Example 11.7. Use Jacobi iteration to transform the following symmetric matrix into diagonal form.

$$\begin{bmatrix} 8 & -1 & 3 & -1 \\ -1 & 6 & 2 & 0 \\ 3 & 2 & 9 & 1 \\ -1 & 0 & 1 & 7 \end{bmatrix}$$

The computational details are left for the reader. The first rotation matrix that will zero out $a_{13} = 3$ is

$$\boldsymbol{R}_1 = \begin{bmatrix} 0.763020 & 0.000000 & 0.646375 & 0.000000 \\ 0.000000 & 0.000000 & 0.000000 & 0.000000 \\ -0.646375 & 0.000000 & 0.763020 & 0.000000 \\ 0.000000 & 0.000000 & 0.000000 & 0.000000 \end{bmatrix}.$$

Calculation reveals that $\boldsymbol{A}_2 = \boldsymbol{R}_1 \boldsymbol{A}_1 \boldsymbol{R}_1$ is

$$\boldsymbol{A}_2 = \begin{bmatrix} 5.458619 & -2.055770 & 0.000000 & -1.409395 \\ -2.055770 & 6.000000 & 0.879665 & 0.000000 \\ 0.000000 & 0.879665 & 11.541381 & 0.116645 \\ -1.409395 & 0.000000 & 0.116645 & 7.000000 \end{bmatrix}.$$

Next, the element $a_{12} = -2.055770$ is zeroed out and we get

$$\boldsymbol{A}_3 = \begin{bmatrix} 3.655795 & 0.000000 & 0.579997 & -1.059649 \\ 0.000000 & 7.802824 & 0.661373 & 0.929268 \\ 0.579997 & 0.661373 & 11.541381 & 0.116645 \\ -1.059649 & 0.929268 & 0.116645 & 7.000000 \end{bmatrix}.$$

After 10 iterations we arrive at

$$\boldsymbol{A}_{10} = \begin{bmatrix} 3.295870 & 0.002521 & 0.037859 & 0.000000 \\ 0.002521 & 8.405210 & -0.004957 & 0.066758 \\ 0.037859 & -0.004957 & 11.704123 & -0.001430 \\ 0.000000 & 0.066758 & -0.001430 & 6.594797 \end{bmatrix}.$$

It will take six more iterations for the diagonal elements to get close to the diagonal matrix

$$\boldsymbol{D} = \operatorname{diag}(3.295699, 8.407662, 11.704301, 6.592338).$$

However, the off-diagonal elements are not small enough, and it will take three more iterations for them to be less than 10^{-6} in magnitude. Then the eigenvectors are the columns of the matrix $\boldsymbol{V} = \boldsymbol{R}_1 \boldsymbol{R}_2 \cdots \boldsymbol{R}_{18}$, which is

$$\boldsymbol{V} = \begin{bmatrix} 0.528779 & -0.573042 & 0.582298 & 0.230097 \\ 0.591967 & 0.472301 & 0.175776 & -0.628975 \\ -0.536039 & 0.282050 & 0.792487 & -0.071235 \\ 0.287454 & 0.607455 & 0.044680 & 0.739169 \end{bmatrix}. \qquad \blacksquare$$