

# An Elegant Proof

math1023

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## P31.4.

Prove the following theorem:

**Theorem 0.1.**  $2^{1/n}$  is irrational for all integers  $n \geq 3$ .

*Hint:* First recall **Fermat's Last Theorem**. Then, set up a proof by contradiction and prove the original theorem.

(a) We recall the following theorem.

**Theorem 0.2.** (Fermat) The equation  $x^n + y^n = z^n$  has no integer solutions for  $x, y, z \neq 0$  and  $n > 2$ .

(The proof is trivial and left as an exercise to the reader)

(b)

**Lemma 0.3.** Suppose  $2^{1/n}$  is rational for some integer  $n \geq 3$ . Then  $\exists a, b \in \mathbb{Z}^{\neq 0}$  for which  $b^n + b^n = a^n$ .

*Proof.* Let  $2^{1/n}$  be rational for some integer  $n \geq 3$ .

Then, there exist nonzero integers  $a$  and  $b$  such that

$$\frac{a}{b} = 2^{1/n}.$$

And so

$$a^n = 2b^n = b^n + b^n.$$

This completes the proof. ■

(d) We will now prove the original statement.

*Proof.* Let a counterexample to the original theorem exist. Then, by our lemma in (b), there exist nonzero integers  $a$  and  $b$  where  $b^n + b^n = a^n$ . However, this equation presents a solution to the equation in part (a) which has no solutions. Hence, we conclude that no counterexamples to the theorem exist, and therefore, the theorem is true. ■

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