Theorem (2.2.4.4)

Let $A \in \mathbb{C}^{m \times m}$. If $\|Ax\|_2 = \|x\|_2$, then A is a unitary matrix.

Solution

This alternative proof of the theorem is based partly on the partial proof on the dedicated course site and partly on a proof from this blog whose notation is a bit different than ours.

The proof is in three parts where the second part is also divided into three. The third part simply uses the results from the first two to conclude the proof.

Part 1:

Assume $x = e_i$ where e_i is a standard basis vector in $\mathbb{C}^m$. Then

$$\begin{aligned}
\|Ax\|_2 &= \|x\|_2 && < \text{condition} > \\
\|Ax\|_2^2 &= \|x\|_2^2 && < \text{algebra} > \\
\|Ae_i\|_2^2 &= \|e_i\|_2^2 && < \text{assume } x = e_i > \\
(Ae_i)^H Ae_i &= 1 && < \text{property of vector } 2-\text{norm, } \|e_i\|_2 = 1 > \\
e_i^H A^H Ae_i &= 1 && < \text{property of Hermitian form} > \\
(a_i^H)a_i &= 1. && < \text{matrix} - \text{vector multiplication} >
\end{aligned}$$

Part 2.a:

Assume $x = e_j + e_k$ where e_j, e_k are standard basis vectors in $\mathbb{C}^m$ and $j \neq k$. Then

$$\begin{aligned}
\|Ax\|_2 &= \|x\|_2 && < \text{condition} > \\
\|Ax\|_2^2 &= \|x\|_2^2 && < \text{algebra} > \\
\|A(e_j + e_k)\|_2^2 &= \|e_j + e_k\|_2^2 && < \text{assume } x = e_j + e_k > \\
\|Ae_j + Ae_k\|_2^2 &= \|e_j + e_k\|_2^2 && < \text{matrix} - \text{matrix mult. distributive} > \\
(Ae_j + Ae_k)^H(Ae_j + Ae_k) &= (e_j + e_k)^H(e_j + e_k) && < \text{vector } 2 - \text{norm property} > \\
((Ae_j)^H + (Ae_k)^H)(Ae_j + Ae_k) &= (e_j^H + e_k^H)(e_j + e_k) && < \text{Hermitian form property} > \\
(a_j^H + a_k^H)(a_j + a_k) &= (e_j^H + e_k^H)(e_j + e_k) && < Ae_j = a_j > \\
a_j^H a_j + a_j^H a_k + a_k^H a_j + a_k^H a_k &= e_j^H e_j + e_j^H e_k + e_k^H e_j + e_k^H e_k && < \text{algebra} > \\
1 + a_j^H a_k + a_k^H a_j + 1 &= 1 + 0 + 0 + 1 && < a_j^H a_j = 1 \text{ by 1st part of proof, } e_j \perp e_k > \\
a_j^H a_k + a_k^H a_j &= 0 && < \text{algebra} > \\
a_j^H a_k + \overline{(a_j^H a_k)} &= 0 && < \overline{x^H y} = y^H x > \\
2\mathfrak{R}(a_j^H a_k) &= 0 && < x + \bar{x} = 2\mathfrak{R}(x) > \\
\mathfrak{R}(a_j^H a_k) &= 0 && < \text{algebra} >
\end{aligned}$$

where $\mathfrak{R}()$ is the real part of a complex number.

Part 2.b:

Assume $x = e_j + ie_k$ where e_j, e_k are standard basis vector in $\mathbb{C}^m$ and $j \neq k$.

$$
\begin{array}{rl}
\|Ax\|_2 = \|x\|_2 & < \text{condition} > \\
\|Ax\|_2^2 = \|x\|_2^2 & < \text{algebra} > \\
\|A(e_j + ie_k)\|_2^2 = \|e_j + ie_k\|_2^2 & < \text{assume } x = e_j + ie_k > \\
\|Ae_j + iAe_k\|_2^2 = \|e_j + ie_k\|_2^2 & < \text{matrix} - \text{matrix mult. distributive} > \\
((Ae_j)^H + (iAe_k)^H)(Ae_j + iAe_k) = (e_j + ie_k)^H(e_j + ie_k) & < \text{vector } 2 - \text{norm property} > \\
(a_j^H - ia_k^H)(a_j + ia_k) = (e_j^H - ie_k^H)(e_j + ie_k) & < \text{Hermitian form property, } Ae_j = a_j, i^H = -i > \\
a_j^H a_j + ia_j^H a_k - ia_k^H a_j - (ia_k^H)(ia_k) = e_j^H e_j + e_j^H ie_k - ie_k^H e_j - ie_k^H(ie_k) & < \text{algebra} > \\
1 + ia_j^H a_k - ia_k^H a_j + 1 = 1 + 0 + 0 + 1 & < a_j^H a_j = 1 \text{ by 1st part of proof, } e_j \perp e_k > \\
ia_j^H a_k - ia_k^H a_j = 0 & < \text{algebra} > \\
ia_j^H a_k - i\overline{(a_j^H a_k)} = 0 & < \overline{x^H y} = y^H x > \\
i(a_j^H a_k - \overline{a_j^H a_k}) = 0 & < \text{algebra} > \\
a_j^H a_k - \overline{a_j^H a_k} = 0 & < \text{algebra} > \\
2\mathfrak{I}(a_j^H a_k) = 0 & < x - \bar{x} = 2\mathfrak{I}(x) > \\
\mathfrak{I}(a_j^H a_k) = 0 & < \text{algebra} >
\end{array}
$$

where $\mathfrak{I}()$ is the imaginary part of a complex number.

Part 2.c:

By Part 2.a and 2.b, $\mathfrak{R}(a_j^H a_k) + \mathfrak{I}(a_j^H a_k) = 0$ when $j \neq k$, which implies

$$a_j^H a_k = 0$$

when $j \neq k$.

Part 3:

By Part 1 and Part 2.c,

$$a_j^H a_k = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{otherwise} \end{cases}$$

which implies

$$A^H A = I$$

and therefore A is a unitary matrix.