Design of Optimized Walking Gaits and Flexible Soles for Humanoid Robots

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***Abstract*—We detailed the WPG algorithms.**

I. Problem formulation

Our objective is to obtain a sole shape Ω and optimized walking gait for the optimized flexible sole. The optimized sole and walking gait reduces (i) the impact force during the heel-strike phase (shock-absorbing) and (ii) foot rotation during the whole foot movement, while following a given resultant force and position of the support reaction $(\boldsymbol{F}_{\text{ZMP}}(t), \boldsymbol{Z}_{\text{ZMP}}(t))$ that minimizes the robot energy consumption. We formulate this problem as an optimization program.

Using these considerations, we defined the cost function as:

$$c = c_{\text{s}} + q c_{\text{wpg}} \tag{1}$$

where:

$$\begin{array}{rcl} c_{\text{s}} & = & \int_{t_0}^{t_h} \sigma_{\max}\left(\boldsymbol{K_c}^{\text{tr}}(\Omega,t)\right)\ dt - w \int_{t_0}^{t_f} \sigma_{\min}\left(\boldsymbol{K_c}^{\text{rot}}(\Omega,t)\right)\ \text{dt} \\ & \stackrel{\text{def}}{=} & c^{\text{tr}} - w\ c^{\text{rot}} \end{array} \tag{2}$$

$$\begin{array}{rcl} c_{\text{wpg}} & = & \int_{t_i}^{t_f} \lambda \|\mathbf{F}_{\text{COM}}(t)\|^2 + (1-\lambda)\,\|\boldsymbol{\Gamma}_a(t)\|^2\ \text{dt}+ \\ & & \epsilon \int_{t_i}^{t_f} (\|\ddot{\mathbf{P}}_{\text{ZMP}_1}(t)\|^2 + \|\ddot{\mathbf{P}}_{\text{ZMP}_2}(t)\|^2)\ \text{dt} \end{array} \tag{3}$$

where q, w, λ, ϵ are the weights of each criteria $\in \mathbb{R}_+$.

In (2) $\boldsymbol{K_c}$ is the Cartesian stiffness matrix, t_h is the end of the heel-strike phase that we define as lasting 10% of the whole contact phase, $\sigma_{\max}$ and $\sigma_{\min}$ denotes respectively the largest and smallest singular value of a matrix, and the two terms correspond respectively to:

1) minimization of the translational part of the cartesian stiffness matrix $\boldsymbol{K_c}$ (maximize the compliance) during the heel-strike phase to reduce the impact force, and
2) maximization of the rotational part of the cartesian stiffness matrix $\boldsymbol{K_c}$ during the whole foot movement to decrease foot rotation in case of perturbations. Low foot rotation enforces the vertical posture and secures the balance of the robot during walking.

Note that K_c and then c_{s} is parametrized by $\boldsymbol{F}_{\text{ZMP}}$ and $\boldsymbol{Z}_{\text{ZMP}}$. We specify a maximum volume $v_{\max}$ for the sole.

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In (3), we minimize the energy consumption. Our criteria is quadratic and as close as possible to the robot energy consumption. From [1], energy consumption taking into account motor and reducer model has the following expression form:

$$E = \sum_j \int_{t_i}^{t_f} (a_j \tau_j^2 + b_j \tau_j \dot{q}_j + c_j \dot{q}_j^2) dt \tag{4}$$

where τ_j is force/torque and $\dot{q}_j$ is the joint j velocity, and a_j, b_j, c_j are coefficients depending on joint j motor and reducer. Using the simple pendulum model proposed in [2], we simplified the expression of the energy consumption into (3).

$\mathbf{F}_{\text{COM}}$ in (3) is the COM force and can be expressed as:

$$\mathbf{F}_{\text{COM}}(t) = m\left(\ddot{\mathbf{P}}_{\text{COM}}(t) + g\vec{z}\right) \tag{5}$$

where $\ddot{\mathbf{P}}_{\text{COM}}$ is the center of mass(COM) acceleration.

$\boldsymbol{\Gamma}_a$ is the torque at the ankle joint a. Considering only the foot in contact and assuming it is at a fixed position during the whole single support phase (SSP), we have:

$$\boldsymbol{\Gamma}_a(t) = \mathbf{F}_{\text{ZMP}}(t) \times (\mathbf{P}_a(t) - \mathbf{P}_{\text{ZMP}}(t)), \tag{6}$$

where $\mathbf{P}_a$ is the ankle position, $\mathbf{F}_{\text{ZMP}}$ is the ground reaction force and $\mathbf{P}_{\text{ZMP}}$ is the ZMP position.

During the double support phase (DSP), the two feet are in contact. Considering independently each foot and assuming that they are at a fixed position during the whole DSP, we have:

$$\begin{array}{l} \boldsymbol{\Gamma}_{a_1}(t) = \mathbf{F}_{\text{ZMP}_1}(t) \times (\mathbf{P}_{a_1}(t) - \mathbf{P}_{\text{ZMP}_1}(t)) \\ \boldsymbol{\Gamma}_{a_2}(t) = \mathbf{F}_{\text{ZMP}_2}(t) \times (\mathbf{P}_{a_2}(t) - \mathbf{P}_{\text{ZMP}_2}(t)) \end{array} \tag{7}$$

During the DSP, we note with subscript '1' the foot that leaves the floor at the end of DSP, and with subscript '2', the foot that comes in contact at the beginning of DSP.

$\ddot{\mathbf{P}}_{\text{ZMP}_1}$ and $\ddot{\mathbf{P}}_{\text{ZMP}_2}$ are respectively the acceleration of ZMP_1 and ZMP_2. These terms are not issued from the energy consumption but allow to generate ZMP_1 and ZMP_2 trajectories in the direction of the walk and to minimize the feet angular acceleration.

II. Walking Pattern Generator

To model the deformation of the sole during the walking, three parameters are linked: (i) sole shape, (ii) position of contact points and their respective applied forces during the contact between the sole and the ground, and (iii) foot orientation. If we know two of these three points, we can deduce the third. In this paper, we decided to generate the contact points with their respective applied forces and then deduce the foot orientation.

We approximated the contact points and their respective applied forces by a ground reaction force generated from a walking pattern generator (WPG) and the zero moment point (ZMP) trajectory. As explain before, in order to design the sole as a shape optimization problem, integrating existing walking pattern generators (PG) is not straightforward due to the link with the foot orientation. Among other technical reasons required by the use of flexible soles (i) we must manage a ZMP under each foot, especially during the double support phase, and (ii) we must have a smooth ZMP to get a smooth orientation. To date, no existing PG fulfils such requirements.

A. *ZMP and COM trajectories*

Based on Morisawa *et al.* [3], we generated the ZMP and the COM trajectories. Nevertheless, the ZMP trajectory is continue and defined as an association of 5^{th} order polynomial functions (minimum jerk invariant theory). To ensure a smooth ZMP trajectory, these polynomials are linked by boundary conditions in position, speed and acceleration.

The ZMP_1 trajectory is discontinue and also defined by 5^{th} order polynomial functions. To ensure a smooth transition from ZMP to ZMP_1 at the beginning of each DSP to obtain a smooth ZMP under each foot during a whole foot step, ZMP and ZMP_1 also verify boundary conditions in position, speed and acceleration.

The ZMP_2 trajectory is discontinue. Using the ZMP definition, we obtain:

$$\mathbf{\Gamma}_{\text{ZMP}}(\mathbf{F}_{\text{ZMP}_1})(t) + \mathbf{\Gamma}_{\text{ZMP}}(\mathbf{F}_{\text{ZMP}_2})(t) = 0 \tag{8}$$

where $\mathbf{\Gamma}_{\text{ZMP}}(\mathbf{f}_{\text{ZMP}_i})$ is the torque of the reaction force $\mathbf{f}_{\text{ZMP}_i}$ under foot $i \in \{1, 2\}$ applied on ZMP during the DSP. Solving (8) in the direction $\vec{y}$, we obtain:

$$x_{\text{ZMP}_2} = x_{\text{ZMP}} - \frac{F_{\text{ZMP}_1}}{F_{\text{ZMP}_2}}(x_{\text{ZMP}_1} - x_{\text{ZMP}}) \tag{9}$$

With the same reasoning in the direction $\vec{x}$, we obtain y_{ZMP_2} by substituting y for x in (9). When solving the optimization problem, the ZMP_2 trajectory have boundary conditions in position, speed and acceleration with the ZMP trajectory at the end of DSP.

Two via points are defined to parametrize the duration of each ZMP and ZMP_1 polynomials. To find the polynomial coefficients of ZMP and ZMP_1 and obtain smoother trajectories, we enforced the boundary conditions in position, speed and acceleration at each via-point. The values of position, speed and acceleration are the optimization parameters.

B. *Optimization parameters*

To obtain the ZMP and COM trajectories, we used a QP optimization based on the cost function (10). The optimization parameters are:

- Position, speed and acceleration at via-points of ZMP
- Feet positions
- Position, speed and acceleration at via-points of ZMP_1.

We imposed:

- Initial and final feet positions
- Initial and final position, speed and acceleration of ZMP
- Initial and final position of COM
- Durations of the SSP and the DSP
- The COM height
- Foot orientation

We can also impose some feet positions to manage the walking path.

To enforce the humanoid stability during walking, the ZMP trajectories must be inside the support convex hull [4][5]. The support convex hull is defined by the area of the support foot. To obtain this area, we added inequality constraints to our PG optimization problem. During the SSP, this condition is linear and can be integrated as inequality constraints into a QP. During DSP, this condition is not linear. However, if ZMP_1 and ZMP_2 trajectories are in the support convex hull defined by the areas of each foot, the stability condition for ZMP trajectories is fulfilled. These conditions are linear and can also be integrated in the QP optimization.

Based on human walking results [6] and to compensate the robot precision [2], we defined a feet support areas with a security margin of $5 - 10\%$ of the foot length. To avoid auto-collisions, we defined a minimal distance between the robot feet (3cm). To avoid stretched legs singularity, we chose a maximum step length (30cm).

III. Conclusion

In this paper we detailed the algorithms of the WPG.

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IV. Walking pattern generator gradient

Every equation will be express in the x-axis direction but the same reasoning can be done in the y-axis direction and concatenate everything at the end.

A. Cost function and Gradient

The cost function of the WPG optimization is:

$$\mathrm{c_{wpg}} = \min\int_{t_i}^{t_f} \left(\lambda\ \|\mathbf{F}_{\text{COM}}(t)\|^2 + (1-\lambda)\|\mathbf{\Gamma}_{an_{SSP}}(t)\|^2 + \frac{1}{2}(1-\lambda)(\|\mathbf{\Gamma}_{an_1}(t)\|^2 + \|\mathbf{\Gamma}_{an_2}(t)\|^2) + \epsilon(\|\ddot{\mathbf{P}}_{\text{ZMP}_1}(t)\|^2 + \|\ddot{\mathbf{P}}_{\text{ZMP}_2}(t)\|^2)\right)\ \mathrm{dt} \quad (10)$$

Where an_{SSP} refers to the ankle of the support foot in SSP and an_i refers to the ankle of the foot "1" or "2" in DSP.

$\mathbf{F}_{\text{COM}}$, $\mathbf{\Gamma}_{an}$ and $\ddot{\mathbf{P}}_{ZMP_i}$ are linear on optimization variables. They can be written as:

$$\mathbf{F}_{\text{COM}} = \mathbf{A}_{\mathbf{F}_{\text{COM}}}X + B_{\mathbf{F}_{\text{COM}}},\ \mathbf{\Gamma}_{an} = \mathbf{A}_{\mathbf{\Gamma}_{an}}X + B_{\mathbf{\Gamma}_{an}},\ \ddot{\mathbf{P}}_{ZMP_i} = \mathbf{A}_{\ddot{\mathbf{P}}_{ZMP_i}}X + B_{\ddot{\mathbf{P}}_{ZMP_i}} \quad (11)$$

X is the vector of optimization variables:

$$X = \begin{bmatrix}\mathbf{x_0^{vp}} & \mathbf{x_0^{\text{COM}}} & \mathbf{x_0^{\text{step}}} & \mathbf{x_0^{vp'}}\end{bmatrix}^T \quad (12)$$

Where $\mathbf{x_0^{vp}}$ represents the boundary conditions of ZMP via-points, $\mathbf{x_0^{\text{COM}}}$ represents the initial and final boundary condition of COM, $\mathbf{x_0^{\text{step}}}$ represents the foot step positions and $\mathbf{x_0^{vp'}}$ represents the boundary conditions of ZMP_1 via-points.

Thus the cost function can be written:

$$\mathrm{c_{wpg}} = \frac{1}{2}X^T\mathbf{H}X + \mathbf{G}X + \frac{1}{2}C \quad (13)$$

Where

$$\begin{aligned}
\mathbf{H} &= \lambda\mathbf{A}_{\mathbf{F}_{\text{COM}}}^T\mathbf{A}_{\mathbf{F}_{\text{COM}}} + (1-\lambda)\mathbf{A}_{\mathbf{\Gamma}_{an_{SSP}}}^T\mathbf{A}_{\mathbf{\Gamma}_{an_{SSP}}} + \frac{1}{2}(1-\lambda)(\mathbf{A}_{\mathbf{\Gamma}_{an_1}}^T\mathbf{A}_{\mathbf{\Gamma}_{an_1}} + \mathbf{A}_{\mathbf{\Gamma}_{an_2}(t)}^T\mathbf{A}_{\mathbf{\Gamma}_{an_2}(t)}) \\
&\quad +\epsilon(\mathbf{A}_{\ddot{\mathbf{P}}_{\text{ZMP}_1}}^T\mathbf{A}_{\ddot{\mathbf{P}}_{\text{ZMP}_1}} + \mathbf{A}_{\ddot{\mathbf{P}}_{\text{ZMP}_2}}^T\mathbf{A}_{\ddot{\mathbf{P}}_{\text{ZMP}_2}}) \\
\mathbf{G} &= \lambda B_{\mathbf{F}_{\text{COM}}}^T\mathbf{A}_{\mathbf{F}_{\text{COM}}} + (1-\lambda)B_{\mathbf{\Gamma}_{an_{SSP}}}^T\mathbf{A}_{\mathbf{\Gamma}_{an_{SSP}}} + \frac{1}{2}(1-\lambda)(B_{\mathbf{\Gamma}_{an_1}}^T\mathbf{A}_{\mathbf{\Gamma}_{an_1}} + B_{\mathbf{\Gamma}_{an_2}(t)}^T\mathbf{A}_{\mathbf{\Gamma}_{an_2}(t)}) \\
&\quad +\epsilon(B_{\ddot{\mathbf{P}}_{\text{ZMP}_1}}^T\mathbf{A}_{\ddot{\mathbf{P}}_{\text{ZMP}_1}} + B_{\ddot{\mathbf{P}}_{\text{ZMP}_2}}^T\mathbf{A}_{\ddot{\mathbf{P}}_{\text{ZMP}_2}}) \\
\mathbf{C} &= \lambda\mathbf{B}_{\mathbf{F}_{\text{COM}}}^T\mathbf{B}_{\mathbf{F}_{\text{COM}}} + (1-\lambda)\mathbf{B}_{\mathbf{\Gamma}_{an_{SSP}}}^T\mathbf{B}_{\mathbf{\Gamma}_{an_{SSP}}} + \frac{1}{2}(1-\lambda)(\mathbf{B}_{\mathbf{\Gamma}_{an_1}}^T\mathbf{B}_{\mathbf{\Gamma}_{an_1}} + \mathbf{B}_{\mathbf{\Gamma}_{an_2}(t)}^T\mathbf{B}_{\mathbf{\Gamma}_{an_2}(t)}) \\
&\quad +\epsilon(\mathbf{B}_{\ddot{\mathbf{P}}_{\text{ZMP}_1}}^T\mathbf{B}_{\ddot{\mathbf{P}}_{\text{ZMP}_1}} + \mathbf{B}_{\ddot{\mathbf{P}}_{\text{ZMP}_2}}^T\mathbf{B}_{\ddot{\mathbf{P}}_{\text{ZMP}_2}})
\end{aligned} \quad (14)$$

The gradient of $cost$ is:

$$\frac{\partial\mathrm{c_{wpg}}}{\partial X} = X^T\mathbf{H} + \mathbf{G} \quad (15)$$

The COM forces are defined by:

$$\mathbf{F}_{\text{COM}} = m\left(\ddot{\mathbf{P}}_{\text{COM}} - g\vec{z}\right) \quad (16)$$

With

$$\ddot{x}_{\text{COM}} = \frac{g}{z_{\text{COM}}}(x_{\text{COM}} - x_{\text{ZMP}}),\ x_{\text{COM}} = \mathbf{A}_{x_{\text{COM}}}X + B_{x_{\text{COM}}},\ x_{\text{ZMP}} = \mathbf{A}_{x_{\text{ZMP}}}X + B_{x_{\text{ZMP}}} \quad (17)$$

Thus

$$f_{\text{COM}}^x = \frac{mg}{z_{\text{COM}}}\left(\mathbf{A}_{x_{\text{COM}}} - \mathbf{A}_{x_{\text{ZMP}}}\right)X + \frac{mg}{z_{\text{COM}}}\left(B_{x_{\text{COM}}} - B_{x_{\text{ZMP}}}\right) \quad (18)$$

The torques in ankle are defined by:

$$\mathbf{\Gamma}_{an_g} = \mathbf{F}_{\text{ZMP}_g} \times (\mathbf{P}_{an_g} - \mathbf{P}_{\text{ZMP}_g}),\ \mathbf{\Gamma}_{an_g}^y = f_{\text{ZMP}_g}^x \cdot h_{an_g} + f_{\text{ZMP}_g}^z \cdot (x_{an_g} - x_{\text{ZMP}g}) \quad (19)$$

With $h_{an_g} =$ height of the ankle g, $g \in \{\text{SSP}, 1, 2\}$ and:

$$\mathbf{F}_{\text{ZMP}} = -\mathbf{F}_{\text{COM}},\ \mathbf{F}_{\text{ZMP}} = \mathbf{F}_{\text{ZMP}_1} + \mathbf{F}_{\text{ZMP}_2},\ \mathbf{F}_{\text{ZMP}_2} = k \cdot \mathbf{F}_{\text{ZMP}},\ x_{\text{ZMP}_g} = \mathbf{A}_{x_{\text{ZMP}_g}}X + B_{x_{\text{ZMP}_g}},\ x_{an_g} = \mathbf{A}_{x_{an_g}}X + B_{x_{an_g}} \quad (20)$$

k is a 5^{th} order polynomial representing the repartition of ZMP force under each foot.

Thus:

$$\begin{aligned}
\mathbf{\Gamma}_{an_{SSP}}^y &= m \cdot h_{an_{SSP}}\left(\mathbf{A}_{x_{\text{COM}}} - \mathbf{A}_{x_{\text{ZMP}}}\right)X - mg\left(\mathbf{A}_{x_{an_{SSP}}} - \mathbf{A}_{x_{\text{ZMP}}}\right)X \\
&\quad + m \cdot h_{an_{SSP}}\left(B_{x_{\text{COM}}} - B_{x_{\text{ZMP}}}\right) - mg\left(B_{x_{an_{SSP}}} - B_{x_{\text{ZMP}}}\right)
\end{aligned} \quad (21)$$

$$\begin{aligned}
\mathbf{\Gamma}_{an_1}^y &= m \cdot h_{an_1} \cdot (1-k)\left(\mathbf{A}_{x_{\text{COM}}} - \mathbf{A}_{x_{\text{ZMP}}}\right)X - mg\left(\mathbf{A}_{x_{an_1}} - \mathbf{A}_{x_{\text{ZMP}_1}}\right)X \\
&\quad + m \cdot h_{an_1} \cdot (1-k)\left(B_{x_{\text{COM}}} - B_{x_{\text{ZMP}}}\right) - mg\left(B_{x_{an_1}} - B_{x_{\text{ZMP}_1}}\right)
\end{aligned} \quad (22)$$

$$\begin{aligned}
\mathbf{\Gamma}_{an_2}^y &= m \cdot h_{an_2} \cdot k\left(\mathbf{A}_{x_{\text{COM}}} - \mathbf{A}_{x_{\text{ZMP}}}\right)X - mg\left(\mathbf{A}_{x_{an_2}} - \mathbf{A}_{x_{\text{ZMP}_2}}\right)X \\
&\quad + m \cdot h_{an_2} \cdot k\left(B_{x_{\text{COM}}} - B_{x_{\text{ZMP}}}\right) - mg\left(B_{x_{an_2}} - B_{x_{\text{ZMP}_2}}\right)
\end{aligned} \quad (23)$$

B. ZMP detailed

During the sequence number j, ZMP is defined by:

$$x_{ZMP}^{(j)}(t) = \sum_{i=0}^{5} a_i^{(j)} (\Delta t_j)^i \tag{24}$$

Where a_i is the i^{th} polynomial coefficient, $\Delta t_j = t - T_j$ and T_j is the time at the beginning of the sequence j.

ZMP polynomials are interpolated between two via-points and verify boundary conditions in position, velocity and acceleration. Thus polynomial coefficients of the sequence j are defined by:

$$\begin{bmatrix} \mathbf{Ca}_{012}^{(j)} & \mathbf{Ca}_{345}^{(j)} \end{bmatrix} \begin{bmatrix} x_0^{(j)} \\ \dot{x}_0^{(j)} \\ \ddot{x}_0^{(j)} \\ x_0^{(j+1)} \\ \dot{x}_0^{(j+1)} \\ \ddot{x}_0^{(j+1)} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & \Delta T_j & (\Delta T_j)^2 & (\Delta T_j)^3 & (\Delta T_j)^4 & (\Delta T_j)^5 \\ 0 & 1 & 2(\Delta T_j) & 3(\Delta T_j)^2 & 4(\Delta T_j)^3 & 5(\Delta T_j)^4 \\ 0 & 0 & 1 & 6(\Delta T_j) & 12(\Delta T_j)^2 & 20(\Delta T_j)^3 \end{bmatrix}^{-1} \begin{bmatrix} x_0^{(j)} \\ \dot{x}_0^{(j)} \\ \ddot{x}_0^{(j)} \\ x_0^{(j+1)} \\ \dot{x}_0^{(j+1)} \\ \ddot{x}_0^{(j+1)} \end{bmatrix} = \begin{bmatrix} a_0^{(j)} \\ a_1^{(j)} \\ \vdots \\ a_5^{(j)} \end{bmatrix} \tag{25}$$

Where $\mathbf{Ca}_{012}^{(j)}$ and $\mathbf{Ca}_{345}^{(j)}$ are 6×3 matrix, $\Delta T_j = T_{j+1} - T_j$ and $x_0^{(j)}$ is the position of the via-point at the beginning of the sequence j.

$$\mathbf{Ca} = \begin{bmatrix} \mathbf{Ca}_{012}^{(1)} & \mathbf{Ca}_{345}^{(1)} & 0 & 0 & \dots & \dots & \dots & 0 \\ 0 & \mathbf{Ca}_{012}^{(2)} & \mathbf{Ca}_{345}^{(2)} & 0 & \dots & \dots & \dots & 0 \\ \vdots & & & \ddots & \ddots & & & \vdots \\ 0 & \dots & \dots & \dots & 0 & \mathbf{Ca}_{012}^{(ns-1)} & \mathbf{Ca}_{345}^{(ns-1)} & 0 \\ 0 & \dots & \dots & \dots & 0 & 0 & \mathbf{Ca}_{012}^{(ns)} & \mathbf{Ca}_{345}^{(ns)} \end{bmatrix}_{6ns \times 3(ns+1)} \tag{26}$$

Where ns is the number of sequence.

The ZMP polynomial coefficients are:

$$\begin{bmatrix} a_0^{(0)} & a_1^{(0)} & \dots & a_5^{(0)} & \dots & a_5^{(ns)} \end{bmatrix}^T = \mathbf{Ca} \cdot \mathbf{x_0^{vp}} \tag{27}$$

With

$$\mathbf{x_0^{vp}} = \begin{bmatrix} x_0^{(1)} & \dot{x}_0^{(1)} & \ddot{x}_0^{(1)} & x_0^{(2)} & \cdots & \ddot{x}_0^{(ns+1)} \end{bmatrix}^T \tag{28}$$

Now we discretize our ZMP. The time discretization is defined by:

$$\mathbf{t}(t,j) = \begin{bmatrix} 1 & \Delta t_j & (\Delta t_j)^2 & (\Delta t_j)^3 & (\Delta t_j)^4 & (\Delta t_j)^5 \end{bmatrix}, \; \mathbf{dt}(j) = \begin{bmatrix} \mathbf{t}(t_j + dt^{(j)}, j) & \mathbf{t}(t_j + 2dt^{(j)}, j) & \cdots & \mathbf{t}(t_{j+1}, j) \end{bmatrix}^T$$

$$\mathbf{M}_t = \begin{bmatrix} \mathbf{t}(t_1, 1) & 0 & \cdots & 0 \\ \mathbf{dt}(1) & 0 & \cdots & \vdots \\ 0 & \mathbf{dt}(2) & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & \cdots & \mathbf{dt}(ns) \end{bmatrix} \tag{29}$$

Where $dt^{(j)}$ is the discretized time period of the phase j.

The ZMP trajectory can be written:

$$x_{\text{ZMP}} = \mathbf{A}_{x_{\text{ZMP}}} X + B_{x_{\text{ZMP}}} = \mathbf{M}_t \cdot \mathbf{Ca} \cdot \mathbf{x_0^{vp}} \tag{30}$$

Thus

$$\mathbf{A}_{x_{\text{ZMP}}} = \mathbf{M}_t \begin{bmatrix} \mathbf{Ca} & 0 \end{bmatrix}, B_{x_{\text{ZMP}}} = 0 \tag{31}$$

C. COM detailed

Based on Morisawa *et al.* [3]:

$$\ddot{x}^{(j)}_{\text{COM}} = \frac{g}{z_{\text{COM}}}(x^{(j)}_{\text{COM}} - x^{(j)}_{\text{ZMP}}) \tag{32}$$

Substituting (24) into (17), the COM position can be expressed by:

$$x^{(j)}_{\text{COM}}(t) = V^{(j)}c_j + W^{(j)}s_j + \sum_{i=0}^{5} A^{(j)}_i (\Delta t_j)^i \tag{33}$$

where:

$$c_j = \cosh(\omega_j \Delta t_j),\ s_j = \sinh(\omega_j \Delta t_j),\ \omega_j = \sqrt{\frac{g}{z^{(j)}_{\text{COM}}}} = \sqrt{g/z_{\text{COM}}},$$
$$A^{(j)}_i = \begin{cases} a^{(j)}_i + \sum_{k=1}^{(5-i)/2} b^{(j)}_{i+2k} a^{(j)}_{i+2k} & \text{for } i = 0 \dots 3 \\ a^{(j)}_i & \text{for } i = 4, 5 \end{cases},\quad b^{(j)}_{i+2k} = \prod_{l=1}^{k} \frac{(i+2l)(i+2l-1)}{w_j^2}$$

$V^{(j)}$ and $W^{(j)}$ are the unknowns of the system. In (33), $a^{(j)}_i$ coefficients are known. This is the difference between (33) and the system of equations in [3].

Equation (33) has $2m$ unknowns with $m = 3n + 2$ phases. These unknowns satisfy the following boundary conditions for the COM position and velocity:

1) Initial

$$x^{(1)}(T_0) = V^{(1)} + A^{(1)}_0 \tag{34}$$

$$\dot{x}^{(1)}(T_0) = W^{(1)} + A^{(1)}_1 \tag{35}$$

2) Relationship between two successive sequences

$$VW^{(j)} + \sum_{i=0}^{5} A^{(j)}_i (\Delta T_j)^i = V^{(j+1)} + A^{(j+1)}_0 \tag{36}$$

$$VW\omega^{(j)} + \sum_{i=1}^{5} iA^{(j)}_i (\Delta T_j)^{i-1} = W^{(j+1)}\omega_j + A^{(j+1)}_1 \tag{37}$$

where

$$VW^{(j)} = V^{(j)}C_j + W^{(j)}S_j$$
$$VW\omega^{(j)} = V^{(j)}\omega_j S_j + W^{(j)}\omega_j C_j,\ C_j = \cosh(\omega_j \Delta T_j),\quad S_j = \sinh(\omega_j \Delta T_j)$$

3) Final

$$x^{(ns)}(T_{ns}) = VW^{(ns)} + \sum_{i=0}^{5} A^{(ns)}_i (\Delta T_{ns})^i \tag{38}$$

$$\frac{dx^{(ns)}}{dt}(T_{ns}) = VW\omega^{(ns)} + \sum_{i=1}^{5} iA^{(ns)}_i (\Delta T_{ns})^{i-1} \tag{39}$$

where $\Delta T_j = T_{j+1} - T_j$.

From the boundary conditions (34)-(38), the total conditions are $2m+2$. Removing COM velocity conditions on initial and final phases (they are solved at the pattern optimization level), $2m$ conditions remain. The unknowns can be calculated then by the following system:

$$\mathbf{G} \cdot \mathbf{y} = \mathbf{N} \cdot \mathbf{x} + \mathbf{H} \cdot \mathbf{l} \tag{40}$$

where

$$
\begin{aligned}
&\mathbf{y} = \left[V^{(1)}\ W^{(1)}\ \cdots\ V^{(j)}\ W^{(j)}\ \cdots\ V^{(ns)}\ W^{(ns)}\right]^T \\
&\mathbf{x} = \mathbf{x_0^{\text{COM}}} = \left[x_{\text{COM}}^{(1)}(T_1)\ \dot{x}_{\text{COM}}^{(1)}(T_1)\ x_{\text{COM}}^{(ns+1)}(T_{ns+1})\ \dot{x}_{\text{COM}}^{(ns+1)}(T_{ns+1})\right]^T \\
&\mathbf{l} = \left[A_0^{(1)}\ \cdots\ A_5^{(1)}\ \cdots\ A_0^{(ns)}\ \cdots\ A_5^{(ns)}\right]^T \\
&\mathbf{G}_{h,q} = \begin{cases} [1\ 0] & \text{for } (h,q)=(1,1) \\ [C_{ns}\ S_{ns}] & \text{for } (h,q)=(ns+2,ns) \\ G_{1,h} & \text{for } q=h-1 \\ G_{2,h} & \text{for } q=h \\ 0 & \text{otherwise} \end{cases} \\
&\mathbf{N}_{h,1} = \begin{cases} \begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix} & \text{for } h=1 \\ \begin{bmatrix}0 & 0 & 1 & 0\end{bmatrix} & \text{for } h=ns+2 \\ \begin{bmatrix}0 & 0 & 0 & 0\end{bmatrix} & \text{otherwise} \end{cases} \\
&\mathbf{H}_{h,q} = \begin{cases} \begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0\end{bmatrix} & \text{for } (h,q)=(1,1) \\ -\left[(\Delta T_{ns})^0\ \ \cdots\ \ (\Delta T_{ns})^5\right] & \text{for } (h,q)=(ns+2,ns) \\ H_{1,h} & \text{for } q=h-1 \\ H_{2,h} & \text{for } q=h \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

with $h=1,\ldots,ns+2$, $q=1,\ldots,ns$, $\mathbf{G}_{1,h} = \begin{bmatrix} C_j & S_h \\ \omega_h S_h & \omega_h C_h \end{bmatrix}$, $\mathbf{G}_{2,h} = \begin{bmatrix} -1 & 0 \\ 0 & -\omega_h \end{bmatrix}$,
$\mathbf{H}_{1,h} = -\begin{bmatrix} (\Delta T_{ns})^0 & (\Delta T_{ns})^1 & \cdots & (\Delta T_{ns})^5 \\ 0 & (\Delta T_{ns})^0 & \cdots & (\Delta T_{ns})^4 \end{bmatrix}$, $\mathbf{H}_{2,h} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$.

From (33), the part related to the ZMP polynomial coefficients is:

$$
\begin{bmatrix} A_0^{(j)} \\ A_1^{(j)} \\ \vdots \\ A_5^{(j)} \end{bmatrix} = \left[\mathbf{A}_A^{(j)}\right] \begin{bmatrix} a_0^{(j)} \\ a_1^{(j)} \\ \vdots \\ a_5^{(j)} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2/w^2 & 0 & 24/w^4 & 0 \\ 0 & 1 & 0 & 6/w^2 & 0 & 120/w^4 \\ 0 & 0 & 1 & 0 & 12/w^2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 20/w^2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_0^{(j)} \\ a_1^{(j)} \\ \vdots \\ a_5^{(j)} \end{bmatrix} \tag{41}
$$

Thus

$$
\mathbf{A}_A = \begin{bmatrix} \mathbf{A}_A^{(1)} & 0 & \cdots & 0 \\ 0 & \mathbf{A}_A^{(2)} & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & \cdots & \mathbf{A}_A^{(ns)} \end{bmatrix} \tag{42}
$$

And

$$
\mathbf{l} = \left[A_0^{(1)}\ \cdots\ A_5^{(1)}\ \cdots\ A_0^{(ns)}\ \cdots\ A_5^{(ns)}\right]^T = \mathbf{A}_A \cdot \mathbf{Ca} \cdot \mathbf{x_0^{vp}} = \left[\mathbf{A}_A \cdot \mathbf{Ca}\quad 0\right] X \tag{43}
$$

By replacing into (40):

$$
\begin{aligned}
\mathbf{y} &= \left[V^{(1)}\quad W^{(1)}\quad \cdots\quad V^{(j)}\quad W^{(j)}\quad \cdots\quad V^{(ns)}\quad W^{(ns)}\right]^T \\
&= \mathbf{G}^{-1}(\mathbf{N}\cdot \mathbf{x_0^{\text{COM}}} + \mathbf{H}\cdot \mathbf{A}_A \cdot \mathbf{Ca} \cdot \mathbf{x_0^{vp}}) = \mathbf{G}^{-1}\left[\mathbf{H}\cdot\mathbf{A}_A\cdot\mathbf{Ca}\quad \mathbf{N}\quad 0\right] X
\end{aligned} \tag{44}
$$

Now we have to discretize the terms in $cosh$ and $sinh$. the discretized $cosh$ and $sinh$ are:

$$
\mathbf{cs}(t,j) = \left[cosh(\omega_j \Delta t_j)\quad sinh(\omega_j \Delta t_j)\right], \mathbf{dcs}(j) = \left[\mathbf{cs}(t_j + dt^{(j)}, j)\quad \mathbf{cs}(t_j + 2dt^{(j)}, j)\quad \cdots\quad \mathbf{cs}(t_{j+1}, j)\right]^T
$$

$$
\mathbf{M}_{cs} = \begin{bmatrix} \mathbf{cs}(t_1,1) & 0 & \cdots & 0 \\ \mathbf{dcs}(1) & 0 & \cdots & \vdots \\ 0 & \mathbf{dcs}(2) & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & \cdots & \mathbf{dcs}(ns) \end{bmatrix} \tag{45}
$$

with (40), (43) and (45), COM can be written as:

$$
\begin{aligned}
x_{\text{COM}} &= \mathbf{A}_{x_{\text{COM}}} X + B_{x_{\text{COM}}} = \mathbf{M}_{cs}\cdot \mathbf{y} + \mathbf{M}_t \cdot \mathbf{l} \\
&= \mathbf{M}_{cs}\cdot\mathbf{G}^{-1}\left[\mathbf{H}\cdot\mathbf{A}_A\cdot\mathbf{Ca}\quad \mathbf{N}\quad 0\right] X + \mathbf{M}_t \cdot \left[\mathbf{A}_A\cdot\mathbf{Ca}\quad 0\right] X \\
&= \left[\mathbf{M}_{cs}\cdot\mathbf{G}^{-1}\cdot\mathbf{H}\cdot\mathbf{A}_A\cdot\mathbf{Ca} + \mathbf{M}_t\cdot\mathbf{A}_A\cdot\mathbf{Ca}\quad \mathbf{M}_{cs}\cdot\mathbf{G}^{-1}\cdot\mathbf{N}\right] X
\end{aligned} \tag{46}
$$

Thus

$$\mathbf{A}_{x_{\text{COM}}} = \begin{bmatrix}\mathbf{M}_{cs} \cdot \mathbf{G}^{-1} \cdot \mathbf{H} \cdot \mathbf{A}_A \cdot \mathbf{Ca} + \mathbf{M}_t \cdot \mathbf{A}_A \cdot \mathbf{Ca} & \mathbf{M}_{cs} \cdot \mathbf{G}^{-1} \cdot \mathbf{N}\end{bmatrix}, B_{x_{\text{COM}}} = 0 \tag{47}$$

D. Ankle position detailed

Ankle positions are equivalent to foot step positions:

$$\mathbf{x_0^{\text{step}}} = \begin{bmatrix} x_1^{\text{step}} & x_2^{\text{step}} & \cdots & x_{nfs}^{\text{step}} \end{bmatrix} \tag{48}$$

Where nfs is the number of foot step.

Foot step positions are expressed as:

$$\mathbf{A}_{x_{an_{SSP}}} = \mathbf{M}_{an_{SSP}} \cdot \mathbf{x_0^{\text{step}}} = \begin{bmatrix} 0 & \mathbf{M}_{an_{SSP}} & 0 \end{bmatrix} X \tag{49}$$

Foot step positions in SSP are:

$$\mathbf{M}_{an_{SSP}}^{(m,n)} = \begin{cases} 1 & \text{for } m = \text{SSP increments} \quad \text{and} \quad n = \text{ corresponding SSP foot step} \\ 0 & \text{otherwise} \end{cases} \tag{50}$$

Foot$_1$ step positions in DSP are:

$$\mathbf{A}_{x_{an_1}} = \mathbf{M}_{an_1} \cdot \mathbf{x_0^{\text{step}}} = \begin{bmatrix} 0 & \mathbf{M}_{an_1} & 0 \end{bmatrix} X \tag{51}$$

$$\mathbf{M}_{an_1}^{(m,n)} = \begin{cases} 1 & \text{for } m = \text{DSP increments} \quad \text{and} \quad n = \text{ corresponding DSP foot}_1 \text{ step} \\ 0 & \text{otherwise} \end{cases} \tag{52}$$

Foot$_2$ step positions in DSP are:

$$\mathbf{A}_{x_{an_2}} = \mathbf{M}_{an_2} \cdot \mathbf{x_0^{\text{step}}} = \begin{bmatrix} 0 & \mathbf{M}_{an_2} & 0 \end{bmatrix} X \tag{53}$$

$$\mathbf{M}_{an_2}^{(m,n)} = \begin{cases} 1 & \text{for } m = \text{DSP increments} \quad \text{and} \quad n = \text{ corresponding DSP foot}_2 \text{ step} \\ 0 & \text{otherwise} \end{cases} \tag{54}$$

With $m = 1, \ldots, nd$, $n = 1, \ldots, nfs$ and $nd = 1 + \sum_{i=1}^{ns} \lceil t_{i+1} - t_i)/dt^{(i)} \rceil$ is the number of the total discretized point.

E. ZMP$_1$ and ZMP$_2$ detailed

During the sequence number $j \in$ DSP, ZMP$_1$ is defined by:

$$x_{ZMP_1}^{(j)}(t) = \sum_{i=0}^{5} a_{1,i}^{(j)} (\Delta t_j)^i \tag{55}$$

As ZMP, ZMP$_1$ polynomials are interpolated between two via-points and verify boundary conditions in position, velocity and acceleration. Thus polynomial coefficients of the sequence $j \in$ DSP are defined by:

$$\begin{bmatrix} \mathbf{Ca}_{1,012}^{(j)} & \mathbf{Ca}_{1,345}^{(j)} \end{bmatrix} \begin{bmatrix} x_0^{(j)} \\ \dot{x}_0^{(j)} \\ \ddot{x}_0^{(j)} \\ x_1^{(j)} \\ \dot{x}_1^{(j)} \\ \ddot{x}_1^{(j)} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & \Delta T_j & (\Delta T_j)^2 & (\Delta T_j)^3 & (\Delta T_j)^4 & (\Delta T_j)^5 \\ 0 & 1 & 2(\Delta T_j) & 3(\Delta T_j)^2 & 4(\Delta T_j)^3 & 5(\Delta T_j)^4 \\ 0 & 0 & 1 & 6(\Delta T_j) & 12(\Delta T_j)^2 & 20(\Delta T_j)^3 \end{bmatrix}^{-1} \begin{bmatrix} x_0^{(j)} \\ \dot{x}_0^{(j)} \\ \ddot{x}_0^{(j)} \\ x_1^{(j)} \\ \dot{x}_1^{(j)} \\ \ddot{x}_1^{(j)} \end{bmatrix} = \begin{bmatrix} a_{1,0}^{(j)} \\ a_{1,1}^{(j)} \\ \vdots \\ a_{1,5}^{(j)} \end{bmatrix} \tag{56}$$

Where $\mathbf{Ca}_{1,012}^{(j)}$ and $\mathbf{Ca}_{1,345}^{(j)}$ are 6×3 matrix, $\Delta T_j = t_{j+1} - t_j$, $x_0^{(j)}$ is the position of the via-point at the beginning of the sequence j and $x_1^{(j)}$ is the position of the via-points at the end of the DSP.

$$\mathbf{Ca}_1^{(r,s)} = \begin{cases} \mathbf{Ca}_{1,012}^{(r)} & \text{for } r = s = \text{DSP sequence} \\ \mathbf{Ca}_{1,345}^{(r)} & \text{for } r = \text{DSP sequence} \quad \text{and } s = ns + \text{corresponding ZMP}_1 \text{ via-point} \\ 0 & \text{otherwise} \end{cases} \tag{57}$$

With $r = 1, \ldots, ns$, $s = 1, \ldots,$ (number of rows of X).

ZMP$_1$ polynomial coefficients are:

$$\begin{bmatrix} a_{1,0}^{(0)} & a_{1,1}^{(0)} & \ldots & a_{1,5}^{(0)} & \ldots & a_{1,5}^{(ns)} \end{bmatrix}^T = \mathbf{Ca}_1 \begin{bmatrix} \mathbf{x_0^{vp}} \\ \mathbf{x_0^{\text{COM}}} \\ \mathbf{x_0^{\text{step}}} \\ \mathbf{x_0^{vp'}} \end{bmatrix} \tag{58}$$

The ZMP_1 trajectory is:

$$x_{\text{ZMP}_1} = \mathbf{A}_{x_{\text{ZMP}_1}} X + B_{x_{\text{ZMP}_1}} = \mathbf{M}_t \cdot \mathbf{Ca}_1 \cdot X \tag{59}$$

Thus

$$\mathbf{A}_{x_{\text{ZMP}_1}} = \mathbf{M}_t \cdot \mathbf{Ca}_1, B_{x_{\text{ZMP}}} = 0 \tag{60}$$

ZMP_2 is obtained with

$$x^{(j)}_{\text{ZMP}_2} = x^{(j)}_{\text{ZMP}_1} - \frac{1}{k^{(j)}}(x^{(j)}_{\text{ZMP}_1} - x^{(j)}_{\text{ZMP}}) \tag{61}$$

Where

$$k^{(j)} = \mathbf{dt}(j) \cdot \begin{bmatrix} \mathbf{Ca}^{(j)}_{012} & \mathbf{Ca}^{(j)}_{345} \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}^T, \ \mathbf{k}^{u,1} = \begin{cases} k^{(r)} & \text{for } r = \text{DSP sequence} \\ 0 & \text{otherwise} \end{cases} \tag{62}$$

The ZMP_2 trajectory is:

$$\begin{aligned} x_{\text{ZMP}_2} &= \mathbf{A}_{x_{\text{ZMP}_2}} X + B_{x_{\text{ZMP}_2}} \\ &= \mathbf{A}_{x_{\text{ZMP}_1}} X + B_{x_{\text{ZMP}_1}} \quad -(\text{diag}(\mathbf{k}))^{-1} \cdot ((\mathbf{A}_{x_{\text{ZMP}_1}} - \mathbf{A}_{x_{\text{ZMP}}})X + (B_{x_{\text{ZMP}_1}} - B_{x_{\text{ZMP}}})) = 0 \end{aligned} \tag{63}$$

Thus

$$\begin{aligned} \mathbf{A}_{x_{\text{ZMP}_2}} &= \mathbf{A}_{x_{\text{ZMP}_1}} - (\text{diag}(\mathbf{k}))^{-1} \cdot (\mathbf{A}_{x_{\text{ZMP}_1}} - \mathbf{A}_{x_{\text{ZMP}}}) \\ B_{x_{\text{ZMP}_2}} &= \mathbf{B}_{x_{\text{ZMP}_1}} - (\text{diag}(\mathbf{k}))^{-1} \cdot (\mathbf{B}_{x_{\text{ZMP}_1}} - \mathbf{B}_{x_{\text{ZMP}}}) \end{aligned} \tag{64}$$

F. Cost concatenation and gradients

The concatenation of *cost* (13) is written:

$$\text{c}_{\text{wpg}} = \frac{1}{2} \begin{bmatrix} X^x \\ X^y \end{bmatrix}^T \begin{bmatrix} H^x & 0 \\ 0 & H^y \end{bmatrix} \begin{bmatrix} X^x \\ X^y \end{bmatrix} + \begin{bmatrix} G^x & G^y \end{bmatrix} \begin{bmatrix} X^x \\ X^y \end{bmatrix} + \frac{1}{2}(C^x + C^y) \tag{65}$$

From (65) and (15), the gradient of the WPG cost is:

$$\frac{\partial \text{c}_{\text{wpg}}}{\partial X} = \begin{bmatrix} X^x \\ X^y \end{bmatrix}^T \begin{bmatrix} H^x & 0 \\ 0 & H^y \end{bmatrix} + \begin{bmatrix} G^x & G^y \end{bmatrix} \tag{66}$$

From (18), the gradient of ZMP forces is:

$$\frac{\partial \mathbf{f}_{ZMP}}{\partial X} = -\frac{\partial \mathbf{f}_{COM}}{\partial X} = -\begin{bmatrix} \frac{mg}{z_{\text{COM}}} (\mathbf{A}_{x_{\text{COM}}} - \mathbf{A}_{x_{\text{ZMP}}}) & 0 \\ 0 & \frac{mg}{z_{\text{COM}}} (\mathbf{A}_{y_{\text{COM}}} - \mathbf{A}_{y_{\text{ZMP}}}) \end{bmatrix} \tag{67}$$

From (30), (59) and (63), the gradients of ZMP, ZMP_1 and ZMP_2 are:

$$\frac{\partial \mathbf{P}_{\text{ZMP}}}{\partial X} = \begin{bmatrix} \mathbf{A}_{x_{\text{ZMP}}} & 0 \\ 0 & \mathbf{A}_{y_{\text{ZMP}}} \end{bmatrix}, \frac{\partial \mathbf{P}_{\text{ZMP}_1}}{\partial X} = \begin{bmatrix} \mathbf{A}_{x_{\text{ZMP}_1}} & 0 \\ 0 & \mathbf{A}_{y_{\text{ZMP}_1}} \end{bmatrix}, \frac{\partial \mathbf{P}_{\text{ZMP}_2}}{\partial X} = \begin{bmatrix} \mathbf{A}_{x_{\text{ZMP}_2}} & 0 \\ 0 & \mathbf{A}_{y_{\text{ZMP}_2}} \end{bmatrix} \tag{68}$$