1.6 Additional homework

Exercise (1.6.1.1.)

Assume $e_j \in \mathbb{R}^n$. Find the 1-, 2-, infinity- and p-norms of e_j.

Solution

$$\|e_j\|_p = \sqrt[p]{|0|^p + \cdots + |0|^p + |1|^p + |0|^p + \cdots + |0|^p} = \sqrt[p]{|1|^p} = 1.$$

Therefore

$$\|e_j\|_1 = \|e_j\|_2 = \|e_j\|_\infty = \|e_j\|_p = 1.$$

Exercise (1.6.1.2.)

Find $\|I\|_1$, $\|I\|_2$, $\|I\|_\infty$, $\|I\|_p$,and $\|I\|_F$.

Solution

We had shown in Exercise (1.4.1.1.) that the induced matrix norm for an identity matrix is 1. Therefore $\|I\|_1 = \|I\|_2 = \|I\|_\infty = \|I\|_p = 1$.

By Exercise (1.3.3.3.) and Exercise (1.6.1.1.),

$$\|A\|_F = \sqrt{\sum_{j=0}^{n-1} \|a_j\|_2^2} = \sqrt{\sum_{j=0}^{n-1} \|e_j\|_2^2} = \sqrt{\sum_{j=0}^{n-1} (1)^2} = \sqrt{n}.$$

Exercise (1.6.1.3.)

Find $\|D\|_1$, $\|D\|_\infty$, $\|D\|_p$,and $\|D\|_F$ where $D \in \mathbb{C}^{n\times n}$ is a diagonal matrix, that is

$$D = \begin{pmatrix} \delta_0 & 0 & \cdots & 0 \\ 0 & \delta_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \delta_{n-1} \end{pmatrix}.$$

Solution

By Exercise (1.3.6.1.), $\|D\|_1 = \max_{j=0}^{n-1} \|d_j\|_1 = \max_{j=0}^{n-1} |\delta_j|$.

By Exercise (1.3.6.2.), $\|D\|_\infty = \max_{j=0}^{n-1} \|\tilde{d}_j\|_1 = \max_{j=0}^{n-1} |\delta_j|$.

Inspired from the above two solutions and utilizing the proof in Exercise (1.3.5.1.), we conjecture

$$\|D\|_p = \max_{j=0}^{n-1} |\delta_j|.$$

Using the method outlined in Remark (1.3.5.4.) , $f(x) = \|Dx\|_p$ and $\alpha = \max_{i=0}^{n-1} |\delta_i|$.

For the first part, we prove max $f(x) \leq \alpha$,

$$\begin{aligned}
\|D\|_p &= \max_{\|x\|_p=1} \sqrt[p]{\|Dx\|_p^p} && < \text{definition of matrix p} - \text{norm} > \\
\|D\|_p^p &= \max_{\|x\|_p=1} \|Dx\|_p^p && < \text{algebra} > \\
&= \max_{\|x\|_p=1} \left\| \begin{pmatrix} \delta_0 \chi_0 \\ \vdots \\ \delta_{n-1}\chi_{n-1} \end{pmatrix} \right\|_p^p && < \text{matrix} - \text{vector multiplication} > \\
&= \max_{\|x\|_p=1} \left(\sum_{i=0}^{n-1} |\delta_i \chi_i|^p \right) && < \text{definition of vector p} - \text{norm, algebra} > \\
&= \max_{\|x\|_p=1} \left(\sum_{i=0}^{n-1} |\delta_i|^p |\chi_i|^p \right) && < \text{algebra} > \\
&\leq \max_{\|x\|_p=1} \left(\sum_{i=0}^{n-1} \left(\max_{j=0}^{n-1} |\delta_i| \right)^p |\chi_i|^p \right) && < \text{algebra} > \\
&= \left(\max_{j=0}^{n-1} |\delta_j| \right)^p \left(\max_{\|x\|_p=1} \sum_{i=0}^{n-1} |\chi_i|^p \right) && < \text{algebra} > \\
&= \left(\max_{j=0}^{n-1} |\delta_j| \right)^p (1) && < \|x\|_p = 1 > \\
\|D\|_p^p &\leq \left(\max_{j=0}^{n-1} |\delta_j| \right)^p && < \text{algebra} > \\
\|D\|_p &\leq \max_{j=0}^{n-1} |\delta_j|. && < \text{algebra} >
\end{aligned}$$

For part 2 (max $f(x) \geq \alpha$), assume that J is the index of the diagonal element δ_J s.t.

$$|\delta_J| = \max_{i=0}^{n-1} |\delta_i|$$

and $e_J \in \mathbb{C}^n$ is a standard basis vector.

Then

$$
\begin{aligned}
\|D\|_p &= \max_{\|x\|_p=1} \|Dx\|_p && < \text{definition of matrix p} - \text{norm} > \\
&\geq \|De_J\|_p && < e_J \text{ specific vector} > \\
&= \|\delta_J e_J\|_p && < \text{ matrix} - \text{vector multiplication} > \\
&= |\delta_J| \|e_J\|_p && < \delta_J \text{ scalar, property of norm} > \\
&= |\delta_J| (1) && < e_J \text{ standard basis vector} > \\
\|D\|_p &\geq \max_{i=0}^{n-1} |\delta_i|. && < \text{definition of } \delta_j >
\end{aligned}
$$

Because $\|D\|_p \geq \max_{i=0}^{n-1} |\delta_i|$ and $\|D\|_p \leq \max_{i=0}^{n-1} |\delta_i|$,

$$\|D\|_p = \max_{i=0}^{n-1} |\delta_i|$$

must be true.

Exercise (1.6.1.4.)

Assume we partition the vector $x \in \mathbb{C}^m$ as $x = \begin{pmatrix} x_0 \\ -- \\ \vdots \\ -- \\ x_k \\ -- \\ \vdots \\ -- \\ x_{M-1} \end{pmatrix}$. Then $\|x_k\|_p \leq \|x\|_p$ where $0 \leq k \leq M - 1$, $M \leq m$, and $1 \leq p \leq \infty$.

Solution

Assume x_k is of size ($k_n - k_0 + 1$), i.e. $x_k = \begin{pmatrix} \chi_{k_0} & \cdots & \chi_{k_n} \end{pmatrix}^T$ and
$0 = 0_0 \leq 0_n < 1_0 \leq 1_1 < \ldots < k_0 \leq k_n < \ldots < (M-1)_0 \leq (M-1)_n = m$ - 1.

In addition,

$$\|x_k\|_p^p = |\chi_{k_0}|^p + \cdots + |\chi_{k_n}|^p = \sum_{j=k_0}^{k_n} |\chi_j|^p.$$

Therefore

$$\|x\|_p = \sqrt[p]{\sum_{i=0}^{m-1} |\chi_i|^p} \qquad < \text{definition of vector p} - \text{norm} >$$

$$\|x\|_p^p = \sum_{i=0}^{m-1} |\chi_i|^p \qquad < \text{algebra} >$$

$$= \sum_{i=0}^{M-1} \sum_{j=i_0}^{i_n} |\chi_j|^p \qquad < \text{algebra} >$$

$$= \sum_{i=0}^{M-1} \|x_i\|_p^p \qquad < \text{definition of } x_i \text{ and vector p} - \text{norm} >$$

$$\|x\|_p^p \geq \|x_k\|_p^p \qquad < \text{algebra} >$$

$$\|x\|_p \geq \|x_k\|_p. \qquad < \text{algebra} >$$

Exercise (1.6.1.5.)

Suppose $A = \begin{pmatrix} 1 & 2 & -1 \\ -1 & 1 & 0 \end{pmatrix}$. Compute $\|A\|_1$, $\|A\|_\infty$, and $\|A\|_F$.

Solution

$\|A\|_1 = 3$, $\|A\|_\infty = 4$, $\|A\|_F = 2\sqrt{2}$.

Exercise (1.6.1.6.)

Suppose $\|\cdot\|: \mathbb{C}^{m \times n} \longrightarrow \mathbb{R}$ is a matrix norm such that $\|A\| = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} |\alpha_{i,j}|$. Prove that $\|\cdot\|$ is a norm and relate it to the vector 1-norm.

Show that $\|A\| = \|A^H\|$ and $\|\cdot\|$ is submultiplicative.

Solution

a) $\|\cdot\|$ is a matrix norm.

$A \neq 0$ implies there is at least one element of A, $\alpha_{p,q} \neq 0$. Then $\|A\| \neq 0$.

$$\|\beta A\| = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} |\beta \alpha_{i,j}| = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} |\beta| |\alpha_{i,j}| = |\beta| \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} |\alpha_{i,j}| = |\beta| \|A\|.$$

Suppose $B \in \mathbb{C}^{m \times n}$.

$$\|A + B\| = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} |\alpha_{i,j} + \beta_{i,j}| \leq \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} (|\alpha_{i,j}| + |\beta_{i,j}|) = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} |\alpha_{i,j}| + \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} |\beta_{i,j}| = \|A\| + \|B\|.$$

Therefore $\|\cdot\|$ is a matrix norm.

b) ||·|| is related to the vector 1-norm.

We can express ||*A*|| as a summation of vector 1-norms of rows or columns of *A*:

$$\|A\| = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} |\alpha_{i,j}| = \sum_{i=0}^{m-1}\left(\sum_{j=0}^{n-1} |\alpha_{i,j}|\right) = \sum_{i=0}^{m-1} \|\tilde{a}_i\|_1.$$

$$\|A\| = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} |\alpha_{i,j}| = \sum_{j=0}^{n-1}\sum_{i=0}^{m-1} |\alpha_{i,j}| = \sum_{j=0}^{n-1}\left(\sum_{i=0}^{m-1} |\alpha_{i,j}|\right) = \sum_{i=0}^{m-1} \|a_j\|_1.$$

c) ||*A*|| = ||A^H||.

If $A = \begin{pmatrix} a_0 & | & \cdots & | & a_{n-1}\end{pmatrix}$, then $A^H = \begin{pmatrix} \bar{a}_0^T \\ -- \\ \vdots \\ -- \\ \bar{a}_{n-1}^T \end{pmatrix}$.

Then $\|A\| = \sum_{i=0}^{n-1} \|a_j\|_1 = \sum_{i=0}^{n-1} \|\bar{a}_j\|_1 = \|A^H\|$.

d) ||·|| is submultiplicative.

Because it is defined using the same formula for all *m* and *n*, ||.|| is consistent.

Suppose $A \in \mathbb{C}^{m\times k}$, $B \in \mathbb{C}^{k\times n}$. Then

$$
\begin{aligned}
\|AB\| &= \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \left| \tilde{a}_i^T b_j \right| && < \text{slice \& dice} > \\
&= \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \left| \sum_{p=0}^{k-1} \alpha_{i,p} \beta_{p,j} \right| && < \text{matrix} - \text{vector multiplication} > \\
&\le \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \sum_{p=0}^{k-1} \left| \alpha_{i,p} \beta_{p,j} \right| && < \text{absolute value obeys triangle inequality} > \\
&= \sum_{p=0}^{k-1} \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} |\alpha_{i,p}| |\beta_{p,j}| && < |\alpha\beta| = |\alpha||\beta|, \text{algebra} > \\
&= \sum_{p=0}^{k-1} \sum_{i=0}^{m-1} \left(|\alpha_{i,p}| \sum_{j=0}^{n-1} |\beta_{p,j}| \right) && < \text{algebra} > \\
&\le \left(\sum_{p=0}^{k-1} \sum_{i=0}^{m-1} |\alpha_{i,p}| \right) \left(\sum_{p=0}^{k-1} \sum_{j=0}^{n-1} |\beta_{p,j}| \right) && < \text{algebra} > \\
\|AB\| &\le \|A\| \|B\|. && < \text{definition of } \|.\| >
\end{aligned}
$$

Alternative proof (@letslearnmath):

$$
\begin{aligned}
\|AB\| &= \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \left| \tilde{a}_i^T b_j \right| && < \text{slice \& dice} > \\
&= \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \left| \tilde{a}_i^H b_j \right| && < |x^T y| = |x^H y| > \\
&\le \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \|\tilde{a}_i\|_2 \|b_j\|_2 && < \text{Cauchy} - \text{Schwartz inequality} > \\
&\le \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \|\tilde{a}_i\|_1 \|b_j\|_1 && < \|x\|_2 \le \|x\|_1 > \\
&= \left(\sum_{i=0}^{m-1} \|\tilde{a}_i\|_1 \right) \sum_{j=0}^{n-1} \|b_j\|_1 && < \text{algebra} > \\
\|AB\| &\le \|A\| \|B\|. && < \text{definition of } \|.\| >
\end{aligned}
$$

Exercise (1.6.1.7.)

Prove $\|A\|_F = \|A^T\|_F$, and $\|A\|_F = \sqrt{\sum_{j=0}^{n-1} \|a_j\|_2^2} = \sqrt{\sum_{j=0}^{m-1} \|\tilde{a}_j\|_2^2}$.

Solution

$\|A\|_F = \sqrt{\sum_{j=0}^{n-1} \|a_j\|_2^2} = \sqrt{\sum_{j=0}^{m-1} \|\tilde{a}_j\|_2^2}$ have been proven in Exercises (1.3.3.1.) and (1.3.3.3.), respectively.

If $A = \begin{pmatrix} a_0 & | & \cdots & | & a_{n-1} \end{pmatrix}$, then $A^T = \begin{pmatrix} a_0^T \\ -- \\ \vdots \\ -- \\ a_{n-1}^T \end{pmatrix}$. Therefore $\|A\|_F^2 = \sum_{i=0}^{n-1} \|a_j\|_2^2 = \|A^T\|_F^2$.

Exercise (1.6.1.8.)

Prove $\|x\|_1 = \|x\|_2 = 1$ if and only if $x = \pm e_j$.

Solution (in the course notes)

We confirm that $\|\pm e_j\|_1 = \|\pm e_j\|_2 = 1$.

Suppose $\|x\|_1 = 1$ but $x \neq \pm e_j$. Therefore $|\chi_j| \leq 1$ for j in $\{0, \ldots, n\text{-}1\}$.

Then $\|x\|_2 = \sqrt{|\chi_0|^2 + \cdots + |\chi_{n-1}|^2} < \sqrt{|\chi_0| + \cdots + |\chi_{n-1}|} = \sqrt{\|x\|_1} = 1$. As this contradicts the conditions, $x = \pm e_j$ is the only possibility.

Exercise (1.6.1.9.)

Prove that if $\|x\|_\nu \leq \beta \, \|x\|_{\mu}1$, then $\|A\|_\nu \leq \beta \, \|A\|_{\nu,\mu}$.

Solution

$$\begin{aligned}
\|A\|_\nu &= \frac{\|Ax\|_\nu}{\|x\|_\nu} && < \text{induced matrix} - \text{norm definition} > \\
&\leq \frac{\beta \|Ax\|_\mu}{\|x\|_\nu} && < \text{the given vector norm equivalence} > \\
\|A\|_\nu &\leq \beta \frac{\|Ax\|_\mu}{\|x\|_\nu}. && < \text{induced matrix} - \text{norm definition} >
\end{aligned}$$