Problem 1: Find the general solution by separation of variables:

$$\frac{dy}{dx} = 2 - y, \qquad y(0) = 0$$

$\frac{dy}{dx} = 2-y$

$\frac{dy}{2-y} = dx$

$u = 2-y$

$du = -dy$

$-du = dy$

$-\int \frac{du}{u} = \int dx$

$-\ln|u| + C_1 = x + C_2$

$-\ln|2-y| = x + C_3$

$\ln|2-y| = -x + C_4$

$2 - y = e^{-x+C_4} = Ce^{-x}$

$-y = Ce^{-x} - 2$

$y = 2 - Ce^{-x}$

$0 = 2 - Ce^{0} = 2 - C = 0 \rightarrow C = 2$

$\boxed{y(x) = 2(1 - e^{-x})}$

Problem 2: Find the general solution by separation of variables:

$$\frac{dy}{dx} = \frac{(y-1)^2}{(x+1)^2}$$

$\frac{dy}{dx} = \frac{(y-1)^2}{(x+1)^2} \rightarrow \frac{dy}{(y-1)^2} = \frac{dx}{(x+1)^2}$

$u = y-1 \quad v = x+1$

$du = dy \quad dv = dx$

$\int \frac{du}{u^2} = \int \frac{dv}{v^2} \rightarrow \frac{-1}{u} + C_1 = \frac{-1}{v} + C_2$

$\frac{-1}{y-1} + C_1 = \frac{-1}{x+1} + C_2 \rightarrow \frac{-1}{y-1} = \frac{-1}{x+1} + C_3 = \frac{-1 + C_3(x+1)}{x+1}$

$-\frac{y-1}{1} = \frac{x+1}{-1+C_3(x+1)} \rightarrow 1 - y = \frac{x+1}{-1+Cx+C} = \frac{x+1}{Cx+C-1} \rightarrow -y = \frac{x+1}{C(x+1)-1} - 1 \rightarrow$

$\boxed{y = \frac{x+1}{1 - C(x+1)} \qquad y = 1 \text{ (const solution)}}$

Problem 3: The rate of change of a certain population is proportional to the square root of its size. Model this situation with a differential equation.

$\frac{dP}{dt} = \alpha\sqrt{P}$

Problem 4:

The rate of change of the velocity of an object is proportional to the square of the velocity. Model this situation with a differential equation.

$\frac{dv}{dt} = \alpha v^2$

Problem 5:

In a population of fixed size S, the rate of change of the number N of persons who have heard a rumor is proportional to the number of those who have not yet heard it. Model this situation with a differential equation.

$\frac{dN}{dt} = \alpha(S - N)$

Problem 6: The amount of a certain medicine in the bloodstream decays exponentially with a half-life of 5 hours. In order to keep a patient safe during a one-hour procedure, there needs to be at least 50 mg of medicine per kg of body weight. How much medicine should be administered to a 60kg patient at the start of the procedure?

$A(t) = Ce^{-t/\tau}$

$\frac{1}{2}\cancel{P} = \cancel{P}e^{-5/\tau}$

$\ln(\frac{1}{2}) = \frac{-5}{\tau} = -\ln(2)$

$\tau = \frac{-5}{-\ln(2)} = 7.2$

need: $\frac{50 \text{ mg}}{1 \text{ kg}} \times 60$ kg @ end of procedure: 3000 mg @ end

$3000 = Pe^{-5/7.2}$

$P = \frac{3000}{e^{-5/7.2}} = \boxed{6007 \text{ mg}}$

Problem 7: Early one morning it starts to snow. At 7AM a snowplow sets off to clear the road. By 8AM, it has gone 2 miles. It takes an additional 2 hours for the plow to go another 2 miles. Let $t = 0$ when it begins to snow, let x denote the distance traveled by the plow at time t. Assuming the snowplow clears snow at a constant rate in cubic meters/hour:

a) Find the DE modeling the value of x.

b) When did it start snowing?

k_1 = rate of snow fall $h(t) = k_1 t$

k_2 = rate of snow cleared $c(t) = k_2 t$

had a pretty tricky time with this one, here is a partial explanation to supplement the solution: https://www.reddit.com/r/learnmath/comments/hcrjec/trying_to_understand_a_simple_differential/

W = width of plow

- in some incremental time (Δt), the plow will take a step of incremental distance Δx
- in this step, the plow cleared $\Delta x\, h(t) W$ m^3 of snow or $\Delta x\, h(t+\Delta t) W$ m^3 of snow (the t vs. $t+\Delta t$ thing is kind of like the 2 sides of a trapezoid in the trapezoidal rule)
- let's use $\Delta x\, h(t) W$

So rate of snow clearence $= \dfrac{\Delta x \cdot k_1 t \cdot W}{\Delta t} \approx \dfrac{dx}{dt} k_1 t W = k_2 \rightarrow \boxed{\dfrac{dx}{dt} = \dfrac{k_2}{k_1 W t}}$

let a = time from when it started snowing to when the plow went out

$$dx = \frac{k_2\, dt}{k_1 W t} \rightarrow \int dx = \frac{k_2}{k_1 W}\int \frac{dt}{t} = \frac{k_2}{k_1 W}\ln(t) + C_1 = x + C_2 \rightarrow x(t) = \frac{k_2}{k_1 W}\ln(t) + C$$

($t > 0$, so no need for $| \ |$)

$$x(a) = 0 = \frac{k_2}{k_1 W}\ln(a) + C \rightarrow C = \frac{-k_2}{k_1 W}\ln(a)$$

$$x(a+1) = \frac{k_2}{k_1 W}\ln(a+1) - \frac{k_2}{k_1 W}\ln(a) = \frac{k_2}{k_1 W}\left[\ln(a+1) - \ln(a)\right] = \frac{k_2}{k_1 W}\ln\left(\frac{a+1}{a}\right) = 2 \rightarrow \frac{k_2}{k_1 W} = \frac{2}{\ln\left(\frac{a+1}{a}\right)}$$

$$x(a+3) = \frac{k_2}{k_1 W}\ln(a+3) - \frac{k_2}{k_1 W}\ln(a) = \frac{k_2}{k_1 W}\left[\ln(a+3) - \ln(a)\right] = \frac{k_2}{k_1 W}\ln\left(\frac{a+3}{a}\right) = \frac{2}{\ln\left(\frac{a+1}{a}\right)}\ln\left(\frac{a+3}{a}\right) = 4$$

$$2\ln\left(\frac{a+3}{a}\right) = 4\ln\left(\frac{a+1}{a}\right) \rightarrow \ln\left(\frac{a+3}{a}\right) = 2\ln\left(\frac{a+1}{a}\right) \rightarrow \frac{a+3}{a} = \left(\frac{a+1}{a}\right)^2$$

$$\frac{a+3}{\cancel{a}} = \frac{(a+1)^2}{a^{\cancel{2}}} \rightarrow \cancel{a^2} + 3a = \cancel{a^2} + 2a + 1 \rightarrow 3a = 2a + 1 \rightarrow a = 1$$

So $t = a$ (7am) $\rightarrow t = 1$, which means $\boxed{\text{the snow started at 6 am}}$

Problem 8: A tank holds 100 liters of water which contains 25 grams of salt initially. Pure water then flows into the tank, and salt water flows out of the tank, both at 5 liters/minute. The mixture is kept uniform at all times by stirring.

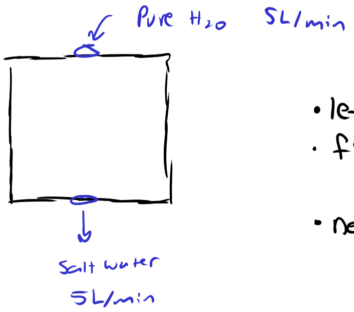
a) Write down the DE with IC for this situation.

b) How long will it take until only 1 gram of salt remains in the tank?

DE for amount of salt in the mixture

initial $V = 100\,L$

$W = 25g$



• let $s(t)$ = amount of salt in tank at time t

• find $\frac{ds}{dt}$

• net change of salt (ds) = $\underbrace{\text{salt in}}_{0}$ − salt out

how do we calculate the amount of salt leaving at each time step?

Volume of H_2O leaving × concentration of salt $\left(\frac{g}{L}\right) = 5 \times \frac{s(t)}{100}$

$$\frac{ds}{dt} = 0 - 5 \times \frac{s(t)}{100} = -0.05\,s \qquad \boxed{\frac{ds}{dt} = -0.05\,s}$$

<this is exponential decay>

$$s(t) = Ce^{-0.05t} \qquad s(0) = Ce^{0} = 25 \quad C = 25$$

b) time until $s(t) = 1$?

$$1 = 25e^{-0.05t} \rightarrow \frac{1}{25} = e^{-0.05t} \rightarrow \ln\left(\frac{1}{25}\right) = -0.05t$$

$$t = \frac{\ln\left(\frac{1}{25}\right)}{-0.05} \approx \boxed{64.38 \text{ min}}$$