A Finite Difference Code for the Navier-Stokes Equations in Vorticity/Stream Function Formulation

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Objectives:

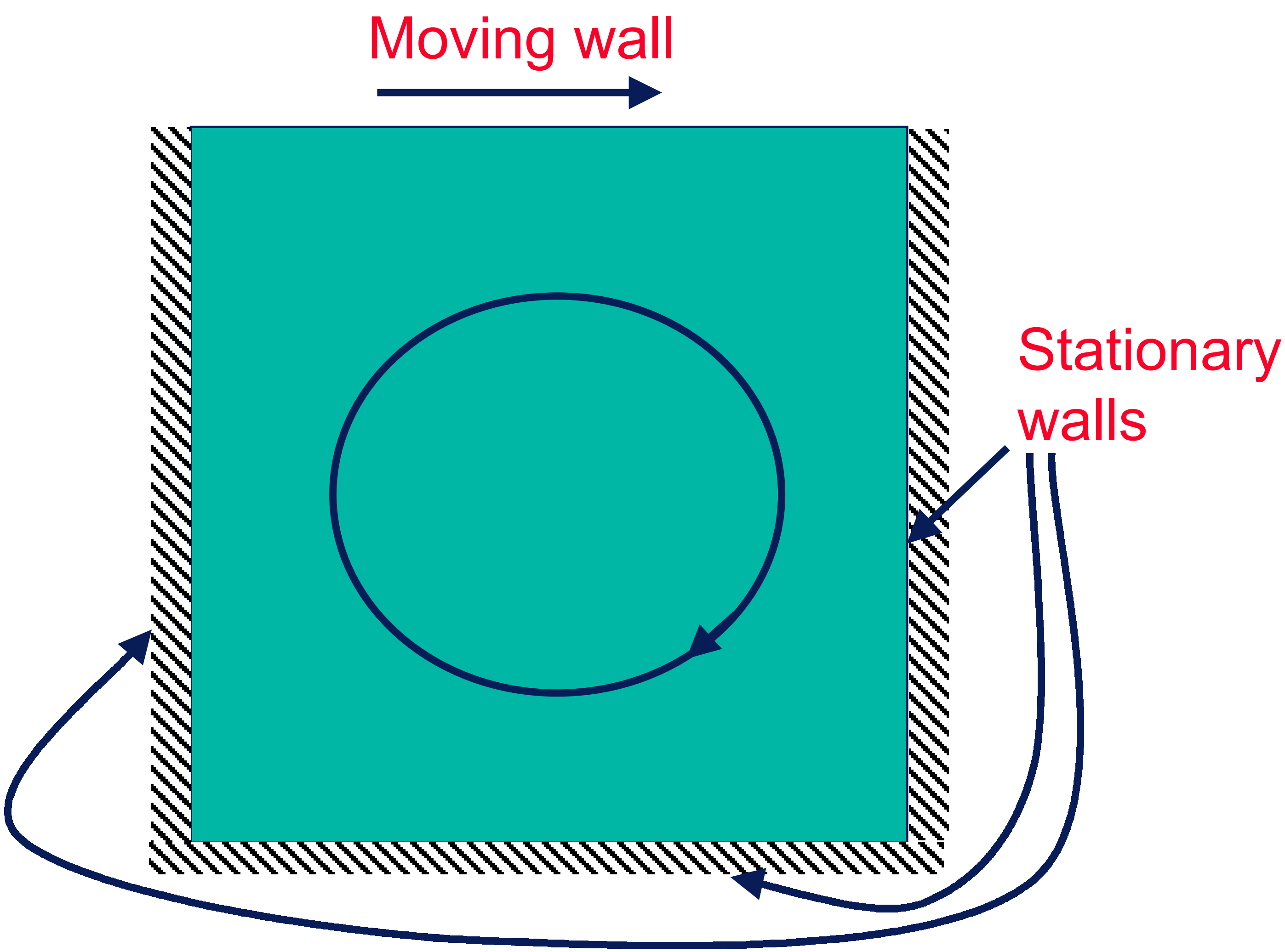
Develop an understanding of the steps involved in solving the Navier-Stokes equations using a numerical method

Write a simple code to solve the “driven cavity” problem using the Navier-Stokes equations in vorticity form

Outline

- The Driven Cavity Problem
- The Navier-Stokes Equations in Vorticity/Stream Function form
- Boundary Conditions
- Finite Difference Approximations to the Derivatives
- The Grid
- Finite Difference Approximation of the Vorticity/Streamfunction equations
- Finite Difference Approximation of the Boundary Conditions
- Iterative Solution of the Elliptic Equation
- The Code
- Results
- Convergence Under Grid Refinement

The Driven Cavity Problem



Nondimensional N-S Equation

Incompressible N-S Equation in 2-D

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

Nondimensionalization

$$\tilde{u}=u/U,\ \tilde{x}=x/L,\ \tilde{t}=tU/L,\ \tilde{p}=p/\rho U^2$$

$$\frac{U^2}{L}\left(\frac{\partial\tilde{u}}{\partial\tilde{t}}+\tilde{u}\frac{\partial\tilde{u}}{\partial\tilde{x}}+\tilde{v}\frac{\partial\tilde{u}}{\partial\tilde{y}}\right)=-\frac{U^2}{L}\frac{\partial\tilde{p}}{\partial\tilde{x}}+\frac{\nu U}{L^2}\left(\frac{\partial^2\tilde{u}}{\partial\tilde{x}^2}+\frac{\partial^2\tilde{u}}{\partial\tilde{y}^2}\right)$$

$$\boxed{\frac{\partial\tilde{u}}{\partial\tilde{t}}+\tilde{u}\frac{\partial\tilde{u}}{\partial\tilde{x}}+\tilde{v}\frac{\partial\tilde{u}}{\partial\tilde{y}}=-\frac{\partial\tilde{p}}{\partial\tilde{x}}+\frac{1}{\mathrm{Re}}\left(\frac{\partial^2\tilde{u}}{\partial\tilde{x}^2}+\frac{\partial^2\tilde{u}}{\partial\tilde{y}^2}\right)}$$

The Vorticity Equation

$$-\frac{\partial}{\partial y}\left[\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)\right]$$

$$\frac{\partial}{\partial x}\left[\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{\partial p}{\partial y}+\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)\right]$$

$$\frac{\partial \omega}{\partial t}+u\frac{\partial \omega}{\partial x}+v\frac{\partial \omega}{\partial y}=\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 \omega}{\partial x^2}+\frac{\partial^2 \omega}{\partial y^2}\right)$$

$$\omega=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}$$

The Stream Function Equation

Define the stream function

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}$$

which automatically satisfies the incompressibility conditions

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

by substituting

$$\frac{\partial}{\cancel{\partial x}} \frac{\cancel{\partial} \psi}{\partial y} - \frac{\partial}{\cancel{\partial y}} \frac{\cancel{\partial} \psi}{\partial x} = 0$$

The vorticity/stream function equations

Substituting

$$u = \frac{\partial \psi}{\partial y}; \quad v = -\frac{\partial \psi}{\partial x}$$

into the definition of the vorticity

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

yields

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\omega$$

The Navier-Stokes equations in vorticity-stream function form are:

Advection/diffusion equation

$$\boxed{\frac{\partial \omega}{\partial t} = -\frac{\partial \psi}{\partial y}\frac{\partial \omega}{\partial x} + \frac{\partial \psi}{\partial x}\frac{\partial \omega}{\partial y} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right)}$$

Elliptic equation

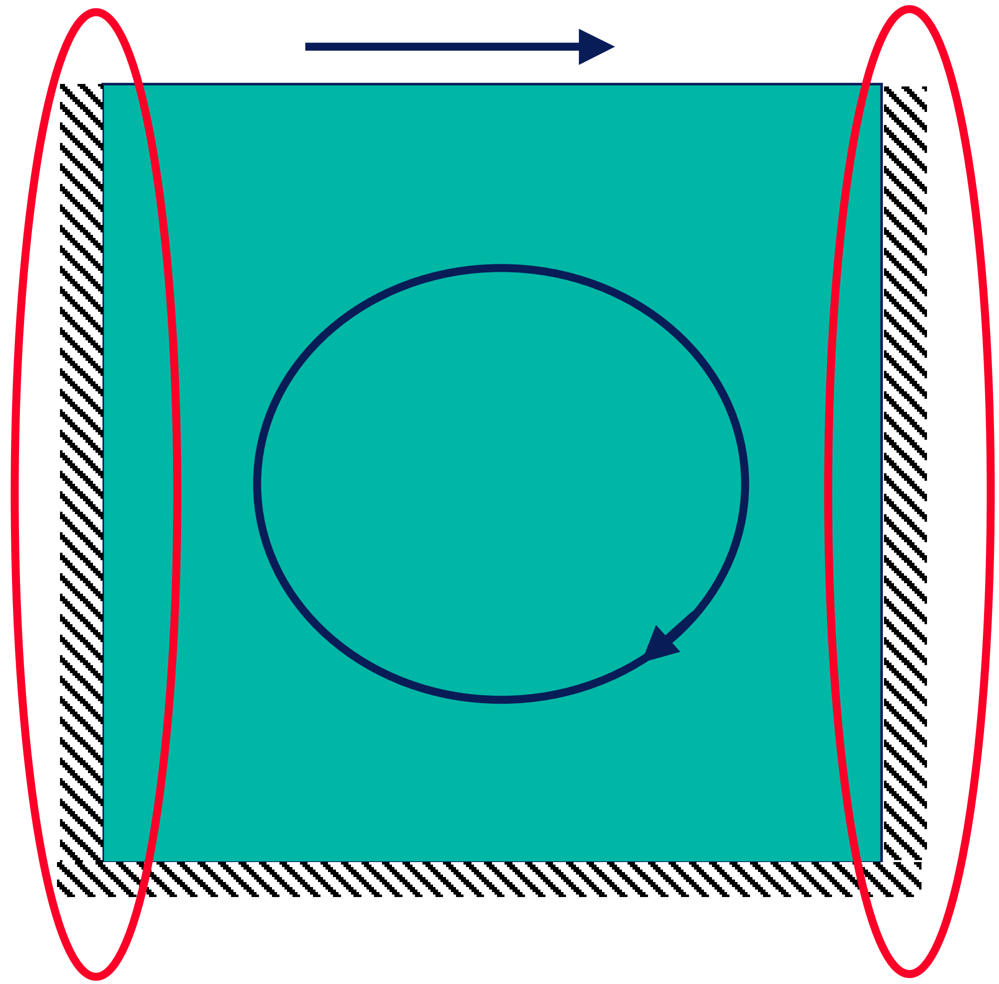
$$\boxed{\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\omega}$$

Boundary Conditions for the Stream Function

At the right and
the left boundary:

$$u = 0 \Rightarrow \frac{\partial \psi}{\partial y} = 0$$

$$\Rightarrow \psi = \text{Constant}$$

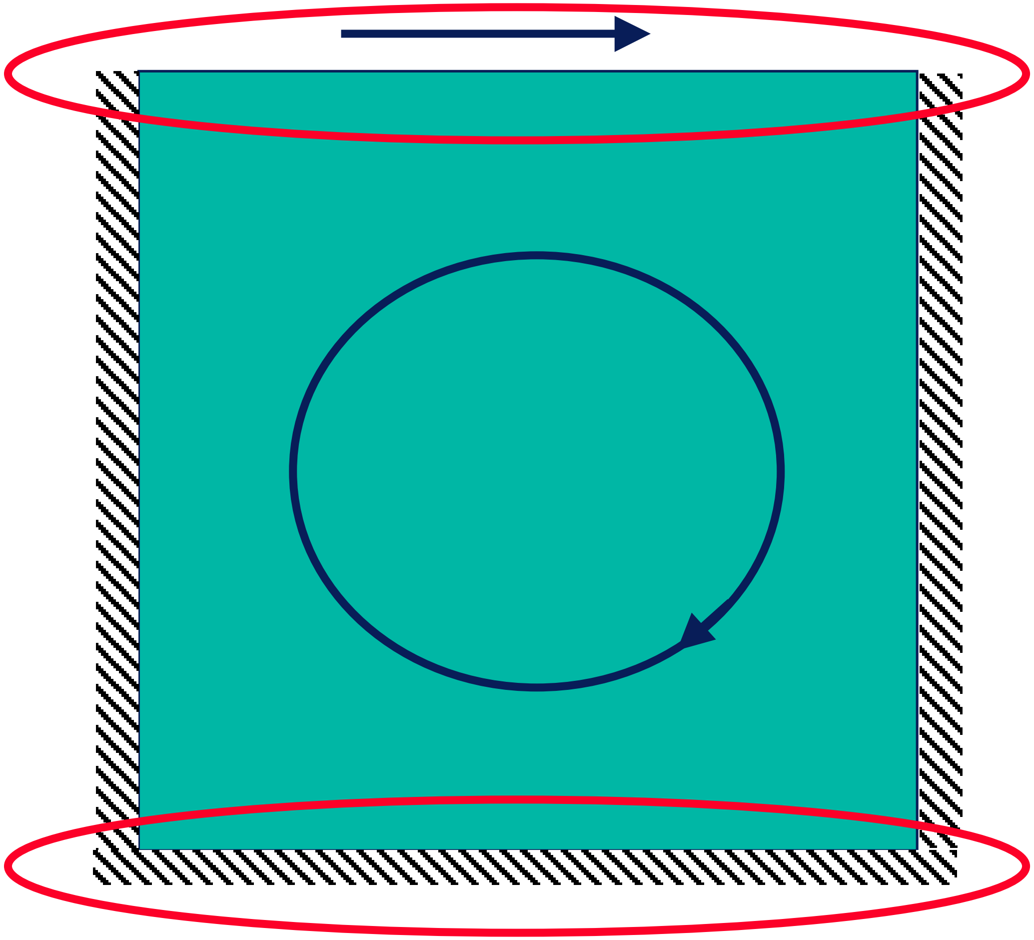


Boundary Conditions for the Stream Function

At the top and the bottom boundary:

$$v = 0 \Rightarrow \frac{\partial \psi}{\partial x} = 0$$

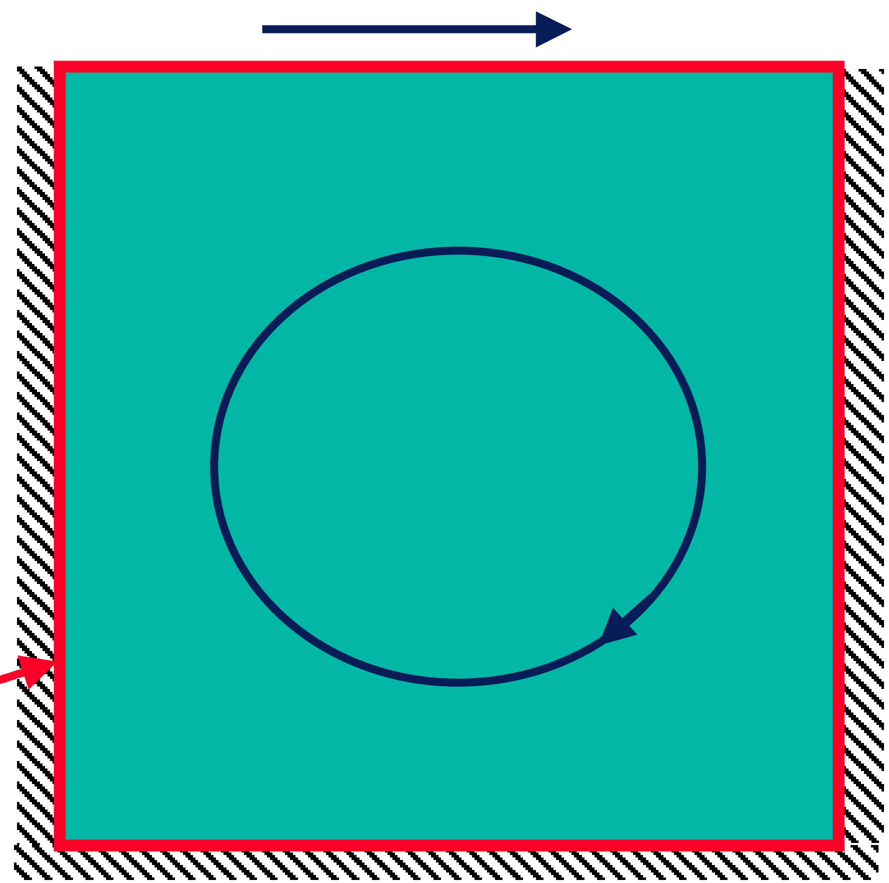
$$\Rightarrow \psi = \text{Constant}$$



Boundary Conditions for the Stream Function

Since the boundaries meet, the constant must be the same on all boundaries

$$\psi = \text{Constant}$$



Boundary Conditions for the Vorticity

The normal velocity is zero since the stream function is a constant on the wall, but the zero tangential velocity must be enforced:

At the right and left boundary:

$$v = 0 \Rightarrow \frac{\partial \psi}{\partial x} = 0$$

At the bottom boundary:

$$u = 0 \Rightarrow \frac{\partial \psi}{\partial y} = 0$$

At the top boundary:

$$u = U_{wall} \Rightarrow \frac{\partial \psi}{\partial y} = U_{wall}$$

Boundary Conditions for the Vorticity

The wall vorticity must be found from the streamfunction. The stream function is constant on the walls.

At the right and the left boundary:

$$\frac{\partial^2\psi}{\partial x^2} + \cancel{\frac{\partial^2\psi}{\partial y^2}} = -\omega \qquad \Rightarrow \qquad \omega_{wall} = -\frac{\partial^2\psi}{\partial x^2}$$

Similarly, at the top and the bottom boundary:

$$\cancel{\frac{\partial^2\psi}{\partial x^2}} + \frac{\partial^2\psi}{\partial y^2} = -\omega \qquad \Rightarrow \qquad \omega_{wall} = -\frac{\partial^2\psi}{\partial y^2}$$

Summary of Boundary Conditions

$$\frac{\partial \psi}{\partial y} = U_{wall}; \quad \omega_{wall} = -\frac{\partial^2 \psi}{\partial y^2}$$

$$\frac{\partial \psi}{\partial x} = 0$$

$$\omega_{wall} = -\frac{\partial^2 \psi}{\partial x^2}$$

$$\psi = \text{Constant}$$

$$\frac{\partial \psi}{\partial x} = 0$$

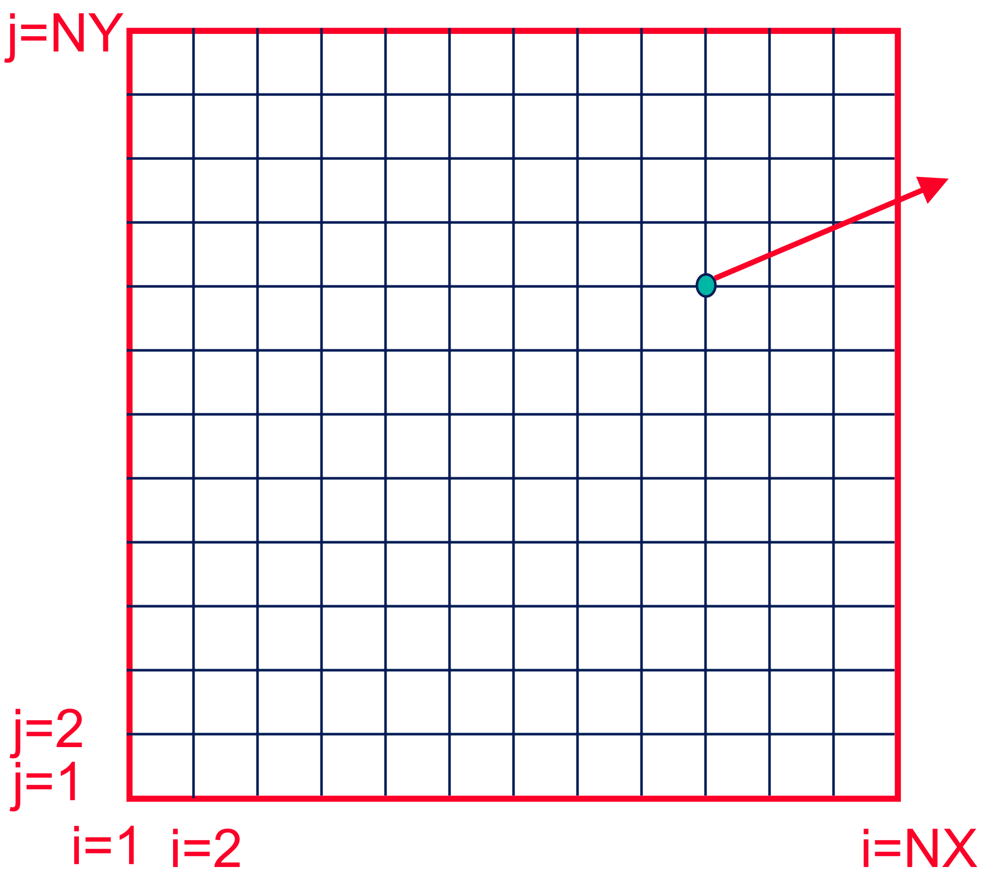
$$\omega_{wall} = -\frac{\partial^2 \psi}{\partial x^2}$$

$$\frac{\partial \psi}{\partial y} = 0; \quad \omega_{wall} = -\frac{\partial^2 \psi}{\partial y^2}$$

To compute an approximate solution numerically, the continuum equations must be discretized. There are a few different ways to do this, but we will use FINITE DIFFERENCE approximations here.

Discretizing the Domain

Uniform mesh (h=constant)

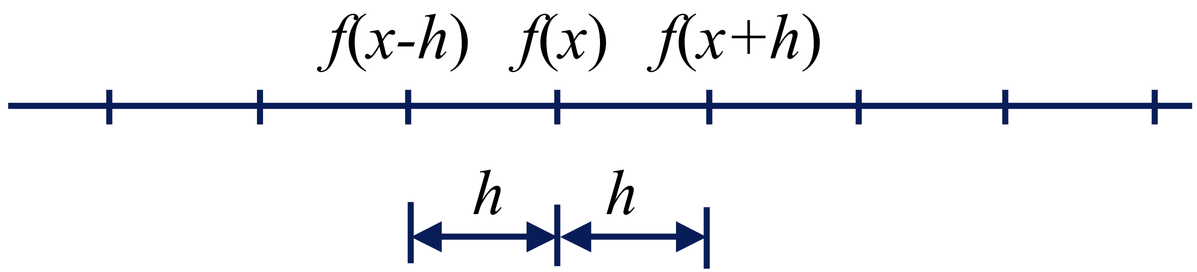


$\psi_{i,j}$ and $\omega_{i,j}$

Stored at each grid point

When using FINITE DIFFERENCE approximations, the values of f are stored at discrete points and the derivatives of the function are approximated using a Taylor series:

Start by expressing the value of $f(x+h)$ and $f(x-h)$ in terms of $f(x)$



Finite difference approximations

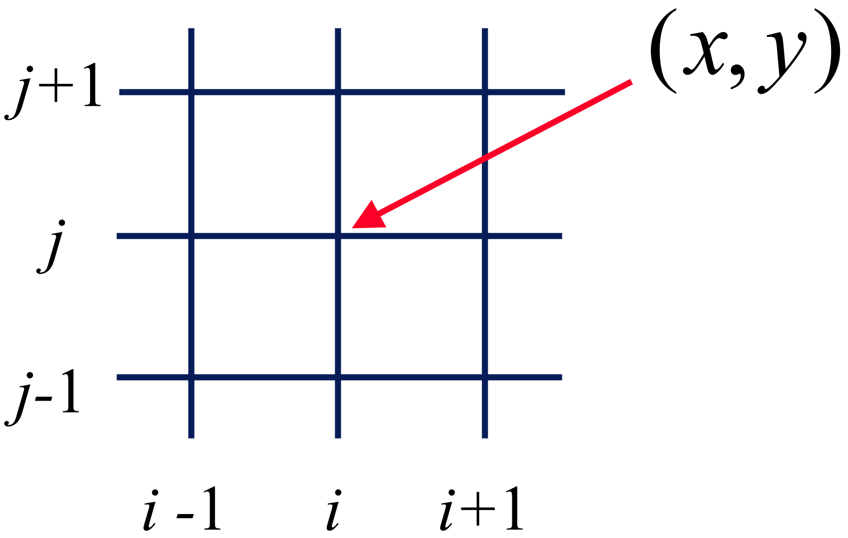
$$\frac{\partial f(x)}{\partial x} = \frac{f(x+h) - f(x-h)}{2h} + \frac{\partial^3 f(x)}{\partial x^3}\frac{h^2}{6} + \cdots$$

$$\frac{\partial^2 f(x)}{\partial x^2} = \frac{f(x+h) - 2f(h) + f(x-h)}{h^2} + \frac{\partial^4 f(x)}{\partial x^4}\frac{h^2}{12} + \cdots$$

$$\frac{\partial f(t)}{\partial t} = \frac{f(t+\Delta t) - f(t)}{\Delta t} + \frac{\partial^2 f(t)}{\partial t^2}\frac{\Delta t}{2} + \cdots$$

2nd order in space, 1st order in time

For a two-dimensional flow discretize the variables on a two-dimensional grid



$$f_{i,j} = f(x,y)$$

$$f_{i+1,j} = f(x+h,y)$$

$$f_{i,j+1} = f(x,y+h)$$

Laplacian

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} =$$

$$\frac{f_{i+1,j}^n - 2f_{i,j}^n + f_{i-1,j}^n}{h^2} + \frac{f_{i,j+1}^n - 2f_{i,j}^n + f_{i,j-1}^n}{h^2} =$$

$$\frac{f_{i+1,j}^n + f_{i-1,j}^n + f_{i,j+1}^n + f_{i,j-1}^n - 4f_{i,j}^n}{h^2}$$

$$\frac{\partial \omega}{\partial t} = -\frac{\partial \psi}{\partial y}\frac{\partial \omega}{\partial x} + \frac{\partial \psi}{\partial x}\frac{\partial \omega}{\partial y} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right)$$

Using these approximations, the vorticity equation becomes:

$$\frac{\omega_{i,j}^{n+1} - \omega_{i,j}^{n}}{\Delta t} =$$

$$-\left(\frac{\psi_{i,j+1}^{n} - \psi_{i,j-1}^{n}}{2h}\right)\left(\frac{\omega_{i+1,j}^{n} - \omega_{i-1,j}^{n}}{2h}\right) + \left(\frac{\psi_{i+1,j}^{n} - \psi_{i-1,j}^{n}}{2h}\right)\left(\frac{\omega_{i,j+1}^{n} - \omega_{i,j-1}^{n}}{2h}\right)$$

$$+\frac{1}{\mathrm{Re}}\left(\frac{\omega_{i+1,j}^{n} + \omega_{i-1,j}^{n} + \omega_{i,j+1}^{n} + \omega_{i,j-1}^{n} - 4\omega_{i,j}^{n}}{h^2}\right)$$

The vorticity at the new time is given by:

$$\omega_{i,j}^{n+1} = \omega_{i,j}^{n} - \Delta t \left[\left(\frac{\psi_{i,j+1}^{n} - \psi_{i,j-1}^{n}}{2h} \right) \left(\frac{\omega_{i+1,j}^{n} - \omega_{i-1,j}^{n}}{2h} \right) - \left(\frac{\psi_{i+1,j}^{n} - \psi_{i-1,j}^{n}}{2h} \right) \left(\frac{\omega_{i,j+1}^{n} - \omega_{i,j-1}^{n}}{2h} \right) + \frac{1}{\mathrm{Re}} \left(\frac{\omega_{i+1,j}^{n} + \omega_{i-1,j}^{n} + \omega_{i,j+1}^{n} + \omega_{i,j-1}^{n} - 4\omega_{i,j}^{n}}{h^2} \right) \right]$$

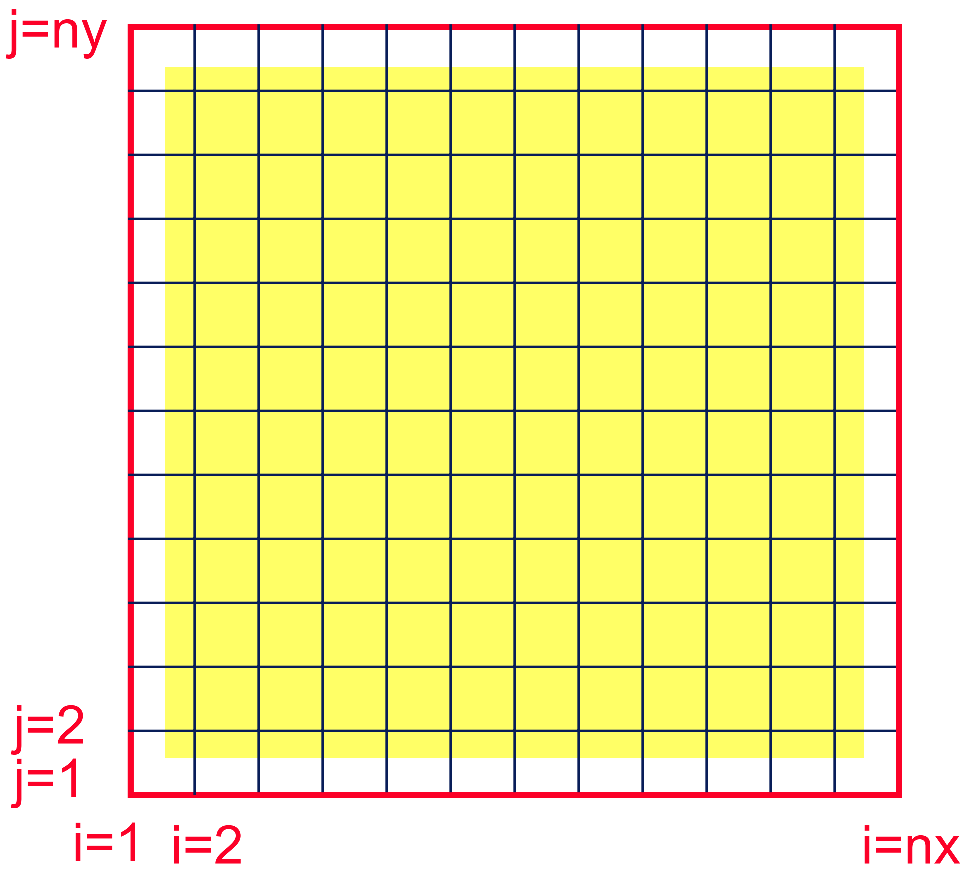
The stream function equation is:

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=-\omega$$

$$\frac{\psi^n_{i+1,j}+\psi^n_{i-1,j}+\psi^n_{i,j+1}+\psi^n_{i,j-1}-4\psi^n_{i,j}}{h^2}=-\omega^n_{i,j}$$

Discretized Domain

Discretize the domain



$$\psi_{i,j} = 0$$

for

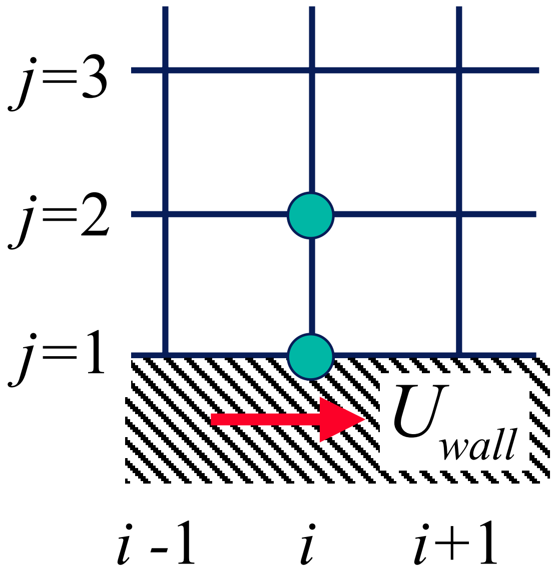
$i = 1$

$i = \mathrm{nx}$

$j = 1$

$j = \mathrm{ny}$

Discrete Boundary Condition



$$\omega_{wall} = \omega_{i,j=1}$$

$$\psi_{i,j=2} = \psi_{i,j=1} + \frac{\partial \psi_{i,j=1}}{\partial y} h + \frac{\partial^2 \psi_{i,j=1}}{\partial y^2} \frac{h^2}{2} + O(h^3)$$

Discrete Boundary Condition

$$\psi_{i,j=2} = \psi_{i,j=1} + \frac{\partial \psi_{i,j=1}}{\partial y} h + \frac{\partial^2 \psi_{i,j=1}}{\partial y^2} \frac{h^2}{2} + O(h^3)$$

Using:

$$\omega_{wall} = -\frac{\partial^2 \psi_{i,j=1}}{\partial y^2}; \quad U_{wall} = \frac{\partial \psi_{i,j=1}}{\partial y}$$

This becomes:

$$\psi_{i,j=2} = \psi_{i,j=1} + U_{wall} h - \omega_{wall} \frac{h^2}{2} + O(h^3)$$

Solving for the wall vorticity:

$$\omega_{wall} = \left(\psi_{i,j=1} - \psi_{i,j=2}\right)\frac{2}{h^2} + U_{wall}\frac{2}{h} + O(h)$$

Discrete Boundary Condition

At the bottom wall $(j=1)$:

$$\omega_{wall} = \left(\psi_{i,j=1} - \psi_{i,j=2}\right)\frac{2}{h^2} + U_{wall}\frac{2}{h} + O(h)$$

Similarly, at the top wall $(j=\text{ny})$:

$$\omega_{wall} = \left(\psi_{i,j=ny} - \psi_{i,j=ny-1}\right)\frac{2}{h^2} - U_{wall}\frac{2}{h} + O(h)$$

At the left wall $(i=1)$:

Fill the blank

At the right wall $(i=\text{nx})$:

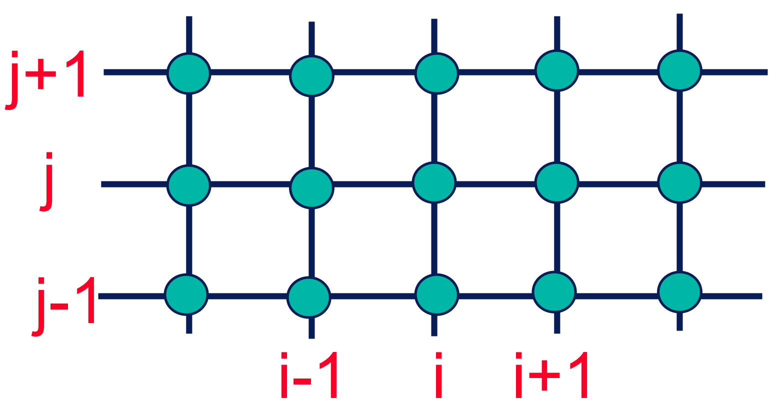
Fill the blank

Solving the elliptic equation:

$$\frac{\psi_{i+1,j}^{n} + \psi_{i-1,j}^{n} + \psi_{i,j+1}^{n} + \psi_{i,j-1}^{n} - 4\psi_{i,j}^{n}}{h^2} = -\omega_{i,j}^{n}$$

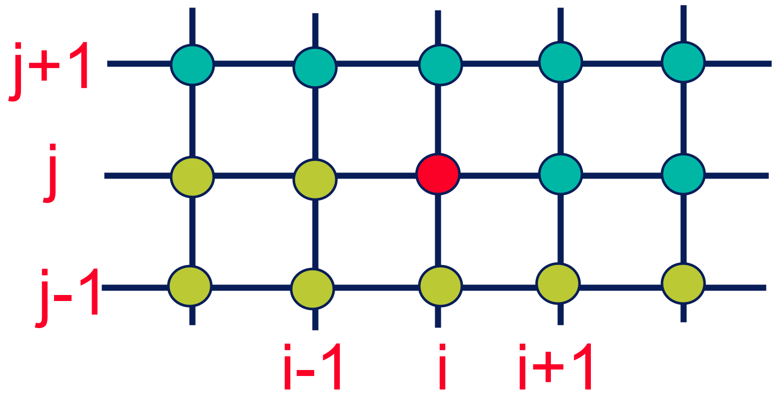
Rewrite as

$$\psi_{i,j}^{n+1} = 0.25\left(\psi_{i+1,j}^{n} + \psi_{i-1,j}^{n} + \psi_{i,j+1}^{n} + \psi_{i,j-1}^{n} + h^2\omega_{i,j}^{n}\right)$$



Solving the Stream Function Equation

If the grid points are done in order, half of the points have already been updated:



Successive Over Relaxation (SOR)

$$\psi_{i,j}^{n+1} = \beta\, 0.25\,(\psi_{i+1,j}^{n} + \psi_{i-1,j}^{n+1} + \psi_{i,j+1}^{n} + \psi_{i,j-1}^{n+1} + h^2\omega_{i,j}^{n})$$

$$+ (1-\beta)\psi_{i,j}^{n}$$

Time Step Control

Limitations on the time step

$$\frac{\nu \Delta t}{h^2} \leq \frac{1}{4} \qquad \frac{(|u|+|v|)\Delta t}{\nu} \leq 2$$

Solution Algorithm

1. Initial vorticity given
2. Solve for the stream function

```
for i=1:MaxIterations
    for i=2:nx-1; for j=2:ny-1
        s(i,j)=SOR for the stream function
    end; end
end
```

3. Find vorticity on boundary

```
v(i,j)=
```

4. Find RHS of vorticity equation

```
for i=2:nx-1; for j=2:ny-1
        rhs(i,j)=Advection+diffusion
end; end
```

5. Update vorticity in interior

```
v(i,j)=v(i,j)+dt*rhs(i,j)
```

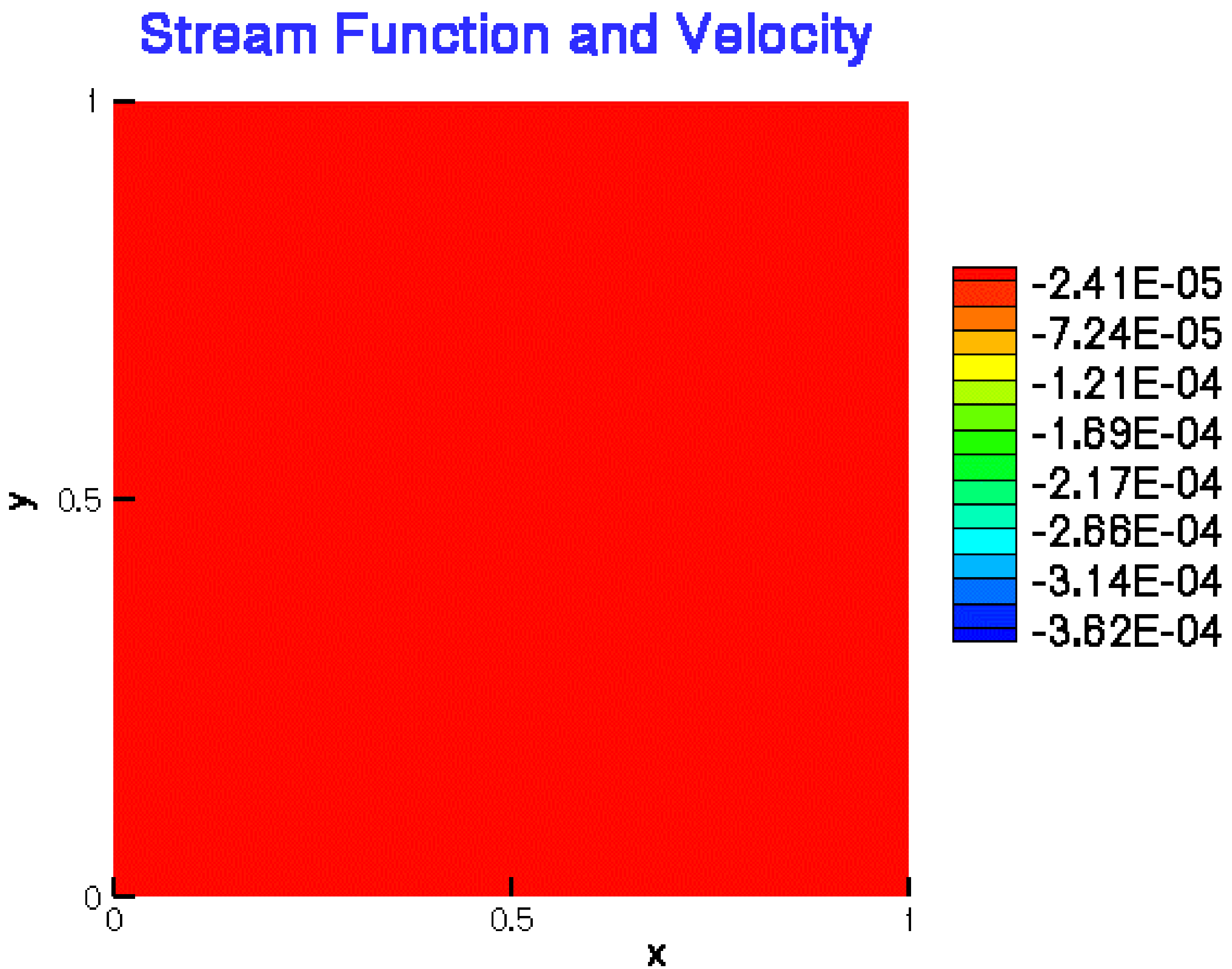
6. $t=t+\Delta t$

MATLAB Code

```
clf;nx=9; ny=9; MaxStep=60; Visc=0.1; dt=0.02;                % resolution & governing parameters
MaxIt=100; Beta=1.5; MaxErr=0.001;                            % parameters for SOR iteration
sf=zeros(nx,ny); vt=zeros(nx,ny); w=zeros(nx,ny); h=1.0/(nx-1); t=0.0;
for istep=1:MaxStep,                                          % start the time integration
 for iter=1:MaxIt,                                            % solve for the streamfunction
   w=sf;                                                      % by SOR iteration
  for i=2:nx-1; for j=2:ny-1
     sf(i,j)=0.25*Beta*(sf(i+1,j)+sf(i-1,j)...
       +sf(i,j+1)+sf(i,j-1)+h*h*vt(i,j))+(1.0-Beta)*sf(i,j);
   end; end;
   Err=0.0; for i=1:nx; for j=1:ny, Err=Err+abs(w(i,j)-sf(i,j)); end; end;
   if Err <= MaxErr, break, end                               % stop if iteration has converged
  end;
  vt(2:nx-1,1)=-2.0*sf(2:nx-1,2)/(h*h);                       % vorticity on bottom wall
  vt(2:nx-1,ny)=-2.0*sf(2:nx-1,ny-1)/(h*h)-2.0/h;             % vorticity on top wall
  vt(1,2:ny-1)=-2.0*sf(2,2:ny-1)/(h*h);                       % vorticity on right wall
  vt(nx,2:ny-1)=-2.0*sf(nx-1,2:ny-1)/(h*h);                   % vorticity on left wall
  for i=2:nx-1; for j=2:ny-1                                  % compute
     w(i,j)=-0.25*((sf(i,j+1)-sf(i,j-1))*(vt(i+1,j)-vt(i-1,j))...   % the RHS
     -(sf(i+1,j)-sf(i-1,j))*(vt(i,j+1)-vt(i,j-1)))/(h*h)...   % of the
     +Visc*(vt(i+1,j)+vt(i-1,j)+vt(i,j+1)+vt(i,j-1)-4.0*vt(i,j))/(h*h);   % vorticity
  end; end;                                                   % equation
  vt(2:nx-1,2:ny-1)=vt(2:nx-1,2:ny-1)+dt*w(2:nx-1,2:ny-1);    % update the vorticity
  t=t+dt                                                      % print out t
  subplot(121), contour(rot90(fliplr(vt))), axis('square');   % plot vorticity
  subplot(122), contour(rot90(fliplr(sf))), axis('square');pause(0.01)   % streamfunction
end;
```

Solution Fields

nx=17, ny=17, dt = 0.005, Re = 10, U_{wall} = 1.0



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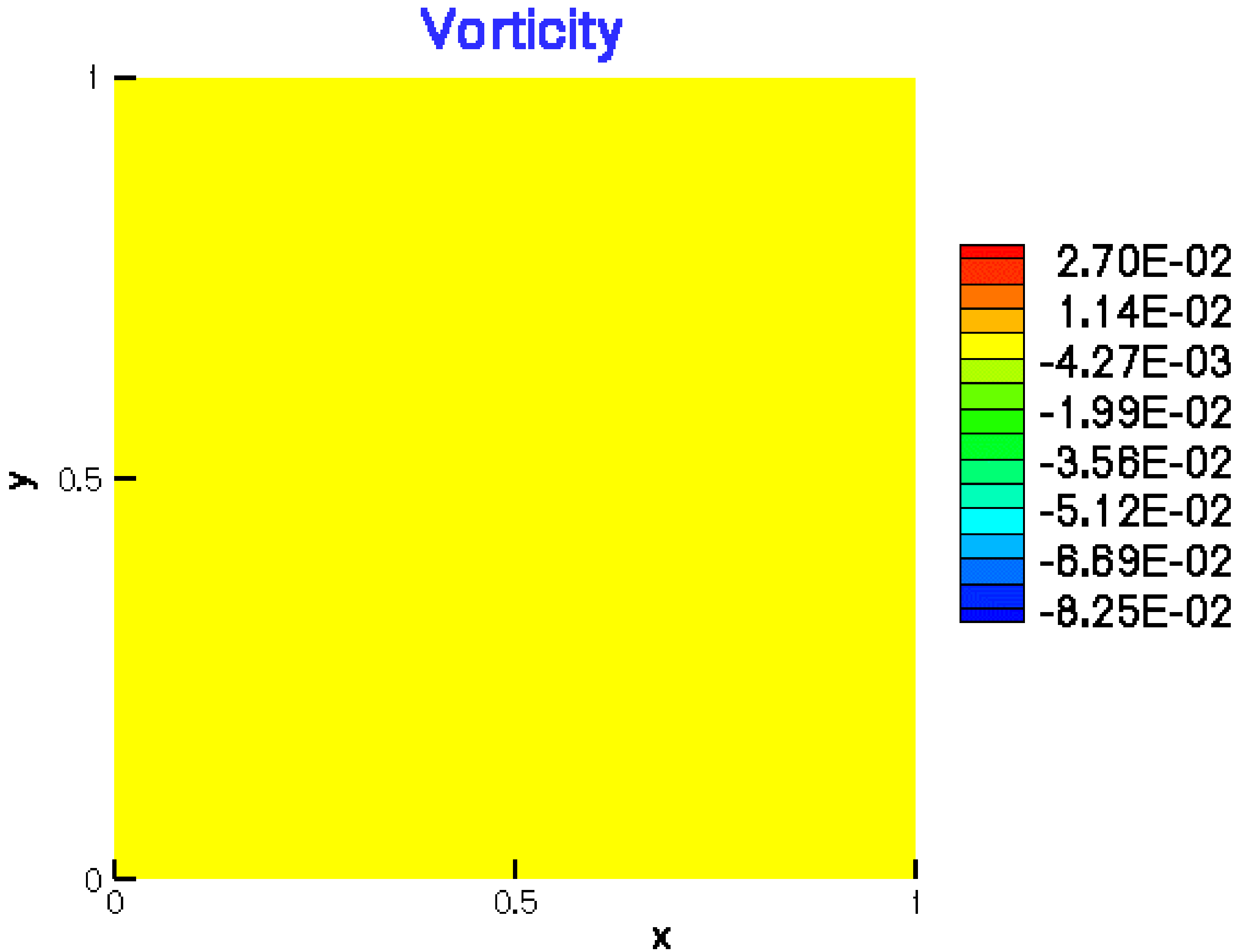
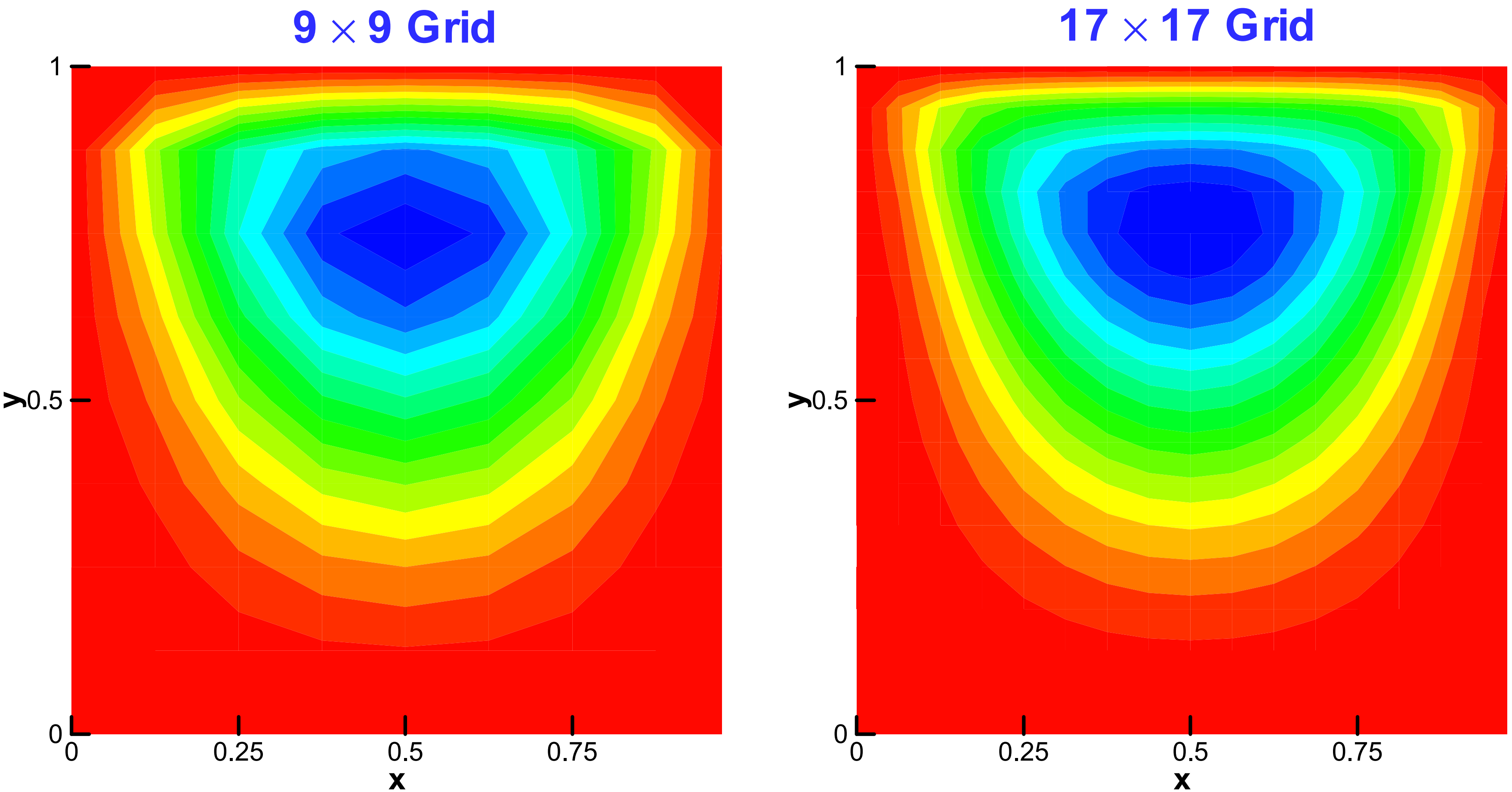


Computational Fluid Dynamics I

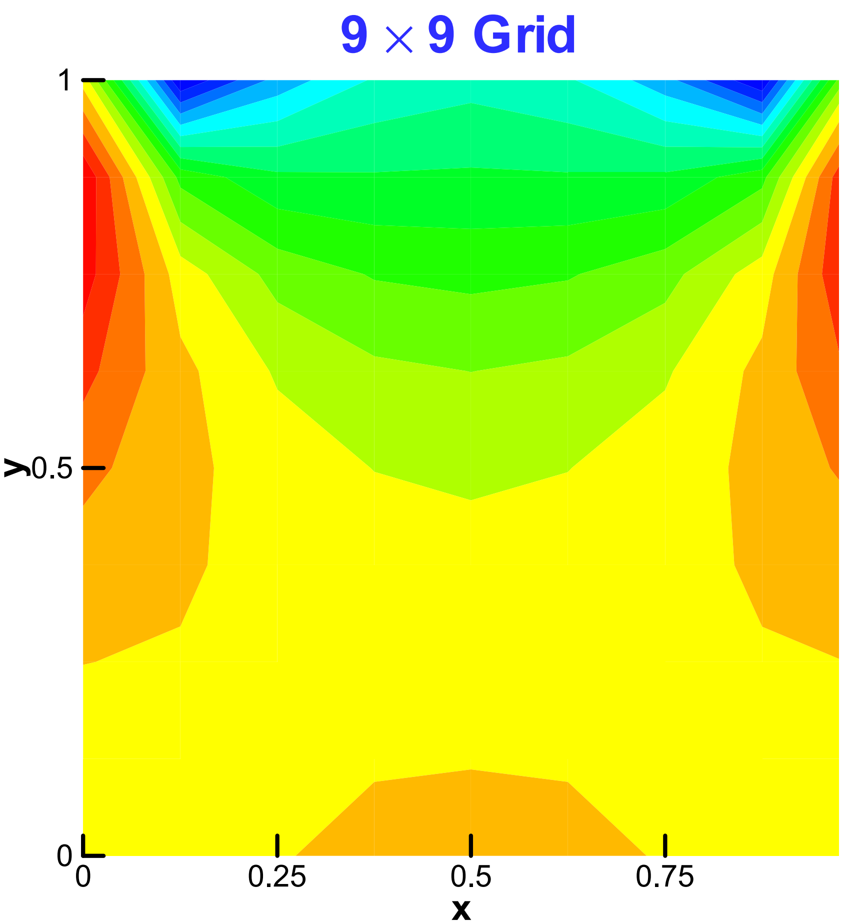
Accuracy by Resolution

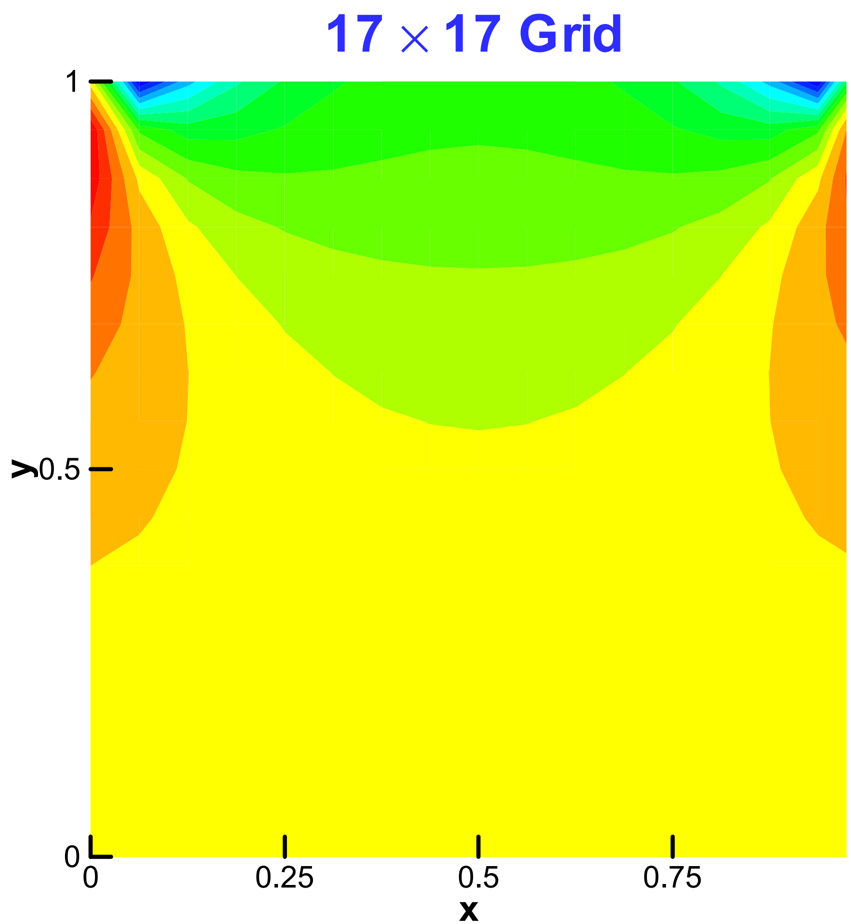
Stream Function at t = 1.2



Accuracy by Resolution

Vorticity at t = 1.2





Mini-project:

Add the temperature equation to the vorticity-streamfunction equation and compute the increase in heat transfer rate:

Convection

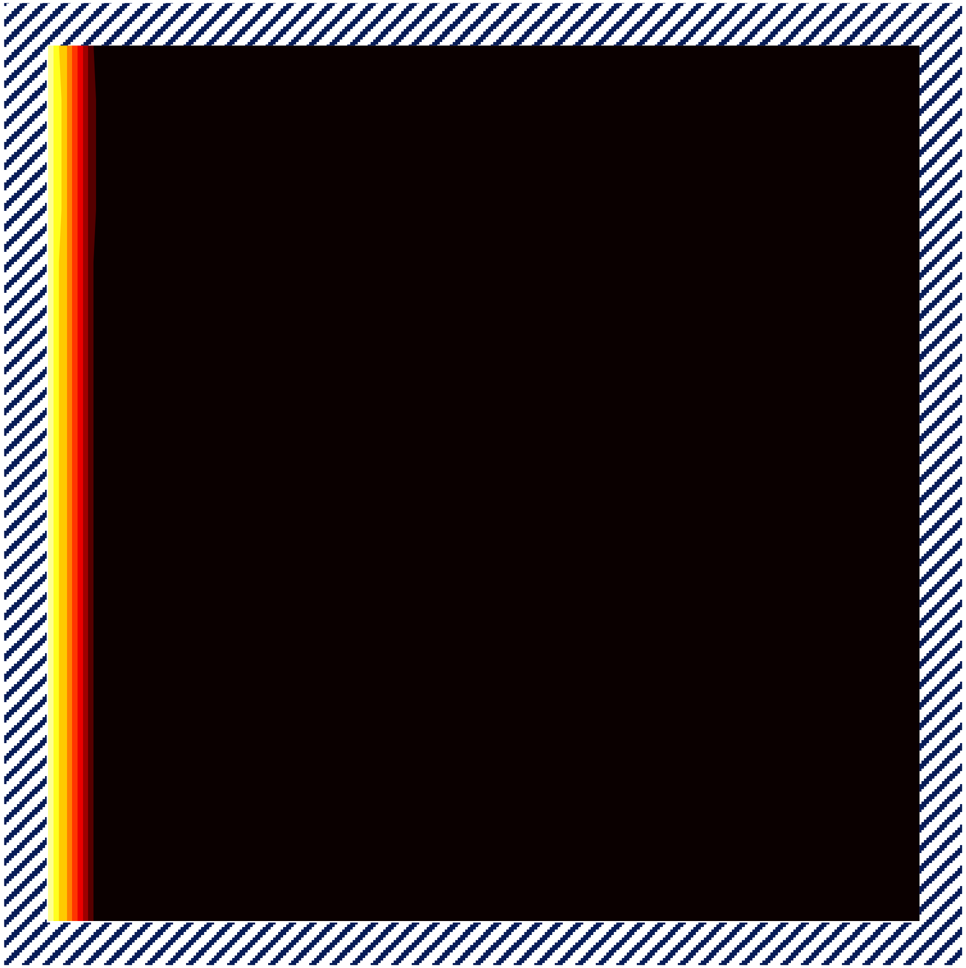
$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

where

$$\alpha = \frac{D}{\rho c_p}$$

Natural convection

Natural convection in a closed cavity. The left vertical wall is heated.



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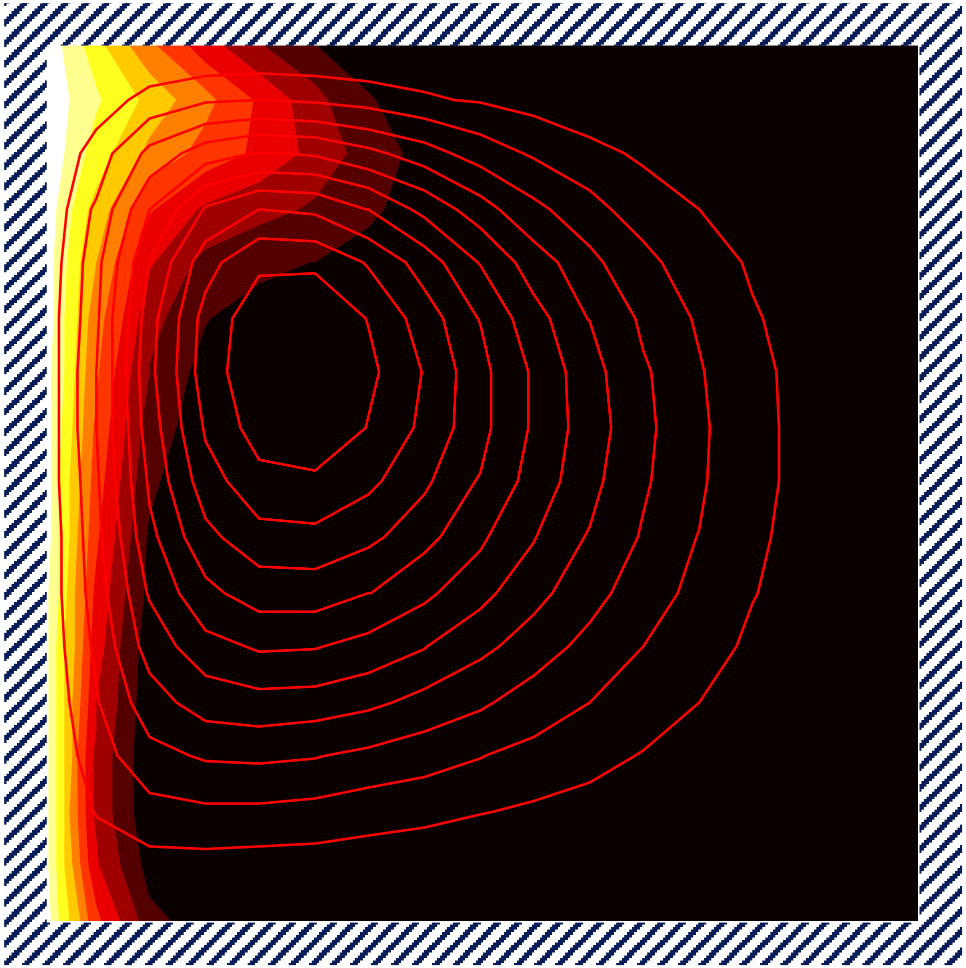

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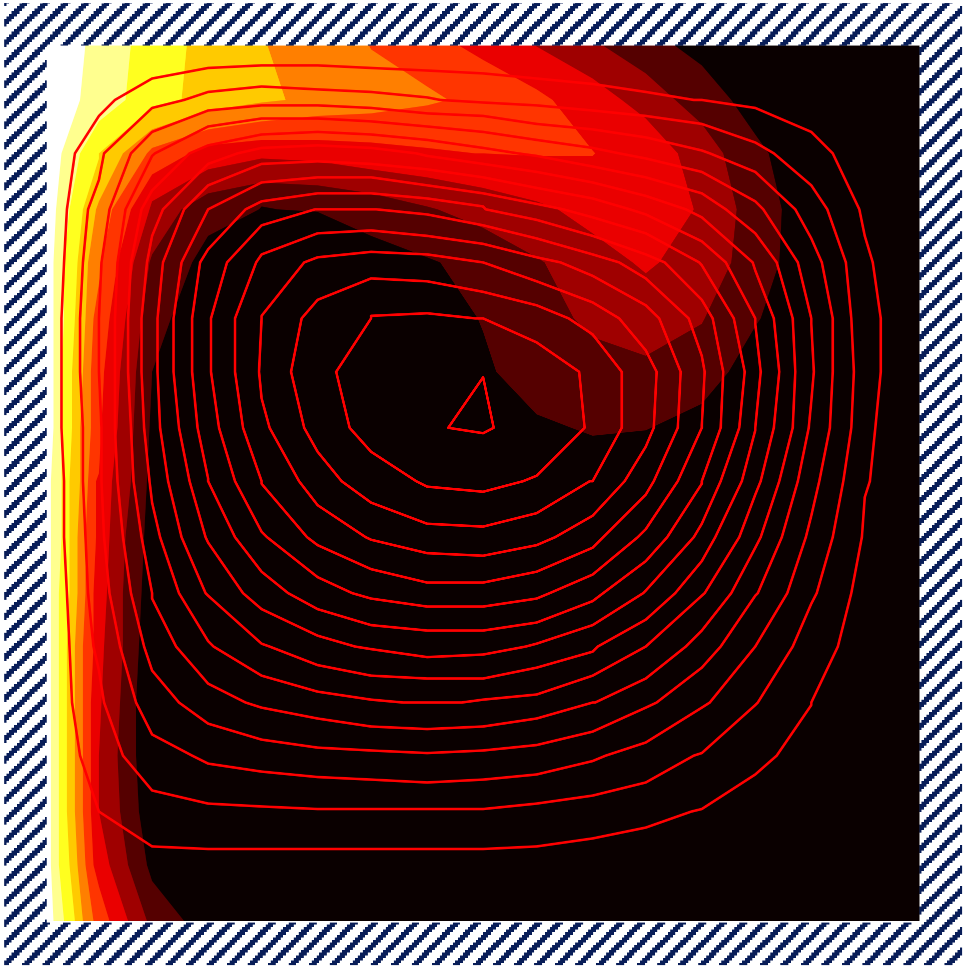

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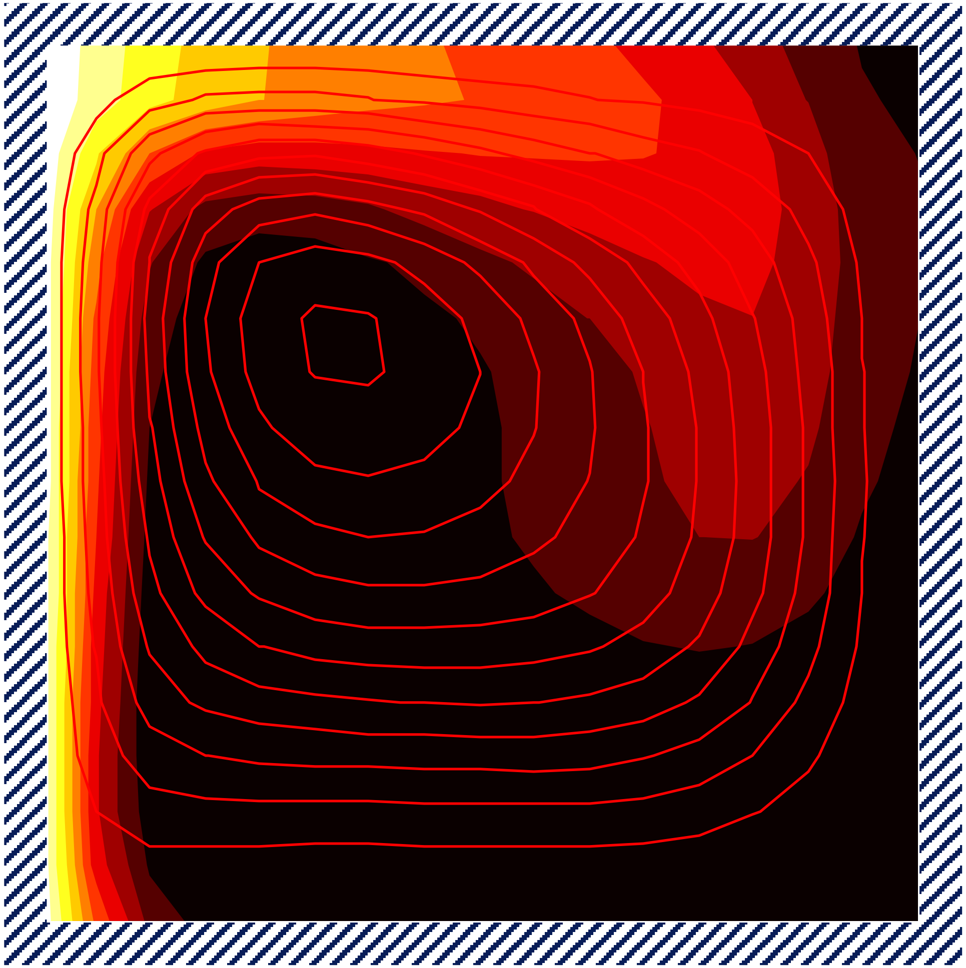

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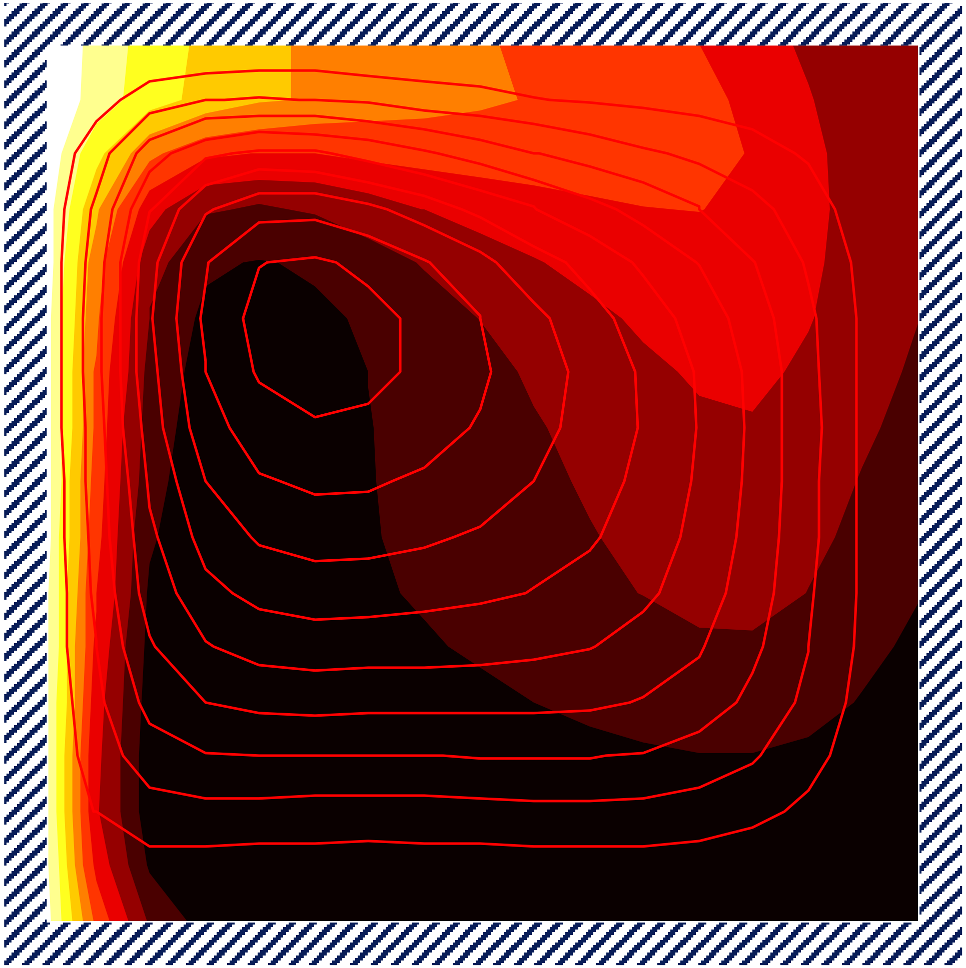

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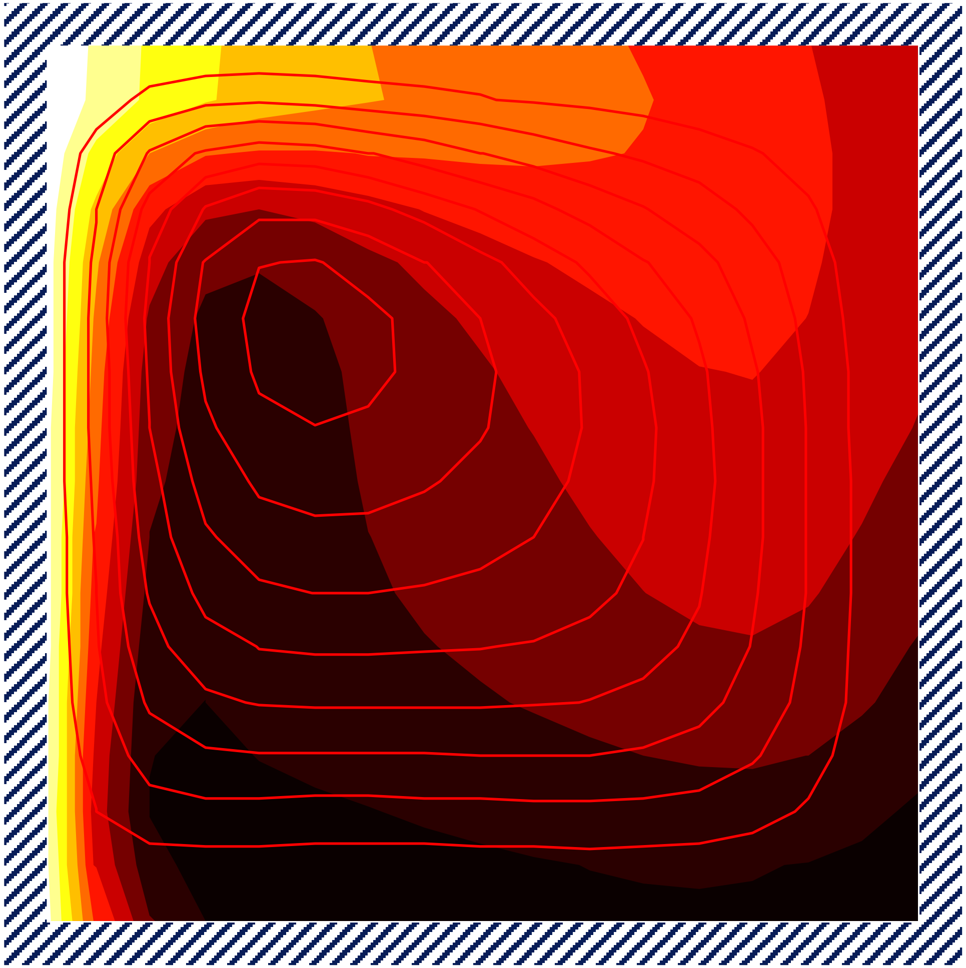

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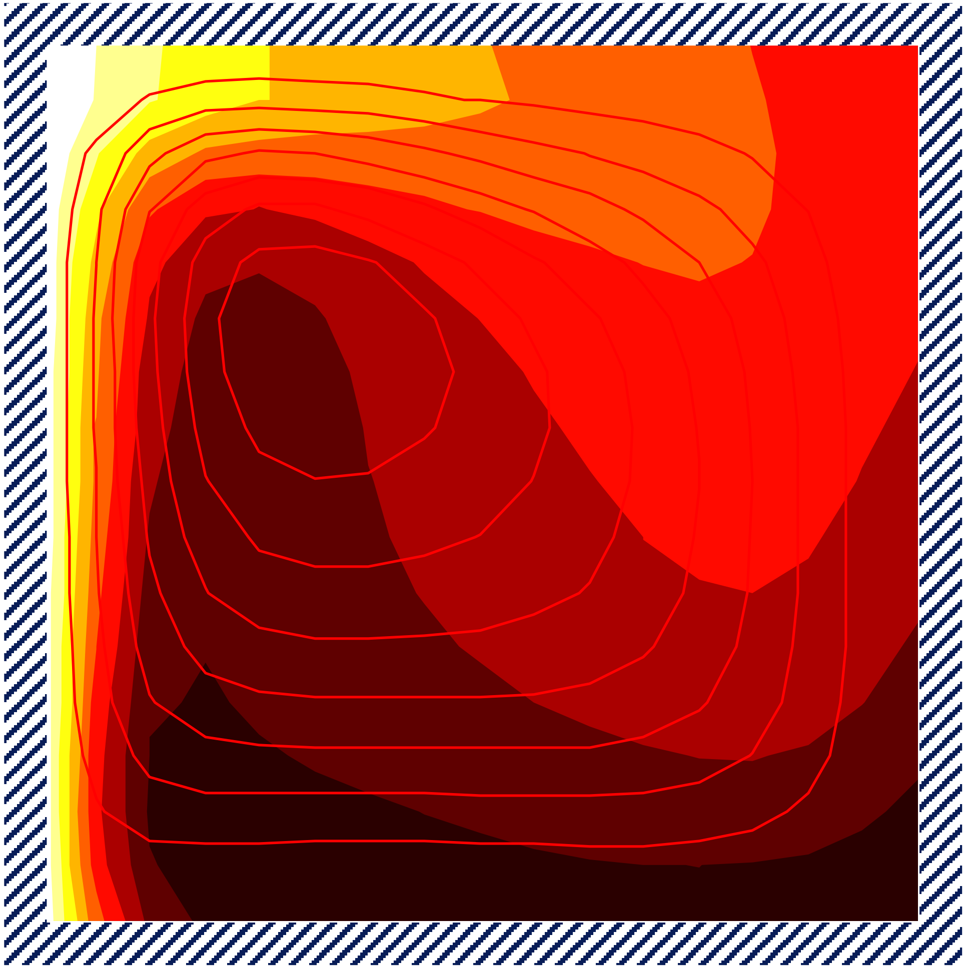

Natural convection