① find the eigenvectors and eigenvalues of $\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$

$$\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \rightarrow \det\begin{pmatrix} 3-\lambda & 1 \\ 0 & 2-\lambda \end{pmatrix} = (3-\lambda)(2-\lambda) - 0 = 0$$

$$6 - 5\lambda + \lambda^2 = 0 \rightarrow \lambda^2 - 5\lambda + 6 = 0 = (\lambda-2)(\lambda-3)$$

so $\lambda = 2, \lambda = 3$

$\lambda = 2$: $$\begin{bmatrix} 3-2 & 1 \\ 0 & 2-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 + x_2 \\ 0 \end{bmatrix}$$

$x_1 + x_2 = 0$

so $x_1 = -1$, $x_2 = 1$ (to within a multiplicative constant)

$\lambda = 3$: $$\begin{bmatrix} 3-3 & 1 \\ 0 & 2-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

So $x_1 = -x_2$, or $x_1 = -1$, $x_2 = 1$

eigenvalues: 2, 3 eigenvectors: $\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

② $$\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \rightarrow \det\left(\begin{bmatrix} 0-\lambda & 1 \\ -2 & -3-\lambda \end{bmatrix}\right) = 0$$

$$(-\lambda)(-3-\lambda) + 2 = 0$$
$$+3\lambda + \lambda^2 + 2 = 0$$
$$\lambda^2 + 3\lambda + 2 = 0$$
$$(\lambda+1)(\lambda+2) = 0$$

→ eigenvalues are $-1, -2$

$\lambda = -1$: $$\begin{bmatrix} 0+1 & 1 \\ -2 & -3+1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} +x_1 + x_2 \\ -2x_1 - 2x_2 \end{bmatrix} = 0$$

$x_1 = -x_2$

$-x_1 - x_2 = 0 \rightarrow x_1 = -x_2$

So eigenvector is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lambda = -2$: $$\begin{bmatrix} 0+2 & 1 \\ -2 & -3+2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 + x_2 \\ -2x_1 - x_2 \end{bmatrix} = 0$$

$2x_1 + x_2 = 0$
$-2x_1 - x_2 = 0$ → $x_2 = -2x_1$

So eigenvector is $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

③ $\begin{bmatrix} 5 & 2 & 0 \\ 2 & 5 & 0 \\ -3 & 4 & 6 \end{bmatrix} \to \det\left(\begin{bmatrix} 5-\lambda & 2 & 0 \\ 2 & 5-\lambda & 0 \\ -3 & 4 & 6-\lambda \end{bmatrix}\right) = 0$

$(5-\lambda)\,[(5-\lambda)(6-\lambda)-0] - 2\,[2(6-\lambda)-0)] + 0 = 0$

$(5-\lambda)(5-\lambda)(6-\lambda) - 4(6-\lambda) = 0$

$(6-\lambda)[(5-\lambda)(5-\lambda)-4] = 0$

$(6-\lambda)[25-10\lambda+\lambda^2-4] = 0$

$(6-\lambda)[\lambda^2-10\lambda+21] = 0$

$(6-\lambda)(\lambda-7)(\lambda-3) = 0 \to$ eigenvalues are $\lambda = 3,\ \lambda = 6,\ \lambda = 7$

$\lambda = 3$: $\begin{bmatrix} 5-3 & 2 & 0 \\ 2 & 5-3 & 0 \\ -3 & 4 & 6-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1+2x_2 \\ 2x_1+2x_2 \\ -3x_1+4x_2+3x_3 \end{bmatrix} = 0 \to$

$x_1 + x_2 = 0$ $\quad \boxed{x_1 = -x_2}$

$x_1 + x_2 = 0$

$-3x_1 + 4x_2 + x_3 = 0$ $\quad$ let $x_1 = 3$

$3x_2 + 4x_2 + 3x_3 = 0$ $\quad x_2 = -3$

$7x_2 + 3x_3 = 0$ $\quad -7(-3) = 3(x_3)$

$\boxed{-7x_2 = 3x_3}$ $\quad x_3 = 7$

an eigenvector is $\begin{bmatrix} 3 \\ -3 \\ 7 \end{bmatrix}$

$\lambda = 6$: $\begin{bmatrix} 5-6 & 2 & 0 \\ 2 & 5-6 & 0 \\ -3 & 4 & \cancel{5}6-6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_1+2x_2 \\ 2x_1-x_2 \\ -3x_1+4x_2 \end{bmatrix} = 0$

$x_1 = -2x_2$ $\quad$ only $x_1 = 0,\ x_2 = 0$

$2x_1 = x_2$ $\quad$ satisfies this

$4x_2 = 3x_1$

an eigenvector is $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\lambda = 7$ $\begin{bmatrix} 5-7 & 2 & 0 \\ 2 & 5-7 & 0 \\ -3 & 4 & 6-7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_1+2x_2 \\ 2x_1-2x_2 \\ -3x_1+4x_2-x_3 \end{bmatrix} = 0$

$x_1 = x_2$

$-3x_1 + 4x_1 - x_3 = 0$

$x_1 - x_3 = 0$

$x_3 = x_1$

an eigenvector is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$