2.5.1. Additional homework

Homework 2.5.1.1.

Prove $U \in \mathbb{C}^{m \times m}$ is unitary if and only if $(Ux)^H(Uy) = x^Hy$.

Solution

***U* unitary ⇒ $(Ux)^H(Uy) = x^Hy$:**

$$\begin{aligned} (Ux)^H(Uy) &= x^H U^H U y & &< \text{Hermitian form property} > \\ &= x^H (I) y & &< \text{unitary matrix property} > \\ &= x^H y\,. & &< \text{identity matrix property} > \end{aligned}$$

$(Ux)^H(Uy) = x^Hy$ ⇒ *U* unitary:

$$\begin{aligned} x^H y &= (Ux)^H U y & &< \text{statement} > \\ x^H y &= x^H U^H U y & &< \text{Hermitian from property} > \\ e_i^H e_j &= e_i^H U^H U e_j & &< \text{substitute } x = e_i, y = e_j > \\ e_i^H e_j &= (U^H U)_{i,j} & &< \text{standard basis vector property} > \\ (U^H U)_{i,j} &= e_i^H e_j & &< \text{symmetric property of equality} > \\ (U^H U)_{i,j} &= \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases} & &< \text{standard basis vector property} > \\ U^H U &= I\,. & &< \text{identity matrix definition} > \end{aligned}$$

As any square matrix which yields an identity matrix when multiplied with its Hermitian form is a unitary matrix, *U* is a unitary matrix.

Homework 2.5.1.2.

Suppose $A, B \in \mathbb{C}^{m \times n}$ whereas $U \in \mathbb{C}^{m \times m}$ and $V \in \mathbb{C}^{n \times n}$ are unitary. Prove $UAV^H = B$ if and only if $U^HBV = A$.

Solution

$UAV^H = B \Rightarrow U^HBV = A$:

$$\begin{aligned} UAV^H &= B & &< \text{statement} > \\ U^H UAV^H V &= U^H BV & &< \text{matrix mult. preserves equality} > \\ (I)A(I) &= U^H BV & &< \text{unitary matrix property} > \\ A &= U^H BV\,. & &< \text{identity matrix property} > \end{aligned}$$

$U^HBV = A \Rightarrow UAV^H = B$:

$$\begin{aligned} U^H B V &= A && \text{< statement >} \\ U U^H B V V^H &= U A V^H && \text{< matrix mult. preserves equality >} \\ (I) B (I) &= U A V^H && \text{< unitary matrix property >} \\ B &= U A V^H. && \text{< identity matrix property >} \end{aligned}$$

Homework 2.5.1.3.

Prove that nonsingular $A \in \mathbb{C}^{n \times n}$ has the condition number $\kappa_2(A) = 1$ if and only if $A = \sigma Q$ where Q is unitary and $\sigma > 0$.

Solution

$A = \sigma Q \Rightarrow \kappa_2(A) = 1$:

$$\begin{aligned} \kappa_2(A) &= \|A\|_2 \|A^{-1}\|_2 && \text{< definition of condition number >} \\ &= \|\sigma Q\|_2 \|(\sigma Q)^{-1}\|_2 && \text{< definiton of } A \text{ >} \\ &= \|\sigma Q\|_2 \left\| \frac{Q^{-1}}{\sigma} \right\|_2 && \text{< matrix inversion property >} \\ &= \sigma \|Q\|_2 \frac{1}{\sigma} \left\| Q^{-1} \right\|_2 && \text{< } \sigma \text{ scalar, induced matrix} - \text{norm property >} \\ &= \sigma (1) \frac{1}{\sigma} (1) && \text{< unitary matrix and matrix 2} - \text{norm property >} \\ &= 1. && \text{< algebra >} \end{aligned}$$

$\kappa_2(A) = 1 \Rightarrow A = \sigma Q$:

$$\begin{aligned} 1 &= \kappa_2(A) && \text{< condition >} \\ &= \|A\|_2 \|A^{-1}\|_2 && \text{< definition of condition number >} \\ &= \|U \Sigma V^H\|_2 \|(U \Sigma V^H)^{-1}\|_2 && \text{< SVD of } A = U \Sigma V^H \text{ >} \\ &= \|U \Sigma V^H\|_2 \|V^H \Sigma^{-1} U\|_2 && \text{< matrix inversion \& unitary matrix properties >} \\ &= \|\Sigma\|_2 \|\Sigma^{-1}\|_2 && \text{< } U, V \text{ unitary matrices, Homework 2.2.4.9 >} \\ 1 &= \sigma_0 \left(\frac{1}{\sigma_{n-1}} \right) && \text{< } |D\|_2 = \max_{i=0}^{m-1} |\delta_i|, \Sigma = \operatorname{diag}(\sigma_0, \ldots, \sigma_{n-1}), \\ & && \quad \Sigma^{-1} = \operatorname{diag}(1/\sigma_0, \ldots, 1/\sigma_{n-1}), \sigma_0 \geq \ldots \geq \sigma_{n-1} \text{ by SVD >} \\ \sigma_0 = \sigma_{n-1} &= \sigma > 0. && \text{< algebra, assume } \sigma = \sigma_0, A \text{ nonsingular >} \end{aligned}$$

Then

$\Sigma = \sigma I.$ < $\sigma_0 \geq \ldots \geq \sigma_{n-1}$ by SVD and $\sigma = \sigma_0 = \sigma_{n-1}$ by the proof above imply $\sigma = \sigma_0 = \ldots = \sigma_{n-1}$ and hence $\Sigma = \mathrm{diag}(\sigma, \ldots, \sigma)$ > (1)

This implies

$$\begin{array}{rcll} A & = & U\Sigma V^H & \text{< SVD of } A \text{ >} \\ & = & U(\sigma I)V^H & \text{< substitute equation (1) >} \\ & = & \sigma U V^H & \text{< } \sigma \text{ scalar, identity matrix property >} \\ & = & \sigma Q. & \text{< } Q = UV^H \text{ unitary because } U, V \text{ unitary >} \end{array}$$

Homework 2.5.1.4.

Suppose $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ are unitary. Then prove that the matrix $\begin{pmatrix} U & | & 0 \\ - & - & - \\ 0 & | & V \end{pmatrix}$ is unitary.

Solution

Matrix $\begin{pmatrix} U & | & 0 \\ - & - & - \\ 0 & | & V \end{pmatrix}$ is a square matrix of $(m+n)\times(m+n)$.

$$\begin{aligned} \begin{pmatrix} U & | & 0 \\ - & - & - \\ 0 & | & V \end{pmatrix}^H \begin{pmatrix} U & | & 0 \\ - & - & - \\ 0 & | & V \end{pmatrix} &= \begin{pmatrix} U^H & | & 0 \\ - & - & - \\ 0 & | & V^H \end{pmatrix} \begin{pmatrix} U & | & 0 \\ - & - & - \\ 0 & | & V \end{pmatrix} && \text{< block matrix transpose >} \\ &= \begin{pmatrix} U^H U & | & 0 \\ - & - & - \\ 0 & | & V^H V \end{pmatrix} && \text{< block matrix multiplication >} \\ &= \begin{pmatrix} I & | & 0 \\ - & - & - \\ 0 & | & I \end{pmatrix} && \text{< unitary matrix property >} \\ &= I. && \text{< de} - \text{partitioning >} \end{aligned}$$

Because any square matrix Q where $Q^H Q = I$ is unitary, matrix $\begin{pmatrix} U & | & 0 \\ - & - & - \\ 0 & | & V \end{pmatrix}$ is unitary.

Homework 2.5.1.5.

$A \in \mathbb{R}^{m \times m}$ is a stochastic matrix if and only if all of its entries are nonnegative and each of its columns add up to 1. Show that a matrix is both unitary and stochastic if and only if it is a permutation matrix.

Solution

A permutation matrix P is a specific rearrangement of the rows (or columns) of an identity matrix without repetition. A permutation matrix has the property $P^T = P^{-1}$.

Unitary and stochastic matrix ⇒ permutation matrix:

Suppose $A = (a_0 \,|\, \ldots \,|\, a_{m-1}) = \begin{pmatrix} \alpha_{0,0} & \cdots & \alpha_{0,m-1} \\ \vdots & \ddots & \vdots \\ \alpha_{m-1,0} & \cdots & \alpha_{m-1,m-1} \end{pmatrix}$.

If A is a stochastic matrix, then $\alpha_{i,j} \geq 0$, where $i, j = \{0, \ldots, m\text{-}1\}$, and $\sum_{i=0}^{m-1} \alpha_{i,j} = 1$ where $j \in \{0, \ldots, m\text{-}1\}$.

If A is unitary, then because it is in $\mathbb{R}^{m \times m}$, it is also orthogonal. Then $a_i^T a_j = \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases}$ which implies $\sum_{i=0}^{m-1} \alpha_{i,j}^2 = 1$ where $j \in \{0, \ldots, m\text{-}1\}$.

If $\sum_{i=0}^{m-1} \alpha_{i,j} = 1$ and $\alpha_{i,j} \geq 0$, then $1 \geq \alpha_{i,j} \geq 0$ for $i, j \in \{0, \ldots, m\text{-}1\}$. However, both $\sum_{i=0}^{m-1} \alpha_{i,j} = 1$ and $\sum_{i=0}^{m-1} \alpha_{i,j}^2 = 1$ are possible at the same time if and only if only a single element of every a_j is equal to 1 and the rest are zero given $1 \geq \alpha_{i,j} \geq 0$. If two or more elements of a_j are non-zero, then either $\sum_{i=0}^{m-1} \alpha_{i,j} \neq 1$ or $\sum_{i=0}^{m-1} \alpha_{i,j}^2 \neq 1$. Therefore all a_j are standard basis vectors in $\mathbb{R}^m$ where $j \in \{0, \ldots, m\text{-}1\}$. However because $a_i^T a_j = 0$ when $i \neq j$, all a_j need to be different. Then $a_j = e_j$ where $j \in \{0, \ldots, m\text{-}1\}$.

If all columns of A are different standard basis vectors and A is a square matrix, then A is a permutation matrix.

Permutation matrix ⇒ unitary and stochastic matrix:

Because a permutation matrix P has the property $P^T = P^{-1}$, $A^T = A^{-1}$. Then because $A^TA = I$, A is a unitary matrix. Because all elements of A is greater or equal to zero and $\sum_{i=0}^{m-1} \alpha_{i,j} = 1$ where $j \in \{0, \ldots, m-1\}$, A is also a stochastic matrix.

Homework 2.5.1.6.

Show that if $\|\cdots\|$ is a norm and A is nonsingular, then $\|\cdots\|_{A^{-1}}$ defined as $\|x\|_{A^{-1}} = \|A^{-1}x\|$ is a norm.

Solution

Positive definite:

$$
\begin{array}{rcll}
x & \neq & 0 & \text{< statement >} \\
A^{-1}x & \neq & 0 & \text{< } \mathcal{N}(A) = 0 \text{ >} \\
\|A^{-1}x\| & \neq & 0. & \text{< } \|\cdots\| \text{ is positive definite >}
\end{array}
$$

Homogeneous:

$$
\begin{array}{rcll}
\|\alpha x\|_{A^{-1}} & = & \|A^{-1}\alpha x\| & \text{< definition of } \|\cdots\|_{A^{-1}} \text{ >} \\
 & = & |\alpha|\ \|A^{-1}x\| & \text{< homogeneity property of } \|\cdots\| \text{ >} \\
 & = & |\alpha|\ \|x\|_{A^{-1}}. & \text{< definition of } \|\cdots\|_{A^{-1}} \text{ >}
\end{array}
$$

Triangle inequality:

$$
\begin{array}{rcll}
\|x+y\|_{A^{-1}} & = & \|A^{-1}(x+y)\| & \text{< definition of } \|\cdots\|_{A^{-1}} \text{ >} \\
 & \leq & \|A^{-1}x\| + \|A^{-1}y\| & \text{< } \|\cdots\| \text{ obeys triangle inequality >} \\
 & \leq & \|x\|_{A^{-1}} + \|y\|_{A^{-1}}. & \text{< definition of } \|\cdots\|_{A^{-1}} \text{ >}
\end{array}
$$

$\|\cdots\|_{A^{-1}}$ is a norm because it meets all three conditions for being a norm.

Homework 2.5.1.7.

Let $A \in \mathbb{C}^{m\times m}$ and $A = U\Sigma V^H$ be its SVD where $\Sigma = \begin{pmatrix} \sigma_0 & 0 & \cdots & 0 \\ 0 & \sigma_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_{m-1} \end{pmatrix}$.

Find which of the following give the condition number of A, $\kappa_2(A)$:

a) $\|A\|_2\ \|A^{-1}\|_2$,

b) σ_0 / σ_{m-1},

c) $u_0^H A v_0 / u_{m-1}^H A v_{m-1}$,

d) $\dfrac{\max\limits_{\|x\|_2=1} \|Ax\|_2}{\min\limits_{\|x\|_2=1} \|Ax\|_2}$.

Solution

All four give the condition number of A.

a) Yes, definition of the condition number.

b) Yes,

$$
\begin{aligned}
\kappa_2(A) &= \|A\|_2(\|A^{-1}\|_2) && < \text{definition of condition number} > \\
&= \|U\Sigma V^H\|_2 \|(U\Sigma V^H)^{-1}\|_2 && < \text{SVD of } A > \\
&= \|\Sigma\|_2 \|V\Sigma^{-1}U^H\|_2 && < U, V \text{ unitary, property of inversion} > \\
&= \|\Sigma\|_2 \|\Sigma^{-1}\|_2 && < U, V \text{ unitary} > \\
&= \sigma_0 \frac{1}{\sigma_{m-1}}. && < \|D\|_2 = \max_{i=0}^{m-1} |\delta_i|, \Sigma^{-1} = \text{diag}(1/\sigma_0, \cdots, 1/\sigma_{m-1)} >
\end{aligned}
$$

c) Yes, because

$$
\begin{aligned}
\frac{u_0^H A v_0}{u_{m-1}^H A v_{m-1}} &= \frac{u_0^H (U\Sigma V^H) v_0}{u_{m-1}^H (U\Sigma V^H) v_{m-1}} && < \mathrm{A = U\Sigma V^H} > \\
&= \frac{(u_0^H U)\Sigma(V^H v_0)}{(u_{m-1}^H U)\Sigma(V^H v_{m-1})} && < \text{associativity of matrix mult.} > \\
&= \frac{(e_0^T)\Sigma(e_0)}{(e_{m-1}^T)\Sigma(e_{m-1})} && < \text{matrix} - \text{matrix mult.,} \\
& && \quad \text{def. of stand. basis vector} > \\
&= \frac{\sigma_0}{\sigma_{m-1}} && < \text{property of standard basis vector} > \\
&= \kappa_2(A). && < \text{see part b) of this homework} >
\end{aligned}
$$

d) Yes, because

$$
\begin{aligned}
\frac{\max\limits_{\|x\|_2=1} \|Ax\|_2}{\min\limits_{\|x\|_2=1} \|Ax\|_2} &= \max_{\|x\|_2=1} \|Ax\|_2 \left(\frac{1}{\min\limits_{\|x\|_2=1} \|Ax\|_2} \right) \\
&= \|A\|_2(\|A^{-1}\|_2) && < \text{definition of matrix } 2-\text{norm,} \\
& && \quad \text{Homework (2.3.5.4.)} > \\
&= \kappa_2(A). && < \text{definition of condition number} >
\end{aligned}
$$

Homework 2.5.1.8.

Theorem 2.2.4.4. stated that given $A \in \mathbb{C}^{m \times m}$, if $\|Ax\|_2 = \|x\|_2$, then A is a unitary matrix. Prove this using the SVD theorem.

Solution

$$
\begin{array}{rcll}
\|Ax\|_2 & = & \|x\|_2 & < \text{given condition} > \\
\|Av_j\|_2 & = & \|v_j\|_2 & < \text{instantiate } x \text{ as } v_j \text{ where } j \in \{0, \ldots, m\text{-}1\}> \\
\|\sigma_j u_j\|_2 & = & \|v_j\|_2 & < \text{by Homework 2.3.6.1., } Av_j = \sigma_j u_j \\
 & & & \quad \text{assuming SVD of } A = U\Sigma V^H,\ \Sigma = \text{diag}(\sigma_0, \ldots, \sigma_{m-1}) > \\
|\sigma_j|\ \|u_j\|_2 & = & \|v_j\|_2 & < \sigma_j \text{ a scalar, norms homogenous} > \\
|\sigma_j|\ (1) & = & (1) & < u_j, v_j \text{ unitary matrix columns} > \\
|\sigma_j| & = & 1 & < \text{algebra} > \\
\sigma_j & = & 1 & < \sigma_j \geq 0 >
\end{array}
$$

where U is partitioned as $(u_0 \mid \ldots \mid u_{m-1})$ and V as $(v_0 \mid \ldots \mid v_{m-1})$.

Therefore $\Sigma = I$ which implies $A = U(I)V^H = UV^H$. Because multiplication of unitary matrices is a unitary matrix, A is a unitary matrix.

Homework 2.5.1.9.

Given $A \in \mathbb{C}^{m \times n}$, prove $\|A\|_2 \leq \|A\|_F$ using the SVD theorem.

Solution

Suppose $A = U\Sigma V^H$ is the SVD of A where U and V are unitary and $\Sigma = \text{diag}(\sigma_0, \ldots, \sigma_{min(m,n)-1})$ with $\sigma_0 \geq \ldots \geq \sigma_{\min(m,n)-1} \geq 0$. Then

$$
\begin{aligned}
\|A\|_2 &= \|U\Sigma V^H\|_2 && < \text{SVD of } A > \\
\|A\|_2^2 &= \|U\Sigma V^H\|_2^2 && < \text{algebra} > \\
&= \|\Sigma\|_2^2 && < U, V \text{ unitary, Homework 2.2.4.9.} > \\
&= \max_{\|x\|_2=1} \|\Sigma x\|_2^2 && < \text{definition of matrix } 2-\text{norm} > \\
&= \max_{\|x\|_2=1} \sum_{j=0}^{n-1} |s_j^T \chi_j|^2 && < \Sigma = (s_0 | \cdots | s_{n-1}), \text{definition vector } 2-\text{norm} > \\
&= \max_{\|x\|_2=1} \sum_{j=0}^{\min(m,n)-1} |\sigma_j \chi_j|^2 && < s_j^T = (0 \ \cdots \ \sigma_j \ 0 \ \cdots \ 0) > \\
&= \max_{\|x\|_2=1} \sum_{j=0}^{\min(m,n)-1} |\sigma_j|^2 |\chi_j|^2 && < \text{absolute value homoegenous} > \\
&\leq \sum_{j=0}^{\min(m,n)-1} |\sigma_j|^2 && < |\chi_{\text{j}}|^2 \leq 1 > \\
&= \sum_{j=0}^{n-1} \|s_j\|_2^2 && < \text{definition of vector } 2-\text{norm} > \\
&= \|\Sigma\|_F^2 && < \text{property of Frobenius norm} > \\
&= \|U\Sigma V^H\|_F^2 && < U, V \text{ unitary, Homework 2.2.4.10} > \\
\|A\|_2^2 &\leq \|A\|_F^2 && < \text{SVD of } A > \\
\|A\|_2 &\leq \|A\|_F. && < \text{square root an increasing function} >
\end{aligned}
$$

Equality is attained for A = $(1 \ \ldots \ 1)^T$.

Homework 2.5.1.10.

Given $A \in \mathbb{C}^{m\times n}$ and A has r singular values, prove $\|A\|_F \leq \sqrt{r}\ \|A\|_2$ using the SVD theorem. (r is not called the rank of A here unlike in the original homework because it is yet to be proven or formally stated that r is equal to A's rank.)

Solution

$$
\begin{aligned}
\|A\|_F &= \|U\Sigma V^H\|_F && < \text{SVD of } A > \\
\|A\|_F^2 &= \|U\Sigma V^H\|_F^2 && < \text{algebra} > \\
&= \|\Sigma\|_F^2 && < \text{U, V unitary, Homework 2.2.4.10} > \\
&= \sum_{j=0}^{n-1} \|s_j\|_2^2 && < \Sigma = (s_0 \mid \cdots \mid s_{n-1}), \text{ property of Frobenius} - \text{norm} > \\
&= \sum_{j=0}^{\min(m,n)-1} |\sigma_j|^2 && < \text{definition vector } 2 - \text{norm} > \\
&= \sum_{j=0}^{r-1} |\sigma_j|^2 && < \Sigma \text{ has } r \text{ singular values} > \\
&\leq \sum_{j=0}^{r-1} |\sigma_0|^2 && < \sigma_0 \geq \ldots \geq \sigma_{r-1} > \\
&= r\,|\sigma_0|^2 && < \text{algebra} > \\
\|A\|_F^2 &\leq r\|A\|_2^2 && < \text{SVD of } A > \\
\|A\|_F &\leq \sqrt{r}\|A\|_2. && < \text{square root an increasing function} >
\end{aligned}
$$

Equality is attained for A = (1 … 1)^T.