Vector spaces

Put simply, a vector space is a collection of any kind of mathematical objects that can be added and multiplied together.

Motivation

All of these problems can be solved using the same toolkit:

$$\begin{bmatrix} 3 & 2 & 0 \\ 1 & 0 & 1 \\ 2 & 3 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \\ 7 \end{bmatrix}$$

$$x_1(3t^2 + 5t - 2) + x_2(0t^2 - t + 6) + x_3(9t^2 + 0t + 1) = 6t^2 + 9t + 2$$

$$x_1 \sin(\pi t) + x_2 \sin(2\pi t) + x_3 \sin(3\pi t) + \ldots = e^{5it}$$

In the first case, we're dealing with column matrices, in the second, polynomials, and the third, functions.

Definition: Field

A **Field** is a set $\mathbb{F}$ of numbers with the property that if $a, b \in \mathbb{F}$, then $a + b$, $a - b$, ab, and a/b are also in $\mathbb{F}$.

With that being said, a **Vector Space** consists of a set V, field $\mathbb{F}$, and two operations:
- *addition*, which takes two vectors $v, w \in V$ and produces a third vector $v + w \in V$.
- *scalar multiplication*, which takes a scalar $c \in \mathbb{F}$ and a vector $v \in V$ and produces a new vector, $cv \in V$.

and also satisfies the following axioms*:

- addition is associative
- zero vector ($u + 0 = u$)
- existance of negatives ($u + (-u) = 0$)
- multiplication is associative
- multiplication is distributive
- unitary: $1u = u$

*Note: I am giving a very abbreviated description of the axioms compared to what you would find in a textbook

Null space

Consider a matrix A:

$$\begin{bmatrix} x_{11} & \dots & x_{1n} \\ \dots & \dots & \dots \\ x_{m1} & \dots & \dots x_{mn} \end{bmatrix}$$

The null space of A is the set of all vectors B such that $AB = 0$

In other words, solve the equation for B:

$$\begin{bmatrix} x_{11} & \dots & x_{1n} \\ \dots & \dots & \dots \\ x_{m1} & \dots & \dots x_{mn} \end{bmatrix} \begin{bmatrix} b_1 \\ . \\ . \\ b_n \end{bmatrix} = 0$$

References

- https://www.math.toronto.edu/gscott/WhatVS.pdf
- https://brilliant.org/wiki/vector-space/