

Homework (1.3.7.3.) - Matrix norm equivalences

Prove the following matrix norm equivalences (order slightly different than in the original question) where $A \in \mathbb{C}^{m \times n}$:

		1	2	3	4
		$\ x\ _1$	$\ x\ _F$	$\ x\ _2$	$\ x\ _\infty$
1	$\ x\ _1$	—	$\ A\ _1 \leq \sqrt{m} \ A\ _F$	$\ A\ _1 \leq \sqrt{m} \ A\ _2$	$\ A\ _1 \leq m \ A\ _\infty$
2	$\ x\ _F$	$\ A\ _F \leq \sqrt{n} \ A\ _1$	—	$\ A\ _F \leq \sqrt{n} \ A\ _2$ (*)	$\ A\ _F \leq \sqrt{m} \ A\ _\infty$
3	$\ x\ _2$	$\ A\ _2 \leq \sqrt{n} \ A\ _1$	$\ A\ _2 \leq \ A\ _F$	—	$\ A\ _2 \leq \sqrt{m} \ A\ _\infty$
4	$\ x\ _\infty$	$\ A\ _\infty \leq n \ A\ _1$	$\ A\ _\infty \leq \sqrt{n} \ A\ _F$	$\ A\ _\infty \leq \sqrt{n} \ A\ _2$	—
(*) The bound can be conditionally improved. See Question #957909 on math.stackexchange.com .					

Solution

(Following left to right, top to bottom order in the table. See the note at the end for $|x^H y| = |x^T y|$.)

(#12) $\|A\|_1 \leq \sqrt{m} \|A\|_F$: (solution exists in course notes)

$$\begin{aligned} \|A\|_1 &\leq \sqrt{m} \|A\|_2 &< \text{proven as \#13} > \\ \|A\|_1 &\leq \sqrt{m} \|A\|_F. &< \|A\|_2 \leq \|A\|_F \text{ (\#32)} > \end{aligned}$$

Alternative solution

$$\begin{aligned} \|A\|_1 &\leq \sqrt{m} \|A\|_2 &< \text{proven as \#13} > \\ \|A\|_1 &\leq \sqrt{m} \max_{\|x\|_2=1} \|Ax\|_2 &< \text{matrix 2-norm definition} > \\ &= \sqrt{m} \max_{\|x\|_2=1} \sqrt{\sum_{i=0}^{m-1} |\tilde{a}_i^T x|^2} &< \text{slice \& dice} > \\ &= \sqrt{m} \max_{\|x\|_2=1} \sqrt{\sum_{i=0}^{m-1} |\tilde{a}_i^H x|^2} &< |x^H y| = |x^T y| > \\ &\leq \sqrt{m} \max_{\|x\|_2=1} \sqrt{\sum_{i=0}^{m-1} \|\tilde{a}_i\|_2^2 \|x\|_2^2} &< \text{Cauchy-Schwartz inequality} > \\ &= \sqrt{m} \sqrt{\sum_{i=0}^{m-1} \|\tilde{a}_i\|_2^2} &< \|x\|_2 = 1 > \\ \|A\|_1 &\leq \sqrt{m} \|A\|_F. &< \text{Frobenius norm definition} > \end{aligned}$$

Equality is attained for $A = (1 \dots 1)^T$.

(#13) $\|A\|_1 \leq \sqrt{m} \|A\|_2$: (solution exists in course notes)

$$\begin{aligned}
 \|Ax\|_1 &\leq \sqrt{m} \|Ax\|_2 &< \|z\|_1 \leq \sqrt{k} \|z\|_2 \text{ for } z \in \mathbb{C}^k > \\
 \frac{\|Ax\|_1}{\|x\|_1} &\leq \sqrt{m} \frac{\|Ax\|_2}{\|x\|_2} &< \|z\|_2 \leq \|z\|_1 \text{ for } z \in \mathbb{C}^k > \\
 \max_{x \neq 0} \frac{\|Ax\|_1}{\|x\|_1} &\leq \max_{x \neq 0} \sqrt{m} \frac{\|Ax\|_2}{\|x\|_2} &< \text{algebra} > \\
 \max_{x \neq 0} \frac{\|Ax\|_1}{\|x\|_1} &\leq \sqrt{m} \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} &< \text{algebra} > \\
 \|A\|_1 &\leq \sqrt{m} \|A\|_2. &< \text{definitions of } \|A\|_1, \|A\|_2 >
 \end{aligned}$$

Equality attained for $A = (1 \dots 1)^T$.

(#14) $\|A\|_1 \leq m \|A\|_\infty$: (solution exists in course notes)

$$\begin{aligned}
 \|A\|_1 &= \max_{x \neq 0} \frac{\|Ax\|_1}{\|x\|_1} &< \text{matrix 1-norm definition} > \\
 &\leq \max_{x \neq 0} \frac{\|Ax\|_1}{\|x\|_\infty} &< \|z\|_\infty \leq \|z\|_1 \text{ for } z \in \mathbb{C}^k > \\
 &\leq \max_{x \neq 0} \frac{m \|Ax\|_\infty}{\|x\|_\infty} &< \|z\|_1 \leq m \|z\|_\infty \text{ for } z \in \mathbb{C}^k > \\
 &= m \max_{x \neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty} &< \text{algebra} > \\
 \|A\|_1 &\leq m \|A\|_\infty. &< \text{matrix infinity-norm definition} >
 \end{aligned}$$

Equality is attained for $A = (1 \dots 1)^T$.

(#21) $\|A\|_F \leq \sqrt{n} \|A\|_1$: (no solution in course notes)

$$\|A\|_F = \sqrt{\sum_{i=0}^{m-1} \sum_{j=0}^{n-1} |\alpha_{i,j}|^2} \quad < \text{Frobenius norm definition} >$$

$$= \sqrt{\sum_{j=0}^{n-1} \|a_j\|_2^2} \quad < \text{see Homework 1.3.3.1. (dedicated course site)} >$$

$$\|A\|_F^2 = \sum_{j=0}^{n-1} \|a_j\|_2^2 \quad < \text{algebra} >$$

$$\leq \sum_{j=0}^{n-1} \|a_j\|_1^2 \quad < \|z\|_2 \leq \|z\|_1 \text{ for } z \in \mathbb{C}^k >$$

$$= \sum_{j=0}^{n-1} \|Ae_j\|_1^2 \quad < \text{matrix-vector multiplication} >$$

$$\leq \sum_{j=0}^{n-1} \|A\|_1^2 \quad < \text{by definition } \|A\|_1 \geq \|Ae_j\|_1 >$$

$$\|A\|_F^2 \leq n \|A\|_1^2 \quad < \text{algebra} >$$

$$\|A\|_F \leq \sqrt{n} \|A\|_1. \quad < \text{square root an increasing function} >$$

Equality is attained for $A = (1 \dots 1)$.

(#23) $\|A\|_F \leq \sqrt{n} \|A\|_2$: (no solution in course notes)

$$\|A\|_F = \sqrt{\sum_{i=0}^{m-1} \sum_{j=0}^{n-1} |\alpha_{i,j}|^2} \quad < \text{Frobenius norm definition} >$$

$$= \sqrt{\sum_{j=0}^{n-1} \|a_j\|_2^2} \quad < \text{see Homework 1.3.3.1 (dedicated course site)} >$$

$$\|A\|_F^2 = \sum_{j=0}^{n-1} \|a_j\|_2^2 \quad < \text{algebra} >$$

$$= \sum_{j=0}^{n-1} \|A e_j\|_2^2 \quad < \text{matrix-vector multiplication} >$$

$$\leq \sum_{j=0}^{n-1} \|A\|_2^2 \quad < \text{by definition } \|A\|_2 \geq \|A e_j\|_2 >$$

$$\|A\|_F^2 \leq n \|A\|_2^2 \quad < \text{algebra} >$$

$$\|A\|_F \leq \sqrt{n} \|A\|_2. \quad < \text{square root an increasing function} >$$

Equality is attained for $A = (1 \dots 1)^T$.

(#24) $\|A\|_F \leq \sqrt{m} \|A\|_\infty$: (no solution in course notes)

$$\|A\|_F = \sqrt{\sum_{i=0}^{m-1} \sum_{j=0}^{n-1} |\alpha_{i,j}|^2} \quad < \text{Frobenius norm definition} >$$

$$= \sqrt{\sum_{i=0}^{m-1} \|\tilde{a}_i\|_2^2} \quad < \text{see Homework 1.3.3.3.} >$$

$$\|A\|_F^2 = \sum_{i=0}^{m-1} \|\tilde{a}_i\|_2^2 \quad < \text{algebra} >$$

$$\|A\|_F^2 \leq \sum_{i=0}^{m-1} \|\tilde{a}_i\|_1^2 \quad < \|z\|_2 \leq \|z\|_1 \text{ for } z \in \mathbb{C}^k >$$

$$\leq \sum_{i=0}^{m-1} \left(\max_{p=0}^{m-1} \|\tilde{a}_p\|_1 \right)^2 \quad < \text{algebra} >$$

$$= \sum_{i=0}^{m-1} (\|A\|_\infty)^2 \quad < \text{see Homework 1.3.6.2.} >$$

$$\|A\|_F^2 \leq m \|A\|_\infty^2 \quad < \text{square root an increasing function} >$$

$$\|A\|_F \leq \sqrt{m} \|A\|_\infty. \quad < \text{algebra} >$$

Equality is attained for $A = (1 \dots 1)^T$.

(#31) $\|A\|_2 \leq \sqrt{n} \|A\|_1$: (solution exists in course notes)

$$\|A\|_2 = \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} \quad < \text{matrix 2-norm definition} >$$

$$\leq \max_{x \neq 0} \frac{\|Ax\|_1}{\|x\|_2} \quad < \|z\|_2 \leq \|z\|_1 \text{ for } z \in \mathbb{C}^k >$$

$$\leq \max_{x \neq 0} \frac{\sqrt{n} \|Ax\|_1}{\|x\|_1} \quad < \|z\|_1 \leq \sqrt{k} \|z\|_2 \text{ for } z \in \mathbb{C}^k >$$

$$= \sqrt{n} \max_{x \neq 0} \frac{\|Ax\|_1}{\|x\|_1} \quad < \text{algebra} >$$

$$\|A\|_2 \leq \sqrt{n} \|A\|_1. \quad < \text{matrix 1-norm definition} >$$

Equality is attained for $A = (1 \dots 1)$ and $x = (1 \dots 1)^T$.

(#32) $\|A\|_2 \leq \|A\|_F$: (solution only through the SVD exists in course notes)

$$\begin{aligned}
 \|A\|_2 &= \max_{\|x\|_2=1} \|Ax\|_2 &< \text{matrix 2-norm definition} > \\
 &= \max_{\|x\|_2=1} \sqrt{\sum_{i=0}^{m-1} |\tilde{a}_i^T x|^2} &< \text{slice\&dice} > \\
 &= \max_{\|x\|_2=1} \sqrt{\sum_{i=0}^{m-1} |\tilde{a}_i^H x|^2} &< |y^T z| = |y^H z| > \\
 &\leq \max_{\|x\|_2=1} \sqrt{\sum_{i=0}^{m-1} \|\tilde{a}_i\|_2^2 \|x\|_2^2} &< \text{Cauchy-Schwartz inequality} > \\
 &= \sqrt{\sum_{i=0}^{m-1} \|\tilde{a}_i\|_2^2} &< \|x\|_2 = 1 > \\
 \|A\|_2 &\leq \|A\|_F. &< \text{Frobenius norm definition} >
 \end{aligned}$$

Equality is attained for $A = (1 \dots 1)$ and $x = (1 \dots 1)^T$.

(#34) $\|A\|_2 \leq \sqrt{m} \|A\|_\infty$: (solution exists in course notes)

$$\begin{aligned}
 \|A\|_2 &= \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} &< \text{matrix 2-norm definition} > \\
 &\leq \max_{x \neq 0} \frac{\sqrt{m} \|Ax\|_\infty}{\|x\|_2} &< \|z\|_2 \leq \sqrt{m} \|z\|_\infty \text{ for } z \in \mathbb{C}^k > \\
 &\leq \max_{x \neq 0} \frac{\sqrt{m} \|Ax\|_\infty}{\|x\|_\infty} &< \|z\|_\infty \leq \|z\|_2 \text{ for } z \in \mathbb{C}^k > \\
 &= \sqrt{m} \max_{x \neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty} &< \text{algebra} > \\
 \|A\|_2 &\leq \sqrt{m} \|A\|_\infty. &< \text{matrix infinity-norm definition} >
 \end{aligned}$$

Equality is attained for $A = (1 \dots 1)^T$.

(#41) $\|A\|_\infty \leq n \|A\|_1$: (no solution in course notes)

$$\begin{aligned}
 \|A\|_\infty &= \max_{x \neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty} &< \text{matrix infinity-norm definition} > \\
 &\leq \max_{x \neq 0} \frac{\|Ax\|_1}{\|x\|_\infty} &< \|z\|_\infty \leq \|z\|_1 \text{ for } z \in \mathbb{C}^k > \\
 &\leq \max_{x \neq 0} \frac{n\|Ax\|_1}{\|x\|_1} &< \|z\|_1 \leq k\|z\|_\infty \text{ for } z \in \mathbb{C}^k > \\
 &= n \max_{x \neq 0} \frac{\|Ax\|_1}{\|x\|_1} &< \text{algebra} > \\
 \|A\|_\infty &\leq n\|A\|_1. &< \text{matrix 1-norm definition} >
 \end{aligned}$$

Equality is attained for $A = (1 \dots 1)$.

(#42) $\|A\|_\infty \leq \sqrt{n} \|A\|_F$: (no solution in course notes)

$$\begin{aligned}
 \|A\|_\infty &\leq \sqrt{n} \|A\|_2 &< \text{proven as \#43} > \\
 \|A\|_\infty^2 &\leq n \|A\|_2^2 &< \text{algebra} > \\
 &= n \max_{\|x\|_2=1} \|Ax\|_2^2 &< \text{matrix 2-norm definition} > \\
 &= n \max_{\|x\|_2=1} \sum_{i=0}^{m-1} |\tilde{a}_i^T x|^2 &< \text{slice \& dice} > \\
 &= n \max_{\|x\|_2=1} \sum_{i=0}^{m-1} |\tilde{a}_i^H x|^2 &< |x^H y| = |x^T y| > \\
 &\leq n \max_{\|x\|_2=1} \sum_{i=0}^{m-1} \|\tilde{a}_i\|_2^2 \|x\|_2^2 &< \text{Cauchy-Schwartz inequality} > \\
 \|A\|_\infty^2 &\leq n \sum_{i=0}^{m-1} \|\tilde{a}_i\|_2^2 &< \|x\|_2 = 1 > \\
 \|A\|_\infty &\leq \sqrt{n} \sqrt{\sum_{i=0}^{m-1} \|\tilde{a}_i\|_2^2} &< \text{square root an increasing function} > \\
 \|A\|_\infty &\leq \sqrt{n} \|A_F\|. &< \text{Frobenius norm definition} >
 \end{aligned}$$

Equality is attained for $A = (1 \dots 1)$.

(#43) $\|A\|_\infty \leq \sqrt{n} \|A\|_2$: (no solution in course notes)

$$\begin{aligned}
 \|A\|_\infty &= \max_{x \neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty} &< \text{matrix infinity-norm definition} > \\
 &\leq \max_{x \neq 0} \frac{\sqrt{n}}{\|x\|_2} \|Ax\|_\infty &< \|z\|_2 \leq \sqrt{k} \|z\|_\infty \text{ for } z \in \mathbb{C}^k > \\
 &= \sqrt{n} \max_{x \neq 0} \frac{\|Ax\|_\infty}{\|x\|_2} &< \text{algebra} > \\
 &\leq \sqrt{n} \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} &< \|z\|_\infty \leq \|z\|_2 \text{ for } z \in \mathbb{C}^k > \\
 \|A\|_\infty &\leq \sqrt{n} \|A\|_2. &< \text{matrix 2-norm definition} >
 \end{aligned}$$

Equality is attained for $A = (1 \dots 1)$ and $x = (1 \dots 1)^T$.

Note

If $x, y \in \mathbb{C}^m$, then $|x^T y| = |x^H y|$.

(Originally posted in a discussion in Section 1.3.8. for Homework 1.3.8.4.)

Proof

$$\begin{aligned}
 |x^T y| &= |\bar{x}^T y| &< \text{statement} > \\
 |x^T y|^2 &= |\bar{x}^T y|^2 &< \text{algebra} > \\
 (x^T y)^H (x^T y) &= (\bar{x}^T y)^H (\bar{x}^T y) &< |\alpha| = \|\alpha\|_2 = \sqrt{\alpha^H \alpha} > \\
 \overline{(x^T y)}^T (x^T y) &= \overline{(\bar{x}^T y)}^T (\bar{x}^T y) &< \text{definition of Hermitian} > \\
 (\bar{y}^T \bar{x})(x^T y) &= (\bar{y}^T x)(\bar{x}^T y) &< \bar{x}\bar{y} = \bar{x}\bar{y}, (AB)^T = B^T A^T > \\
 \bar{y}^T (\bar{x} x^T) y &= \bar{y}^T (x \bar{x}^T) y &< \text{associativity of matrix multiplication} > \\
 \bar{y}^T (\bar{x} x^T) y &= \bar{y}^T (\bar{x} x^T)^H y &< \text{definition of Hermitian} > \\
 y^H (X) y &= y^H (X)^H y &< X = \bar{x} x^T; \text{fun fact : } X \text{ is Hermitian, i.e. } X = X^H > \\
 y^H X y &= y^H (X^H y) &< \text{associativity of matrix multiplication} > \\
 y^H X y &= y^H (y^H X)^H &< (A^H)^H = A, (AB)^H = B^H A^H > \\
 y^H X y &= (y^H X y)^H. &< \text{equality as a real number confirmed because both sides} \\
 &&\text{are scalar } (1 \times 1) \text{ and their conjugates are equal} >
 \end{aligned}$$