An Elegant Proof

math1023

John Smith

July 31, 2020

P31.4.

Prove the following theorem:

Theorem 0.1. $2^{1/n}$ is irrational for all integers $n \geq 3$.

Hint: First recall **Fermat's Last Theorem.** Then, set up a proof by contradiction and prove the original theorem.

(a) We recall the following theorem.

Theorem 0.2. (Fermat) The equation $x^n + y^n = z^n$ has no integer solutions for $x, y, z \neq 0$ and $n > 2$.

(The proof is trivial and left as an exercise to the reader)

(b)

Lemma 0.3. Suppose $2^{1/n}$ is rational for some integer $n \geq 3$. Then $\exists a, b \in \mathbb{Z}^{\neq 0}$ for which $b^n + b^n = a^n$.

Proof. Let $2^{1/n}$ be rational for some integer $n \geq 3$.
Then, there exist nonzero integers a and b such that

$$\frac{a}{b} = 2^{1/n}.$$

And so

$$a^n = 2b^n = b^n + b^n.$$

This completes the proof. ■

(d) We will now prove the original statement.

Proof. Let a counterexample to the original theorem exist. Then, by our lemma in (b), there exist nonzero integers a and b where $b^n + b^n = a^n$. However, this equation presents a solution to the equation in part (a) which has no solutions. Hence, we conclude that no counterexamples to the theorem exist, and therefore, the theorem is true. ■