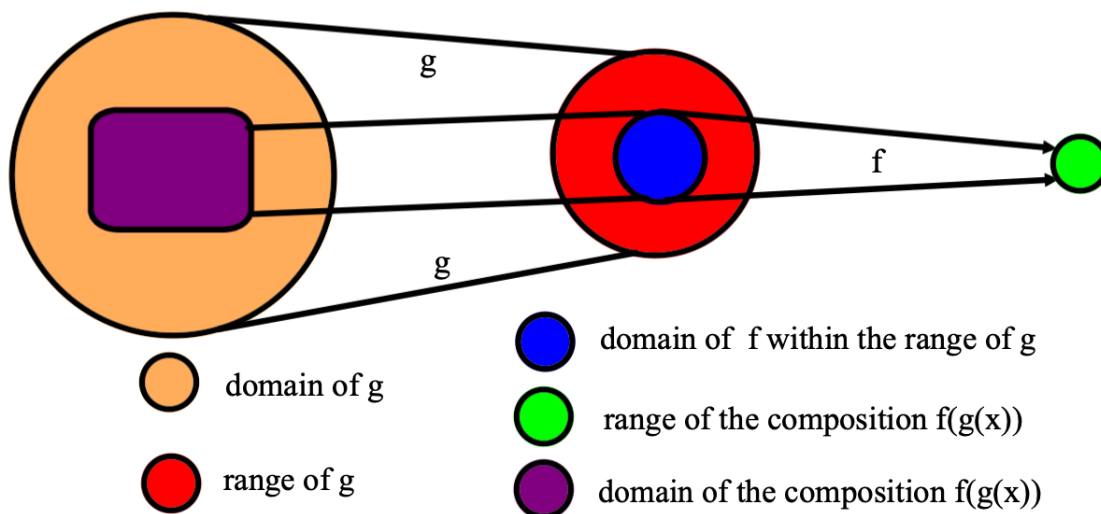

domain of f within the range of g

range of the composition f(g(x))

domain of the composition f(g(x))

domain of g

range of g

Figure 13: The domain of a composition $f \circ g$

The domain of the composition consists of all x-values in the domain of g such that their output under g belongs to the domain of f, hence ensuring that g and f are defined simultaneously. Students should be able to create a composition using elementary functions and correctly identify the elementary functions that make up a given composite.

Example

Write $l(x) = \sqrt{\sin(1 - x^3)}$ as a composition of elementary functions.

Solution

We could use three or four functions, depending on how elementary one expects the expressions to be. The innermost operation is the cubic expression x^3. The remaining operations, in order, are: subtract the input from one, take the sine, and finally, take the square root. We write out these rules as separate functions:

$$k(x) = x^3$$
$$j(x) = 1 - x$$
$$i(x) = \sin x$$
$$h(x) = \sqrt{x}$$

The reader can verify that $l(x) = (h \circ i \circ j \circ k)(x)$.

Example

If we compose an odd function with an even function, is the resulting composition odd, even, or neither?

Solution

Suppose f is odd and g is even. Since $g(x)$ is even, the composition $f \circ g$ is even because:

$$g(x)) = g(-x)$$

$$f(g(x)) = f(g(-x))$$

In this situation, the function $f(x)$ does not need to have any special symmetry; it could be odd, even, or neither, and the overall composition would still be even.

5.5   Inverse of a Function

A function is a mapping that sends every x value from the domain space to at most one output $f(x)$ in the range space. The inverse 'mapping' of f, written as f^{-1}, undoes the effect of f. For example, if f takes a to b, f^{-1} takes b to a:

$$f : x \rightarrow y$$

$$x \leftarrow y : f^{-1}$$
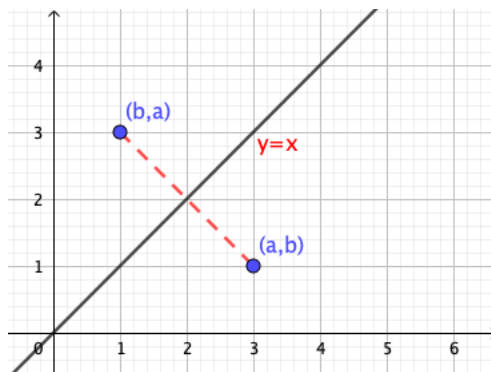
Figure 14: Inverse twins: swapping x and y-coordinates

Algebraically, the inverse mapping of $y = f(x)$ is determined by swapping the variables x and y, so that we have $x = f(y)$. Each point (a, b) contained in f, will have an 'inverse' point (b, a) contained in the inverse mapping f^{-1}. Geometrically, the slope of the line connecting these twin points is given by $m = \frac{b-a}{a-b} = -1$, hence the path is perpendicular to the line $y = x$. Since the two points are also equidistant from the identity function $y = x$, we conclude that the geometric transformation used to find the inverse is a reflection about the line $y = x$, also referred to as the $45°$ line. For example, the inverse mapping of $y = x^2$ is $x = y^2$.
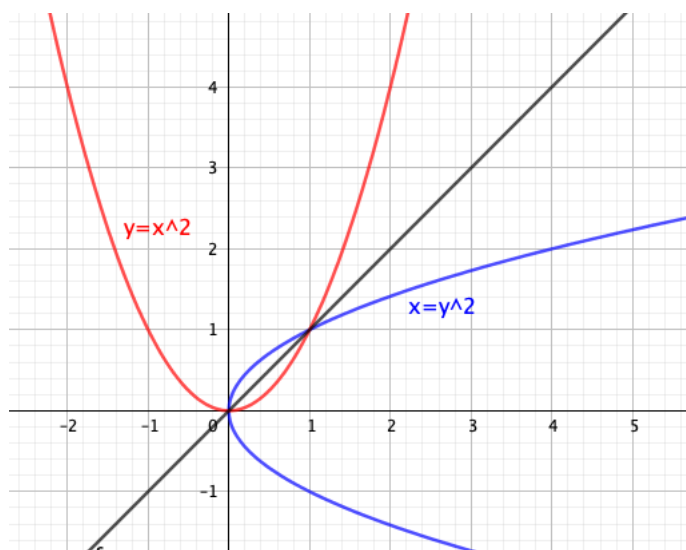
Figure 15: Graphs of $y = x^2$ and its inverse mapping, $x = y^2$

Graphically, we observed that the inverse mapping is a reflection about the line $y = x$. When reflected in this fashion, a horizontal line with equation $y = k$ turns into a vertical line with equation $x = k$. Therefore, to determine whether or not the inverse mapping of a function $y = f(x)$ is also a function (in other words, passes the vertical line test), it suffices to apply the horizontal line test to the graph of the original function $f(x)$. For example, the inverse of $y = x^2$ is not a function because the original function fails the horizontal line test. On the other hand, a function like $f(x) = x^3$, which passes the horizontal and vertical line tests, has an inverse mapping $x = y^3$ that is also a function. We write the inverse as $f^{-1}(x) = \sqrt[3]{x}$.
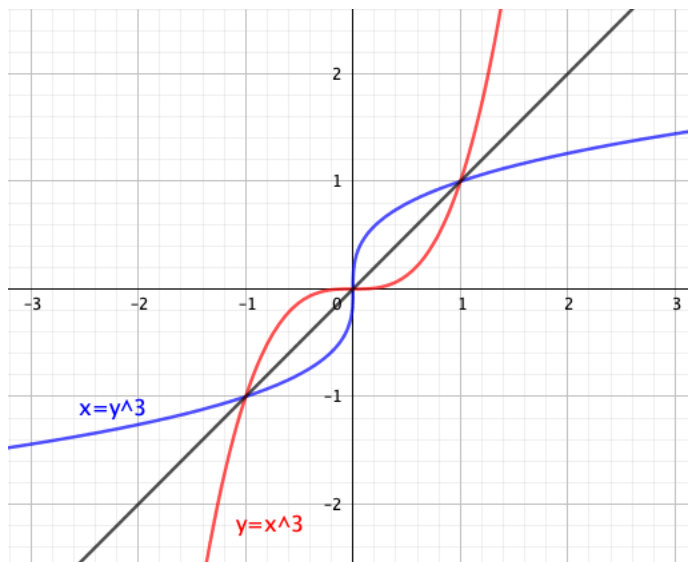

Figure 16: Graph of $y = x^3$ and its inverse mapping, $x = y^3$

Fact: If a function $f(x)$ is monotonic (strictly increasing or strictly decreasing), then its inverse mapping is also a function.

(The monotone behavior implies that the graph passes the horizontal line test.)

Example

Find the inverse of $y = \sin(x^3 - 1)$ algebraically. Plot the original function and its inverse on the same window.

$$y = \sin(x^3 - 1)$$

$$x = \sin(y^3 - 1)$$

$$\arcsin(x) = y^3 - 1$$

$$\arcsin(x) + 1 = y^3$$

$$y = \sqrt[3]{\arcsin(x) + 1}$$

We use technology to plot both functions. Note that the inverse mapping in (red) has a restricted domain, showing only the portion of the graph for which the original curve and its inverse pass the vertical line test.
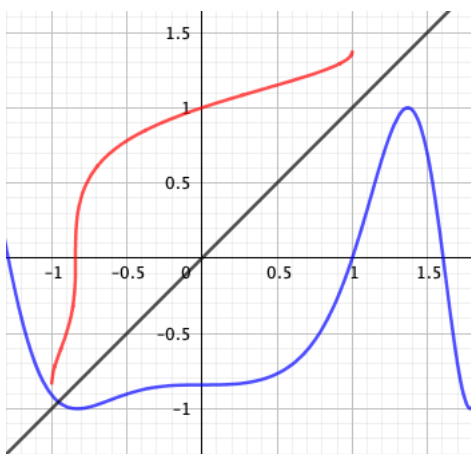

Figure 17: Graphs of $y = \sin(x^3 - 1)$ and its inverse mapping $y = \sqrt[3]{\arcsin(x) + 1}$

When working with inverse mappings, we restrict the domain to ensure that the inverse is a function. For example, the domain of the trigonometric function $f(x) = \sin(x)$ is restricted to $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$, since the graph is monotonic on this interval. The range $[-1, 1]$ becomes the domain of $y = \arcsin x$, the inverse function. The graphs of these two functions are shown:
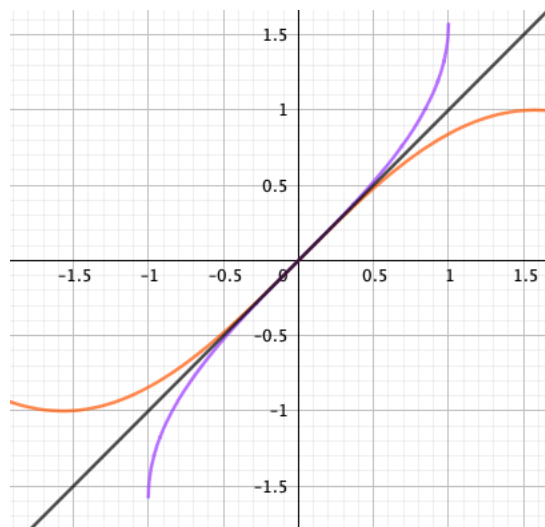
Figure 18: Graphs of $y = \sin(x)$ and its inverse mapping $y = \arcsin x = \sin^{-1} x$

5.6   Inverse of a Composition

Let f and g describe the flights Madrid – Rome and Boston – Madrid, respectively. Assuming this is the desired route from Boston to Rome, we apply $f \circ g$ to a passenger who wishes to travel from Boston to Rome. The inverse of this composition will be a composition of inverses, in reverse order:
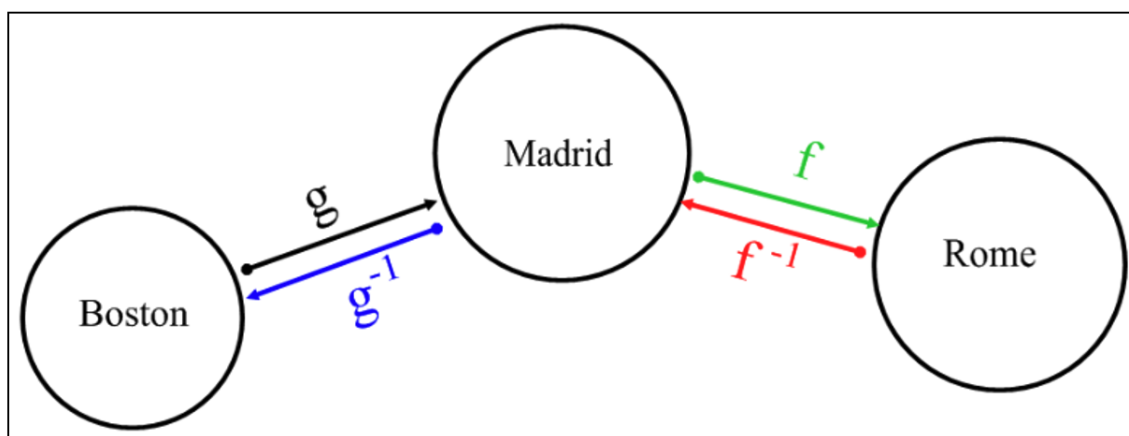

Figure 19: The inverse of a composition

$$\text{FACT} : (f \circ g)^{-1} = g^{-1} \circ f^{-1}$$

Example

The inverse of the composition $y = (3x - 1)^3$ is the composition of $g^{-1} = \frac{x+1}{3}$ with $f^{-1} = x^{\frac{1}{3}}$. In this example, the elementary functions are $g(x) = 3x - 1$ and $f(x) = x^3$.

Example

Suppose you are given the following information about two functions f and g.

x	f	g
-1	4	3
3	5	6
4	-4	9
6	3	8

Use information from the table to evaluate the following quantities:

$a. f^{-1}(-4)$ $\qquad$ $b. f(g(3)$ $\qquad$ $c. (f + g)(3)$ $\qquad$ $d. (fg)(3)$ $\qquad$ $e. (g \circ f)(-1)$ $\qquad$ $f. (g^{-1} \circ f)(6)$

Solution

$a. f^{-1}(-4) = -1$

$b. f(g(3) = f(6) = 3$

$c. (f + g)(3) = f(3) + g(3) = 5 + 6 = 11$

$d. (fg)(3) = f(3)g(3) = 5 \times 6 = 30$

$e. (g \circ f)(-1) = g(f(-1) = g(4) = 9$

$f. (g^{-1} \circ f)(6) = g^{-1}(f(6) = g^{-1}(3) = -1$

5.7   Property of Inverse Functions

Two functions f and g are inverses of each-other if and only if $f(g(x)) = x = g(f(x))$. Algebraically, it suffices to verify that both compositions give the identity function, which means the two effects undo each-other when composed in either order.

Example

Show that $f(x) = \frac{1+x^5}{2}$ and $g(x) = \sqrt[5]{2x - 1}$ are inverses of each-other.

Solution

$f \circ g = f(g(x)) = \frac{1+(g(x))^5}{2} = \frac{1+(\sqrt[5]{2x-1})^5}{2} = \frac{1+2x-1}{2} = \frac{2x}{2} = x$

We can similarly verify the other direction:

$g \circ f = g(f(x)) = \sqrt[5]{2f(x) - 1} = \sqrt[5]{2\frac{1+x^5}{2} - 1} = \sqrt[5]{1 + x^5 - 1} = \sqrt[5]{x^5} = x$

Since both compositions are equal to the identity function $y = x$, we conclude that the pair are inverses of each-other. We confirm graphically with the aid of technology:
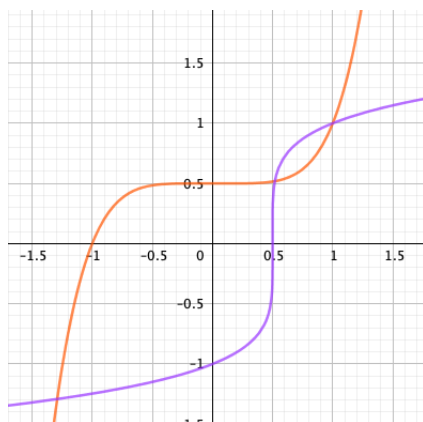

Figure 20: The graphs of are symmetric about $y = x$

6   Polynomials

6.1   Discussion and Sketching

Polynomials are functions of the form $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_{n-1} x^{n-1} + a_n x^n$, where n is a non-negative integer. We start with some vocabulary and definitions:

1) a_0 through a_n are real numbers. They are called coefficients.

2) Since $f(0) = a_0$, a_0 represents the y-intercept.

3) a_n, the coefficient in front of the largest power of x, is called the leading coefficient.

4) The largest exponent, n, is the degree of the polynomial.

5) The expression $a_n x^n$ is called the leading term.

Polynomial graphs are smooth throughout their domain, which consists of all real numbers. When discussing polynomials, we first identify the degree, the leading coefficient, and the y-intercept. If possible, we find the x-intercepts (zeros) by solving the equation $f(x) = 0$. Additionally, we also comment on the end-behavior of a polynomial by completing these two sentences:

1. As x goes to $-\infty$, $f(x)$ goes to ...

2. As x goes to ∞ , $f(x)$ goes to ...

Graphically, the two sentences above describe what the function does for extreme values of x. More specifically, as long as the degree is greater than 0, the $f(x)$ values will either go to ∞ or $-\infty$. The leading term decides end-behavior (see table below).

Degree \ Leading Coefficient	Positive	Negative
Even	As $x \to \infty, y \to \infty$ As $x \to -\infty, y \to \infty$ Example: $p(x) = \frac{1}{3}x^4 - x + 1$	As $x \to \infty, y \to -\infty$ As $x \to -\infty, y \to -\infty$ Example: $p(x) = -x^6 + \sqrt{2}x^2$
Odd	As $x \to \infty, y \to \infty$ As $x \to -\infty, y \to -\infty$ Example: $p(x) = 7x^7 + x^2 - 10$	As $x \to \infty, y \to -\infty$ As $x \to -\infty, y \to \infty$ Example: $p(x) = -x^{101} + 2x^{100}$

The end-behavior of polynomials

6.2 Factor Theorem

Factor Theorem: $x = a$ is zero of a polynomial $p(x)$ if and only if $(x - a)$ is a factor $p(x)$.

Example

Discuss and sketch $p(x) = x^3 - 7x + 6$.

Solution Start by plugging in small numbers such as -2, -1, 1, and 2. If we can find a number that makes $p(x) = 0$, we can use the Factor Theorem to identify a first factor. In this case, $p(1) = 0$, so $(x - 1)$ is one of the factors. We perform long or synthetic division to reduce $p(x)$, and then easily factor the quotient, since it is a quadratic expression:

$$p(x) = x^3 - 7x + 6 = (x - 1)(x^2 + x - 6) = (x - 1)(x + 3)(x - 2)$$

Continuing with the discussion of $p(x)$, it has an odd degree $(n = 3)$ and a positive leading coefficient $(a_3 = 1)$, so the graph must start in the third quadrant and end in the first. All the factors appear an odd number of times (this is called odd multiplicity), hence the graph intersects the horizontal axis at every intercept: $-3, 1$, and 2. Please note: if a zero has an even multiplicity, then the horizontal axis serves as a tangent to the polynomial at that point. i.e. the graph 'bounces' off without intersecting.

Before we put all the details into a sketch, we also note that the y-intercept is 6, so the point $(0, 6)$ is part of the graph. Combining the intercepts and end-behavior, we conclude with a graph of $p(x)$ below.
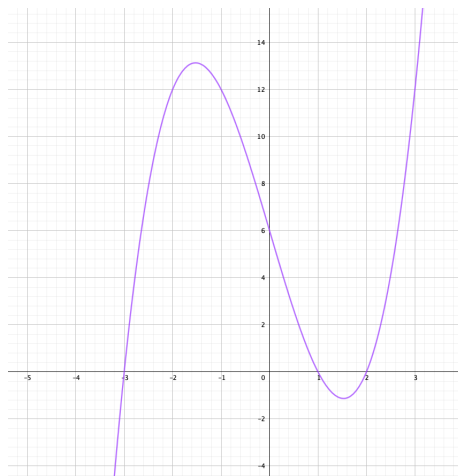
Figure 21: The graph of $p(x) = x^3 - 7x + 6$

6.3 Remainder Theorem

Remainder Theorem: Let $x = a$ be a constant. Suppose we divide a polynomial $p(x)$ by a linear expression $(x - a)$ and we find a remainder expression $r(x)$. The value of the remainder at $x = a$ is the same as as that of the polynomial $p(x)$. Simply put: $p(a) = r(a)$.

Example

When dividing $p(x) = x^3 - 7x + 6$ by $(x + 1)$, without performing the polynomial division, we know that that the remainder $r(-1) = p(-1) = 12$.

Long or synthetic division can be used to confirm:

$$\frac{p(x)}{d(x)} = \frac{x^3 - 7x + 6}{x + 1} = x^2 - x - 6 + \frac{12}{x - 1}$$

7 Rational Functions

7.1 Sketching Guidelines

Rational functions are ratios of polynomial expressions. Here are some examples:

$$r(x) = \frac{1}{x}, y = \frac{1 - x^3}{2 + x}, f(x) = \frac{x^7 - \sqrt{2}x^2}{2 - 0.1x^{10}}$$

The behavior of the rational function $r(x) = \frac{P(x)}{Q(x)}$ is closely related to properties of the polynomials $P(x)$ and $Q(x)$.

The discussion of a rational function involves several steps that conclude with a neat sketch satisfying all the observations. Below we assume that the rational function is of the form $r(x) = \frac{P(x)}{Q(x)}$.

1) If possible, factor $P(x)$ and $Q(x)$ completely.

2) The domain of $r(x)$ consists of all real numbers for which $Q(x)$ is nonzero. Therefore, solve for $Q(x) = 0$, and then state the domain of $r(x)$ as all reals except the solutions you found.

3) The x-intercepts are the zeros of the numerator $P(x)$, provided that they are in the domain, of course.

4) The y-intercept can be evaluated by setting $x = 0$. If $x = 0$ makes $Q(x) = 0$, then there is no y-intercept.

5) For x values that make both the numerator and the denominator zero (i.e. $P(x)$ and $Q(x)$ share a factor, by the Factor Theorem), the graph of $r(x)$ will have a hole. In calculus, this point is known as a removable discontinuity. For example, we observe that $f(x) = \frac{1-x^2}{x-1}$ has a hole at $x = 1$, due to the shared factor $(x - 1)$. Graphing devices such as, for example, the TI-84, may fail to reveal such details on the graph unless you try to evaluate at the point.

6) The rational function has a vertical asymptote at all x-values that make only $Q(x)$ zero. Using the Factor Theorem, we can restate this: If $(x - a)$ is a factor of $Q(x)$ but not of $P(x)$, then $x = a$ serves as a vertical asymptote. The graph of $r(x)$ will approach one of the infinities when x nears the vertical asymptote.

7) The degrees of the polynomials $P(x)$ and $Q(x)$ determine whether the rational function $r(x)$ has a horizontal asymptote. Let the degrees of $P(x)$ and $Q(x)$ be m and n, respectively, and let their leading coefficients be a_m and a_n, respectively.

a) If $m < n$, the horizontal asymptote is $y = 0$.

b) If $m = n$, the horizontal asymptote is $y = \frac{a_m}{b_n}$.

c) If $m > n$, there is no horizontal asymptote.

8) If the degrees of $P(x)$ and $Q(x)$ are such that $m = n+1$, then we need to determine the equation of the slant (or oblique) asymptote. The equation is simply equal to the quotient $mx + b$ resulting from the polynomial division:

$$r(x) = \frac{P(x)}{Q(x)} = mx + b + \frac{\text{R}}{Q(x)}$$

The slant and the horizontal asymptote describe the end-behavior of a rational function. The graph gets infinitely close to such asymptotes for extreme values of x, and could possibly intersect

it locally. On the other hand, the graph of a rational function never intersects a vertical asymptote.

Before we discuss an example, it is worth noting that when sketching a rational function without the aid of technology, in addition to the steps above, collecting additional points in a two-column table can help improve the accuracy of the final sketch.

Example

Discuss and sketch: $r(x) = \frac{x^4 - 1}{-x^3 + 3x^2 - 4}$.

Solution

We first factor the numerator.

$$r(x) = \frac{x^4 - 1}{-x^3 + 3x^2 - 4} = \frac{(x^2 - 1)(x^2 + 1)}{-x^3 + 3x^2 - 4} = \frac{(x - 1)(x + 1)(x^2 + 1)}{-x^3 + 3x^2 - 4}$$

After plugging in small values such as $-2, -1, 1, 2$ into $Q(x)$, we find that $Q(2) = 0$, therefore $(x - 2)$ is a factor of $Q(x)$. Long or synthetic division yields a quadratic factor:

$$
\begin{array}{r}
-x^2 + x + 2 \\
x - 2 \overline{)\,-x^3 + 3x^2 - 4} \\
...R = 0
\end{array}
$$

The fully factored form is:

$$r(x) = \frac{(x - 1)(x + 1)(x^2 + 1)}{(x - 2)(2 - x)(x + 1)}$$

The domain consists of all real numbers except -1 and 2.

There is only one x-intercept: $(1, 0)$.

The y-intercept is $(0, \frac{1}{4})$

There is a hole at $x = -1$, since $(x + 1)$ is a shared factor.

There is one vertical at asymptote at $x = 2$.

The degree of the numerator is greater than that of the denominator, hence there is no horizontal asymptote. We determine the equation of the slant asymptote by performing long division, since the degree of $P(x)$ is exactly one unit greater than that of $Q(x)$:

$$r(x) = -x - 3 - \frac{-9x - 13}{(x - 2)^2}$$

The slant asymptote has equation $y = -x - 3$. The graph of the rational function should approach this line for extreme values of x. We plot the function with the aid of technology to confirm the details:
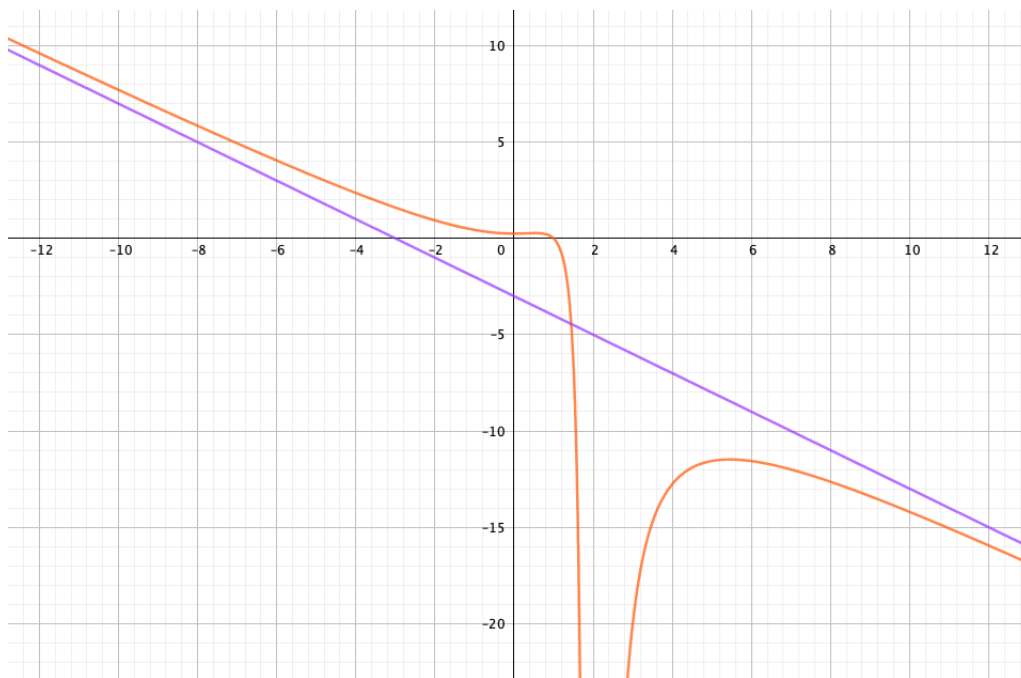

Figure 22: Plot of $r(x)$ and its slant asymtote $y = -x - 3$

Example

Find a rational function that satisfies the conditions: vertical asymptotes at $x = 1, x = -2$; horizontal asymptote at $y = 3$; and the graph passes through $(0, 4)$.

Solution

The rational function must have $(x - 1)(x + 2)$ in the denominator only, and its numerator must be of the form $3x^2 + Ax + B$. We choose $A = 0, B = -8$ to satisfy the y-intercept condition. The following equation satisfies the desired properties:

$$r(x) = \frac{3x^2 - 8}{(x - 1)(x + 2)}$$

8   Exponential and Logarithmic Functions

8.1   Growth and Decay

Exponential growth and decay models fit the equation $y = f(t) = ab^t$, where t represents time, b is the base, and a is the initial population (growth model) or the initial amount of radioactive material (decay model).

$$y(0) = y_0 = ab^0 = a$$

We can rewrite the equation, using Euler's constant e as the base:

$$y = y_0 b^t = y_0 (e^k)^t = y_0 e^{kt}$$

$$k < 0 \Rightarrow \text{Exponential Decay}$$

$$k > 0 \Rightarrow \text{Exponential Growth}$$

In the case of the exponential decay model, we define half-life t_H as the time value at which the amount $y(t)$ is exactly half the initial amount y_0:

$$\frac{1}{2} y_0 = y_0 e^{kt_H}$$
$$\frac{1}{2} = e^{kt_H}$$
$$\ln \frac{1}{2} = kt_H$$
$$t_H = \frac{\ln \frac{1}{2}}{k}$$
$$t_H = \frac{-\ln 2}{k}$$

In the case of the exponential growth model, we define doubling time t_D as the time value at which the amount $y(t)$ is exactly double the initial population y_0. Using similar steps, we find:

$$t_D = \frac{\ln 2}{k}$$

Example

The population of a certain species is modeled by $y(t) = 1000e^{0.02t}$. Determine the doubling time.

Solution

The initial population is 1000 individuals, so we could use the rule of thumb above, or solve directly:

$$2000 = 1000e^{0.02t} \Rightarrow t_D = \frac{\ln 2}{0.02} \approx 34.657 \text{ units of time}$$

Example

If the half-life of a certain radioactive element is five years, find the amount of time it takes for the original amount to be reduced by 75%.

Solution

At half-life, we have:

$$5 = \frac{-\ln 2}{k}$$
$$k = \frac{-\ln 2}{5}$$

At the time when the amount is reduced by 75%, 25% of y_0 remains, so we write:

$$0.25y_0 = y_0 e^{kt}$$
$$\ln \frac{1}{4} = kt$$
$$-2\ln 2 = -\frac{\ln 2}{5}t$$
$$t_{25\%} = 10$$

As expected, it takes 10 years for y_0 to be reduced to $0.25y_0$.

8.2 Working with Logarithms

Logarithmic and exponential operations are inverses of one-another. Justifications to the properties below requires the use of the laws of exponents.

First, we go over a few properties of logarithmic operations:

$$\log(AB) = \log A + \log B \tag{1}$$

$$\log(\frac{A}{B}) = \log A - \log B \tag{2}$$

$$\log a^m = m \log a \tag{3}$$

$$\log_a b = \frac{\log_c b}{\log_c a} \tag{4}$$

$$a^{\log_a m} = m \tag{5}$$

$$\log_a a^x = x \tag{6}$$

Property (4), which is the change of base property, is particularly useful when using technology to plot a logarithmic function whose base is neither 10 nor e.

The logarithmic function $y = \log_a x$, where $a > 0$, has domain $x > 0$ and x-intercept $(1, 0)$.

When $0 < a < 1$, the graph is decreasing at a decreasing rate throughout its domain. When $a > 1$, the graph is increasing at a decreasing rate throughout its domain.
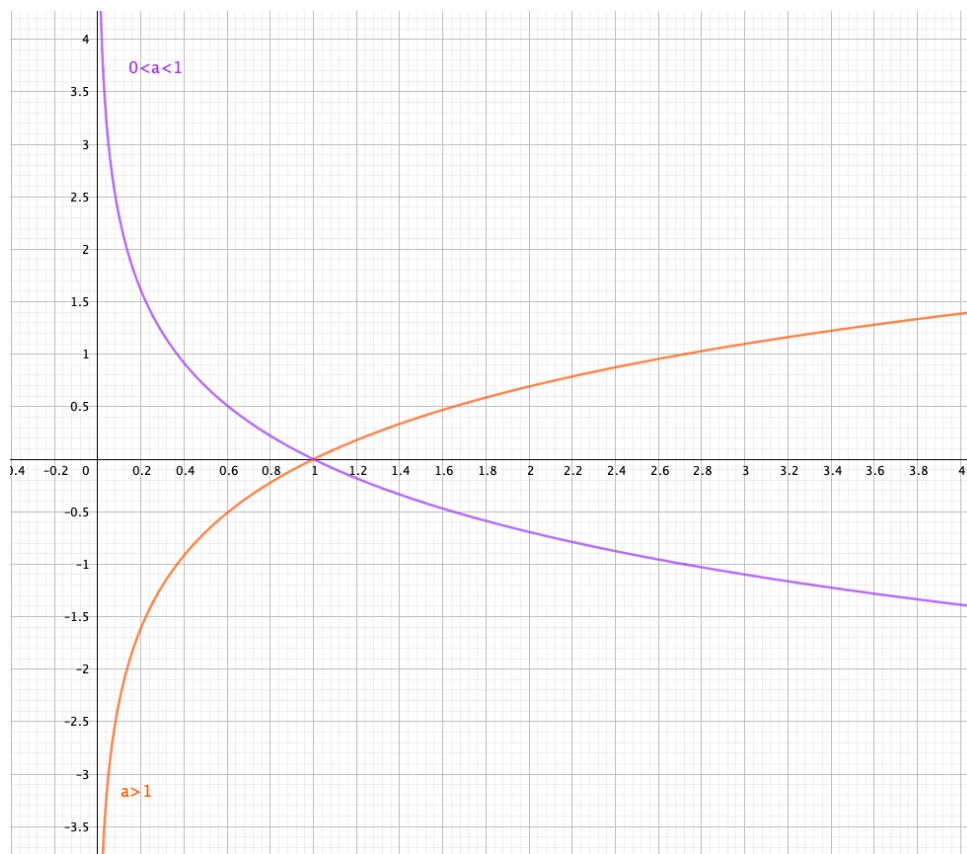
Figure 23: The graphs of the $f(x) = \log_a x$

Example

Solve the equation $2\log_2 x - \log_2(x+1) = 1$.

Solution The domain of this equation is determined by intersecting the domains of the individual logarithmic expressions. The first log is defined for $x > 0$, whereas the second log is defined for $x > -1$. Thus the domain is $x > 0$. We combine the left side of the equation into one logarithm, and then express the equation in its exponential form:

$$2\log_2 x - \log_2(x+1) = 1$$
$$\log_2 x^2 - \log_2(x+1) = 1$$
$$\log_2\left(\frac{x^2}{x+1}\right) = 1$$
$$\frac{x^2}{x+1} = 2^1$$
$$x^2 = 2x + 2$$
$$x^2 - 2x - 2 = 0$$
$$x = \frac{2 \pm \sqrt{12}}{2}$$
$$x = \frac{2 \pm 2\sqrt{3}}{2}$$
$$x = 1 \pm \sqrt{3}$$

We exclude the extraneous solution and conclude that the only solution is $x = 1 + \sqrt{3}$.

Example

Find the inverse function of $y = \log(x-2)$.

Solution

The given function passes the horizontal line test, so we know that its inverse mapping will also be a function. We find the inverse algebraically by swapping x and y, and then solving for y:

$$y = \log(x-2)$$
$$x = \log(y-2)$$
$$10^x = y - 2$$
$$10^x + 2 = y$$
$$f^{-1}(x) = 10^x + 2$$

The domain and range of the original function are $(2, \infty)$ and $(-\infty, \infty)$, respectively. The graph verifies that the range of the original function is the domain of the inverse, and vice versa.
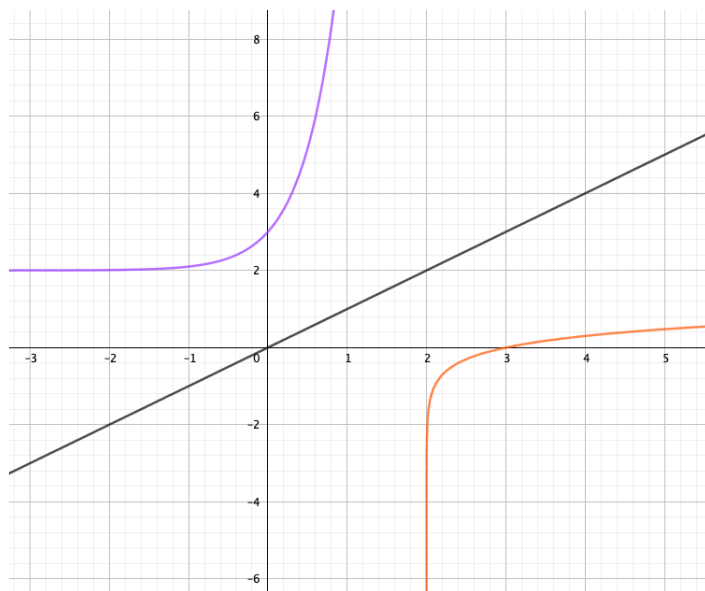

Figure 24: The graphs of the $y = \log(x - 2)$ and $y = 10^x + 2$.

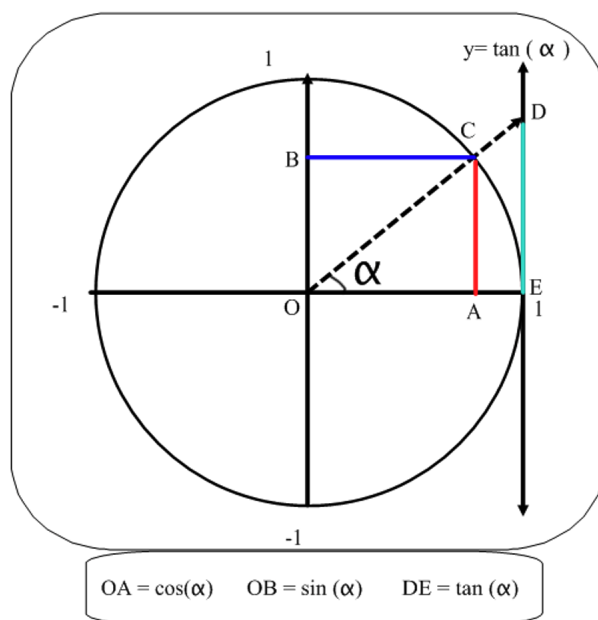# 9   Trigonometry

9.1   The Unit Circle

Figure 25: The unit circle and trig ratios.

$$\tan \alpha = \frac{\text{adj.}}{\text{opp.}} = \frac{AC}{OA} = \frac{DE}{OE} = DE \Rightarrow DE = \tan \alpha$$

In the diagram above, we think of the vertical line $x = 1$ as the tangent axis. As such, it can be used to describe the sign and magnitude of the tangent of any angle. The point $(1,0)$ is the origin; the positive direction points upward, just like the y-axis; the negative direction points downward. To determine the tangent of an angle α, we extend the line containing the terminal side, through the origin if in quadrants II or III, until it intersects the tangent axis $x = 1$. For example, angles in the first and third quadrants have positive tangents, whereas angles in the second and forth quadrants have negative tangents. The tangent is not defined at the north or south poles $(\pm 90°)$ and is equal to zero at the far east $(0°)$ and far west $(180°)$ points on the unit circle.

The radian measure of an angle is equal to the distance traveled along the unit circle from the point $(1, 0)$ to the tip of the terminal side. Positive angles describe motion in the counter-clock-wise direction, and negative angles describe motion in the clock-wise direction. A full rotation requires $2\pi r = 2\pi \approx 6.28$ units traveled, so a counter-clock-wise angle corresponds to 2π radians. Smaller radian measures are obtained by dividing the full 'distance traveled' into smaller parts.

To convert from degrees to radian, multiply by $\frac{\pi}{180}$.

To convert from radian to degree, multiply by $\frac{180}{\pi}$.

The chart below summarizes key angles along the unit circle in radian measure, and their sine and cosine values.
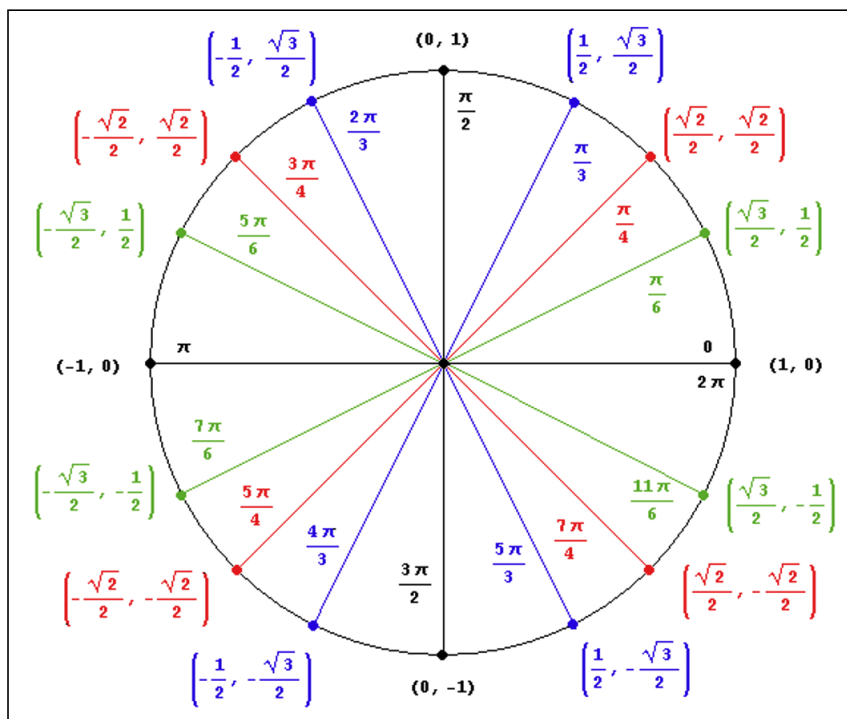

Figure 26: Angles in radian measure: cosine and sine values.

9.2   Trigonometric Ratios

Students who are preparing to study calculus should be comfortable with sketching and describing in full detail (zeros, domain, range, asymptotes) the six trigonometric ratios $\sin x, \cos x, \tan x, \csc x, \sec x, \cot x$, and the three inverse trigonometric functions $\arcsin x, \arccos x, \arctan x$.
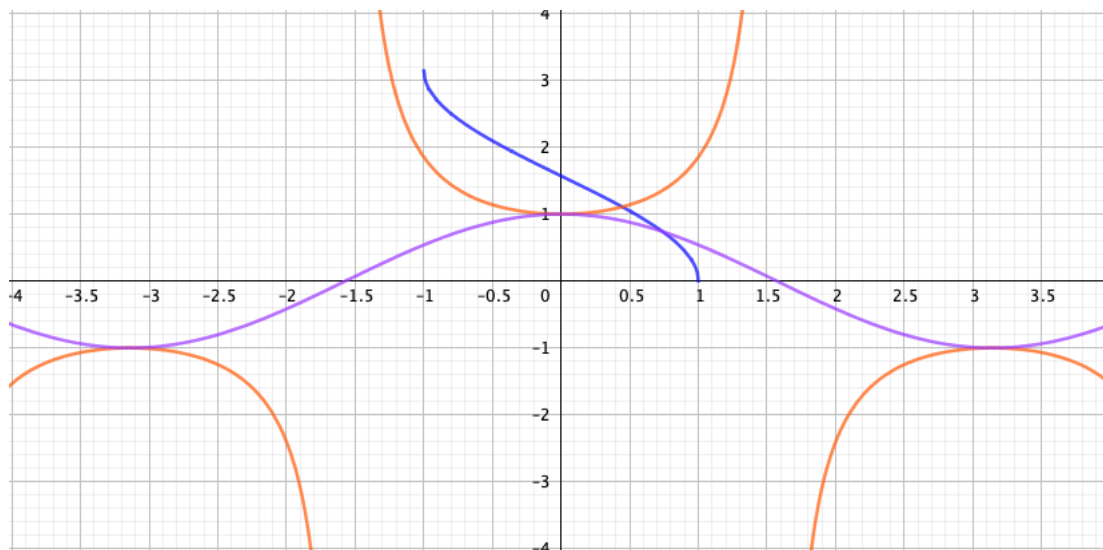
Below we summarize all nine functions, three at a time.

Figure 27: Cosine, Secant, Arccosine.

Cosine Domain:$(-\infty, \infty)$ Range:$[-1, 1]$ Zeros:$\frac{\pi}{2} + k\pi$

Secant Domain:$\mathbb{R} - \{\frac{\pi}{2} + k\pi\}$ Range:$(-\infty, 1] \cup [1, \infty)$

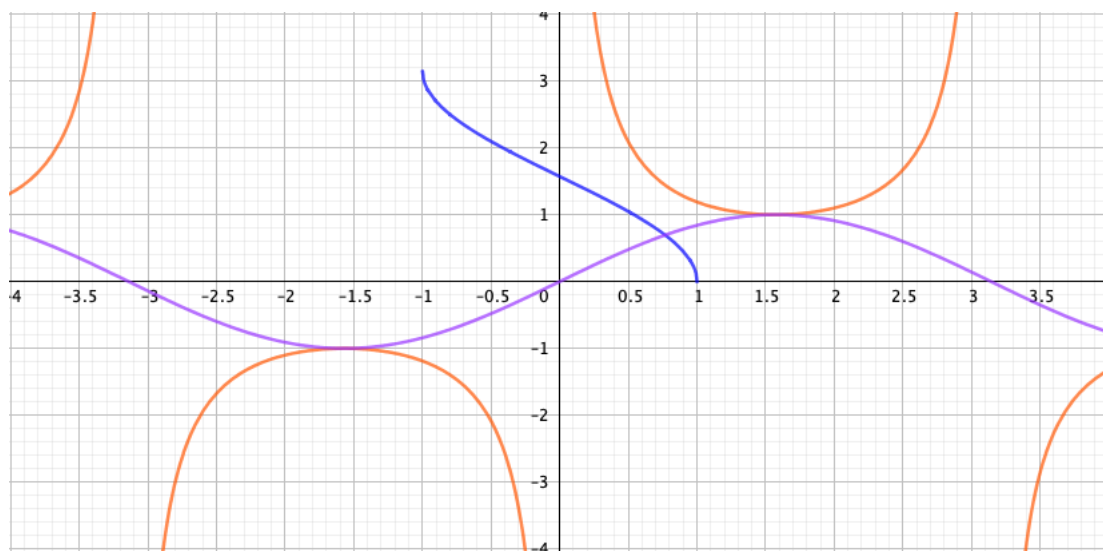**Arccosine** Domain:$[-1, 1]$ Range:$[0, \pi]$

Figure 28: Sine, Cosecant, Arcsine.

Sine Domain:$(-\infty, \infty)$ Range:$[-1, 1]$ Zeros:$k\pi$

Cosecant Domain:$\mathbb{R} - \{k\pi\}$ Range:$(-\infty, 1] \cup [1, \infty)$

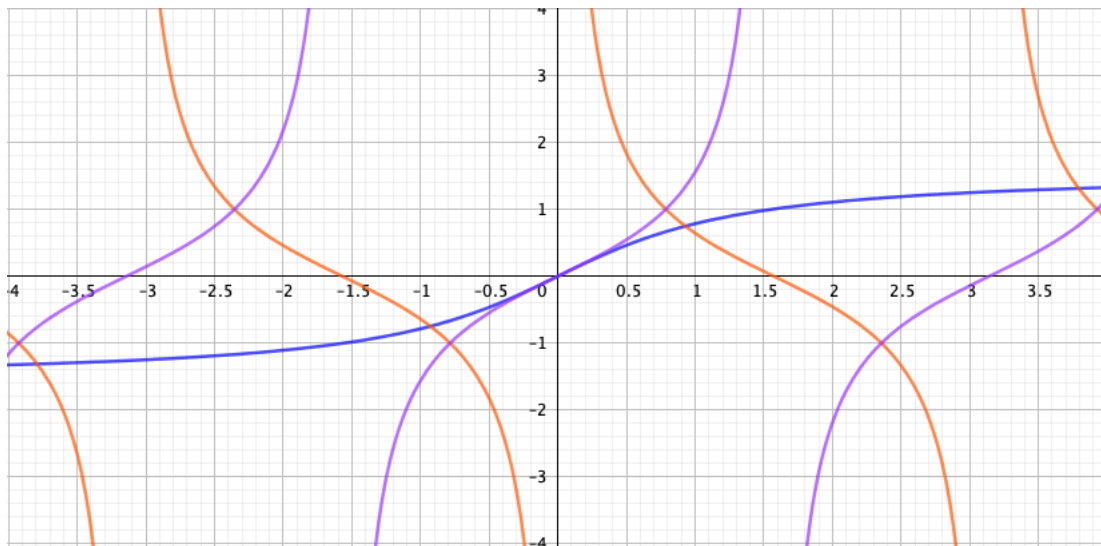**Arcsine** Domain:$[-1, 1]$ Range:$[-\frac{\pi}{2}, \frac{\pi}{2}]$

Figure 29: Tangent, Cotangent, Arctangent.

Tangent Domain:$\mathbb{R} - \{\frac{\pi}{2} + k\pi\}$ Range:$(-\infty, \infty)$ Zeros:$k\pi$

Cotangent Domain:$\mathbb{R} - \{k\pi\}$ Range:$(-\infty, \infty)$

Arctangent Domain:$(-\infty, \infty)$ Range:$(-\frac{\pi}{2}, \frac{\pi}{2})$ Horizontal asymptotes: $y = \pm\frac{\pi}{2}$

9.3 Trig Formulas and Identities

Applying the Pythagorean Theorem to any angle x on the unit circle, we get:

$$\cos^2 x + \sin^2 x = 1$$

Dividing by $\cos^2 x$ and $\sin^2 x$ gives two new identities:

$$1 + \tan^2 x = \sec^2 x$$

$$\cot^2 x + 1 = \csc^2 x$$

We ignore the double angle formula for tangent and instead highlight the sine and cosine of the double angle:

$$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$$

$$\cos(2\alpha) = \cos^2 \alpha - \sin^2 \alpha$$

$$\cos(2\alpha) = 2\cos^2\alpha - 1$$

$$\cos(2\alpha) = 1 - 2\sin^2\alpha$$

Transformed versions of the double angle formulas tend to be especially useful in calculus:

$$\cos^2\alpha = \frac{\cos(2\alpha) + 1}{2}$$

$$\sin^2\alpha = \frac{1 - \cos(2\alpha)}{2}$$

The sine and cosine of the sum of two angles can be of use as well:

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$
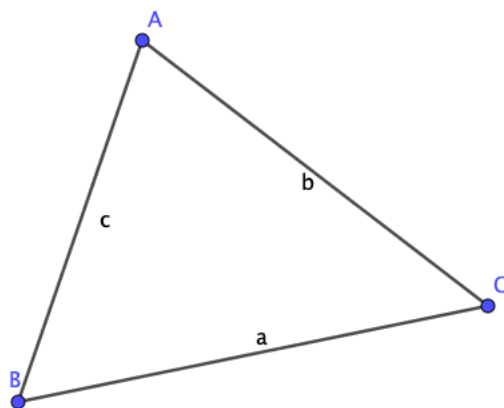
$$\sin(x + y) = \sin x \cos y + \sin y \cos x$$

Figure 30: A triangle with angles A, B, C and sides a, b, c.

The Law of Sines states that in any triangle with angles A, B, C and sides of length a, b, c, the following holds true:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

The Law of Cosines states that in any triangle with angles A, B, C and sides of length a, b, c, if A is the angle formed by sides b and c, then:

$$a^2 = b^2 + c^2 - 2bc \cos A$$

10 Conic Sections

10.1 Ellipses and Circles

The standard equation of an ellipse centered at a point (h, k) is given by:

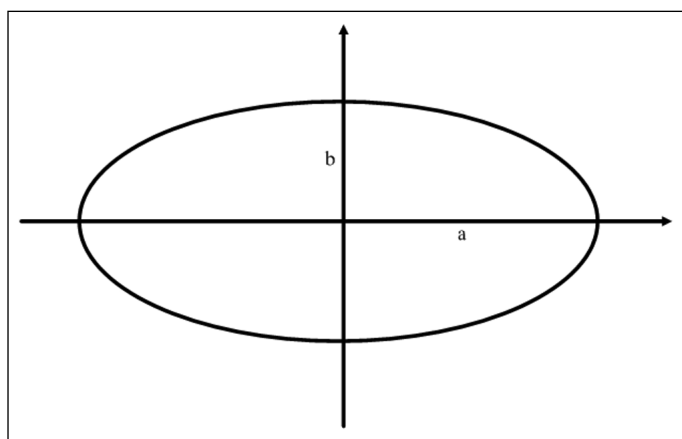$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

Figure 31: An ellipse centered at the origin.

If $a > b$, the ellipse's major and minor axes are $2a$ and $2b$, respectively. Note that when $a = b$, the ellipse is simply a circle. The general equation for a circle centered at a point (h, k) is:

$$(x-h)^2 + (y-k)^2 = r^2$$

Students should be comfortable with completing the square in a mixed equation in order to identify the equation of a curve as that of a circle or an ellipse.

Example

Write the equation $x^2 + 9y^2 + 36y + 2x = -28$ in standard form and identify the curve described by this equation.

Solution

We complete the square for each variable:

$$x^2 + 9y^2 + 36y + 2x = -28$$

$$x^2 + 2x + 9(y^2 + 4y) = -28$$

$$x^2 + 2x + 1 + 9(y^2 + 4y + 4) = -28 + 1 + 36$$

$$(x + 1)^2 + 9(y + 2)^2 = 9$$

$$\frac{(x + 1)^2}{9} + \frac{(y + 2)^2}{1} = 1$$

The equation describes an ellipse centered at $(-1, -2)$ with a horizontal major axis of length 6 and a vertical minor axis of length 2.

10.2   Sector of a Circle

The area of the sector of a circle with radius r and angle α is determined by setting the ratio of the angles (in radian) equal to the ratio of the corresponding areas:

$$\frac{\text{sector area}}{\text{are of the circle}} = \frac{A}{\pi r^2} = \frac{\alpha}{2\pi}$$
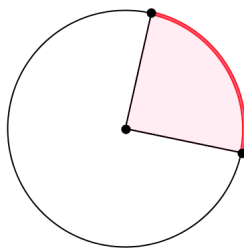
$$A = \frac{1}{2}\alpha r^2$$

Figure 32: Sector of a circle.

We follow similar steps to derive a formula for the length of the arc of a sector:

$$\frac{\text{length of arc}}{\text{circumference}} = \frac{L}{2\pi r} = \frac{\alpha}{2\pi}$$

$$L = \alpha r$$

When measuring the arc length or the area of a sector, we use radian measure for the angle.

Example

Find the arc length and the area of a sector with radius 3 meters and angle 1.

Solution

We assume that the angle is 1 radian.

$A = \frac{1}{2}\alpha r^2 = \frac{1}{2} \times 1 \times 3^2 = \frac{9}{2}$ square meters.

The arc length of the sector, excluding the radii, is given by $L = \alpha r = 3$ meters.