Eigenvectors and Eigenvalues

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Sources:
https://www.youtube.com/watch?v=PFDu9oVAE-g

similar topics:

- linear systems
- determinates
- change of basis

One way to think about matricies is to picture them as linear transformations. Think about a 2d coordinate system, with $\hat{x}$ and $\hat{y}$ unit vectors. The matrix

$$\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$$

will take the $\hat{x}$ vector (originally $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$) and map it to the coordinate $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$ in the new coordinate system and take the $\hat{y}$ vector (originally $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$) and map it to the coordinate $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ in the new coordinate system.

When undergoing this transformation, most vectors in the the space will be rotated and / or stretched / shrinked.

For example, the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ will get mapped to $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$:

$$\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 * 1 + 1 * 1 \\ 0 * 1 + 2 * 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

However take, for example, the vector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$:

$$\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 * -1 + 1 * 1 \\ 0 * 1 + 2 * 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

The resultant vector after the linear transformation is a stretched version of the original vector

The name we give to these special vectors are *eigenvectors*

The factor by which the eigenvector is streteched is called the *eigenvalue*

Other discussion points / applications

- Consider a 3d rotation - the axis that does not get rotated during that transformation is (by definition) the axis of rotation. So finding the eigenvector of a 3d rotation gives you the axis of rotation
- The eigenvectors of the covariance matrix of a dataset give the principal components (PCA), where the eigenvalues describe how much variance is in each principal component (i.e. the relative importance of each component)
- usually the eigenvectors / eigenvalues let you get at the heart of what a linear transformation really does

How to calculate the eigenvectors and eigenvalues

$$A\vec{v} = \lambda\vec{v}$$

where A = matrix of interest, $\vec{v}$ = eigenvector, λ = eigenvalue

rewrite it as:

$$A\vec{v} = (\lambda I)\vec{v}$$

(where I is the identity matrix)

some manipulation:

$$A\vec{v} - (\lambda I)\vec{v} = (A - \lambda I)\vec{v} = 0$$

For this to be true, the linear transformation associated with $A - \lambda I$ must squish $\vec{v}$ to 0. This is equivalent to saying $\det(A - \lambda I) = 0$