Preparing for AP Calculus AB

– Some Notes –

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1. Introduction [top]

The following notes are intended to provide a discussion and summary of selected pre-requisite topics for students interested in enrolling in a full-year Advanced Placement Calculus AB course. The content addresses a number of topics that students have found challenging in recent years. Please note that this document does not cover all the pre-requisite material for AP Calculus AB, nor does it replace a full-year honors pre-calculus experience.

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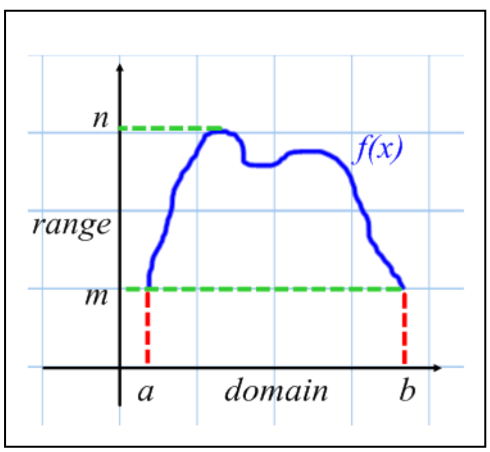
2. Union and Intersection of Sets, Domain and Range of a Function [top]

A set is a collection of things, usually numbers. Sets can be finite or infinite. We describe them by listing the elements, in words, or by means of a mathematical statement such as, for example, an inequality or an equation that the elements must satisfy. A universal set is used in most branches of mathematics. The universal set in single-variable calculus is the infinite set of real numbers.

A few definitions…

- $A \cup B$ is the union of two sets A and B, consisting of elements that belong to A or B, or possibly both.
- $A \cap B$ is the intersection of two sets A and B, consisting of elements that belong to A and B.
- Φ is the empty set.

The domain of a function consists of real numbers on the horizontal axis for which the function exists. Graphically, we look for a collection of $x-$values for which the graph exists. Visually, the domain the can be determined by projecting the graph of the function onto the horizontal axis.



We define the range of a function in a similar fashion. It consists of real numbers on the vertical axis for which the function exists. On the graph, we look for a collection of $y-$values for which the graph exists. Visually, the range can be determined by projecting the graph of the function onto the $y-$axis. When commenting on domain and range, we try to use terms like: union, intersection, open and closed interval.

3. Interval Notation [top]

On the real number line, given two points $x = a$ and $x = b$ such that $a < b$, define:

Notation	Read as…
$(a,b) = \{x \in \mathbb{R} \mid a < x < b\}$	An open interval from a to b.
$[a,b) = \{x \in \mathbb{R} \mid a \le x < b\}$	A half-open interval, closed at a, open at b.
$[a,b] = \{x \in \mathbb{R} \mid a \le x \le b\}$	A closed interval from a to b.
$(-\infty,\infty)$	Describes the entire real number line.
$\mathbb{R}$	The set of Real Numbers.
$\in$	"Belongs to", indicates membership to a set.
$\forall$	"for every" or "for all".
$\exists$	"there exists".
$\Leftrightarrow$	"if and only if" (bi-conditional connective).
$\lfloor x \rfloor$	The greatest integer function.
$\wedge$	"and".
$\vee$	"or".
$\subseteq$	"is a subset of".

4. Distance Between Two Points [top]

On the real number line, we use the operation of absolute value to define the distance between two points $x = a$ and $x = b$:

$$d(a,b) = |b - a| = |a - b|$$

The English sentence "x is within 3 units of 5" translates into the mathematical statement:

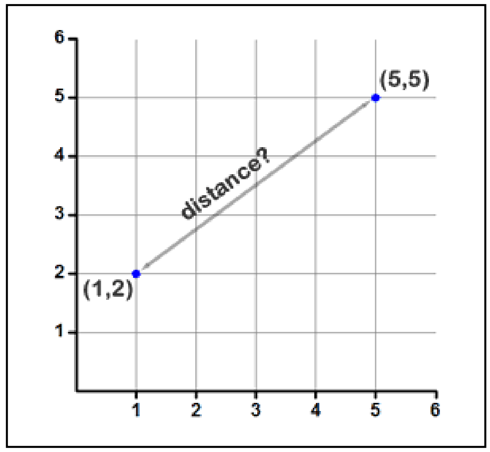
$$|x - 5| < 3$$

, and it describes the open interval (2, 8). We verify by solving the inequality:

$$-3 < x - 5 < 3$$
$$2 < x < 8$$

On the Cartesian plane, the Pythagorean theorem is used to define and determine the distance between two points $A=(x_1,y_1)$ and $B=(x_2,y_2)$.

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$
$$A=(1,2)$$
$$B=(5,5)$$
$$d=\sqrt{(5-1)^2+(5-2)^2}=\sqrt{25}=5$$



The right triangle with legs (x_2-x_1), (y_2-y_1) has hypotenuse $d=|AB|$, so we have:

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

Example: A circle of radius r centered at the origin is defined as the set of all points (x,y) on the two-dimensional plane whose distance from the origin is r. We determine the equation of the circle by applying the distance formula to the variable point (x,y) and the origin $(0,0)$:

$$r=\sqrt{(x-0)^2+(y-0)^2}$$
$$\Rightarrow x^2+y^2=r^2$$

5. Factoring Patterns and the Quadratic Formula [top]

Difference of Squares $a^2-b^2=(a-b)(a+b)$	Square of the Sum and Difference $a^2+2ab+b^2=(a+b)^2$ $a^2-2ab+b^2=(a-b)^2$
Difference of Cubes $a^3-b^3=(a-b)(a^2+ab+b^2)$	Sum of Cubes $a^3+b^3=(a+b)(a^2-ab+b^2)$
Factoring a Minus Sign $a-b=-(-a+b)=-(b-a)$	Difference of even powers $a^{2m}-b^{2m}=(a^m-b^m)(a^m+b^m)$

Expressions with Fractional Exponents

It is often easier to factor the term with the least exponent. Somehow these problems are "cooked" so that the "left-overs" have integer exponents, as shown in this example:

$$\frac{x^{-1/2}+2x^{3/2}-x^{-3/2}}{4x^{-3/2}}=\frac{x^{-3/2}[x+2x^{3}-1]}{4x^{-3/2}}=\frac{x+2x^{3}-1}{4}$$

Splitting the Middle Term: ax^2+bx+c

Assuming the expression is factorable, we first find two integers p,q such that $p+q=b \quad \wedge \quad pq=ac$, and then rewrite the quadratic expression as a four-term sum: $ax^2+px+qx+c$. Grouping the terms two by two leads to complete factoring.

Example: Factor $2x^2-x-3$ by splitting the middle term.

$$\begin{cases} p+q=-1 \\ pq=2\cdot(-3)=-6 \end{cases} \Rightarrow p=2, q=-3$$

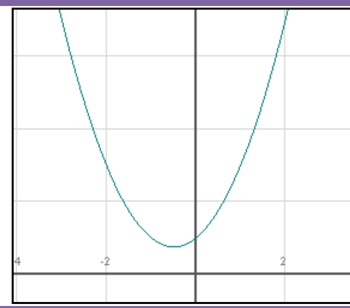
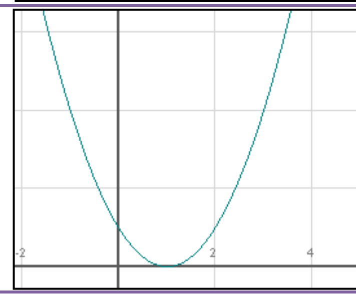
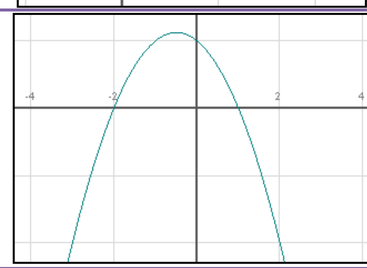
$$\begin{aligned}
&2x^2-x-3= \\
&2x^2+(2x-3x)-3= \\
&(2x^2+2x)+(-3x-3)= \\
&2x(x+1)-3(x+1)= \\
&(2x-3)(x+1)
\end{aligned}$$

The Quadratic Formula

Consider the equation: $ax^2+bx+c=0$. We assume $a\neq 0$.

$$\begin{aligned}
&ax^2+bx+c=0 \\
&a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)=0 \\
&x^2+\frac{b}{a}x=-\frac{c}{a} \\
&x^2+2\frac{b}{2a}x+\left(\frac{b}{2a}\right)^2=-\frac{c}{a}+\left(\frac{b}{2a}\right)^2 \\
&\left(x+\frac{b}{2a}\right)^2=\frac{-4ac+b^2}{2a}\Rightarrow {}_1x_2=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-b\pm\sqrt{D}}{2a}
\end{aligned}$$

The factored form of $ax^2 + bx + c$ using the zeros from the quadratic formula is given by $a(x - x_1)(x - x_2)$. There are three possible scenarios:

Discriminant and Zeros	Graph
$D < 0$: **not factorable; no zeros; the parabola never meets the horizontal axis.** $y = x^2 - 3x + 6$	
$D = 0$: **one factor, repeated; one zero; the parabola is tangent to the horizontal axis.** $y = (x-1)^2$	
$D > 0$: **two distinct factors; two distinct zeros; the parabola intersects the horizontal axis twice.** $y = (1-x)(x-2)$	

6. Transformations: A Table Summary [top]

We summarize the geometric transformations of a function $y = f(x)$:

New Form	Description
$f(x) + a$	Vertical translation (or shift) by a units up $(a > 0)$ or down $(a < 0)$
$a \cdot f(x)$	Vertical stretch $(a > 1)$ or shrink $(0 < a < 1)$
$-f(x)$	Reflection over the horizontal axis
$\|f(x)\|$	Portions of the graph below the $x-$axis reflect about $y = 0$
$f(x - a)$	Horizontal shift by a units to the right $(a > 0)$ or to the left $(a < 0)$
$f(a \cdot x)$	Horizontal shrink $(a > 1)$ or stretch $(0 < a < 1)$ by factor a
$f(-x)$	Reflection over the vertical axis
$f(\|x\|)$	Portions of the graph to the left of $y-$axis reflect about $x = 0$

Example: Apply these transformations to $y = 1 - x$ in the given order

- Horizontal stretch by a factor of 2 $\quad y = 1 - \frac{x}{2}$
- Reflect over horizontal axis $\quad y = -(1 - \frac{x}{2}) = \frac{x}{2} - 1$
- Vertical shift by five units upward $\quad y = \frac{x}{2} - 1 + 5 = \frac{x}{2} + 4$

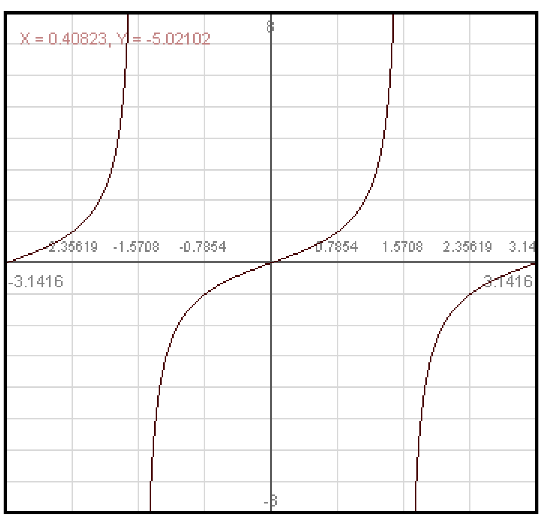
7. Functions: Odd, Even, or Neither [top]

Odd functions have graphs with symmetry about the origin. Algebraically, $f(x)$ is odd if and only if $f(-x) = -f(x)\ \forall x$. Examples: $y = x^3$, $f(x) = -x + k, h(x) = \sin x, l(x) = x^{2n-1}$.

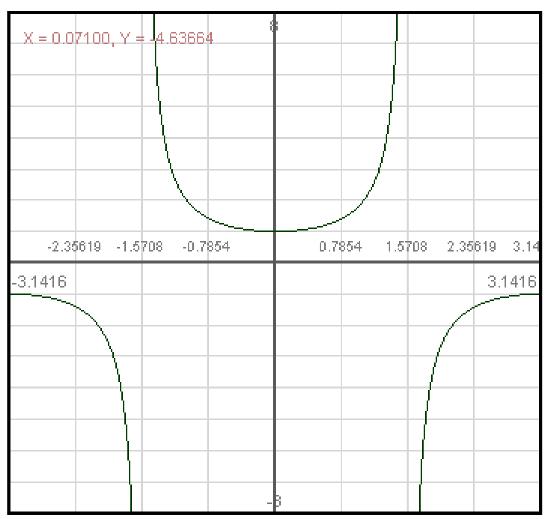
Even functions have graphs with symmetry about the y-axis. Algebraically, $f(x)$ is even if and only if $f(x) = f(-x)\ \forall x$. Examples: $y = x^2$, $f(x) = k, h(x) = \cos x, l(x) = x^{2m}$.

Example: Show that: $f(x) = x^5 - x$ is an odd function.
We need to prove that $f(-x) = -f(x) \quad \forall x$. Starting with the left side, we write:
$f(-x) = (-x)^5 - (-x) = -x^5 + x = -(x^5 - x) = -f(x)$
So we have shown that the algebraic definition of an odd function holds true.

Graphically, the graphs of odd and even functions exhibit symmetries about the origin and the vertical axis throughout their domains, respectively. For example, here are sketches of the tangent and secant functions:



$f(x) = \tan x : Odd$



$f(x) = \sec x : Even$

Functions that do not satisfy either type of special symmetry are simply labeled 'neither'.

Example: Is there a function that is both even and odd?

Assume that there exists a function f that is both odd and even. Let (a,b) be a point on the graph of f, so that $f(a)=b.$ By the property of even functions, it follows that point $(-a,b)$ must also be on the graph, since $f(a)=f(-a)$. By the property of odd functions, it follows that point $(-a,-b)$ must also be on the graph, since $f(-a)=-f(a)$. Hence, we must have a function that passes through points $(-a,b)$ and $(-a,-b)$, failing the Vertical Line Test whenever b is nonzero. We conclude that the horizontal line $f(x)=0$ is the only function that is both odd and even.

8. Algebra of Functions [top]

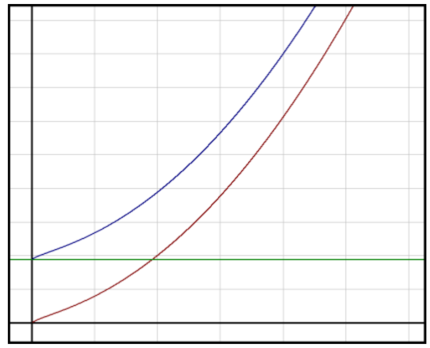
Let $f(x)$ and $g(x)$ be two functions with domains A and $B,$ respectively. We define:

New Function	Name	Domain
$f+g=(f+g)(x)=f(x)+g(x)$	Direct Sum	$A\cap B$
$f-g=(f-g)(x)=f(x)-g(x)$	Difference	$A\cap B$
$fg=(fg)(x)=f(x)\cdot g(x)$	Product	$A\cap B$
$f/g=(f/g)(x)=f(x)/g(x)$	Quotient	$A\cap B \quad (g(x)\neq 0)$

Geometrically, the graphs of new functions above are obtained by performing arithmetic operations on pairs of y-values at each independent value x in the domain $A\cap B,$ and plotting the new $y-$coordinates.

For example, in firm theory models in microeconomics, the total cost function is the direct sum of two "old" functions: fixed costs (a constant function) and variable costs. We write $C(x)=F(x)+V(x)$, or simply $C(x)=F+V(x)$. Graphically, the total cost function is a vertical translation of the variable cost function by F units upward.

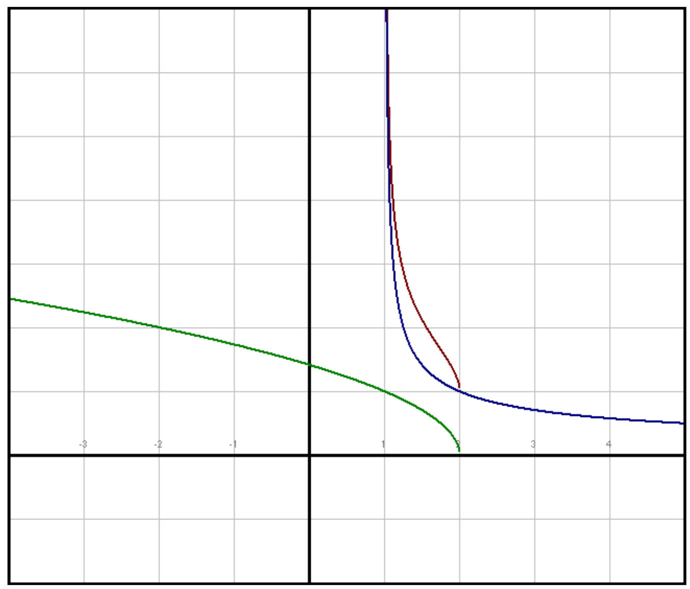
$F(x) = K$ (in green) = Fixed Costs are incurred regardless of level of production
$V(x) =$ Variable Cost (in red) passes through the origin. The variable cost is zero when there is no output.
$C(x) =$ Total Cost (in blue) is the direct sum of the fixed and variable cost curves.



Example: Find the domain of $f(x) = \sqrt{2-x} + \frac{1}{\sqrt{x-1}}$. Express the answer in interval notation.

$$2 - x \geq 0 \leftrightarrow x \leq 2 \leftrightarrow (-\infty, 2]$$
$$x - 1 > 0 \leftrightarrow x > 1 \leftrightarrow (1, \infty)$$
$$(-\infty, 2] \cap (1, \infty) = (1, 2]$$

The direct sum (red) is only defined when the graphs of the algebraic expressions that are being added (green and blue) exist simultaneously.

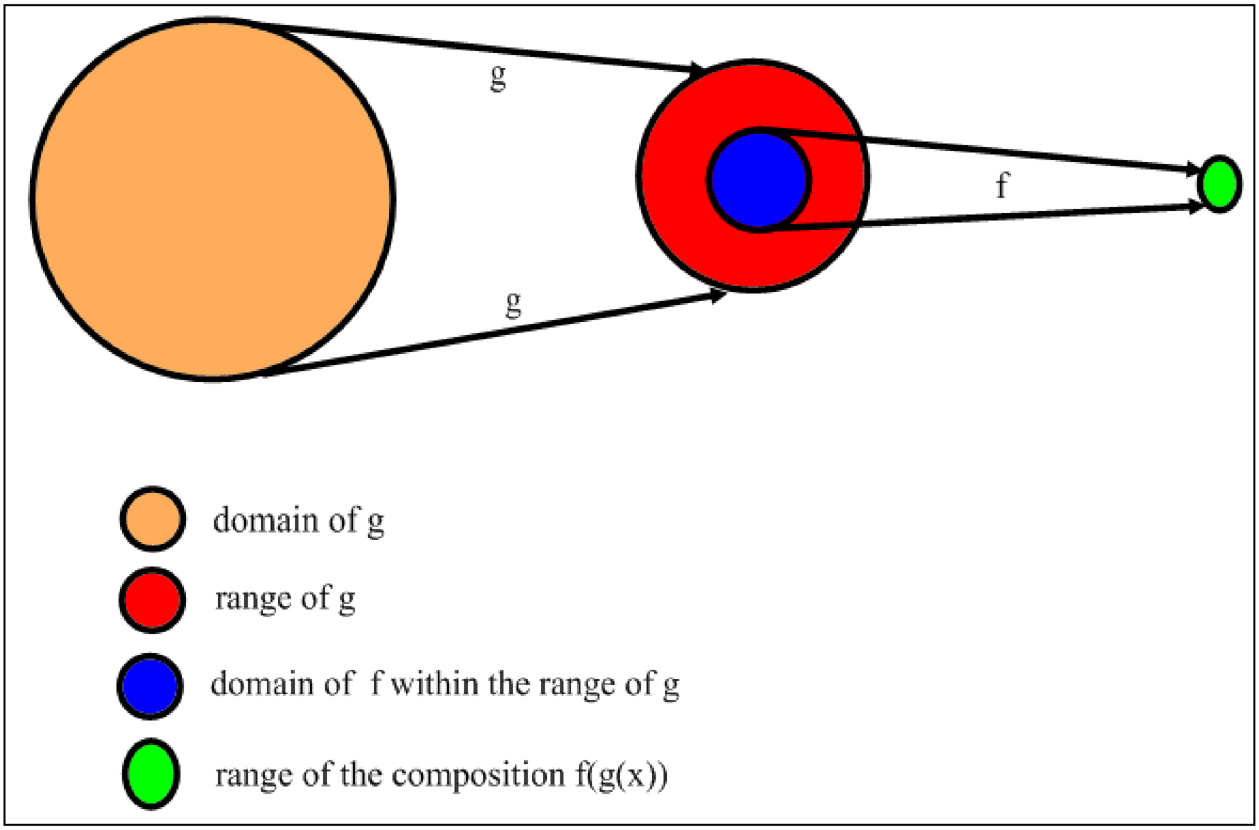


9. Composition of Functions [top]

The composition of $f(x)$ with $g(x)$ is defined as $f \circ g = (f \circ g)(x) = f(g(x))$. g is the inner function, whereas f is called the outer function. Think of the inner function as the rule that is applied to the input first, and the outer as the algebraic rule that is applied last.

$$g \quad "\rightarrow"$$
$$f \quad "\Rightarrow"$$
$$x \rightarrow g(x) \Rightarrow f(g(x))$$

The domain of the composition consists of all values in the domain of g such that their output under g is in the domain of f, hence ensuring that both f and g are defined.



One should be able to compose elementary functions and identify the elementary functions that make up a given composite.

Example: Write $l(x)=\sqrt{\sin(1-x^3)}$ as a composition of elementary functions.

We could use three or four functions, depending on how *elementary* one expects the expressions to be. The innermost operation is the cubic. The remaining operations, in order, are: subtract the input from one, take the sine, and finally, take the square root. We write these rules out as separate functions:

$$k(x)=x^3$$
$$j(x)=1-x$$
$$i(x)=\sin x$$
$$h(x)=\sqrt{x}$$

Verify that $l(x) = (h \circ i \circ j \circ k)(x)$.

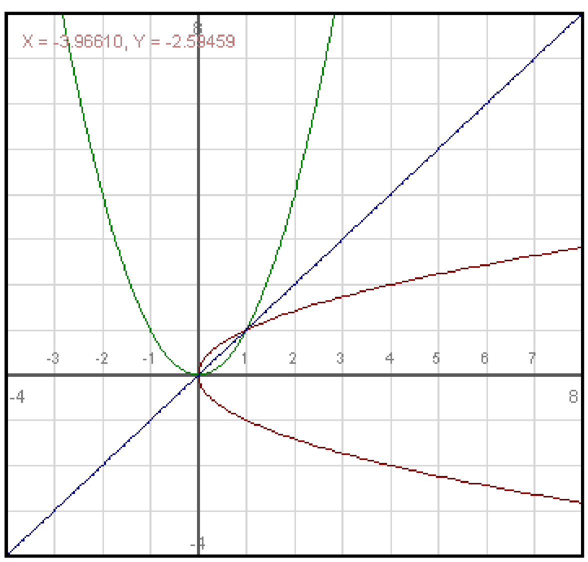
Example: Using specific examples and proofs, investigate the parity (even, odd, or neither) of the composition, sum, difference, product, and quotient of two functions f and g. For example, if one composes an odd function with an even function, is the resulting composition odd, even, or neither? Summarize your findings in a report.

10. The Inverse of a Function [top]

A function f is a mapping that sends x values to $y = f(x)$ such that there is at most one value in the range space for each element in the domain space. The inverse of f, written as f^{-1}, undoes the effect of f:

$$f : x \to y$$
$$x \leftarrow y : f^{-1}$$

Algebraically, the inverse relation of $y = f(x)$ is determined by swapping the variables, so that we have $x = f(y)$. For each point (a,b) contained in f, the "inverse" point (b,a) is contained in the inverse mapping f^{-1}. The slope of the line connecting these two points is given by $m = \frac{b-a}{a-b} = \frac{-(a-b)}{a-b} = -1$, hence perpendicular to the $y = x$ line. Because the two points are also equidistant from the identity function, we conclude that the geometric transformation used to find the inverse is a reflection about the forty-five degree line. For example, the inverse mapping of $y = x^2$ is $x = y^2$.



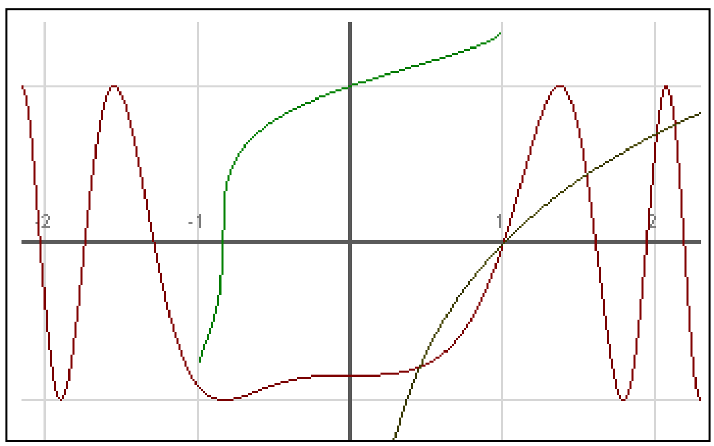
Graph of $y = x^2$ and its inverse mapping, $x = y^2$

Graphically, the inverse is a reflection about the line $y = x$. Note that when reflected about $y = x$, a horizontal line becomes vertical. Hence, to determine whether or not the inverse mapping of a function f is also a function (in other words, passes the vertical line test), it suffices to apply the horizontal line test to the original function f. The inverse of $y = x^2$ is not a function because the original function fails the horizontal line test.

Fact: If f is monotonic (strictly increasing or strictly decreasing), then its inverse mapping f^{-1} is also a function.

Example: Find the inverse of $y = \sin(x^3 - 1)$ algebraically, and then plot the original function and its inverse on the same window.

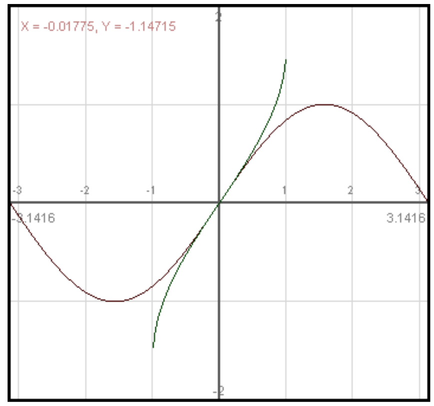
$$\begin{aligned}
&y = \sin(x^3 - 1) \\
&\Rightarrow x = \sin(y^3 - 1) \\
&\Rightarrow \arcsin x = y^3 - 1 \\
&\Rightarrow \arcsin x + 1 = y^3 \\
&\Rightarrow y = \sqrt[3]{\arcsin x + 1}
\end{aligned}$$



In the example above, note that the inverse mapping is a function whenever the original function passes the horizontal line test. In the picture above, this test only passes for $x-$values between -1 and 1.

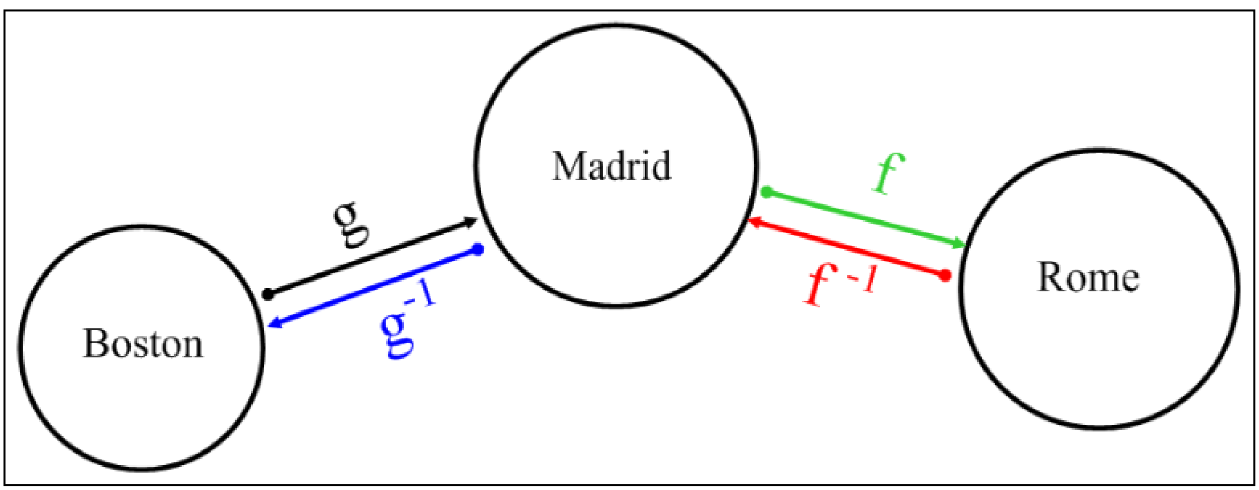
When dealing with inverses, we typically restrict the domain to ensure that the inverse is a function.

For example, the domain of the sine function is restricted to $[\frac{-\pi}{2}, \frac{-\pi}{2}]$since the graph is monotonic on this interval. The range $[-1,1]$ becomes the domain of arcsine, the inverse function. Graphs of the sine and arcsine functions:



11. The Inverse of a Composition [top]

Let f and g describe the flights Madrid – Rome and Boston – Madrid, respectively. Assuming there are no other routes from Boston to Rome, one must apply $f \circ g$ to a passenger who wishes to travel from Boston to Rome. The inverse of this composition will be a composition of inverses, in reverse order:



Fact: $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$

Example: The inverse of the composition $y = (3x-1)^3$is the composite of $g^{-1}(x) = \frac{x+1}{3}$ with $f^{-1}(x) = x^{\frac{1}{3}}$. So we write, $y^{-1}(x) = \frac{x^{1/3}+1}{3}$

Example: Suppose you are given the following information about two functions f and g.

x	f	g
-1	4	3
3	5	6
4	-4	9
6	3	8

Determine the following:

$a.\ f^{-1}(-4)$ $\quad b.\ f(g(3))$ $\quad c.\ (f+g)(3)$ $\quad d.\ (fg)(3)$ $\quad e.\ (g\circ f)(-1)$ $\quad f.\ g^{-1}\circ f(6)$

$f^{-1}(4)=-1$

$f(g(3))=f(6)=3$

$(f+g)(3)=f(3)+g(3)=5+6=11$

$(fg)(3)=f(3)\cdot g(3)=5\cdot 6=30$

$(g\circ f)(-1)=g(f(-1))=g(4)=9$

$(g^{-1}\circ f)(6)=g^{-1}(f(6))=g^{-1}(3)=-1$

12. The Property of Inverse Functions [top]

Two functions f and g are inverses of one another if and only $f(g(x))=x=g(f(x))$. Algebraically, it is sufficient to verify that both compositions give the identity function, which means the two effects undo each-other when composed in either order.

Example: Show that $f(x)=\dfrac{1+x^5}{2}, g(x)=\sqrt[5]{2x-1}$ are inverses of each-other.

$$f\circ g=f(g(x))=\frac{1+(g(x))^5}{2}=\frac{1+(\sqrt[5]{2x-1})^5}{2}=\frac{1+2x-1}{2}=x$$

$$f\circ g=g(f(x))=\sqrt[5]{2f(x)-1}=\sqrt[5]{2\frac{1+x^5}{2}-1}=\sqrt[5]{1+x^5-1}=x$$

13. Polynomials: Discussion and Sketching [top]

Polynomials are functions of the form $p(x)=a_0+a_1x+a_2x^2+a_3x^3+...+a_nx^n$. Their graphs are smooth throughout their domain, which consists of all real numbers. The leading term, a_nx^n, is the term with largest power of x. The degree and leading coefficient of the polynomial are n and a_n, respectively. The numbers $a_0, a_1, ..., a_n$ are the coefficients of the polynomial, and they are all real constants. n is a non-negative integer.

When discussing polynomials, we comment in as much detail as possible on the following: degree, leading coefficient, zeros, $y-$intercept, and end-behavior. The end-behavior of a polynomial is described in two statements:

1. As x goes to ∞, y goes to ….
2. As x goes to $-\infty$, y goes to ….

Graphically, to describe the end-behavior of a function amounts to observing what happens to the graph for extreme values of x. In the case of polynomials, there are four possibilities (assuming degree greater than one), summarized in the following chart:

Degree \ Leading Coefficient	**Positive**	**Negative**
Even	As $x \to \infty, y \to \infty$ As $x \to -\infty, y \to \infty$ Example: $p(x) = \frac{1}{3}x^4 - x + 1$	As $x \to \infty, y \to -\infty$ As $x \to -\infty, y \to -\infty$ Example: $p(x) = -x^6 + \sqrt{2}x^2$
Odd	As $x \to \infty, y \to \infty$ As $x \to -\infty, y \to -\infty$ Example: $p(x) = 7x^7 + x^2 - 10$	As $x \to \infty, y \to -\infty$ As $x \to -\infty, y \to \infty$ Example: $p(x) = -x^{101} + 2x^{100}$

With polynomials of degree greater than two, one must be comfortable with the Factor Theorem, helpful in determining the zeros.

Factor Theorem: $x = a$ is a zero of a polynomial $p(x)$ iff $(x-a)$ is a factor of $p(x)$.

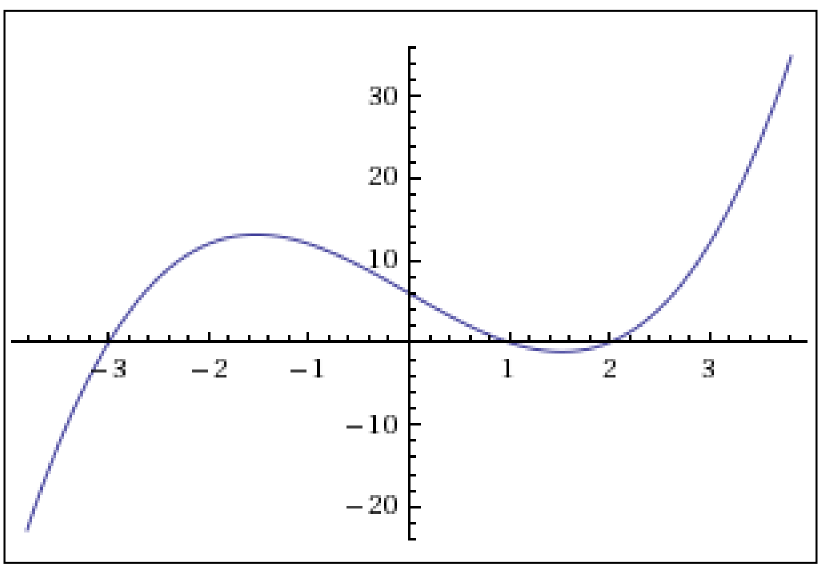
Example: Discuss and sketch $p(x) = x^3 - 7x + 6$.

Evaluate small numbers such as -2, -1, 1, and 2. Note that $p(1) = 0$. By the Factor Theorem, $(x-1)$ is one of the factors. Performing Long or Synthetic Division determines the quotient, which, in this example, is the quadratic expression $x^2 + x - 6$.

$$p(x) = x^3 - 7x + 6 = (x-1)(x^2 + x - 6) = (x-1)(x+3)(x-2)$$

The given polynomial has an odd degree and positive leading coefficient, so its graph must start in the third quadrant and end in the first. All the factors appear an odd number of times (this is called odd multiplicity), hence the graph intersects the horizontal axis at all three intercepts: -3, 1, and 2. More generally, if a zero has an even multiplicity, then the horizontal axis serves as a tangent to the polynomial at that point.

We could use the $y-$intercept and additional points for added accuracy to sketch the graph. Wolfram Alpha can verify the sketch.



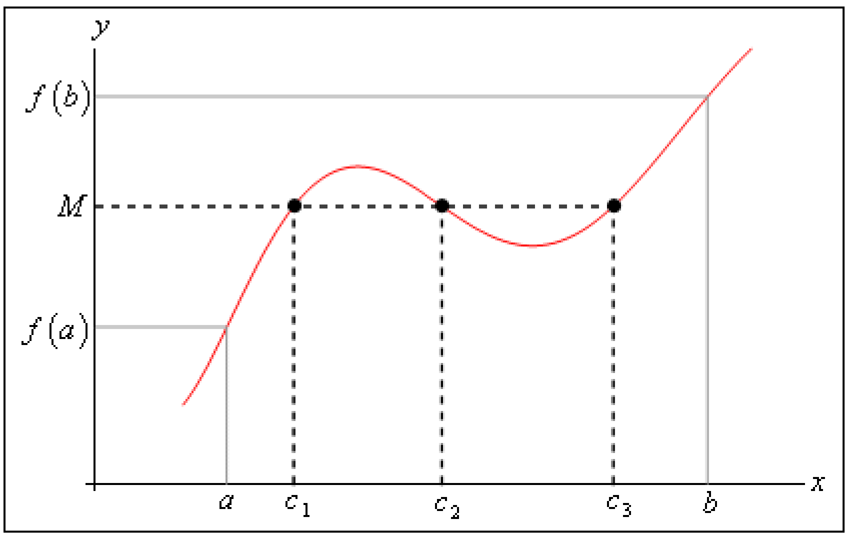
14. Intermediate Value Theorem [top]

Let f be a continuous function on the closed interval $[a,b]$, with $f(a)=K, f(b)=L$, and let $y=M$ be an arbitrary point in the interior of the image of $[a,b]$ under f. Then there exists at least one value $x=c$ in the interior of $[a,b]$ such that $f(c)=M$.

IVT can be applied to continuous functions (especially polynomials) to approximate their zeros. In such cases, M is equal to zero, while K and L have opposite signs. The interval $[a,b]$ is repeatedly "shrunk" by choosing pairs of points that give $y-$values of opposite signs, hence capturing the zero(s) in each step.

Example: What does the IVT say about the zero(s) of $f(x)=1-x^3+2x$?

Observe that $f(1)$ and $f(2)$ have opposite signs, and since all polynomials are continuous, IVT guarantees the existence of at least one zero in the open interval $(1,2)$. One could test values such as, for example, $\frac{4}{3}$ and $\frac{5}{3}$. Of the three resulting intervals investigate the one(s) that have $y-$values of opposite signs at the end points, then repeat the process.



The Intermediate Value Theorem

Fact: The graphs of odd degree polynomials start on the third quadrant and end in the first, or start in the second and end in the fourth. IVT guarantees that all such polynomials have at least one real zero.

15. Rational Functions: Discussion and Sketching [top]

Rational functions are ratios of polynomials. Examples: $r(x)=\frac{1}{x}$, $y=\frac{1-x^3}{2+x}$, $f(x)=\frac{x^7-x^2}{2-x^{10}}$. The behavior of $r(x)=\frac{P(x)}{Q(x)}$ is closely related to properties of the polynomials $P(x)$ and $Q(x)$ that make up the function.

The discussion of a rational function typically consists of the following steps:

- Factor $P(x)$ and $Q(x)$ completely if they are not already factored.

- The **domain** of the rational function consists of all real numbers except the zeros of $Q(x)$. We write $dom[r(x)] = \{x \in \mathbb{R} \mid Q(x) \neq 0\}$.
- The $x-$**intercepts** are the zeros of the numerator $P(x)$, provided that they are in the domain, of course.
- The $y-$**intercept** can be evaluated by setting $x = 0$.
- Note that for numbers that make both the numerator and the denominator zero (so $P(x)$ and $Q(x)$ share a factor), the graph of a function has a hole. This is known as a **jump discontinuity**. For example, we observe that $y = \frac{1-x^2}{x-1}$ has a hole at $x = 1$, due to the shared factor. Graphing devices such as, for example, the TI-83, fail to reveal such details on the graph unless you try to evaluate at the point.
- The rational function has a **vertical asymptote** for each linear factor of $Q(x)$ that is not shared with the numerator. If $(x-a)$ is a factor of the denominator only, then $x = a$ is a vertical asymptote.
- The degrees of the polynomials help determine if the rational function has a **horizontal asymptote**. Let the degrees of $P(x)$ and $Q(x)$ be m and n, respectively, and let their leading coefficients be a_m and b_n, respectively.
  - ✓ If $m < n$, then the horizontal asymptote is $y = 0$.
  - ✓ If $m = n$, then the horizontal asymptote is $y = \frac{a_m}{b_n}$.
  - ✓ If $m > n$, then there is no horizontal asymptote.
- If the degrees are such that $m = n+1$, then we need to determine the equation $y = mx + b \; (m \neq 0)$ of the **slant (or oblique) asymptote**. This equation is simply equal to the quotient of the polynomial division:

$$Q(x)\overset{\displaystyle mx+b}{\overline{\big)\; P(x)}}$$

  Note that the slant and the horizontal asymptote describe the end behavior of a rational function. The graph gets infinitely close to such asymptotes for extreme values of x, and could possibly intersect it locally, near the center of the two-dimensional plot. On the other hand, the graph of a function never intersects a vertical asymptote.

- Collecting additional points in a two-column table will help us draw an accurate graph of the rational function.

Example: Discuss and sketch $r(x)=\dfrac{x^4-1}{-x^3+3x^2-4}$.

We first need to factor $P(x)$ and $Q(x)$ completely. The numerator is a difference of squares. The denominator will require using the Factor Theorem.

$$r(x)=\frac{x^4-1}{-x^3+3x^2-4}=\frac{(x^2-1)(x^2+1)}{-x^3+3x^2-4}=\frac{(x-1)(x+1(x^2+1)}{-x^3+3x^2-4}$$

Testing small values such as -2, -1, 1, 2 in $Q(x)=-x^3+3x^2-4$, we find that $Q(2)=0$, hence $(x-2)$ is a factor. So we perform long division,

$$\begin{array}{r} -x^2+x+2 \\ x-2\overline{\big)-x^3+3x^2-4} \\ \dots R=0 \end{array}$$

then factor the quadratic quotient:

$$r(x)=\frac{x^4-1}{-x^3+3x^2-4}=\frac{(x-1)(x+1)(x^2+1)}{(x-2)(2-x)(x+1)}$$

The domain consists of all real numbers except -1 and 2.

There is only one $x-$intercept, equal to 1.

The $y-$intercept is $r(0)=\frac{1}{4}$.

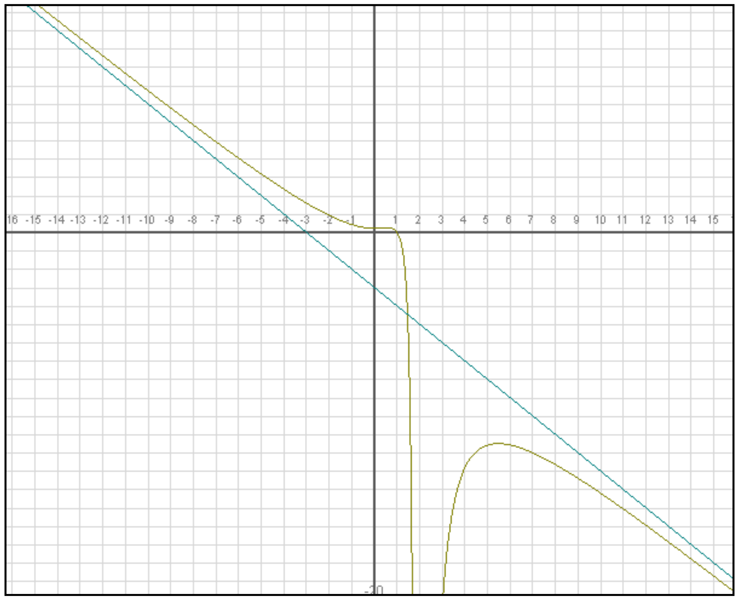
There is a hole at $x=-1$, since this value has a corresponding shared factor.

There is only one vertical asymptote, given by $x=2$.

The degree of the numerator is (one unit) greater than the degree of the denominator, hence there is no horizontal asymptote, but we can determine the equation of a slant asymptote, since $m=n+1$.

$$\begin{array}{r} -x-3=y \\ -x^3+3x^2-4\overline{\big)\;\;x^4-1} \\ \dots R=-0 \end{array}$$

The equation of the slant asymptote is $y = -x - 3$. The graph of the rational function should approach this line in the long run. We plot the function using GraphCalc to compare our work:



Example: Find a rational function that satisfies the conditions: vertical asymptotes at $x = 1, x = -2$; horizontal asymptote at $y = 3$; and passes through (0, 4).

The rational function must have $(x-1)(x+2)$ in the denominator only, and its numerator must be of the form $3x^2 + Ax + B$. Choose $A = 0, B = -8$, to satisfy the intercept condition.

$$r(x) = \frac{3x^2 - 8}{(x-1)(x+2)}$$

16. Growth and Decay: Half-Life and Doubling Time [top]

Exponential growth and decay models fit the equation $y = f(t) = ab^t$, where t represents time, b is the base, and a is in the initial population (growth model) or the initial amount of radioactive material (decay model).

$$y(0) = y_0 = ab^0 = a$$

We can rewrite the equation, using Euler's constant as the base:

$$y = y_0 b^t = y_0 \left(e^k\right)^t = y_0 e^{kt}$$

$$k < 0 \Rightarrow \text{Exponential Decay}$$

$$k > 0 \Rightarrow \text{ Exponential Growth}$$

Half-Life is the time value at which the initial amount is exactly half the initial amount y_0. Algebraically, we write:

$$\begin{aligned}
&\tfrac{1}{2} y_0 = y_0 e^{kt_{HALF-LIFE}} \\
&\tfrac{1}{2} = e^{kt_{HALF-LIFE}} \\
&\ln \tfrac{1}{2} = kt_{HALF} \\
&t_{HALF-LIFE} = \frac{\ln \frac{1}{2}}{k} \\
&t_{HALF-LIFE} = \frac{-\ln 2}{k}
\end{aligned}$$

Using similar steps, the doubling time, which is the amount of time it takes for the population to double, can be determined for the exponential growth model:

$$t_{DOUBLING} = \frac{\ln 2}{k}$$

Example: The population of a certain species is modeled by $y = 1000e^{0.02t}$. Determine the doubling time.

The initial population is 1000 individuals, so we could use the rule of thumb above, or solve directly: $2000 = 1000e^{0.02t} \Rightarrow t = \dfrac{\ln 2}{0.02} \approx 34.657$ units of time.

Example: If the half-life of a certain radioactive element is five years, find the amount of time it takes for the original amount to be reduced by 75%.

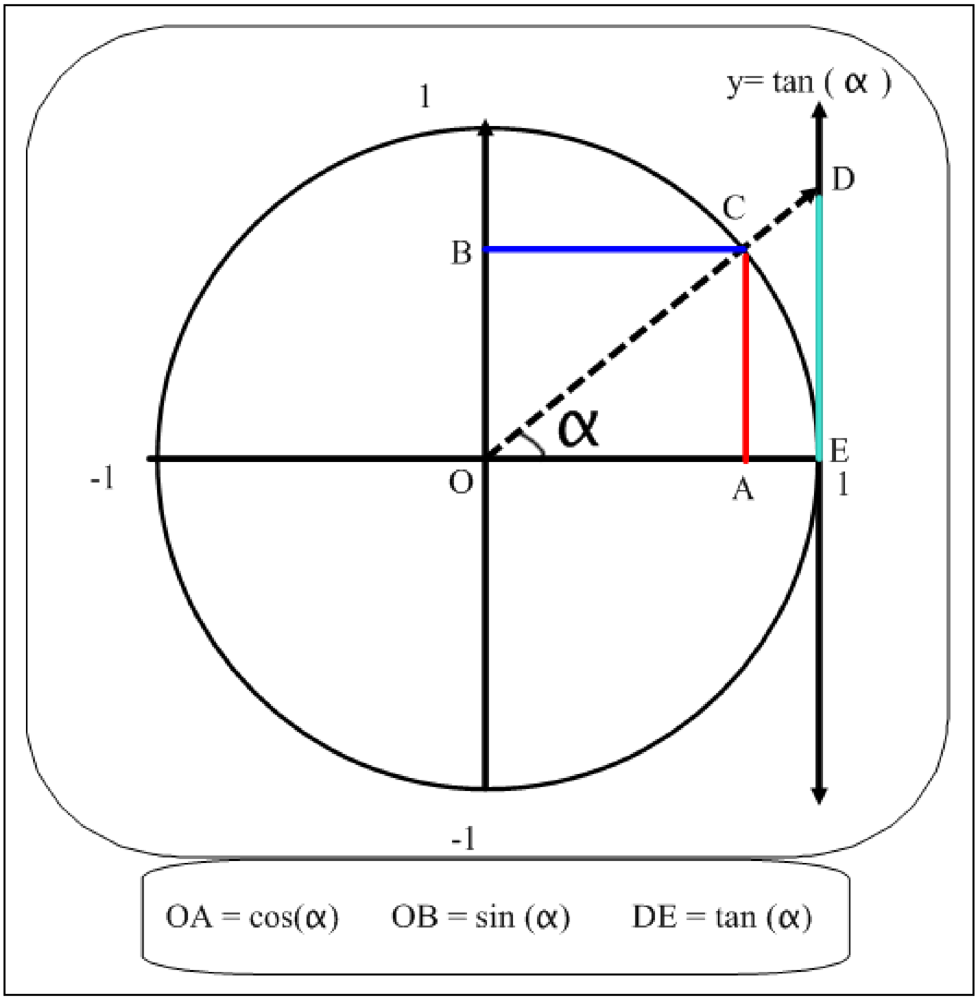
At half life, we have: $5 = \dfrac{-\ln 2}{k}$, so $k = \dfrac{-\ln 2}{5}$.

At the time when the amount is reduced by 75%, we have:

$0.25y_0 = y_0 e^{kt}$
$\ln\frac{1}{4} = kt$
$-2\ln 2 = -\frac{\ln 2}{5}t$
$t_{25\%} = 10$

As expected, it takes 10 years for y_0 to be reduced to $0.25y_0$.

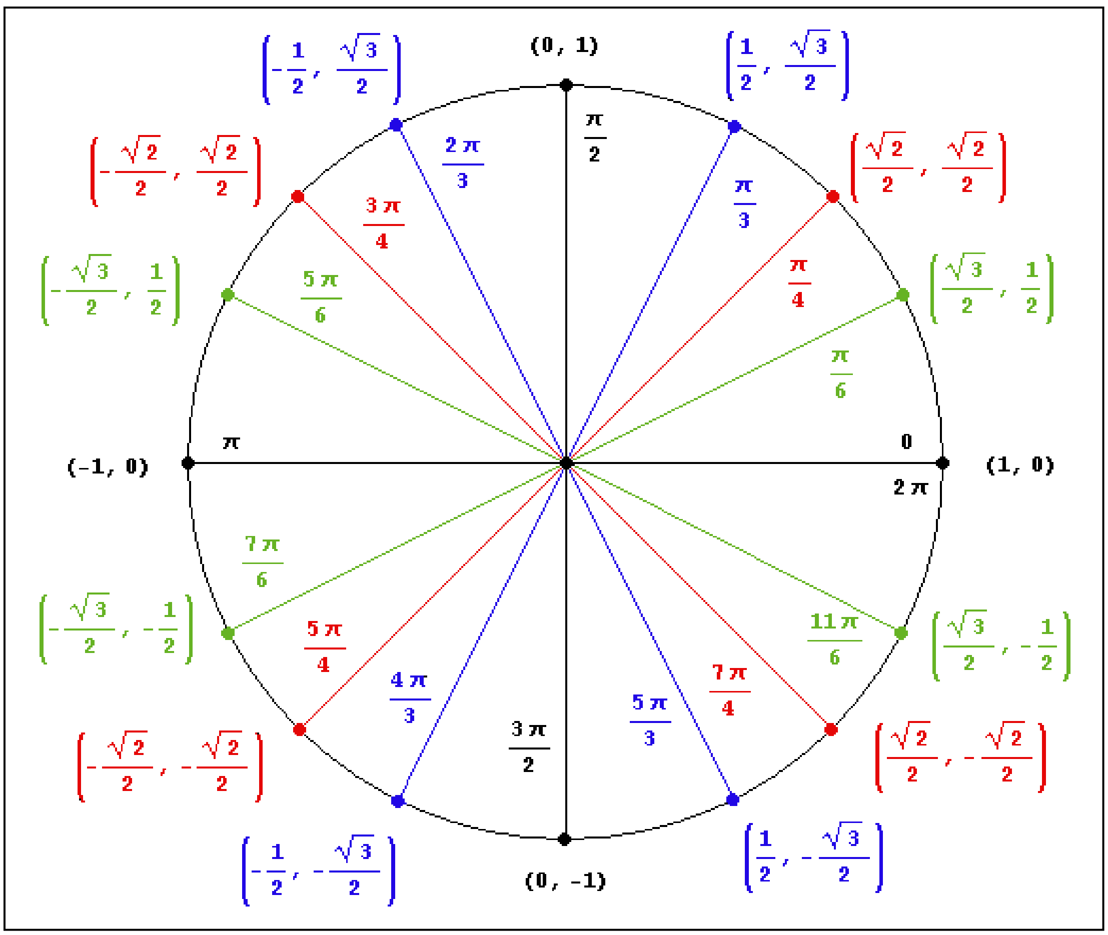
17. Unit Circle and Trigonometric Ratios [top]



The vertical line at $x=1$ can be used to describe the sign and magnitude of the tangent of various angles. The positive direction points upward, just like the $y-$axis. For example, angles in the first and third quadrants have positive tangents, whereas angles in the second and forth have negative tangents. To determine the tangent, we extend the line containing the terminal side until it intersects the tangent axis. Note that the tangent is not defined at the North Pole and is equal to zero at the Far East point $(\tan 90^\circ = DNE, \tan 0^\circ = 0)$.

The radian measure of an angle is equal to the distance traveled along the unit circle, starting from (1, 0). Positive angles describe motion in the counter-clock-wise direction, and negative angles describe motion in the clock-wise direction. A full rotation requires $2\pi r = 2\pi = 6.28\ldots$ units traveled, so a 360° counter-clock-wise angle corresponds to 2π radians.

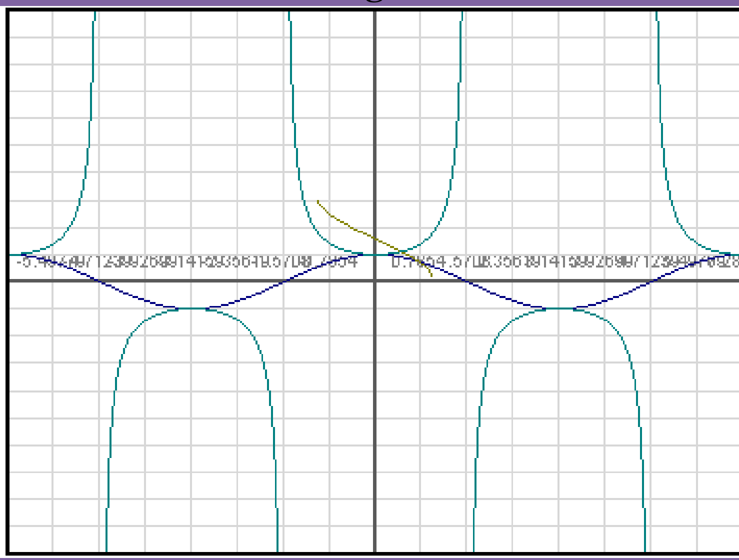
The chart below summarizes key angles along the unit-circle, and their sine and cosine values.



Prior to taking calculus, students should be comfortable with sketching and describing in full detail (zeros, domain, range, asymptotes) the six trigonometric ratios

$(\sin x, \cos x, \tan x, \csc x, \sec x, \cot x)$ and the three inverse trigonometric functions $(\arcsin x, \arccos x, \arctan x)$.

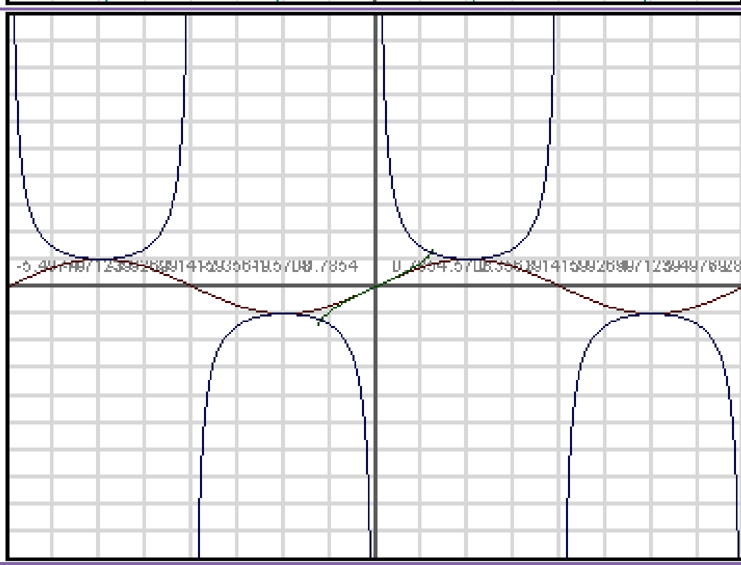
Trigonometric Functions: A summary



Cosine
Domain: $(-\infty, \infty)$
Range: [-1, 1]
Zeros: $\frac{\pi}{2} + k\pi$

Secant
Domain: $\mathbb{R} - \{\frac{\pi}{2} + k\pi\}$
Range: $(-\infty, -1] \cup [1, \infty)$

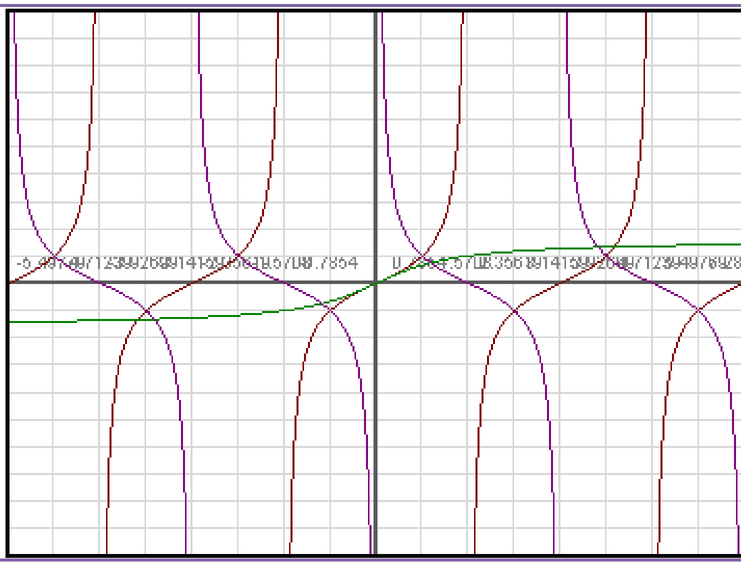
Arc-Cosine
Domain: [-1, 1]
Range: $[0, \pi]$



Sine
Domain: $(-\infty, \infty)$
Range: [-1, 1]
Zeros: $k\pi$

Cosecant
Domain: $\mathbb{R} - \{k\pi\}$
Range: $(-\infty, -1] \cup [1, \infty)$

Arc-Sine
Domain: [-1, 1]
Range: $[\frac{-\pi}{2}, \frac{\pi}{2}]$



Tangent
Domain: $\mathbb{R}-\{k\pi+\frac{\pi}{2}\}$
Range: $(-\infty,\infty)$
Zeros: $k\pi$

Cotangent
Domain: $\mathbb{R}-\{k\pi\}$
Range: $(-\infty,\infty)$

Arc-Tangent
Domain: $(-\infty,\infty)$
Range: $(\frac{-\pi}{2},\frac{\pi}{2})$

18. Trigonometric Formulae and Identities [top]

Applying the Pythagorean Theorem to any angle on the unit circle, we get:

$$\cos^2 x+\sin^2 x=1$$

Dividing by $\cos^2 x$ and $\sin^2 x$ gives two important identities:

$$1+\tan^2 x=\sec^2 x$$
$$\cot^2 x+1=\csc^2 x$$

The Double Angle formulae:

$$\sin(2\alpha)=2\sin\alpha\cos\alpha$$
$$\cos(2\alpha)=\cos^2\alpha-\sin^2\alpha$$
$$\to\cos(2\alpha)=2\cos^2\alpha-1$$
$$\to\cos(2\alpha)=1-2\sin^2\alpha$$

The last four identities are more useful in calculus in transforming the right sides to the left sides. For example:

$$\cos^2\alpha=\frac{\cos(2\alpha)+1}{2}$$
$$\sin^2\alpha=\frac{1-\cos(2\alpha)}{2}$$

The sine and cosine of the sum of two angles are especially useful when deriving the formulae for the instantaneous rates of change of trig functions:

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$
$$\sin(x+y) = \sin x \cos y + \sin y \cos x$$

The Law of Sines states that in any triangle ABC:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

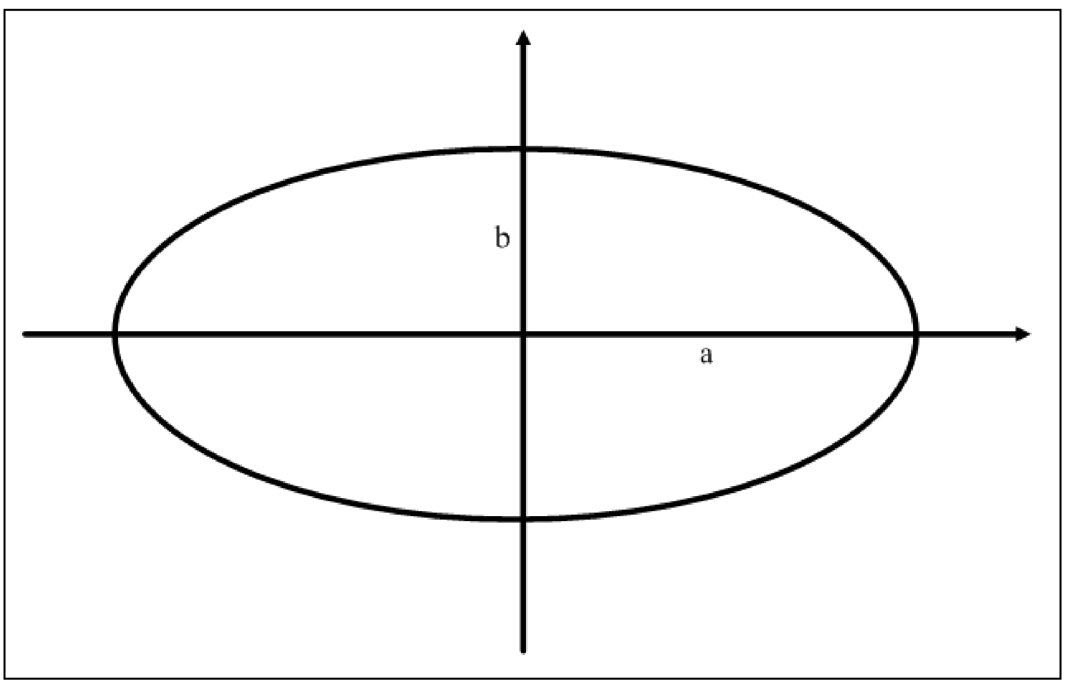
The Law of Cosines states that in any triangle ABC:

$$a^2 = b^2 + c^2 - 2bc \cos A$$

19. Ellipses, Circles, Arc Length and Area of a Sector [top]

The standard equation of an ellipse centered at (h, k) is given by:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$



Its major and minor axes are $2a$ and $2b$, respectively. Note that when a and b are equal, the ellipse is merely a circle. A circle's general equation is:

$$(x-h)^2 + (y-k)^2 = r^2$$

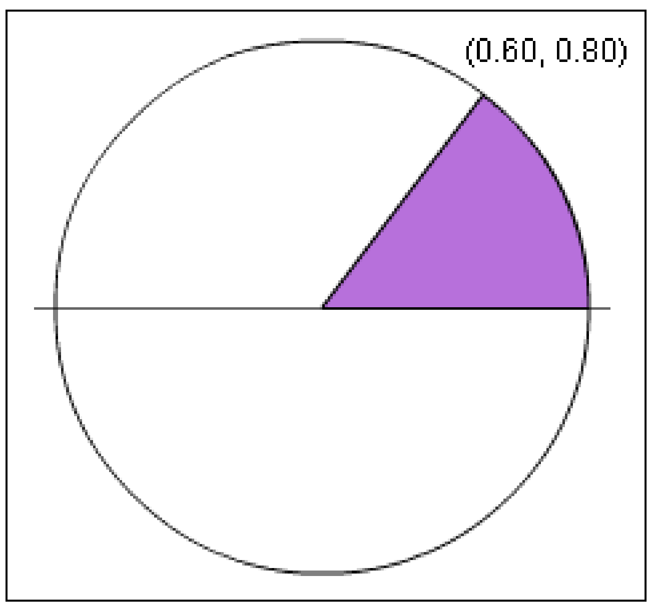
Students should be able to determine the parameters of a circle or an ellipse by completing the square for both variables x and y.

Example: Write the equation $x^2+9y^2+36y+2x=-28$ in standard form.

$$x^2+9y^2+36y+2x=-28 \Rightarrow x^2+2x+9(y^2+4y)=-28$$
$$\Rightarrow x^2+2x+1+9(y^2+4y+4)=-28+1+36 \Rightarrow (x+1)^2+9(y+2)^2=9 \Rightarrow \frac{(x+1)^2}{9}+\frac{(y+2)^2}{1}=1$$

Given a circle with radius r, the area of the sector with angle α can be determined by setting the ratio of angles (in radian) equal to the ratio of the areas:

$$\frac{A}{\pi r^2}=\frac{\alpha}{2\pi}$$
$$A=\frac{1}{2}\alpha r^2$$



We follow similar steps to derive the length of the arc:

$$\frac{L}{2\pi r}=\frac{\alpha}{2\pi}$$
$$L=\alpha r$$

Example: Find the arc length and the area of a sector with radius 3m and angle 1.

One should assume the angle measure is in radian, unless noted otherwise.
$A = \frac{1}{2} \cdot \alpha \cdot r^2 = \frac{1}{2} \cdot 1 \cdot 3^2 = \frac{9}{2}$ square meters.

The arc length of the sector, excluding the radii, is given by $L = \alpha r = 3$ meters.

20. Logarithms: Properties, Functions, and Equations [top]

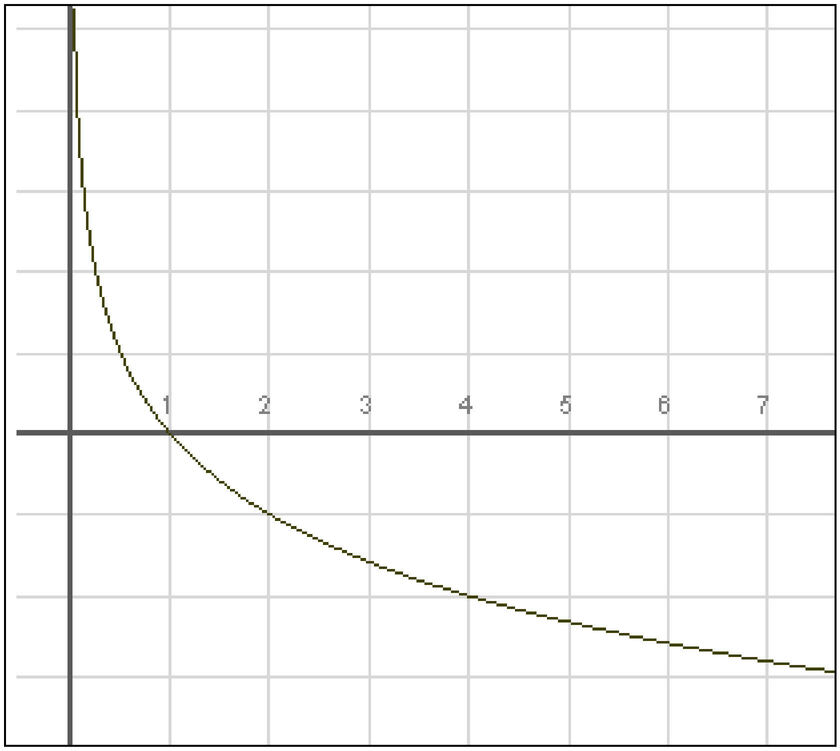
Logarithmic and exponential operations are inverses of one-another. Justification to the properties below requires the use of the laws of exponents.

A few properties…

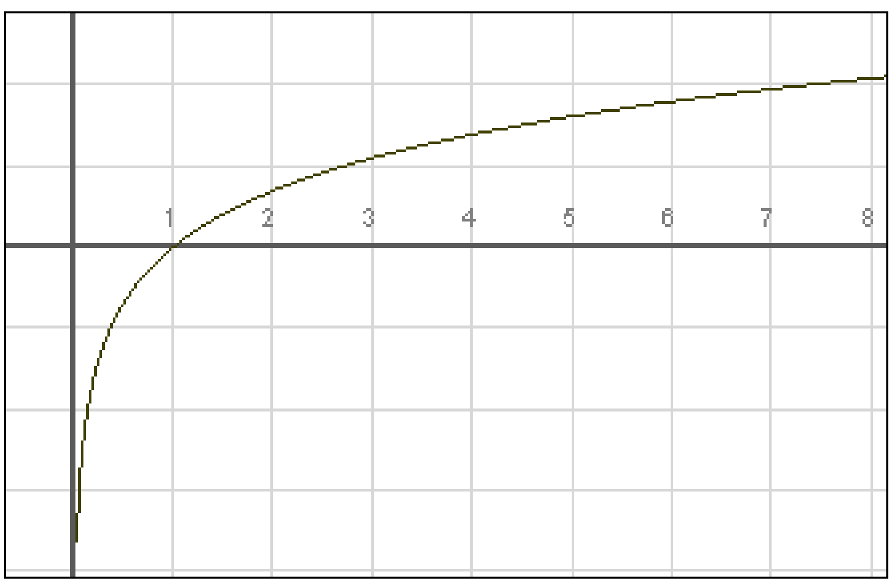
$$\begin{aligned}
&1. \log(AB) = \log A + \log B \\
&2. \log(\tfrac{A}{B}) = \log A - \log B \\
&3. \log a^m = m \log a \\
&4. \log_a b = \frac{\log_c b}{\log_c a} \\
&5. a^{\log_a m} = m \\
&6. \log_a a^x = x
\end{aligned}$$

The logarithmic function $y = \log_a x$ $(a > 0)$ has domain $(0, \infty)$ and $x-$intercept (1, 0).

If the base a is between 0 and 1, the graph is decreasing at a decreasing rate throughout its domain:



If the base a is greater than 1, the graph is increasing at a decreasing rate throughout its domain:



Note that the change of base property (property 4) is particularly useful when using technology to plot a logarithmic function who base is neither 10 nor e.

Example: Solve the equation $2\log_2 x - \log_2(x+1) = 1$.

Observe that the domain of the solution is $x > 0$, the intersection between $x > 0$ and $x > -1$, so that both logarithmic operations are meaningful. We combine the left hand side into logarithm, and then express the equation in its exponential form:

$$\log_2 x^2 - \log_2(x+1) = 1$$
$$\log_2 \frac{x^2}{x+1} = 1$$
$$\frac{x^2}{x+1} = 2$$
$$x^2 = 2x + 2$$
$$x^2 - 2x - 2 = 0$$
$$x = \frac{2 \pm \sqrt{12}}{2} = \frac{2 \pm 2\sqrt{3}}{2} = 1 \pm \sqrt{3}$$

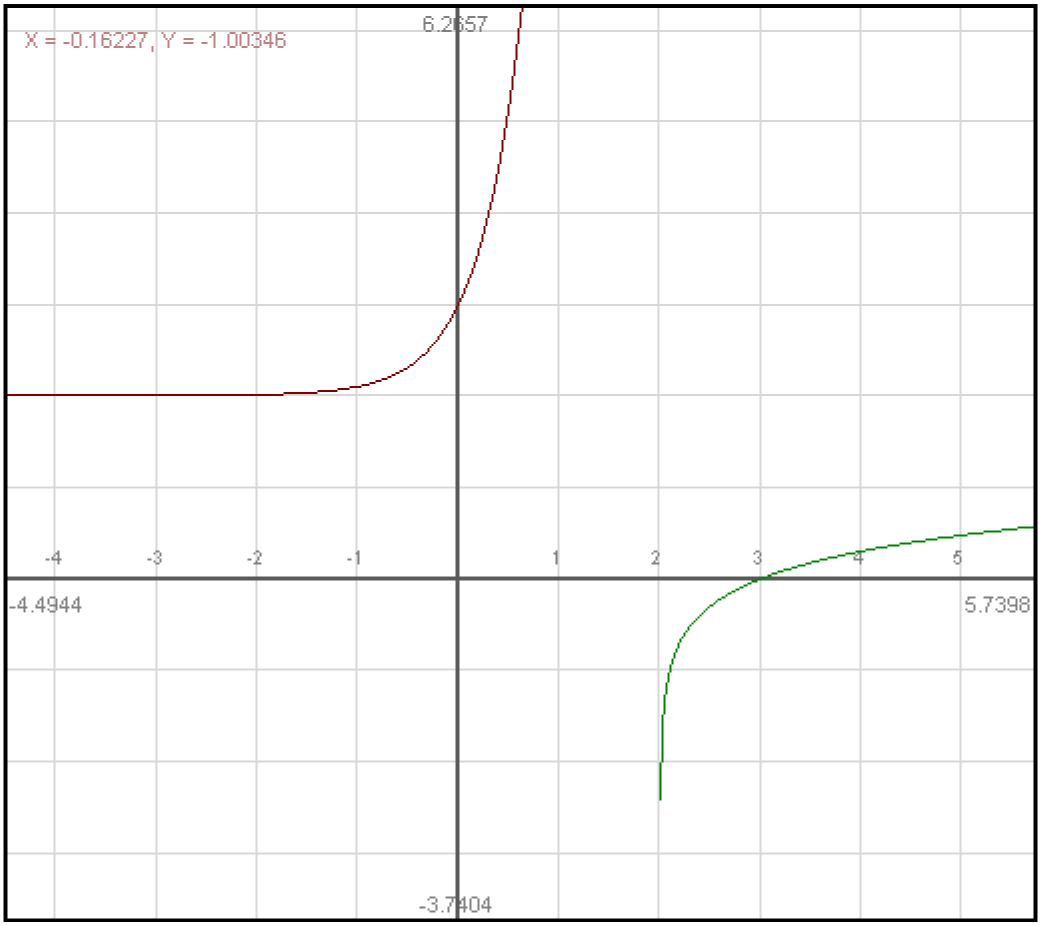
We exclude the extraneous solution and conclude that the only solution is $x = 1 + \sqrt{3}$.

Example: Find the inverse of $y = \log(x-2)$.

The given function passes the horizontal line test, hence we know that its inverse mapping will also be a function. We find the inverse algebraically:

$$y = \log(x-2)$$
$$x = \log(y-2)$$
$$10^x = y - 2$$
$$10^x + 2 = y$$
$$f^{-1}(x) = 10^x + 2$$

Note that the domain and range of the original function are $(2, \infty)$ and $(-\infty, \infty)$, respectively. The graph verifies that the range of the original function is the domain of the inverse, and vice-versa:



The graphs of $f(x) = \log(x-2)$ and its inverse, $f^{-1}(x) = 10^x + 2$

21. Key concept: Limit [top]

The limit of a function $f(x)$ as x approaches a is written as:

$$\lim_{x \to a} f(x) = M .$$

In calculus, $x \to a$ implies that x gets infinitely close to a but does not necessarily reach it. For example, a particle with a starting position one meter away from a wall is programmed to detect its distance to the wall and reduces it in half with each jump. In theory, the particle will never reach the wall, since no matter how small the distance is, it can always be divided in half. We write:

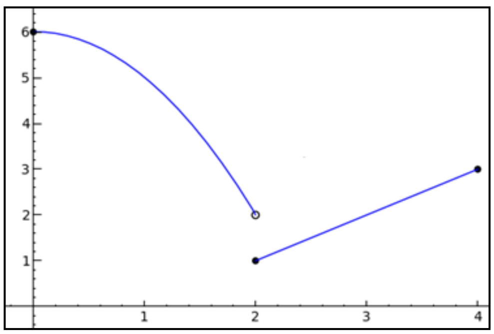
$$\lim_{position \to wall} position = wall ,$$

and observe that $position \neq wall$.

One-sided Limits

$\lim_{x \to a^-} f(x)$ is the limit from the left. When it exists, the left-sided limit represents the tendency of $y-$values as x approaches a from the left.

$\lim_{x \to a^+} f(x)$ is the limit from the right. When it exists, the right-sided limit represents the tendency of $y-$values as x approaches a from the right.



A piece-wise function whose one-sided limits from the left and right are $y=2$ and $y=1$, respectively.

$$f(x)=\begin{cases} -x^2+6 & 0 \le x < 2 \\ x-1 & 2 \le x \le 4 \end{cases}$$

Fact: $\lim_{x \to a} f(x)$ exists if and only if $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$.

Limits Algebraically

We apply various algebraic techniques such as factoring and rationalization to determine the limit. A common mistake is to plug in $x=a$ even though it leads to division by zero. In calculus, most of the limits have a zero in the denominator in their original form.

Example: $\lim_{x \to 3} \frac{x^2-9}{3-x} = \lim_{x \to 3} \frac{(x-3)(x+3)}{3-x} = \lim_{x \to 3} \frac{-(x+3)}{1} = -6$

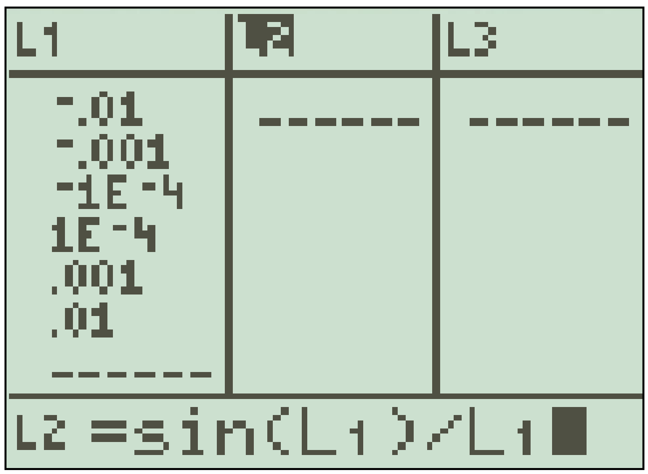
We are able to cancel the shared factor $(3-x)$ since $x \to 3$ implies that $x \neq 3$.

Limits Numerically

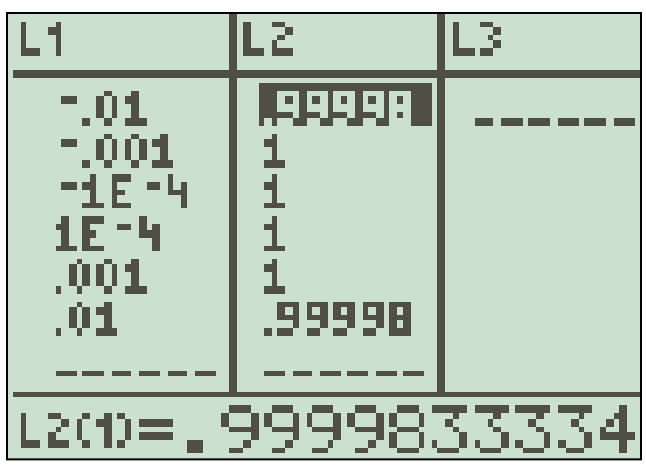
Graphing devices or computers can be used to investigate difficult limits. It suffices to evaluate the expression $f(x)$ for x values *near* $x = a$. A neighborhood of a is simply a small open interval containing $x = a$. On a TI Graphing Calculator such as, for example, the TI-83 Plus, one can investigate a limit of the form:

$$\lim_{x \to 0} \frac{\sin x}{x}$$

by entering fast-approaching numbers from both sides of $x = 0$, in increasing order, hence making it possible to compare the one-sided limits.



With the cursor on L2, one can enter a formula using the argument L1. Pressing **Enter** populates the list, as shown below:



Are the one-sided limits equal to one another? We confidently conclude that the overall limit of the given expression is equal to one.

Example: Numerically estimate Euler's Constant e.

Limits Graphically

Similar to the numerical investigation, graphing devices and computers can be used to trace a certain algebraic expression in the neighborhood of the point of interest. It is important to focus on the one-sided limits and zoom in sufficiently to capture any elaborate details about the graph.

Limits Verbally

A 'big number / little number' or 'faster versus slower growth' justification often earns solid points in the free response section of the AP Calculus exam. For example:

$$\lim_{x \to \infty} \frac{e^x}{x^2} = \infty$$

, since the exponential function grows faster than the power in the long run.

Feedback & Corrections