An Excursion In Particle Physics

Anurag Pallaprolu

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Advisor : Dr. Namit Mahajan, Theoretical Physics Division
PHYSICAL RESEARCH LABORATORY
NAVRANGPURA, AHMEDABAD

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This report contains 69 pages and 29 diagrams.[1]

[1] The bibliography can be accessed easily by Ctrl-F and punching in the three/four letter code. For example, Ctrl+F+FEY should get you to the reference in the subsequent search queries.

1 Prologue

If you want a thing done well,
try doing it yourself.

Napoleon Bonaparte

The present document is an attempt at a coherent description of nature and reality based on my study of the theory of quantum fields and particles. The report is by no means complete nor rigorous but is a concoction of the fundamental ideas on which the theory runs on. It is not possible to study this edifice of physical sciences, which took nearly a century in the making, in two months and it was necessary for me to learn the foundations proper to even attempt a scratch on its surface. However, as it often happens, the foundations sometimes lead to surprising connections with other branches of the science and this struck me as something that is amazing about field theory. I would be general in calling it field theory as only a few minor prescriptions are needed to switch from what we call separately as "quantum field theory", "statistical mechanics", "condensed matter theory", "many body theory", "thermal field theory", and so on. The term "many body physics" might sound as the best candidate to cover all the aforementioned sciences but then I am getting semantic here.

I started my study with two questions which I was seeking answers to, not at an academic level, but at a much deeper level regarding the consistency of human existence and thought. As philosophical as this may sound, I could add some technical flavour and list out the questions as follows

1 We know very well that the Schrodinger prescription of non relativistic quantum mechanics of single body states is inherently symmetric with respect to time and shows certain "nice" properties, but we clearly know that physical reality is not. The question is hence, at what particle number (say N) does the approximation of quantum mechanics break down to the classical prescription where things suddenly become irreversible? Can one write down a quantum mechanical explanation to chaotic phenomena such as turbulence? I ended up with this question after getting to know about Stephen Wolfram's ideas about cellular automata showing complex structures at large iterations [WOL]. What Wolfram was fundamentally driving at was that when his automata, which runs on seemingly simple rules, ends up generating chaotic and fractal patterns, isn't the universe ultimately made up of such small machines which run on (unknown) rules and generate the physical reality we see today? One cannot positively respond to this statement simply based on Wolfram's computation because one cannot verify the unlikely fact that the fundamental rules of the game are really based on cellular automata (even though they do describe the evolution of multicellular life remarkably, see [GAR]).

2 Putting the complexity of systems in reality aside, the other issue at hand which was partially suppressed in the previous question, is regarding time which in itself presents a greater paradox. One must surely have learnt about the Second Law of Thermodynamics, which, in all of its grandeur, states that the universe is ultimately irreversible by quoting that a certain quantity called "entropy" is always positive. There is a gargantuan history behind this debate as to how valid this law really is and I would say that the most important contribution done to partially improve our understanding of the mystery is due to Ludwig Boltzmann [BOL], who stated that entropy is *almost* always positive and that the true basis of physical reality cannot be stated with the Second Law as a fundamental rule. Random fluctuations prescribed by no rule whatsoever seem to be the logical way for our universe to start, but present day cosmologists love to give popular talks about the universe beginning in a low entropy state known as the Big Bang. I would discuss in much more completeness regarding this point at the very end of this report where I would make much more sense after the initial formalism has been described.

In order to answer the first question I had to study the theory of fields from the start and this would occupy the second section of the report. The first section of the report partially describes the simulation and operation of particle storage rings and accelerators as an engineering prelude and a tribute to all the brilliant scientists who spent countless hours on the analysis of particle collisions which would seem pointless if not for the theory which was built in tandem and vice versa. In addition to this I performed an experiment in data science which involves analysing ionospheric reflection plots, commonly known as ionograms, and their corresponding scaling using computer vision ideas. This was done out of sheer interest and bemusement that I felt when I saw my colleagues working in the atmospheric physics division while they were scaling these noisy plots manually.

I would like to thank the selection committee at Physical Research Laboratory, Ahmedabad for giving me this opportunity and I would like to thank Dr. Namit Mahajan for supplying me with extremely clarifying explanations whenever I came up with any questions and for handling my (extremely) silly questions with admirable amounts of patience, and for giving me the freedom and support to study the subject at my own pace. I would like to thank Namitha Suresh for reading the initial draft and pointing out typos and Aarzoo Sharma for (calling my quotations humorous) checking up on my situation regularly (even though she's a thousand miles away). Finally, I would like to thank my mother for listening patiently as I explained the theory to her every evening on the phone and for her opinions too (even though she isn't a physicist, her ideas were remarkably sensible), and I would like to thank my father for his unbounded support throughout these two months.

I would like to conclude this section with a conversation between Prof. Steven Weinberg and Prof. Philip Candelas [KRA], as cited in a conversation between Prof. Weinberg and Prof. Richard Dawkins:

A year or so ago, while Philip Candelas (of the physics department at Texas) and I were waiting for an elevator, our conversation turned to a young theorist who had been quite promising as a graduate student and who had then dropped out of sight. I asked Phil what had interfered with the ex-student's research. Phil shook his head sadly and said, "He tried to understand quantum mechanics."

A. Pallaprolu
Ahmedabad, India.

2 Particle Accelerators and Storage Rings

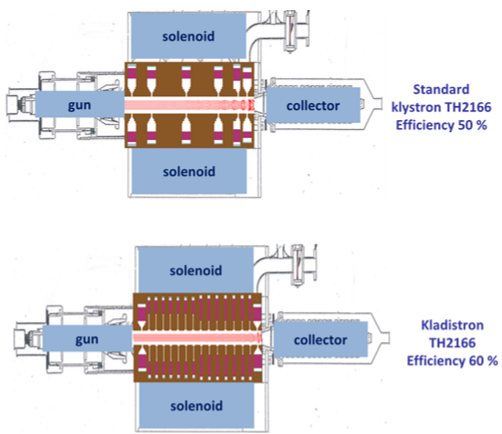
2.1 Introduction

I have done a terrible thing, I have postulated a particle that cannot be detected.

Wolfgang Pauli, The Neutrino Hypothesis

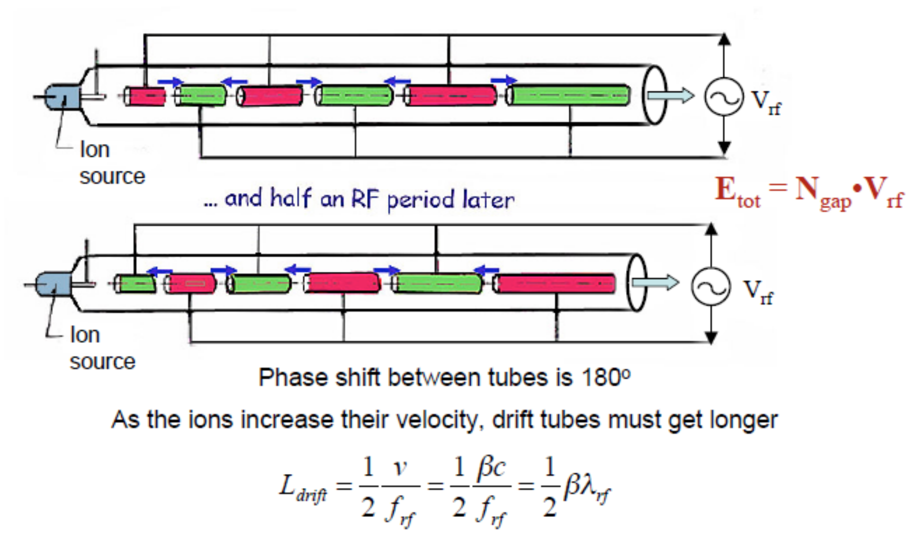
The concept of accelerating objects (particles) to high speeds and observing the results is not something new to experimental physics. One could trace the origins of this exercise straight to Galileo who, to quote Prof. Leon Lederman [KRA], probably operated the Leaning Tower of Pisa as the first "particle" accelerator to test out his ideas on falling bodies. The explanation as to why one needs to continually push to higher speeds, and hence higher energies, to explore more of matter is simple in itself. The forces get stronger as you go closer to the fundamental entities and this requires you to increase the impact energy to higher and higher values to unlock these components. Speaking technically (for once), what Galileo used could be called a "linear" accelerator in the way the object (rocks, metallic spheres) was accelerated over a linear trajectory. Modern day particle accelerators have a circular topology in which the particles are continually energized to revolve in these tracks at higher and higher speeds using magnets which are laid all around the track to guide the beam and maintain stability.

Thence, one can see that the idea of studying such a system is really an exercise in electromagnetism (no real field theory here) but it is rather a complex one (look at the LHC) and hence there are many steps taken by engineers and physicists alike to simplify the process, and it is very interesting to see this happen. If there was one rule I had to stress regarding the operation of these machines, it would be that which is taught in basic electrodynamics courses, that the *electric* field is always responsible for accelerating or retarding a charged particle and it is the *magnetic* field which is responsible for navigating the particle through a given trajectory. The magnetic field *cannot* add to the particle's energy. The reasons as to why this happens should be postponed for the sake of continuity. The earliest model which showed any resemblance to the modern day machine is the phenomenal experiment carried out by Rutherford and his team in 1911 where he bombarded a gold foil with an α beam. I do not consider this experiment as an exact origin to the modern idea, I would rather consider the invention of the Klystron by the Varian brothers which was made public in 1937 as the appropriate forerunner and the device is still used in the power subsystems of most accelerator complexes. The Klystron introduces the idea of using alternating frequency to "bunch" and modulate particle beams to create an effective amplification, which was what was needed as the it was used to amplify RF signals to transmit at UHF. The image below is taken from the EuCARD2 team's website where there is a comparison between the standard Klystron tube and their improved "Kladistron" idea [PLO]



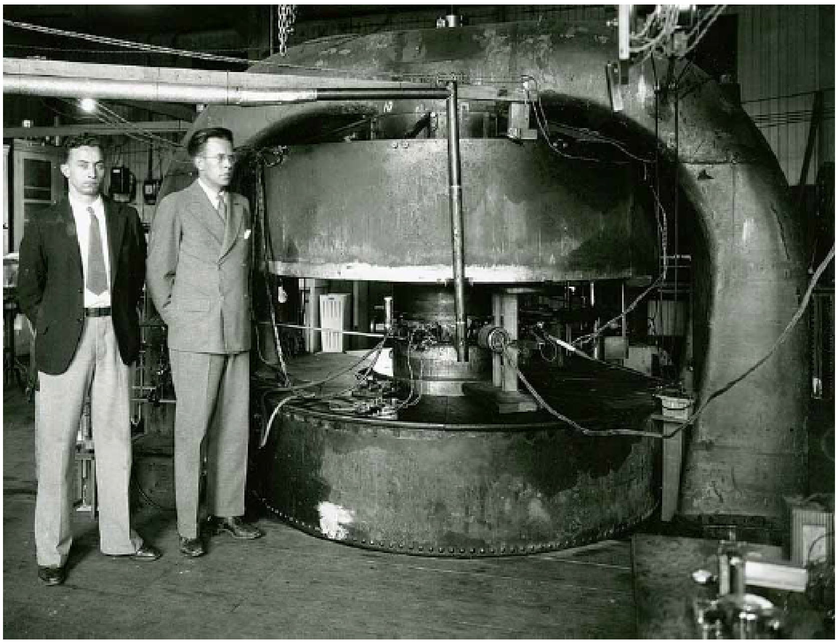
The rigorous classification for the different types of accelerators available at present is rather blurry and rather useless for the operating principle behind them is mostly the same. I give a bird's eye view of the history of the accelerator here, and to really do justice, one must go through this journey paper by paper, and most of this data is coherently placed in the first chapter of [LEE] and I would ask you to refer to this for further exploration. After Rutherford's experiment, William Coolidge developed his famous Coolidge tubes which formed an integral part of electronics at the time but the real breakthrough had to wait for John Cockroft and Ernst Walton's high energy device, the Cockroft-Walton electrostatic accelerator [COC], which energized beams upto voltages of 1MV and which led to the world's first ever artificial transmutation of naturally occurring elements. (Incidentally, James Chadwick presented the existence of a neutron a few pages after this article in the journal). This was tremendous news. Rutherford broke the Coulomb barrier and now the nuclear barrier has been broken, or so it was thought. The idea of Cockroft and Walton was worked upon by the Dutch electrical engineer Robert Van De Graaf and the eponymous generator could produce hair-raising voltages of upto 25 MV.

The design of the commercial grade linear accelerators or linacs, started with the work of Rolf Wideroe, another electrical engineer at RWTH Aachen, who, along with Luis Alvarez, presented the idea of linear resonance and this inspired Ernest O. Lawrence and David H. Sloan to come up with the concept of a cyclotron, see [LAW], [SLO]. The concept of the ion linac can be explained using the slide shown below, taken from [BAR].



Every odd electrode pair accelerates initially while every even one retards the particle entering the slot (and hence the similarity with the Klystron). After half cycle, the line voltage is flipped and now the odd pairs start to decelerate and the even electrodes accelerate. Let us call a specific odd electrode as 1 and an even electrode as 2. If one could design electrodes of such mechanical dimension that by the time ions entering 1 reach 2 the field at 2 reverses, then by the time it reaches the next electrode (say 3) the particle is now bursting with twice the acceleration it had initially. As attractive at it looks, the number of electrodes one would need would become unphysical as the speeds to which the particles are accelerated to gets higher (a back of the envelope calculation gives approximately 33000 everyday 120V generators to get the particle to 0.8c).

The modification Lawrence and Sloan proposed was to instead bend the particle beam into a circle and use the same idea for beam energizing. This was the primitive cyclotron and Prof. Lawrence is seen in the photograph below with the hand made 25 inch cyclotron.



The next breakthrough did not take long as Wideroe's idea (which was initially dismissed by his instructor, eventually leading to him missing the Nobel Prize) was worked on by N. Christofilos (see [BAR]) leading to the "induction" linac and eventually the seminal paper was written by Donald Kerst and Robert Serber while they were busy at the Manhattan Project [KER], and hence the Betatron was formed (the name betatron was chosen as the machine worked perfectly for high speed electrons, the beta particles of Becquerel). Quoting directly from the paper, the basic idea is a rather simple one. If I placed an electron in a radially symmetric magnetic field properly and gave it a momentum p, it would rotate in a perfectly circular trajectory where

$$p = \frac{eHr}{c}$$

Now, Faraday tells us that if the magnetic flux through a circular cross section increased then we would have a tangential electric field which would accelerate the particle

$$E_t = \frac{\dot{\phi}}{2\pi rc}$$

Thus if one increased H then the electron momentum would proportionately increase if we had to fix the radius. Now the tangential force on the electron is given by

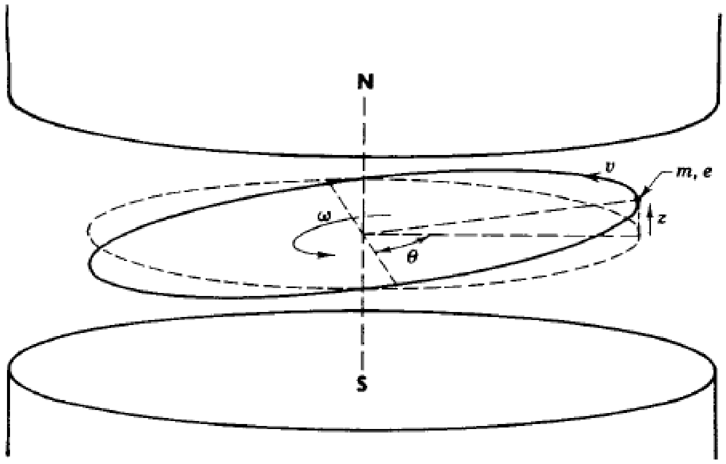
$$F_t = \dot{p} = eE_t = \frac{e\dot{\phi}}{2\pi rc}$$

$$\implies p = \frac{e(\phi - \phi_0)}{2\pi rc}$$

$$r = r_0 \implies H = \frac{\phi - \phi_0}{2\pi r_0^2}$$

Thus, if we had to maintain the same radius, the rate of change of flux must be *twice* that of the uniform H case. This is the betatron principle, originally discovered by Wideroe.

I shall now explain the general motion of an electron and the orbit parameters when placed in an almost uniform magnetic field perpendicular to the track. The image below should make the coordinate convention apt and it is taken from the monumental work of M. Stanley Livingston [LIV].



It is clear that there are three coordinates and hence for a (possibly relativistic) description of the situation one needs to know the momenta of the electron in these dimensions. The radial force can be computed as the difference in the centrifugal effect and the curvature due to the z component of the magnetic field

$$\frac{d}{dt}(m\dot{r}) = mr(\dot{\theta})^2 - er\dot{\theta}B_z$$

Note that m does not refer to the rest mass of the electron. And similarly, following the discussion before,

$$\frac{d}{dt}(mr^2\dot{\theta} - \frac{e\Phi}{2\pi}) = \frac{dE}{d\theta} - \frac{dL}{d\theta}$$

where E is the electric field energy (since $\tau = \frac{P}{\omega}$) and similarly $\frac{dL}{d\theta}$ is the loss in torque due to radiation. It is clear that one has to continually pump electrical energy for the particle to remain in the circular orbit or the particle would come to rest due to the leakage. Finally if the magnetic field is not really symmetric, that is, if it has a radial component B_r, then it would try to change the plane of rotation of the electron (that is rotation in the z plane, refer figure)

$$\frac{d}{dt}(m\dot{z}) = er\dot{\theta}B_r$$

With this, one could, in principle solve a given configuration of the accelerator, except the problem being that the solutions are complicated and we would, like always, want an equivalent mathematical description which helps us compute things easier (one can almost guess what this is, yes, you should look at energies). I would honestly suggest [LEE] for further reading about the history and construction of particle accelerators as it has been dealt with in much detail there. In conclusion we have

1 One of the fundamental building blocks of the accelerator is the magnetic beam lensing apparatus which almost always uses superconducting *multipole* magnets and hence a bare bones analysis of these configurations is essential. Added to this is the cost, material, mechanical design, machining, production, assembly and testing, all of which come after this preliminary analysis.

2 As explained, any electromagnetic structure can be fully worked out in principle using Maxwell equations (for classical orbits) and analytical solutions become unmanageable fairly quickly even for the simplest cases.

2.2 Beth's Formulation Of Magnetic Fields

I have a kind of magnetic attraction to situations of violence.

Wole Soyinka, Nobel Prize in Literature, 1986

Richard A. Beth from the Brookhaven National Laboratory came up with an efficient formalism in which the x and y components of the magnetic field $\bar{H}$ are taken as the real and imaginary parts of a complex valued "magnetic field" [BET]. To see how this works out, recall the following theorem from complex analysis (if you don't, you can always refer to the excellent introduction to the subject [CHU]) which is well known by the name of the Cauchy-Riemann equation. Let $z = x + iy$ be our complex plane, and let

$$F(z) = A(x, y) + iB(x, y)$$

be an analytic function of z, that is it is a very nice function with no poles et cetera in our domain. Then we have the following relation among A and B

$$\frac{\partial A}{\partial x} = \frac{\partial B}{\partial y}$$

$$\frac{\partial A}{\partial y} = -\frac{\partial B}{\partial x}$$

Now, notice something remarkable here. I differentiate the first equation partially with x first and then the second one with y. Then add both the left and right hand sides. The right hand side cancels out thanks to the analyticity condition on F we have imposed initially which allows the x and y differential operators to commute. Thence, we have

$$\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} = 0$$

$$\frac{\partial^2 B}{\partial x^2} + \frac{\partial^2 B}{\partial y^2} = 0$$

These are the age old Laplace equations for A and B. Thus, if we have two solutions of the Laplace equation in two dimensions, we can construct an equivalent (analytic, nice, whatever) function $F(z)$ which stores this data.

Beth's insight comes in observing the fact that in situations where the current distribution (assuming we are doing electrodynamics in two dimensions, say x and y) in an electromagnetic configuration is known say, $\sigma(x, y)$, then we can write the two Maxwell equations (again in whichever units)

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = 4\pi\sigma(x, y)$$

$$\frac{\partial H_y}{\partial y} + \frac{\partial H_x}{\partial x} = 0$$

Thinking slightly imaginatively, he constructs the complex function

$$F(z) = (H_y - 2\pi\sigma x) + i(H_x + 2\pi\sigma y)$$

In the limit of zero current distribution, it becomes the expression

$$F(z) = H_y + iH_x$$

Not only is this formulation reducing the number of functions to work with, it also surprisingly stores all the moments of the magnetic field (!). Expanding $F(z)$ in the general situation around $z = 0$(that is actually around $x = 0$ and $y = 0$),

$$F(z) = \sum_{k=1}^{k=\infty} F_k \frac{z^{k-1}}{k!}$$

In this expansion (the so-called "Multipole Expansion", see [GRI]), one can easily relate the coefficients to the magnetic moments of the configuration. Ideally, the process would be to construct the function $F(z)$ using the formula above and then taking derivatives with z to see the coefficients turn out as the multipole values, F_1 is the dipole moment, F_2 is the quadrupole moment and in general F_k is the 2kth moment.

As if this method needs any more convincing, one could take sample cases, say a wire carrying current and see what the method has to say. It can be seen that for a wire placed along the z-axis, one can see that the magnetic field configuration in the $z = 0$ plane can be computed by plugging the corresponding values to get

$$F(z) = \frac{2I}{z}$$

and it is clear that the function is not really nice at the origin. So in the ϵ exclusion magic that mathematicians do, we could define an origin excluded domain to still do useful things with F. The pole at the origin is not bad news, rather, it is a confirmation of the well known Ampere rule. Run the Cauchy residue theorem on $F(z)$ (again, for people with bad memory, see [CHU]), and you have

$$\oint F(z)dz = 2\pi i \sum_k F(z_k)$$

$$\implies \oint F(z)dz = 4\pi i I$$

This is a recurring theme in Physics as opposed to Mathematics. Poles and divergences are not bad, rather in most of the cases they provide insight into the physical scenario one is dealing with and using theorems linked by mathematical logic one could link physical situations to each other, like above. So, our next idea is to see what happens when we use the Cauchy integral theorem to this new complex magnetic field we constructed. For this, let us rewrite $F(z)$ as

$$F(z) = H_y + iH_x - 2\pi i\sigma(x - iy)$$

$$\implies F(z) = H_y + iH_x - 2\pi i\sigma z^*$$

Notice that, since $Z^*(x, y) = x - iy$ satisfies the conditions of the Cauchy-Riemann conditions, it is trivially analytic. And hence, the integral theorem [CHU] tells us that for any contour P, we have for the function $H(x, y) - 2\pi i\sigma z^*$

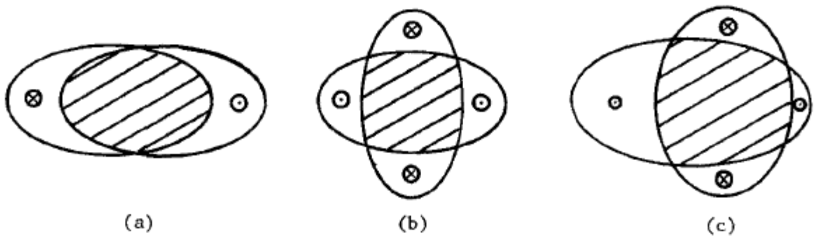
$$F(z') = i\sigma \oint_P \frac{z^*}{z - z'} dz$$

If you are wondering what happened to the H terms, I would tell you that I am integrating over a contour which is the cross section of a current carrying conductor and fields are almost constant (but appear *discontinuous*) and we would have, comparing left and right hand sides, for points outside and inside the conductor

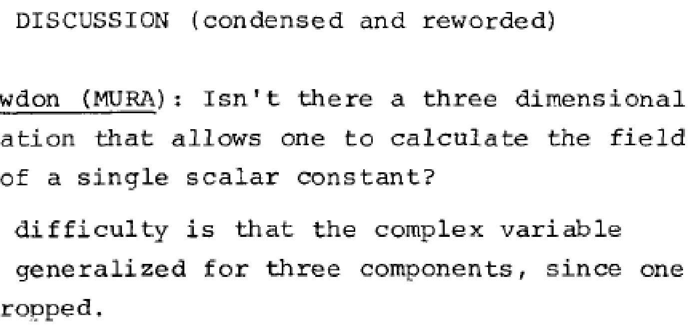
$$H_{outside} = F(z)$$

$$H_{inside} = F(z) + 2\pi i z^*$$

When the point z' is chosen cleverly in the contour integral, one could solve electromagnetic configurations such as conductors with arbitrary cross sections in a rather straightforward way using the complex magnetic field procedure. In the paper [BET], he discusses the case of the elliptical cross section and how one could use the fact that since electromagnetism (so far) is linear, we could use the superposition rule of linear systems to compute magnetic fields for dipole and quadrupole cross sections by simply superposing these elliptical cylinders. The image shown below, taken from [BET], should exemplify this point, which comes of much use while mechanically constructing the superconducting magnets [RAB]



I would ask you to refer to [BET] and [LEE] for extensions (the so called *Beth's Current Sheet Theorem*) and problems one could work out for oneself (along with some standard electrostatic trickery like the Method Of Images).



DISCUSSION (condensed and reworded)

S. C. Snowdon (MURA): Isn't there a three dimensional generalization that allows one to calculate the field in terms of a single scalar constant?

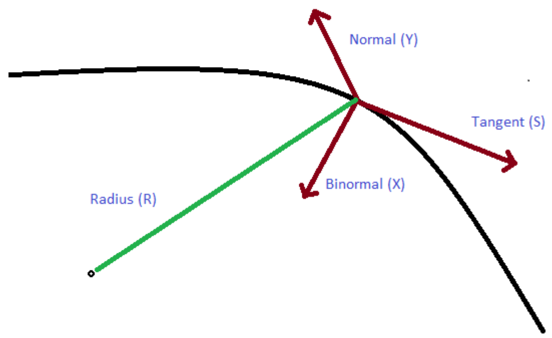
Beth: The difficulty is that the complex variable cannot be generalized for three components, since one must be dropped.

2.3 The Hamiltonian Formalism

Concrete is, essentially, the color of bad weather.

William Rowan Hamilton

At the end of 2.1, I mentioned that the energy formulation might save us from the complex equations of motion. Since the particle in an accelerator is always in a moving frame of reference it is prudent to set our axes on the particle and then calculate. In differential geometry this frame is called the Frenet-Serret frame and the equations which describe the motion of the axes are known as the Frenet-Serret equations. I have made a bad drawing of how it might look for a particle in a curvilinear orbit:



The equations for the frame are nothing but the equations relating the three unit vectors in the three directions, and this is done by noting the following identity

$$\mathrm{r}_0 . \frac{d\mathrm{r}_0}{ds} = 0$$

which is a basic fact in vector calculus that the derivative of a vector is always perpendicular to the vector. Now this also implies that the quantity $\hat{r_0}$ is also perpendicular to the derivative term and hence this term is an ideal candidate for

$$\hat{s} = \frac{d\mathrm{r(s)}}{ds}$$

Why? Because from the definition of the radius of curvature $\rho(s)$ we have

$$\hat{x}(s) = -\rho(s)\frac{d\hat{s}(s)}{ds}$$

It might look like the variable s, that is, the distance moved over the arc, might have replaced the position of the time variable t, and this symmetry is manifest when we discuss about the Hamiltonian in this frame. Finally

$$\hat{z}(s) = \hat{x}(s) \times \hat{s}(s)$$

Taking derivatives of the last two equations with respect to s we have the famous Frenet-Serret equations (all of them, actually)

$$\frac{d\hat{x}(s)}{ds} = \frac{1}{\rho(s)}\hat{s}(s) + \tau(s)\hat{z}(s)$$

$$\frac{d\hat{z}(s)}{ds} = -\tau(s)\hat{x}(s)$$

Now that we are done setting up the frame, we have the following problem at hand. We can write the equation of a charged particle in an electromagnetic field $(\Phi, \bar{A})$

$$\mathcal{H} = e\Phi(x, y, z) + c[m^2c^2 + (p^\mu - eA^\mu)^2]^{\frac{1}{2}}$$

and from this we can write the equations of motion (the *Hamilton Equations*)

$$\dot{x}^{\mu} = \frac{\partial \mathcal{H}}{\partial p_{\mu}}$$

$$\dot{p}^{\mu} = \frac{-\partial \mathcal{H}}{\partial x_{\mu}}$$

The issue is apparent. These equations are frame dependent, and they are presently in the Cartesian frame. We need to perform a *canonical transformation* to get this to the Frenet-Serret (also known as the TNB) frame. The theory of canonical transformations is explained best in the classic [GOL] and I would ask you to refer the relevant sections. To put it in simple terms, there are specifically four kinds of canonical transformations which are transformations on the phase space which leaves the Hamilton equations invariant. Referring to the table in [GOL], we can say in this case we have to do the transformation of the third kind, namely, the generating function is of the form

$$F_3(\bar{p}, x, s, z) = -p^{\mu}.(r_{\mu}(s))$$

$$\bar{r}(s) = \bar{r}_0(s) + x\hat{x}(s) + z\hat{z}(s)$$

and then the canonical momenta follow by differentiating the generating function with respect to the corresponding new canonical variable, thereby we have

$$\tilde{p}_s = -\frac{\partial F_3}{\partial s} = (1 + \frac{x}{\rho(s)})\bar{p}.\hat{s}$$

$$\tilde{p}_x = -\frac{\partial F_3}{\partial x} = \bar{p}.\hat{x}$$

$$\tilde{p}_z = -\frac{\partial F_3}{\partial z} = \bar{p}.\hat{z}$$

and our new Hamiltonian looks like

$$\mathcal{H}(x, z, s) = e\Phi(x, z, s) + [m^2 + \frac{(\tilde{p}_s - eA_s)^2}{(1 + \frac{x}{\rho})^2} + (\tilde{p}_x - eA_x)^2 + (p\tilde{z} - eA_z)^2]^{\frac{1}{2}}$$

where the components $A_{\mu} = \bar{A}.\hat{x}_{\mu}$. There is one more fact to be clarified before one calls it a day, the fact that I have not yet reduced the equations of motion to a sufficiently solvable stage (rather I am just where I ended 2.1, the Euler-Lagrange equations or the Hamilton equations would lead me straight to the three erstwhile derived ones, but I have something more to work with here rather than a dead end), and I see that s has taken the place of t, so naively assuming that t and $\mathcal{H}$ are canonically conjugate (this is not rigorous, but in a first glance, so isn't the time-energy uncertainity), I could say

$$\mathcal{H}_{new} = -\tilde{p}_s$$

To explain this, let us define a general Hamiltonian $\mathcal{H}(q_1, q_2, q_3, .., q_f, p_1, p_2, ..., p_f, t)$ and one can see that $t = q_0$ and $\tilde{P}_0 = -\mathcal{H}$. I invent a zero Hamiltonian

$$\mathcal{H}'(q_0, q_1, q_2, ..., q_f, \tilde{p}_0, \tilde{p}_1, \tilde{p}_2, ..., \tilde{p}_f) = \mathcal{H} + \tilde{p}_0 = 0$$

which is really useless except for the fact that it's sole existence brands it as a Hamiltonian, that is, it follows Hamilton equations. So, in this situation, if we wanted to, say, make a q_k "equivalent" to time, we take this dummy Hamiltonian and write it with $\tilde{p}_k$ as the subject

$$\mathcal{H}'(q_0, q_1, ..., q_f, p_0, p_1, ..., p_f) = 0$$

$$\tilde{p}_k = -\mathcal{K}(q_0, q_1, ..., q_f, ...p_f)$$

And then we can conclude that the new prescription, say $\mathcal{K}^* = \mathcal{K} + \tilde{p}_k$ is enough to retain the structure but shifting out t for q_k, and with this out of the way, the equality $\mathcal{H} = -\tilde{p}_s$ is rather self explanatory. With this explained, we can now take the Frenet-Serret Hamiltonian and we can write it with $\tilde{p}_s$ as the subject to give us

$$\tilde{p}_s = eA_s h + h\sqrt{p^2 - p_\perp^2}$$

where I have set $c = 1$ to save ink and also you must note that there are no tildes on the momenta in the right hand side for they are physical momenta and not canonically conjugate ones (refer [GOL]). Also $p_\perp = p_x^2 + p_y^2$ and $h = 1 + \kappa_0 x + \kappa_0 y$ is the reduced curvature function. Now, I make assumptions. Let the particle be going in a closed orbit, where it deviates only slightly from the exact equilibrium circular orbit (I will prove the existence of non circular closed orbits in a moment) and hence I may assume that the momenta in the directions of the normal and the binormal are non relativistic and I say $x' = \frac{dx}{ds} \approx \frac{p_x}{p_s}$, $y' = \frac{dy}{ds} = \frac{p_y}{p_s}$, and that most of the actual momentum $p = \sqrt{||\bar{p}||^2} \approx p_s$. I then have

$$\tilde{p}_s = eA_s h + hp_s\sqrt{1 - x'^2 - y'^2}$$

$$\tilde{p}_s = p(\frac{eA_s h}{p} + h\sqrt{1 - x'^2 - y'^2})$$

I make things even simpler to work in the scenario where I set p to 1 (people fancily call this process normalizing the momentum, the idea is to find all constants and set them to 1 to further save ink) and recalling the facts about shifting s to t, we see that

$$\mathcal{K} = -\tilde{p}_s = -eA_s h - h\sqrt{1 - x'^2 - y'^2}$$

And we are done. Using the Hamilton equations on this, we get simplified second order equations in x and y which can be solved by using any standard numerical routine, say the Runge-Kutta iteration. In cases where p itself varies, we can write up to a first order perturbation δ that

$$p = p_0(1 + \delta)$$

$$\implies \mathcal{K} = -\frac{eA_s h(1 - \delta)}{p_0} - h\sqrt{1 - x'^2 - y'^2}$$

For a model which can be solved by hand, we could neglect the square root term by saying that the beam is varying in the x and y directions ever so slightly (people fancily call this the paraxial approximation, also used in geometric optics, but more on that in the next section) and get an effective Hamiltonian

$$\mathcal{K} = -\frac{eA_s h(1 - \delta)}{p_0} - (\kappa_{0x} + \kappa_{0y} + 1)$$

Remembering that the Hamiltonian is in variables x, y, x', y' and using the Landau gauge $B_y = -\frac{\partial A_s}{\partial x}$ and $B_x = \frac{\partial A_s}{\partial y}$ I present to you the equations of motion of a particle in an accelerator with a small perturbation closed orbit

$$x'' + \frac{eB_y h(1 - \delta)}{p_0} - \kappa_{0x} = 0$$

$$y'' - \frac{eB_x h(1 - \delta)}{p_0} - \kappa_{0y} = 0$$

2.4 Closed Orbits

Called a star's orbit to pursue,
What is the darkness, star, to you?

Friedrich Nietzche

You might still be wondering whether the differential equations which were the result of the yakkity yak in the last section are solvable. The answer is yes and the two paraxial equations are what mathematicians call *Hill's equations* named after George William Hill who analysed the lunar perigee based on the mean motions of the Sun and the Moon [HIL]. In the most general form they are written as

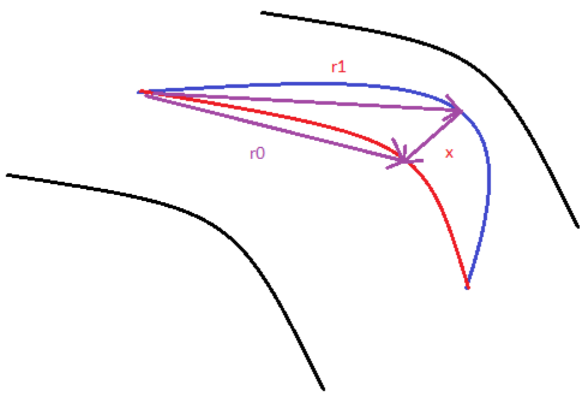
$$\frac{d^2x}{ds^2} + F(s)x = K$$

The special cases in which the function $F(s)$ is periodic are known by many names in the literature: such as the *Mathieu Equation* or the *Meissner Equation.* The solutions to Hill's equation are treated using what is known as *Floquet theory*, a subject which I would be using for my thesis next semester while solving the perturbation problem in the case of oscillating potentials. A first attack at the solution might be observed by taking the Fourier space representation of $F(s)$ and then solving the resulting series term by term (the first order solution being that of Mathieu and the second order solution being that of Meissner). I would not be discussing any of the further topics (such as Hill determinants et cetera) for reducing the problem to a problem of quadratures is good enough for an approximate solution.

Regarding the problem of closed orbits, I refer back to [GOL] page 89, and note a very interesting theorem due to Joseph Bertrand. It involves the solutions of what we call central force problems where there is always a radial potential, and hence a radial force between two bodies. Let us say we have a force law of the form $f(r)$. Then Bertrand's theorem states that the only orbits for which there is a periodic solution is when

$$\frac{d(lnf)}{d(lnr)} = \beta^2 - 3$$

where β is a *rational* number. This might look like there are a great number of force laws which should give us closed orbits but then the restriction of β being rational eventually cuts down many of those. For the inverse square case one obtains an elliptical orbit as expected. I now refer to the paper by Courant and Snyder [COU] which has become an industrial approach to magnetic field design for beam focusing. In this, there is a simple derivation which concludes in the authors establishing the fact that multiple closed orbit solutions exist without ever referring to Bertrand's theorem. What they claim is that for the orbits which are closed, they must enclose the *maximum flux* when compared to *neighbouring orbits* of the *same circumference.* The proof is simple. Look at the diagram below (and sigh at my drawing skills)



In this we have two orbits, the red one, which is supposed to be a random closed orbit and the blue one which is the perturbed orbit. $\bar{x}$ is the perturbation as shown, and with this one could write from simple vector algebra that

$$\bar{r}_0(s) = \bar{r}_1(s) + \bar{x}(s)$$

and then computing the circumference for a path is done using the familiar line integral. We calculate the the circumference for the above orbit as follows

$$C_0 = \oint \bar{r}_0.\bar{\kappa}ds = \oint_C \bar{r}_1(s).\bar{\kappa}ds + \oint \bar{x}(s).\bar{\kappa}ds$$

Denoting the original circumference by C, we have the following

$$C_0 = C + \oint \bar{x}(s).\bar{\kappa}ds$$

and for both the curves to have the same circumference the line integral of $\bar{x}(s)$ must go to zero, that is

$$\oint \bar{x}(s).\bar{\kappa}ds = 0$$

If we can prove that for paths which are a perturbation away the flux difference is minimal, we have proved that the flux attains a maximum around C_0. The flux can be worked out with a little vector math

$$\delta\Phi = \oint \bar{B}.\hat{n}ds = \oint \bar{B}.(\hat{s} \times \hat{x})ds = \oint \hat{x}.(\bar{B} \times \hat{s})ds$$

The curvature κ can be written as a function of the magnetic field, and it is easy to see that

$$\bar{\kappa} = \frac{e\bar{B} \times \hat{s}}{pc} \implies \bar{B} \times \hat{s} = \frac{pc\bar{\kappa}}{e}$$

Plugging this in the earlier integral for the flux, we see that

$$\delta\Phi = \oint \frac{\bar{x}}{||\bar{x}||}.\frac{pc\bar{\kappa}}{e}ds = \frac{pc}{||\bar{x}||e} \oint \bar{x}(s).\bar{\kappa}ds$$

Clearly if the left hand side(s) have to be zero, the right hand integral must vanish and this implies the curves have the same circumference. Hence, proved. The idea of using the Bertrand theorem struck me when I was reading [COU] and the diagram presented there was similar to the one in [GOL]. Recalling the theorem, we have

$$\frac{d(lnf)}{d(lnr)} = \beta^2 - 3$$

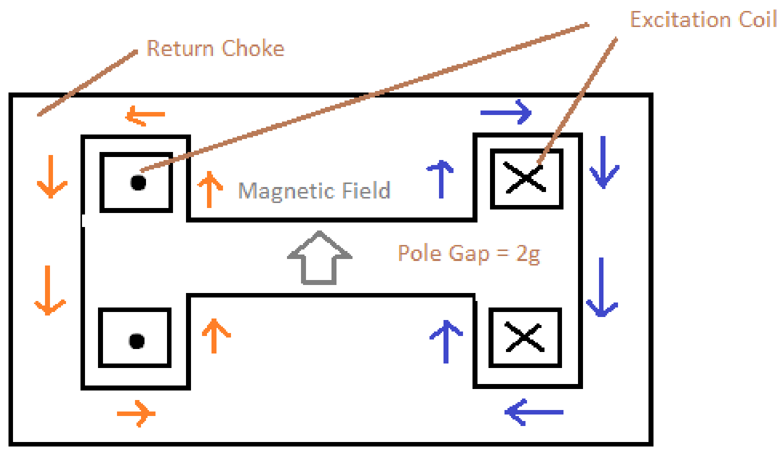
$$\implies \frac{rdf}{fdr} = \beta^2 - 3$$

The force in this case is the Lorentz force and we have

$$f = e(\bar{v} \times \bar{B}) = e(\frac{ds}{dt}\hat{s} \times \bar{B})$$

In an orthogonal axisymmetric magnetic field, one could make two simplifications: $\bar{B}$ and $\hat{s}$ are perpendicular and that $||B|| = gr$, and plugging this one can easily see that β is ± 2, essentially the inverse square problem (notice that we are drawing the similarity by comparing the β parameters) and the trajectory is probably circular.

Of course a rather general description can be given by not assuming the axisymmetric case (keeping orthogonality) and by writing $||B||$ as a power series in r, thereby getting back to the multipole expansion and computing the derivatives around equilibrium. In physical reality, one would place specific constructions of magnets (usually superconducting, at around 1 degree Kelvin) and the structure of these magnets determines the limit of the multipole expansion. For instance a simple dipole magnet construction is shown below, commonly known as a H-dipole, the archetype.



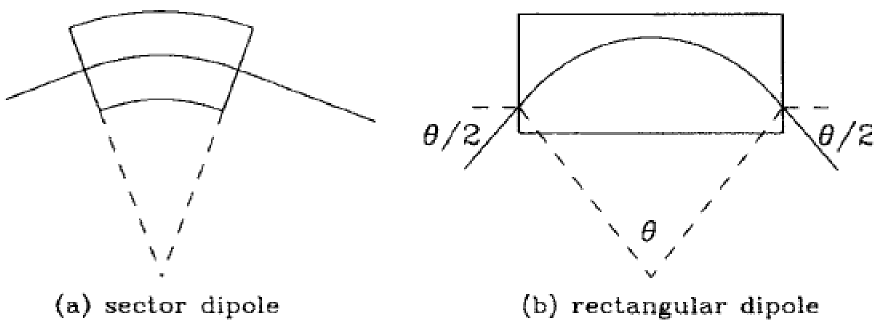
Using the Ampere law, we can see that the line integral of the magnetic field must be proportional to the current through (any of the) excitation coil(s). Since the return choke is mostly made of a ferromagnetic material, we can assume that the permeability is infinite, but the equation for the finite μ_r case is simply

$$B_{gap} \times 2g + \int \frac{\bar{B}.d\bar{l}}{\mu_r} = \mu_0 I_{total}$$

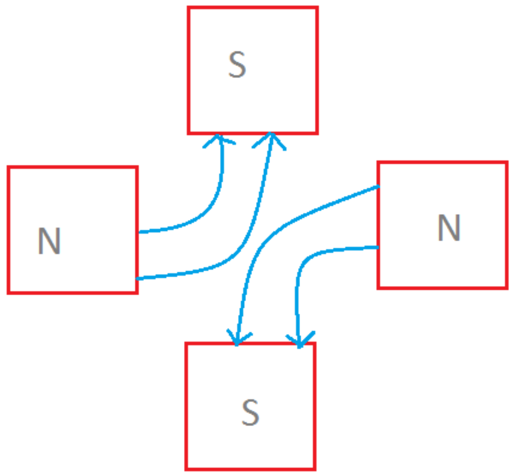
And now plugging $\mu_r = \infty$ and observing that $I_{coil} = \frac{I_{total}}{2}$ we have the following

$$I_{coil} = \frac{gB}{\mu_0}$$

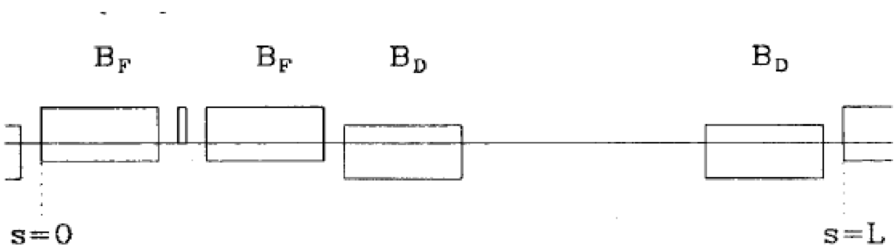
where g is the gradient. Whatever it may be, one can see that the dipole magnet is only good for bending beams and not to do anything with narrowing or spreading the beam, that is, every particle faces the same gradient in a local patch. The images below, taken from [LEE] show what are called as sector magnets which essentially bend the beam over a sector of angle θ



To unlock the features of focusing and defocusing one must go to the next order in the multipole expansion. It is here that one can relate the theory much to that of geometric optics. In geometric optics, when we assume that the rays entering a lens (in this case the quadrupole magnet) are in a small angle around the axis (also called paraxial approximation there) we could directly hit the matrix formulation and compute the focal length and other parameters. I won't discuss the details here, but refer to [LEE] or [GHA] for interesting comparisons with optics. Therefore, in real world designs, one usually approximates the quadrupole magnet, which looks actually like in the figure shown below, with the disgusting field lines exaggerated only for explanatory clarity



[LEE] presents a section of the Fermilab booster lattice which represents what is called a FODO lattice. A FODO lattice is a fancy abbreviation of FOcusing DefOcusing lattice. It is a periodic lattice and the complete booster is made up of 24 or so sections of this lattice.



It might seem like an inappropriate place to end discussions regarding these machines but I am restricted to do so, for this only serves as an introduction to display that the core ideas behind the LHC are actually based on basic principles which are taught to sophomores, and that to get to the bigger picture one only has boundless amounts of literature to explore through. In the next section, I would brief about the present day computer simulation techniques available for such calculations and then I would begin the discussion on quantum field theory. For generic study on particle accelerator physics, I would request you to go through [LEE], [WIE] and [BAI]. For information regarding construction of iron-dominated magnets, one could refer to the volume by Stanford's Jack Tanabe [TAN] and the document by (again, SLAC's) George Fisher [FIS]. A few patents whose schematics are worth noticing for improvements in our basic designs are also given in the bibliography (see [ROS], [KER1], [BLE], [WID] and [WES]) for those who are interested in the mechanical construction of the components.

2.5 Computer Simulation

It's not a Turing machine, but a machine of a different kind.

Richard Feynman, Simulating Physics With Computers

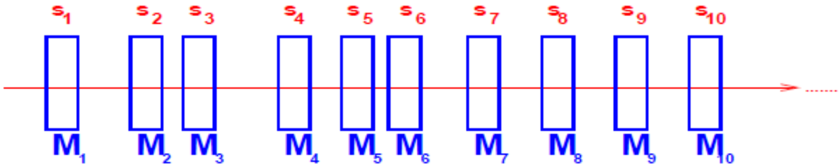
For people who cannot afford a particle accelerator in their backyard, the question of computer simulation is of paramount importance as it is probably the most sensible (it's completely free) way to see whether a design works or not. Many groups participate in this exercise of developing computer libraries which allow users to specify the parameters of their design and watch the computer tell you how it is going to look. As tempting as it may sound to admit that computers simulate physical reality completely the possibility of constructing a computer which can do so perfectly in uncertain, the philosophical rebuttal to this being the fact computers would then have to simulate themselves, and so on (for an enlightening discussion on related matters, please see [FEY]).

There are many simulation packages available, and it is easy to write one provided you account for components not only like the bending dipole magnets, the lensing quadrupole magnets but also standard components like drift guide tubes, RF cavities, orbit correctors and so on. The industry standard is MAD, an abbreviation for Methodical Accelerator Design, and its extended edition, MAD-X. This is a scripting language developed by CERN which allows users to enter parameters in a sequential manner and computes optical functions and additional parameters (Twiss parameters, et cetera, refer [TWI] for further details) . A few years before, PTC, which stands for Polymorphism Tracking Code (the reason for the naming is irrelevant for the present discussion, refer [FOR] for further information) was operated independent of MAD and only recently has it been integrated to form the MAD-X platform. A typical program would look like the following, taken from the official tutorial [HER]

```
length = 14.3;
B = 8.33;
PTOT = 7.0E12;
ANGLHC = B * clight * length/PTOT;
MBLHC: SBEND, L = Length, ANGLE = anglhc;
```

For those who feel sick at the thought of learning a new language, PyORBIT is a package for the Python programming language (the implementation white paper providing the necessary technical information [SHI]) which is built on the erstwhile ORBIT code [SHI1]. These tracking codes are known under a larger classification of what are known as symplectic tracking codes. A last look at [GOL] confirms that the word symplectic might refer to Hermann Weyl's matrix formulation of the Hamiltonian formalism and this is confirmed in the seminal paper [TAL], in which the description of the program TEAPOT is given (an updated version known as ETEAPOT is described in [TAL1]). However, alternate formulations are possible in which familiar ideas of Floquet theory might be used to completely avoid facing the Hamiltonian or the canonical phase space picture [FOR1].

Ultimately, the computer uses whichever formalism possible to analyse the machine. It divides the machine (technically called the lattice) into many sub-lattices involving individual electromagnetic structures through which the particle motion is governed by Hill type differential equations. For instance, similar to the FODO lattice shown in the previous page, one could construct a lattice involving various magnets as follows (picture taken from [HER1])



It is not necessary that the bending of the beam is done smoothly as the arc parameter varies, and this provides us even more possibilities of computer codes, the so called kick-codes which completely discretize the circumference and model it using polygons. Recalling our analogy with geometric optics, the rest of the computation is just matrix multiplications. The Simple Accelerator Modelling with Matlab is an excellent reference for nascent accelerator designing as it is made for MATLAB and can be readily deployed for application, and the user manual is in itself an amazing guide to computational design procedure [WOLS].

There are parallel programming options also for improving speeds over large design simulations and people at the Paul Scherrer Institute have developed the OPAL framework for this [ADE]. As simple the scripting looks the mathematics and computation that goes behind the syntactic sugar is intense and [REE] is a good guide for diving deep into what makes these symplectic codes. Further references include [FOR2], [HOL], for the C++ people, MERLIN is a free beam tracking package which comes compatible[MOL], and for the FORTRAN people, MAGNUM is a propreitary package for electromagnetic structure calculations [BOT].

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3 Quantum Field Theory

3.1 Introduction

The girl I find who wants to talk about quantum theory in a bar is the one I want to marry.

Brandon Boyd, lead singer of Incubus

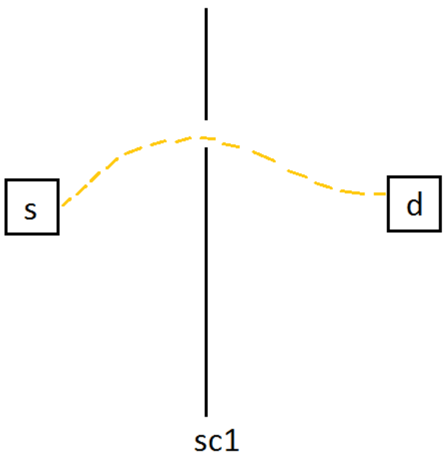
I must admit, to be honest, that an average human being would never need quantum field theory for survival. The subject of number theory, which was thought to be useless for most of the beginning of the twentieth century (to quote G. H. Hardy, "The Theory of Numbers has always been regarded as one of the most obviously useless branches of pure mathematics", see [HAR] for a classic) has found extensive use in modern day cryptography (the RSA cryptosystem is built on the difficulty of prime factoring), but unlike the theory of numbers, which is in itself, a creation of our own intuition, classical physics is something that we are innately born with. One of the masters of field theory, Gerard(us)'t Hooft, states that [GER] quantum mechanics is simply a mathematical tool invented by us to compute things regarding the classical world we live in, and for one to get the right answer, one just takes the classical part of any quantum mechanical calculation. But it is quite obvious to the layperson that he needn't even do classical field theory. Again, to quote another legend, Sidney Coleman [COL], a deer doesn't need to go through the entire Hamilton Jacobi formalism and calculate action-angle variables to plan its escape trajectory from a cheetah, but it simply does so for it was meant to do so. The philosophical question lurking here is in regard to which set of rules living beings are innately imparted with, but that is for another time.

Most of the quantum field theory that I have studied is from the excellent book by Anthony Zee [ZEE] (who was incidentally a graduate student of Sidney Coleman), which is very readable for an advanced undergraduate and my discussion would hence obviously be similar to the book. There are many standard textbooks available, to name a few of the best-sellers, [PES], [ITZ], [CHE], and the three volume work by Weinberg [WEI]. To establish continuity, I assume you know non relativistic quantum mechanics up to time dependent perturbation theory (if not, go straight to [GRI] or [SAK]) and special relativity (an amazing introduction to this can be found in [MER]).

The theory of non relativistic quantum mechanics was not exactly consistent as a fundamental theory of nature as the other contender, special relativity, was speeding ahead (pun not intended) as a more viable tenet. And as expected, this conflict caused many to question the validity of quantum mechanics (a debate which is going on even now, although most take 't Hooft's stance). To be concise, there were two issues: one, quantum mechanics runs on the idea that every process has a non zero probability, more famously known by the name of the Totalitarian principle (due to Murray Gell-Mann, see [GEL]) that *everything not forbidden is compulsory.* But then Einstein comes along and states that nothing in the world can cross the speed of light and with the help of Minkowski, draws the famous light cone picture one sees in every textbook of relativity. The condition of an upper bound on the speed of any object can be seen as being equivalent to the fact that the probability that a particle would move from one spacetime point to another is zero if the points are spacelike separated. But the totalitarian principle still allows us to assign a probability, and a simple calculation done in [PES] shows that there is one (which decays as e^{-m}). The second issue (I said two issues), is again Einstein's notion of the interconversion of mass and energy (the atomic bomb, blah, blah), which creates a rather inescapable situation for Schrodinger's equation. Traditionally seen as a one particle equation, it fails to explain why when an electron is placed in a well, there is pair production noted when the well's width is reduced to the Compton wavelength of the electron, for it should be right for any width of the well.

Thus, we need an upgrade to the old theory to account for the above two (primary) and many other issues and quantum field theory is the candidate, specially borne out of this marriage between quantum mechanics and special relativity. It turns out that this is nothing new, and that condensed matter physicists were doing the same for a really long time, just under the title of "many body physics", as explained in the Prologue, and hence bolstered people's hopes that QFT was indeed the right theory (until the unification of gravitational fields came by, but this wasn't the first hiccup, for an interesting and delightful exposition of the history, see [CLO]).

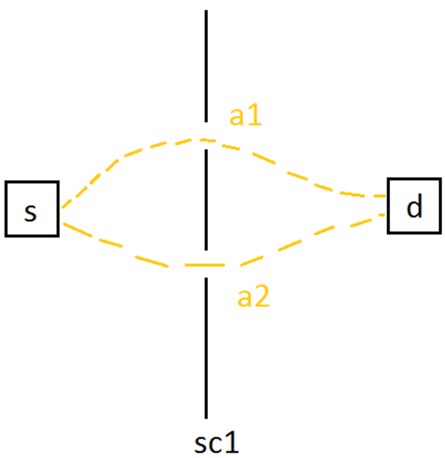
Now, I will describe a thought experiment, which is due to Richard Feynman [FEY], although the original idea is traced back to Paul Dirac's works. Let us assume that we have a source of photons (any quantum mechanical particle will do, bullets won't) and a screen with one slit, and a detector on a wall behind the screen.



The probability a photon makes it to the detector assuming there is no electromagnetism in our world here, let us call this as $\mathcal{A}$, is given by

$$\mathcal{A} =< d|sc1 >< sc1|s >$$

Now, I cut another slit in the screen and label the slits a_1 and a_2. The situation is as shown below



$$\mathcal{A} =< d|a1 >< a1|s > + < d|a2 >< a2|s >$$

and similarly, if I had another screen with three slits in front of this screen, say b_1, b_2, b_3, I could write down the expression for the probability of detecting a photon as

$$\mathcal{A} =< d|a1 >< a1|b1 >< b1|s > + < d|a1 >< a1|b2 >< b2|s > +..............+ < d|a2 >< a2|b3 >< b3|s >$$

Here comes the philosophy. If I had infinite such screens (each of them infinitesimally thin) and each screen had infinite slits (each of them infinitesimally small), we are talking about empty space. Empty as it may be, our calculation is not zero! That is,

$$\mathcal{A} =< d|s >= \sum_{paths} < d|P|s >$$

where P is an operator which varies over different paths chosen for the photon to travel in space. This is (zen) something magnificent, and is known in the literature by the name of the Path Integral formalism of quantum mechanics, or the Sum Over Histories formalism as one has to add up all possible histories of the photon getting from s to d to get the probability that it would actually get there, unberknowst of its path. The insight this formalism provides us can be seen by doing a the following calculation, which also incidentally is what lattice gauge theorists do (to one point). It is slightly involved, so bear with me.

Let us take a particle in vacuum, initially at a point q_I, which goes to another point q_F in time T. If the particle had a Hamiltonian $\mathcal{H} = \frac{p_\mu p^\mu}{2m}$, then the amplitude of vacuum propagation is given as

$$\mathcal{A} =< q_F|\hat{U}|q_I >$$

$$\mathcal{A} =< q_F|e^{-i\mathcal{H}T}|q_I >$$

The familiar property of completeness can be used to write this amplitude as the sum over its histories, and as Prof. Vijay Shenoy of IISc would say [SHE], stare hard in the space in between the ket and the operator, and you see a set of complete states appear

$$\mathcal{A} = \int dq_1 < q_F|e^{-i\mathcal{H}T/2}|q_1 >< q_1|e^{-i\mathcal{H}T/2}|q_I >$$

$$\mathcal{A} = \int dq_1 dq_2 < q_F|e^{-i\mathcal{H}T/3}|q_2 >< q_2|e^{-i\mathcal{H}T/3}|q_1 >< q_1|e^{-i\mathcal{H}T/3}|q_I >$$

Let us do this for N more times and we have the following

$$\mathcal{A} = \int dq_{N-1}dq_{N-2}....dq_2dq_1 < q_F|e^{-i\mathcal{H}T/N}|q_{N-1} >< q_{N-1}|e^{-i\mathcal{H}T/N}|q_{N-2} >$$

$$......... < q_2|e^{-i\mathcal{H}T/N}|q_1 >< q_1|e^{-i\mathcal{H}T/N}|q_I >$$

Calling $\frac{T}{N}$ as δt for convenience, I focus my attention on one of the mini amplitudes in the integration

$$\mathcal{A}_j =< q_{j+1}|e^{-i\mathcal{H}\delta t}|q_j >=< q_{j+1}|e^{-i\frac{p^2}{2m}\delta t}|q_j >$$

I observe that since the operator is a function of $\hat{p}$, the momentum operator, I could expand it in an exponential series and iteratively let it attack the ket, but if I stare hard again, I can insert a set of complete momentum states also (their normalization is 2π not 1, so I have an extra factor too).

$$\mathcal{A}_j = \int \frac{dp}{2\pi} < q_{j+1}|e^{-i\frac{p^2}{2m}\delta t}|p >< p|q_j >$$

The term $< p|q >$ is known to be e^{-ipq} from a famous theorem due to Michel Plancherel who derived it for Fourier transforms [PLA]. Pushing this and now picking the momentum eigenvalues, I have

$$\mathcal{A}_j = \int \frac{dp}{2\pi} e^{\frac{-ip^2\delta t}{2m}} e^{ip(q_{j+1}-q_j)}$$

This is a Gaussian integral, except there is an ugly linear term sitting. Completing the square and pulling the p independent constant out, the result is

$$\mathcal{A}_j = \sqrt{\frac{-i2\pi m}{\delta t}} e^{\frac{im(q_{j+1}-q_j)^2}{2\delta t}}$$

Plugging these terms into our original N dimensional integral over all paths, and relabeling $q_F = q_N$ and $q_I = q_0$, and tending $\delta t \to 0$,

$$\mathcal{A} = \int dq_{N-1} dq_{N-2} .... dq_1 \sqrt{\frac{-i2\pi m}{\delta t}} e^{\frac{im}{2} \int_0^T \dot{q}^2 dt}$$

This is where lattice gauge theorists stop, as advertised before. They discretize space to N parts and work this integral out. You probably are wondering how the square root of an imaginary term is relating to a real probability. There is some math which I have skipped, which I will explain in a moment, but the short answer is I could do a continuous transformation to imaginary time and get rid of all the imaginary terms. For the rigorous answer, we have to tend $N \to \infty$ to cover all possible paths and along with the constant, we replace all the integration variables with the grand notation $\mathcal{D}q(t)$

$$\mathcal{A} = \int \mathcal{D}q(t) e^{i \int_0^T \frac{1}{2} m \dot{q}^2 dt}$$

Surprise, surprise! The exponent is the integral over the kinetic energy. Recollect from classical physics that the only term which ever happened to be the integral of energy over time is the action and this action is the time integral of the Lagrangian. So, if we had a particle, say a photon in an electromagnetic field, our answer wouldn't be much different, except we replace the kinetic energy term sitting in the exponent with the Lagrangian (density)! Our final answer, with any external potential, is as follows

$$\mathcal{A} = \int \mathcal{D}q(t) e^{i \int_0^T \mathcal{L}(q)}$$

If there is any other simplification you can see here, you should probably publish it. This is the core idea of the quantum theory of fields. Evaluating this integral. As involved the derivation was, the use of this integral is limited to not many cases and this is what makes doing field theory hard.

Oh, by the way, the quantity $\mathcal{A}$ is called the *vacuum expectation value*, and is called VEV in much of the literature, and is denoted by Z usually.

3.2 Mathematics

There was a footpath leading across fields to New Southgate, and I used to go there alone to watch the sunset and contemplate suicide. I did not, however, commit suicide, because I wished to know more of mathematics.

Bertrand Russell

I already computed one Gaussian in the previous section and for those who trusted my mathematical abilities, here is the actual formula for the standard Gaussian,

$$\int_{-\infty}^{\infty} dx e^{-ax^2} = \sqrt{\frac{\pi}{a}}$$

This will not be sufficient for evaluating multivariable Gaussian integrals and I need some more formalism, which was thankfully worked out half a century ago by the extremely under-appreciated Italian theoretical physicist Gian-Carlo Wick [WIC], and now I am able to sit here and chug monstrous integrals with ease. Wick also contributed to the subject of the continuous time transformation I had mentioned previously, known famously as the Wick rotation, which I will come to in a moment. Consider an integrable space with a Gaussian measure defined on it. With this I mean, given any function $f(x)$, the integral $\int f(x)e^{-ax^2}dx$ is well defined. Now, if I wanted to compute the average of the function $f(x) = x^{2n}$ for a general n, that is, I need the quantity (I changed the constant in the exponent because I already know this stuff and it shouldn't matter)

$$< x^{2n} >= \frac{\int_{-\infty}^{\infty} x^{2n} e^{-\frac{1}{2}ax^2} dx}{\int_{-\infty}^{\infty} e^{-\frac{1}{2}ax^2} dx}$$

Integration by parts would be a first attack and this would lead to a messy calculation but the answer comes out all right, but a much clever way out would be to cheat and use induction. Trying it out for the cases $n = 1, 2, 3$ I see a pattern, and I generalize this pattern to give you the following formula:

$$< x^{2n} >= \frac{\int_{-\infty}^{\infty} x^{2n} e^{-\frac{1}{2}ax^2} dx}{\int_{-\infty}^{\infty} e^{-\frac{1}{2}ax^2} dx} = \frac{1}{a^n}(2n-1)(2n-3)....5.3.1$$

The rightmost side looks really nice but it can be put into nicer terms by completing the factorial and one has the following

$$< x^{2n} >= \frac{1}{a^n}(2n-1)(2n-3)....5.3.1 = \frac{1}{a^n}\frac{(2n-1)!}{2^n(n-1)!}$$

This expression has a combinatorial explanation, and this genius is what Wick is usually credited to. The factorial expression is the number of ways to make pairs (so called Wick contractions) out of $2n$ x's and is symbolically written as $< xxx...x >$ where you can make $\frac{(2n-1)!}{2^n(n-1)!}$ such pairs and each pairwise integral is equal to $\frac{1}{a}$, that is

$$< xxx...x >= \sum_{pairs} < xx >< xx >< xx > ... < xx >$$

The higher order generalization of Wick's theorem is what we are most interested in, and for this let us assume that $\bar{x}$ is a N dimensional vector and let A be an N by N real symmetric matrix. What is the value of $< x_a x_b x_c ... x_l >$?(where the number of x_i is not important but I should be able to do it for any number). I basically want

$$\frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}....\int_{-\infty}^{\infty} dx_1 dx_2 dx_3 ....dx_N e^{-\frac{1}{2}\bar{x}.\hat{\mathcal{A}}.\bar{x}^T} x_a x_b x_c ... x_l}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}....\int_{-\infty}^{\infty} dx_1 dx_2 dx_3 ....dx_N e^{-\frac{1}{2}\bar{x}.\hat{\mathcal{A}}.\bar{x}^T}}$$

If you are freaking out at the expression, set N to 1 and restore peace by noting that everything is the same as that on the last page, but here integration by parts becomes a thing to laugh about and instead we will use the same analogy Wick used (although it can be proven rigorously, see [KUM] or [ZEE]) and state the following equality

$$< x_a x_b x_c ....x_l >= \sum_{Wick} (A^{-1})_{\alpha\beta}(A^{-1})_{\gamma\delta}......(A^{-1})_{\epsilon\mu}$$

The label "Wick" stands for all contractions possible. You can understand why the A^{-1} comes into the picture by comparing it with the single dimensional case where we had $\frac{1}{a^n}$ and this is of the same dimension(in the previous case the number of contractions was easy to count, that's all there is to Wick contractions). The Greek indices are a permutation of the set [a, b, c, ... , l]. As an example, let us calculate $< x_1 x_2 x_3 x_4 >$, the so called *four point correlation function*:

$$< x_1 x_2 x_3 x_4 >= (A^{-1})_{1,2}(A^{-1})_{3,4} + (A^{-1})_{1,3}(A^{-1})_{2,4} + (A^{-1})_{1,4}(A^{-1})_{2,3}$$

As one can see, the number of inverse matrix elements that enter for a general n point correlation function is $C(n, 2)$.

The vacuum expectation value($\mathcal{A}$ or Z or whatever) is clearly a contour integral and now I answer the query raised in the last section. How are we sure that the path integral will converge? Well, for one, since we are ultimately doing field theory, we need to expect the path integral to somehow lead us to the Euler Lagrange equations for classical Lagrangians. All of this is effected by doing another mathematical trick known as the Wick rotation. It is simple, you take the time axis and you rotate it with the transformation $-it \rightarrow t$. Running this argument from the beginning of the path integral derivation in the previous section, it is easy to see that our vacuum expectation value now looks like

$$\mathcal{A} = \int \mathcal{D}q(t) e^{-\int_0^T dt(\frac{1}{2}m\dot{q}^2 + \hat{V}(q))}$$

and et voila! We have a real integral! Our sanity is restored as we reconcile ourselves with the fact that the vacuum amplitude is truly real and here to stay. As for the Euler Lagrange equations, we use the steepest descent method (for a revision refer [ARF] or [BOA]) and it becomes the condition for the convergence. But one might still argue regarding the validity of the Wick rotation, and this can be extended to the fact that as long as one does not cross any poles or other ugliness while rotating the axis, the Wick theorem holds true. For the more mathematical of you, a reference to the *Osterwalder-Schrader Theorem* [SCH] will suffice.

Regarding the mathematical consistency of the entire formalism I just discussed, and for an interesting digression to the axiomatization of physics (Hilbert's Tenth Problem), I would ask you to check [WIG] out regarding the Wightman axioms developed by Arthur S. Wightman, and for further reading on axiomatic systems (so called formal logic systems), I would suggest [FRA], [GOD] for an interesting read about the Godel theorem.

The ideas presented so far are now ready for an extension to dealing with many particle systems. As a consequence, it is sufficient if I know the Lagrangian of the many particle system to compute the vacuum expectation value and thence calculate expectation values using Wick theorems. In the continuum (classical?) limit, I would instead be looking for a Lagrangian density, and my action functional looks like

$$S = \int_0^T dt \int d^d x \mathcal{L}(\phi)$$

I like to call this spactime as a (d+1) spacetime for obvious reasons. Not any random function of the field can qualify as a Lagrangian for it has many tests to go through (which I will not explain) and one of the primary tests is that of Lorentz invariance. Primarily due to this facet, Lagrangian densities usually end up looking like (I am talking about a 2+1 spacetime here, you can draw an analogy with the Ginzburg Landau theory)

$$\mathcal{L}(\phi) = \sigma(\frac{\partial \phi}{\partial t})^2 - \rho((\frac{\partial \phi}{\partial x})^2 + (\frac{\partial \phi}{\partial y})^2).....$$

Instead of writing all the coordinates explicitly, I will use the summation convention and I will write the following instead

$$\mathcal{L}(\phi) = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2.....$$

You might be wondering what happened to the odd powered terms, and that is why I have placed the ellipsis. They form a part of the Lagrangian which contribute to the *interaction* of the fields (I am getting ahead of the story, but whatever) and the part which I have written above is the *free field* Lagrangian. This is what is called working in the *harmonic paradigm.* For notational clarity, when I say $(\partial\phi)^2$, I do not mean $(\frac{\partial \phi}{\partial x_\mu})^2$ but to the following

$$(\partial\phi)^2 = \partial^\mu\partial_\mu\phi = (\frac{\partial \phi}{\partial t})^2 - (\frac{\partial \phi}{\partial x})^2$$

To solve the quantum harmonic oscillator problem in this paradigm is nothing but to set the number of space dimensions to 0, that is, to work in a 0+1 dimensional spacetime. For a classical field, as explained, you anyway end up with the Euler Lagrange equations. So, everything looks good so far.

3.3 Genesis

How they survive so misguided is a mystery.
Repugnant is a Creature who would squander the ability,
To lift an eye to heaven, conscious of it's fleeting time here.

Maynard James Keenan, Right In Two.

I can handle particles in vacuum but then this isn't really interesting and useful. Let $J(\bar{x})$ be a function which we call the "source" of our quantum particles. The simplest model I can build with this function is a linear model in which due to the presence of this source, I would have to add a term to my Lagrangian which linearly couples with the field. That is, my free field Lagrangian now becomes

$$\mathcal{L}(\phi) = \frac{1}{2}((\partial\phi)^2 - m^2\phi^2) + J(\bar{x})\phi(\bar{x})$$

To clarify, consider a field as a quantity which takes in the spacetime coordinates and chugs a number out and we are amplifying this chug with another function of the spacetime point. I now write the vacuum expectation value to begin my analysis of this configuration

$$\mathcal{A} = \int \mathcal{D}\phi e^{i\int d^4x[\frac{1}{2}((\partial\phi)^2 - m^2\phi^2) + J(\bar{x})\phi(\bar{x})]}$$

Notice that the first summand in the exponent looks really susceptible to integration by parts (the integral in the exponent, not the overall field integration, and also, we are in 3+1 spacetime for now). After the integration by parts, the expression becomes (after a little reshuffling)

$$\mathcal{A} = \int \mathcal{D}\phi e^{i\int d^4x[-\frac{1}{2}\phi(\partial^2+m^2)\phi + J(\bar{x})\phi(\bar{x})]}$$

The first term in between the field variables is the well known Klein-Gordon term. There is an interesting extension to the Wick theorem which I had skipped, and I will mention it here (proof is by induction, of course, set N to 1 and do the elementary integral, then extend it to two dimensional matrices, assume true for N, prove for $N+1$)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}.....\int_{-\infty}^{\infty} dx_1 dx_2....dx_N e^{i(\frac{1}{2}\bar{x}.\hat{A}.\bar{x}^T + \bar{J}.\bar{x})} = \sqrt{\frac{(2\pi i)^N}{||\hat{A}||}} e^{\frac{-i}{2}\bar{J}.\hat{A^{-1}}.\bar{J}^T}$$

The integrands look similar and we can exploit this similarity to solve our problem. But where are the vectors? Well, we ultimately are doing quantum mechanics, and this is all happening in a Hilbert space, and the fields ϕ become the state vectors. It is immediate to see that the Klein-Gordon term becomes a matrix operator which is operating on a field to the left and to the right with the "source" coupling linearly to the field implying that it creates further field states. Therefore writing the continuum as a discrete representation, we have

$$\mathcal{A} = \lim_{N\to\infty}\int\int\int ....\int dq_1 dq_2 dq_3 ... dq_N e^{i(\frac{1}{2}\bar{q}.\hat{A}.\bar{q} + \hat{J}.\bar{q})}$$

I have removed the transpose signs for they can be placed whenever needed by verifying the matrix multiplication dimension for consistency. The integral is the same as the extension to Wick's theorem stated just now. Applying the equation instantly, I see that (suppressing the limit)

$$\mathcal{A} = \sqrt{\frac{(2\pi i)^N}{||\hat{A}||}} e^{\frac{-i}{2}\hat{J}.\hat{A}^{-1}.\hat{J}}$$

But wait, what is $\hat{A}^{-1}$? If my derivation is right, it is the inverse of the matrix represented by the Klein Gordon term $-(\partial^2 + m^2)$. From elementary matrix theory, I know that $A.A^{-1} = I$ or $A_{ij}A^{-1}_{jk} = \delta_{ik}$ where δ is the Kronecker delta. But now, jumping back to the real situation of a continuum, we have to replace the Kronecker delta with its cousin the Dirac delta function and hence, we need to find the inverse, which in this case is a continuous function

$$-(\partial^2 + m^2)F(x^\mu - y^\mu) = \delta^{(4)}(x^\mu - y^\mu)$$

If I can solve this equation for F, I can effectively call it the inverse of the Klein-Gordon term and I am done with my solution. To no surprise, this has been solved over a century ago by the British mathematician George Green, and hence are called as Green's functions (for the original paper, see [GRE] and for an excellent introduction to the subject see [RIC]). The straightforward way to deal with these equations is to take the (four dimensional) Fourier transform on both sides and move to k space, the Dirac delta becoming unity,

$$(k^2 - m^2)F(k) = 1$$

$$\implies F(k) = \frac{1}{k^2 - m^2}$$

This is what I was looking for, and now I plug it back into the original expression. The suppressing of the limit can be explained by taking the term with the limit as a constant (in the sense of being independent of the field configuration) and hence I call it $\mathbb{C}$ and I have

$$\mathcal{A} = \mathbb{C}e^{-\frac{i}{2}\hat{J}.\hat{A}^{-1}.\hat{J}} = \mathbb{C}e^{-\frac{i}{2}\int\int d^4xd^4yJ(\bar{x})F(\bar{x}-\bar{y})J(\bar{y})}$$

Isn't the term $\mathbb{C}$ technically infinite? Yes. It is, and that is because it encapsulates the probability amplitude of transitions from vacuum to vacuum which can be uncountably many when the number of fluctuations become large. Therefore, this term is also sometimes written as $\mathcal{A}(J = 0)$. It would not make sense to argue about how the equations are running with an infinite sitting in front of them because then we won't be able to do anything with this theory.

Another interesting point is to see that we started with a single "source" of particles but we somehow ended up with a quadratic function of J in the exponent (!) This is something amazing. The mathematics of the theory tells us that we cannot sensibly analyse a system with a source continually pumping particles without assuming another "sink" which essentially saps out the particles to return vacuum. The "sink" can be thought of as generating anti-particles if you may like but then again, I am getting ahead of the story as usual. The correct explanation for the automatic construction of the sink is due to our choice of working in the quadratic (or) the harmonic paradigm. For ease of discussion, I define

$$\mathcal{W}(J) = \frac{-1}{2}\int\int d^4xd^4yJ(\bar{x})F(\bar{x}-\bar{y})J(\bar{y})$$

There seems to be another issue, if you are quick to point. This term is in the exponent and this term has a point where it cannot be defined (that is at $k = m$). To resolve this issue, I say I avoid the point where there is a difficulty by scooping out an ϵ neighbourhood out and then I smile and say that everything is okay in my domain now. Mathematically

$$m^2 \to m^2 - i\epsilon$$

$$\implies F(k) = \frac{1}{k^2 - m^2 + i\epsilon}$$

The imaginary unit is to make the overall exponential damp so that I do not have any blow ups at infinity where nothing ever happens.

The function F, which I have been calling as Green's function till now, is also called by many other names. Some call it a propagator for it shows the particle where is the source to sink path (x to y) and some call it a correlation function (links two spacetime points). With that bit of mathematical patchwork done, I can now inverse Fourier transform F to plug into $\mathcal{W}$ and I have the following

$$F(\bar{x}-\bar{y}) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik.(\bar{x}-\bar{y})}}{k^2 - m^2 + i\epsilon}$$

Using the four vector modulus expression for the wave vector in the denominator (it is nothing but the momentum four vector, if you are confused) and I have

$$\int \frac{d^4k}{(2\pi)^4} \frac{e^{ik.(\bar{x}-\bar{y})}}{k^2 - m^2 + i\epsilon} = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik.(\bar{x}-\bar{y})}}{k_0^2 - (||\bar{k}||^2 + m^2 - i\epsilon)} = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik.(\bar{x}-\bar{y})}}{k_0^2 - (\omega_k^2 - i\epsilon)}$$

A reference back to [CHU] from the older Bibliography, and I notice that this is a standard contour integral but with two poles at $k_0 = \pm\sqrt{\omega_k^2 - i\epsilon}$. Factoring the denominator and running the residue theorem over one of the integration variables $d^4k = dk_0 d^3k$ I have

$$F(\bar{x}) = -i \int \frac{d^3k}{(2\pi)^3 2\omega_k} (e^{-i(\omega_k t - \bar{k}.\bar{x})}\Theta(x_0) + (e^{i(\omega_k t - \bar{k}.\bar{x})}\Theta(-x_0))$$

Did you see what just happened? We have essentially computed the effect of a disturbance in the quantum field by introducing a source and this gave us a plane wave solution travelling with ω_k. This is true genesis, all because of the quadratic dispersion and the harmonic paradigm, once again. Some people call this as second quantization, and some people (see [WEI1]) want this terminology to be banned.

Now I am ready to show you that if I actually place a definitive source and a sink in the system, I can work out the dynamics of the source and the sink using the theory. To state the $\mathcal{W}$ functional again

$$\mathcal{W}(J) = -\frac{1}{2} \int \frac{d^4k}{(2\pi)^4} J^*(k) \frac{1}{k^2 - m^2 + i\epsilon} J(k)$$

The conjugates have appeared because I have placed my transposition symbols correctly in the discrete model and redid the entire calculation. I write $J(k) = J_1(k) + J_2(k)$ and say that for now J_2 is the sink and J_1 is the source. There is no difference mathematically, but for clarity, let it be so. I then have

$$\mathcal{W}(J_1, J_2) = \frac{-1}{2} \int \frac{d^4k}{(2\pi)^4} (J_1^* + J_2^*) \frac{1}{k^2 - m^2 + i\epsilon} (J_1 + J_2)$$

The terms which come out after completing the product are similar, but with $J_1^* J_2$ and its conjugate and $J_1^* J_1$. I will neglect the latter (self energy) term and I will focus on the cross terms for they involve both the indices and might tell me something about the dynamics (both the cross terms are similar, in one 1 is the source and 2 is the sink and vice versa, it really doesn't matter, so I multiply the entire thing by 2). If there is a significant overlap in the cross terms, when I remove ϵ to see the reality, I have a pole at $k = m$ and this implies that there is a resonance (the integral blows up), a particle is born going from the source to the sink. It is said to be "on" the "mass shell" and is called a "real particle" to distinguish it from those fluctuations which have $k^2 < m^2$ which are said to be "off" the "mass shell" and hence called "virtual particles". This is all just technicality and does not improve our understanding whatsoever.

To simplify my universe further, I say that these sources are localized, that is, they are themselves Dirac delta functions, thence, $J_1(x) = \delta^{(4)}(x - x_1)$ and $J_2(x) = \delta^{(4)}(x - x_2)$. This is also good for me for now I can write these functions also in their Fourier representation and get the following expression (after noting the ingenious trick $f(x) = \mathcal{F}^{-1}(\mathcal{F}(f(x))$ and flipping the variables inside, this computation in [ZEE] stumped me for a while)

$$\mathcal{W}(J) = -\int\int dx_1^0 dx_2^0 \int \frac{dk^0}{2\pi} e^{ik_0.(x_1^0 - x_2^0)} \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\bar{k}.(\bar{x}_1 - \bar{x}_2)}}{k^2 - m^2 + i\epsilon}$$

I can see some delta functions sitting inside and I capitalize on this by integrating over x_2^0

$$\mathcal{W}(J) = -\int dx_1^0 \int \frac{d^3k}{(2\pi)^3} dk^0 \delta(k^0) \frac{e^{i\bar{k}.(\bar{x}_1 - \bar{x}_2)}}{k^2 - m^2 + i\epsilon}$$

The delta function takes all k^0 terms to zero and I have the following

$$\mathcal{W}(J) = -\int dx_1^0 \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\bar{k}.(\bar{x_1} - \bar{x_2})}}{||k^2|| + m^2}$$

(There are no poles here and the infinitesimal term ϵ can be ignored altogether). The time integral in the front looks like a pain and is probably divergent in every sense of the word. This is when I become a physicist. Note that we started all of this from the path integral where we expanded e^{-iHT} into many paths and so the vacuum expectation value should somehow look like e^{-iET}. The integral in the front won't blow up if I am dealing with the propagation in some finite time T. With this, I compare the exponents to see

$$-iET = -i\mathcal{W}(J)$$
$$\implies E = \frac{\mathcal{W}(J)}{T}$$

This kills of the first term, and I only need to do the three dimensional integral. Boot up your favourite computer algebra system and fire in the integral. You should be amazed at the simplicity of the answer.

$$E = \frac{-e^{m||x_1 - x_2||}}{4\pi||x_1 - x_2||}$$

This is the famous result of Hideki Yukawa who worked this out for neutral pions (which follow similar structure of the Lagrangian) derived in 1935 (see [YUK]). This is spectacular. A spinless massive particle traveling from the source to sink leads to an attractive force (gradient of the potential). I did the following calculation for a generic D space dimension and ended with the conclusion that the Yukawa theory will not work for D > 3 (the integral diverges). Thankfully, for D = 3, it does.

Set $m = 0$ and you see that it is the Coulomb potential and combined with the fact that the photon is massless, it looks like this is not a coincidence. (Whatever happened to the self energy terms. I bet they're infinite.)

3.4 Gauge Invariance

Photons have mass? I didn't even know they were Catholic.

Woody Allen

The calculation I have done in the previous section, the Yukawa formula, was derived for neutral pions and these particles are massive and spinless. Now, if you have to deal with any other massive spinless boson, the process would be the same but you would have to replace the Lagrangian with the appropriate one. For the photon, the Lagrangian is derived by constructing a covariant term involving just the Faraday tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and is given by

$$\mathcal{L}_{EM} = \frac{-1}{4} F^{\mu\nu} F_{\mu\nu}$$

According to [ZEE], there is a trick due to Sidney Coleman who assumed that the photon was initially massive and modified the Lagrangian to look like the following

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} m^2 A^\mu A_\mu$$

You would then do the same calculation (integration by parts, flip the variables, calculate the inverse matrix element, call it the propagator, integrate for the energy) and would ultimately set $m \to 0$. You only have to pray that the answer won't blow up, and the answer won't, not because Coleman said so, but because of another fundamental idea, known as *gauge invariance.*

I will follow the discussion in the paper [FLI] which will lead us straight to the exciting topic of ghost fields. You might have heard of the Coulomb and Lorentz gauges and also of the term "fixing" a gauge. The use of the term "fixing" is traditionally to modify the vector potential so that Maxwell equations remain invariant. A gauge transformation is defined to be a transformation

$$A'_\mu(x) = A_\mu(x) + \frac{1}{e} \partial_\mu \lambda(x)$$

Plug this into the covariant Maxwell formulation and you will see the same equations (the term e is placed for this reason). It turns out that this isn't really as random as it seems, and one can imagine all these A_μ's generated to be in a set and observe that two of these would be identical to Maxwell if one is the gauge transformation of the other. A_μ's which are like this are said to be on the same gauge orbit (more fancy language, please). Of course, if they cannot be related, then they are on different orbits. It is obvious that there are infinitely many vector potential candidates on any given gauge orbit. The problem comes up when we try to run our formalism with this idea. Consider the vacuum expectation

$$\mathcal{A} = \int \mathcal{D} A_\mu e^{i \int dt \mathcal{L}_{EM}}$$

The issue is apparent. For a given gauge orbit, I have to integrate over infinitely many field configurations! The idea is to, therefore, differentiate between orbits and integrate over one potential in each orbit only. This way we don't go crazy and it sounds sensible to not count the same thing twice. Again, for making things sound even more obscure, there are complete and incomplete gauge choices. A complete gauge choice is done when you pick one and only one field from each orbit and when you (accidentally) pick more than one in a few orbits, you have made an incomplete gauge choice. These choices are mathematically written as

$$G[A] = \partial^\mu A_\mu(x) + \beta(x)$$

where β tracks our orbit and the function(al) G becomes zero every time we hit a gauge.

What is the point of introducing G? Well, two Russians Ludvig Faddeev and Victor Popov tried to fix the issue of these gauge theories first in 1967 [FAD] and they ended up fixing a lot of things by noting the following identity

$$\int \mathcal{D}\lambda[\delta(G[A'])||\frac{\delta G[A']}{\delta\lambda}||] = 1$$

where the moduli represent the determinant. There are a few point here. A' is the gauge transformed potential and λ is the gauge fix as shown on the previous page. The first term is a delta function on a function, which [FLI] calls it a delta functional, but I am not sure of the terminology. It doesn't do anything spectacular but sets all occurrences of the function it is sitting on to zero (or the zero function if you're a pedant). Shove this expression straight into the expectation value by observing that you are only multiplying both sides by 1.

$$\mathcal{A} = \int_{GO} \mathcal{D}A_\mu \int \mathcal{D}\lambda\delta(G[A'])||\frac{\delta G[A']}{\delta\lambda}||e^{i\int dt\mathcal{L}_{EM}}$$

The subscript GO stands for our integration being done on a specific gauge orbit only once. Note that

$$G[A'] = \partial^\mu A'_\mu(x) + \beta(x)$$

$$\implies G[A'] = \partial^\mu A_\mu + \frac{1}{e}\partial^\mu\partial_\mu\lambda + \beta(x)$$

$$\implies \frac{\delta G[A']}{\delta\lambda} = \frac{1}{e}\partial^\mu\partial_\mu$$

If you are wondering whether I have cheated you in the above steps, let me explain. The quantity G is a functional and differentiating this with a function λ should give me the operator. Taking the derivative is just the fancy way of observing the change in a quantity versus another quantity and in this case the variation is a constant operator. If you need some more convincing, refer to [FLI], where he gives the following "plausibility argument" for the single gauge point case, where the determinant degenerates to $\frac{dG}{d\lambda}$ and you have a $\delta(G(A'))$ sitting upfront.

All of this was to just get rid of the determinant from the integral as it is clear that it is not dependent on the gauge or the potential. Pulling it out we have

$$\mathcal{A} = ||\frac{\delta G[A']}{\delta\lambda}|| \int_{GO} \mathcal{D}A'_\mu\mathcal{D}\lambda\delta(G[A'])e^{i\int dt\mathcal{L}'_{EM}}$$

where I have written the gauge versions of everything as they are all gauge invariant. Now, I set λ to the zero function (I am still doing a gauge transformation here, so I am fine) and I have the following

$$\mathcal{A} = ||\frac{\delta G[A']}{\delta\lambda}|| \int \mathcal{D}\lambda \int_{GO} \mathcal{D}A_\mu\delta(G[A])e^{i\int dt\mathcal{L}_{EM}}$$

You can stop here to see what has happened, or multiply both sides with the (constant) Gaussian distribution

$$M = \int \mathcal{D}\beta e^{-i\int d^4x\frac{\beta^2}{2\chi}}$$

$$\therefore M\mathcal{A} = \int \mathcal{D}\beta \int_{GO} \mathcal{D}A_\mu\delta(\partial^\mu A_\mu(x) + \beta(x))e^{-i\int d^4x\frac{\beta^2}{2\chi}}e^{i\int dt\mathcal{L}_{EM}}$$

The delta functional fires now and you paste $-\partial^\mu A_\mu$ wherever you see β. I will write the final answer here

$$\mathcal{A} = \frac{1}{M}\int_{GO} \mathcal{D}A_\mu e^{i\int d^4x\mathcal{L}_{EM} - \frac{(\partial^\mu A_\mu)^2}{2\chi}}$$

You can see what happened now. We have a sensible convergent integral for the vacuum expectation value. But the exponent has changed and this means we have an effective gauge fixed Lagrangian density given by

$$L_{GF} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{(\partial^\mu A_\mu)^2}{2\chi}$$

There are no more issues related to infinite degrees of freedom as we now have a parameter χ which reminds us that we have fixed the gauge and it's alright. Writing the action from this density and doing the familiar integration by parts trick we did in the previous section, we get the exponent to be exactly of the $\bar{x}.\hat{A}.\bar{x}$ form and the photon propagator inverse looks like

$$(F^{\mu\nu}(k))^{-1} = g^{\mu\nu}\partial^2 + \partial^\mu\partial^\nu(\frac{1}{\chi} - 1)$$

Notice that this is not invertible if χ wasn't there, that is, if we never thought of fixing the gauge. What exactly does χ stand for physically? This answer needs the Noether theorem. The Noether theorem (due to Emmy Noether) relates symmetries of spacetime to corresponding conserved quantities. Popular examples include translational invariance leading to momentum conservation, rotational invariance leading to angular momentum conservation and time translation invariance leading to energy conservation. Well, does the gauge invariance lead to anything being conserved? The answer is yes. Running the Noether theorem here gives us the fundamental charge conservation principle (which Maxwell already found) and the *Ward-Takahashi Identities*, for which I would ask you to refer to [ZEE]. The process of introducing G is known in the literature as the Feynman-'t Hooft-Landau gauge.

But this wasn't what Faddeev and Popov fixed. Their paper is based on Yang-Mills fields, about which I shall describe briefly. The theory of Quantum Chromodynamics, which deals with the interactions of quarks and gluons (the strong force) is a Yang-Mills theory. Maxwell's electrodynamics, unlike QCD, is an abelian field theory in which the group on which gauge transformations happen is abelian. Chen Ning Yang and Robert Mills tried to extend gauge invariance to non abelian groups and ended up with a generic description of the strong force, and, to be exact, QCD is a SU(3) non abelian Yang-Mills theory. What problems does losing abelian-ness cause? Firstly, the Lagrangian is no longer nice, but will contain higher tensors which break commutativity. And as you could have guessed, the derivative of G type terms will not be independent of the integration variables and we are in for a stump.

The solution to the above problem is done with another trick due to Faddeev and Popov [FAD], in which this time, we introduce anti-commutative variables (Grassmann variables, after Hermann Gunther Grassmann) to beat the non abelian-ness out of the integral (see [FLI]), and in the process, introduce fields and propagators which really don't exist and hence are called *Faddeev-Popov ghost fields*. One such feature of these fields (and their quanta) is that the Faddeev-Popov ghost particles do **not** follow the spin statistics theorem and thence are actually physically unreal.

In the next section, I will develop the monumental formalism due to Feynman and Schwinger which should give me a basis to discuss further excursions. The next section will probably be the last technical section with regards to continuity.

3.5 Interaction

My ballpoint pen is my laboratory.

Julian Schwinger, Nobel Laureate.

I will now extend my universe, which so far had vacuum and particle sources, to now allow for different particles to interact with each other. This happens when I add a quartic term to the Lagrangian in the exponent. Why quartic? For fun. There is nothing that stops me from adding a cubic perturbation, but then I did it with the quartic when I was learning, and I will present it here. Also, I will be working with a general theory (not the Yukawa mesons again) and hence I will be using non standard terminology. My Lagrangian currently looks like

$$\mathcal{L} = \frac{-1}{2}\bar{q}^T.\hat{A}.\bar{q} - \frac{\lambda}{4!}q^4 + \bar{J}^T.\bar{q}$$

As you can see, I have retained the particle source (or else there is no point of interaction) and the free field Lagrangian is in the matrix $\hat{A}$. This model is called the ϕ^4 scalar theory in literature. Also, this confused me for some reason, but $q^4 = \sum_\mu q_\mu^4$ where the sum is over the fields and hence, I would be much more clear in writing

$$\mathcal{L} = \frac{-1}{2}\bar{q}^T.\hat{A}.\bar{q} - \frac{\lambda}{4!}\sum_\mu q_\mu^4 + \bar{J}^T.\bar{q}$$

As always, I need to start with my vacuum expectation value expression and insert this new Lagrangian in it to see what happens. Doing so (after a Wick rotation), I have

$$\mathcal{A}(J) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}....\int_{-\infty}^{\infty} dq_1dq_2dq_3...dq_N e^{\frac{-1}{2}\bar{q}^T.\hat{A}.\bar{q} - \frac{\lambda}{4!}\sum_\mu q_\mu^4 + \bar{J}^T.\bar{q}}$$

$$\mathcal{A}(J) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}....\int_{-\infty}^{\infty} dq_1dq_2dq_3...dq_N e^{\frac{-1}{2}\bar{q}^T.\hat{A}.\bar{q} + \bar{J}^T.\bar{q}} e^{-\frac{\lambda}{4!}\sum_\mu q_\mu^4}$$

Now, notice the exponent. If I forget about the quartic perturbation for a moment, and differentiate the integrand with J four times, I would end up with the exponent and a fourth power of q. This is the whole idea of perturbative field theory. Writing this explicitly I have

$$\mathcal{A}(J) = e^{\frac{-\lambda}{4!}\sum_\mu(\frac{\partial}{\partial J_\mu^T})^4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}....\int_{-\infty}^{\infty} dq_1dq_2dq_3...dq_N e^{\frac{-1}{2}\bar{q}^T.\hat{A}.\bar{q} + \bar{J}^T.\bar{q}}$$

I can write it outside the integral for it clearly does not depend on the field. Recalling the Wick theorem used in the Genesis section, I can replace the integral with its closed form and I have the following expression

$$\mathcal{A}(J) = \sqrt{\frac{(2\pi)^N}{||\hat{A}||}} e^{\frac{-\lambda}{4!}\sum_\mu(\frac{\partial}{\partial J_\mu^T})^4} e^{\frac{1}{2}\bar{J}^T.\hat{A}^{-1}.\bar{J}}$$

You might be wondering where the imaginary terms have disappeared to, but then, I had already mentioned that I performed a Wick rotation at the beginning and that is where things changed, so I hope this is clear. The next step is easy to see, to let the differential sitting in the exponent eat the function of J after it. This can be seen by expanding the exponential in its power series

$$\mathcal{A}(J) = \mathbb{C}\sum_{n}^{\infty}\frac{1}{n!}(\frac{-\lambda}{4!})^n(\sum_{\mu}(\frac{\partial}{\partial J_\mu^T})^4)^n e^{\frac{1}{2}\bar{J}^T.\hat{A}^{-1}.\bar{J}}$$

I will write $\frac{\mathcal{A}(J)}{\mathbb{C}}$ as the new 'normalized' vacuum expectation value and from now onwards refer to it as just $\mathcal{A}(J)$. Each term of the expansion attacks the J dependent exponent and gives us various terms of the power series. This is due to Julian Schwinger, and we refer to it as the Schwinger Picture. The higher you expand, the better an approximation of the interaction you get. To see exactly where or how it happens, I will do this in another way. Instead of separating the interaction from the Lagrangian, I will separate the source term. That is

$$\mathcal{A}(J) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}....\int_{-\infty}^{\infty} dq_1 dq_2 dq_3 ... dq_N e^{\frac{-1}{2}\bar{q}^T.\hat{A}.\bar{q}-\frac{\lambda}{4!}\sum_\mu q_\mu^4} e^{\bar{J}^T.\bar{q}}$$

Now, simply expand this source term exponential in a power series, and I have

$$\mathcal{A}(J) = \sum_{s=0}^{\infty}\frac{1}{s!}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}....\int_{-\infty}^{\infty} dq_1 dq_2 dq_3 ... dq_N e^{\frac{-1}{2}\bar{q}^T.\hat{A}.\bar{q}-\frac{\lambda}{4!}\sum_\mu q_\mu^4}(J^\mu q_\mu)^s$$

The multinomial sum $(J^\mu q_\mu)^s$ can be written in general as follows, separating the field independent source terms with the field powers. This is an extremely important step and I hope this too, is clear.

$$\mathcal{A}(J) = \sum_{s=0}^{\infty}\frac{1}{s!}J_1^\mu J_2^\mu J_3^\mu ....J_s^\mu\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}....\int_{-\infty}^{\infty} dq_1 dq_2 dq_3 ... dq_N e^{\frac{-1}{2}\bar{q}^T.\hat{A}.\bar{q}-\frac{\lambda}{4!}\sum_\mu q_\mu^4} q_{\mu,1}q_{\mu,2}q_{\mu,3}...q_{\mu,s}$$

If this bewilders you, or you are tired of waiting to see where the interaction is, there is good news. Let me take a small case for discussion, the $s = 2$ term without the sources or the outside factorial. I will call this term $G^{(2)}_{a_1,a_2}$ and so,

$$G^{(2)}_{a_1,a_2} = \int\int dq_{a_1}dq_{a_2} e^{\frac{-1}{2}\bar{q}^T.\hat{A}.\bar{q}-\frac{\lambda}{4!}\sum_\mu q_\mu^4} q_{a_1}q_{a_2}$$

To make it even easier, let us take the no interaction case, that is $\lambda = 0$ and I have

$$G^{(2)}_{a_1,a_2} = \int\int dq_{a_1}dq_{a_2} e^{\frac{-1}{2}\bar{q}^T.\hat{A}.\bar{q}} q_{a_1}q_{a_2}$$

I have done this integral before! Go back to the Mathematics section and flip to the first Wick theorem (the one with the contractions) and you can see that this is the *two point correlation function*, you just contract on two variables. That is why I chose the esoteric notation for this expression.

$$G^{(2)}_{a_1,a_2} = \int\int dq_{a_1}dq_{a_2} e^{\frac{-1}{2}\bar{q}^T.\hat{A}.\bar{q}} q_{a_1}q_{a_2} = (\hat{A}^{-1})_{a_1 a_2}$$

Aha! This is nothing but the free field propagator/Green's function from a_1 to a_2 and I can confidently proceed, now that it is giving sane results for degenerate cases. I will evaluate the four point correlation function now, but this time with the interaction term. (What about the odd powers of q? Well, you would contract on the even terms and you would end up with a semi-painful Gaussian which you can evaluate anyway.)

$$G^{(4)}_{a_1,a_2,a_3,a_4} = \int\int\int\int dq_{a_1}dq_{a_2}dq_{a_3}dq_{a_4} e^{\frac{-1}{2}\bar{q}^T.\hat{A}.\bar{q}-\frac{\lambda}{4!}q^4} q_{a_1}q_{a_2}q_{a_3}q_{a_4}$$

I write $q^4 = \sum_n q_n^4$ and I have

$$G^{(4)}_{a_1,a_2,a_3,a_4} = \int\int\int\int dq_{a_1}dq_{a_2}dq_{a_3}dq_{a_4}e^{\frac{-1}{2}\vec{q}^T.\hat{A}.\vec{q}[1-\frac{\lambda}{4!}\sum_n q^4+O(\lambda^2)]}q_{a_1}q_{a_2}q_{a_3}q_{a_4}$$

Write the exponential as a power series again, and

$$G^{(4)}_{a_1,a_2,a_3,a_4} = \int\int\int\int dq_{a_1}dq_{a_2}dq_{a_3}dq_{a_4}e^{\frac{-1}{2}\vec{q}^T.\hat{A}.\vec{q}}q_{a_1}q_{a_2}q_{a_3}q_{a_4} - \frac{\lambda}{4!}\sum_n q_{a_1}q_{a_n}q_{a_2}q_{a_n}q_{a_3}q_{a_n}q_{a_4}q_{a_n} + O(\lambda^2)$$

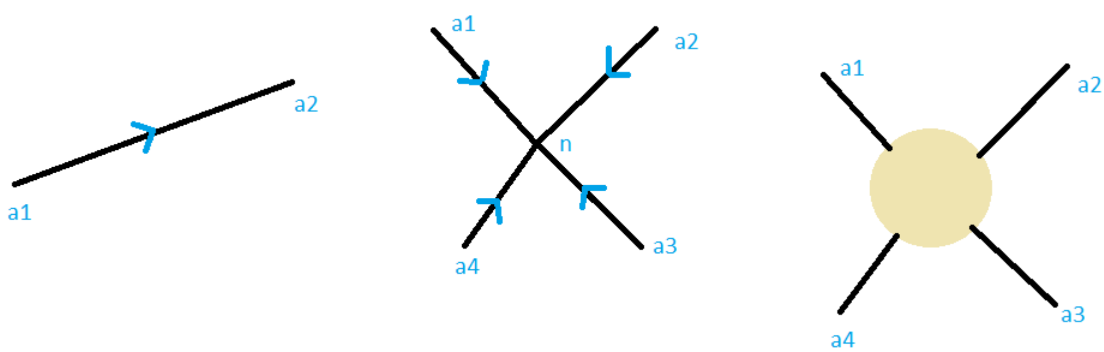
The first term is what I did in the Mathematics section, and it is the four point correlation function $(A^{-1})_{1,2}(A^{-1})_{3,4} + (A^{-1})_{1,3}(A^{-1})_{2,4} + (A^{-1})_{1,4}(A^{-1})_{2,3}$. The second term is where things become interesting, as I have to essentially compute the following expectation, obviously using the Wick theorem in full force

$$< q_{a_1}q_{a_n}q_{a_2}q_{a_n}q_{a_3}q_{a_n}q_{a_4}q_{a_n} >$$

It's a combinatorial game now and I have to pick pairs to contract on. Since there are four subscripts which are the same and hence if I assumed they were different, I would have 4! extra terms which will now all be the same, so my answer multiplies with this factor and it cancels with the denominator (oh the ingenuity). There might be a question regarding contracting over the same indices or over say a_1 and a_2, but I will put that aside for now, and will get back to it in a moment. So the only terms I have left which are contractible are the terms involving pairs of one from a_1, a_2, a_3, a_4 and a_n, a_n, a_n, a_n and I can do this in only one way, whichever way, to get

$$< q_{a_1}q_{a_n}q_{a_2}q_{a_n}q_{a_3}q_{a_n}q_{a_4}q_{a_n} >= \sum_n (\hat{A}^{-1})_{a_1a_n}(\hat{A}^{-1})_{a_2a_n}(\hat{A}^{-1})_{a_3a_n}(\hat{A}^{-1})_{a_4a_n}$$

Now look at the three diagrams below, one for the two point case (with no interaction) and the other two for the four point case (with the interaction).



These pictures are called Feynman diagrams after Richard Feynman who, in Schwinger's own words, brought quantum field computations to the masses with these doodles and the rules that you prescribe to them. In conclusion, I have the following series for my vacuum expectation value ([ZEE] calls this the Wick picture, in contrast to the Schwinger picture)

$$\mathcal{A}(J) = \sum_{s=0}^{\infty}\frac{1}{s!}J_1^{\mu}J_2^{\mu}J_3^{\mu}....J_s^{\mu}G^{(s)}_{a_1,a_2,a_3,...,a_s}$$

To convince you further, let me do a complete continuum version of what I just did, say, for the scalar theory of the Yukawa meson. The only new thing here is to put integrals instead of summations or dot products. That's all. The initial (normalized) vacuum expectation value is given by

$$\mathcal{A}(J) = \int \mathcal{D}\phi e^{i\int d^4x[\frac{1}{2}((\partial\phi)^2 - m^2\phi^2) - \frac{\lambda}{4!}\phi^4 + J\phi]}$$

Viewing this in the Wick picture, and splitting the exponent, I have

$$\mathcal{A}(J) = \int \mathcal{D}\phi e^{i\int d^4x[\frac{1}{2}((\partial\phi)^2 - m^2\phi^2) - \frac{\lambda}{4!}\phi^4]} e^{i\int d^4x J\phi}$$

Now, I Wick rotate only the second exponential and leave the first exponential as it is for now and I get the integral to look like

$$\mathcal{A}(J) = \int \mathcal{D}\phi e^{i\int d^4x[\frac{1}{2}((\partial\phi)^2 - m^2\phi^2) - \frac{\lambda}{4!}\phi^4]} e^{\int d^4x' J\phi}$$

The prime here indicates that this is the rotated spacetime we are integrating in, and since the integrand is very nice, I don't think I have any issues here regarding mathematical consistency. Now, consider the following equation

$$\int d^4x' J\phi = \sum_s^{\infty} J(x_s^\mu)\phi(x_s^\mu)$$

The more denser my summation domain is, the closer I get to the integral and in this case the upper limit refers to precisely that. Noticing this, the next step is the same as the one shown in the last page, but this time with continuous functions instead of discrete series.

$$\mathcal{A}(J) = \sum_{s=0}^{\infty} \frac{1}{s!} J(x_1^\mu)J(x_2^\mu)J(x_3^\mu)...J(x_s^\mu) \int \mathcal{D}\phi e^{i\int d^4x[\frac{1}{2}((\partial\phi)^2 - m^2\phi^2) - \frac{\lambda}{4!}\phi^4]} \phi(x_1^\mu)\phi(x_2^\mu)\phi(x_3^\mu).....\phi(x_s^\mu)$$

The right hand side can now be written as a weighted sum over Green's functions and I have the same diagrams I drew before. This might seem like the Schwinger Picture is useless (I only mentioned it in the beginning), but it is of prime importance in answering the question about the coupling of a_n to a_n terms in the expectation value. Running the continuum Schwinger picture on the present case, I have to only replace the integral with the $\mathcal{W}$ in the exponent, and I will work only upto $O(\lambda)$.

$$\mathcal{A}(J) = e^{-\frac{i}{4!}\lambda\int d^4w(\frac{\delta}{\delta J(w)})^4} e^{-\frac{i}{2}\int\int d^4x_1 d^4x_2 J(x_1)^* F(x_1 - x_2) J(x_2)}$$

The $O(\lambda)$ term comes from the first term in the power series of the exponential feeding on the term in the power series of the following exponential where I have the *fourth* power of J. It might help you out if you write the series and see this happen for real. Writing only the relevant terms,

$$\mathcal{A}(J)_\lambda = -\frac{i\lambda}{4!}\int d^4w(\frac{\delta}{\delta J(w)})^4 \times \frac{i^4}{4!2^4}[\int\int d^4x_1 d^4x_2 J(x_1)^* F(x_1 - x_2) J(x_2)]^4$$

There is a nice trick due to Bernhard Riemann which helps us write the powers of integrals as separate integrals themselves. This is nothing special, and a proof of this idea just need some rearrangement of partial sums, and I will state it here as follows with an example

$$[\int dx x^2]^2 = \int\int dx dy x^2 y^2$$

With this, the two integral signs in the multiplicand of $\mathcal{A}(J)_\lambda$ become eight and I will have four propagators linking them all, holding the bulky thing together. For completeness, I will write it in full, although I never evaluated this completely

$$\mathcal{A}(J)_\lambda = \frac{i\lambda}{4!4!2^4}\int d^4w(\frac{\delta}{\delta J(w)})^4 \int\int\int\int\int\int\int\int d^4x_1d^4x_2d^4x_3d^4x_4d^4x_5d^4x_6d^4x_7d^4x_8$$

$$J^*(x_1)F(x_1-x_5)J(x_5)J^*(x_2)F(x_2-x_6)J(x_6)J^*(x_3)F(x_3-x_7)J(x_7)J^*(x_4)F(x_4-x_8)J(x_8)$$

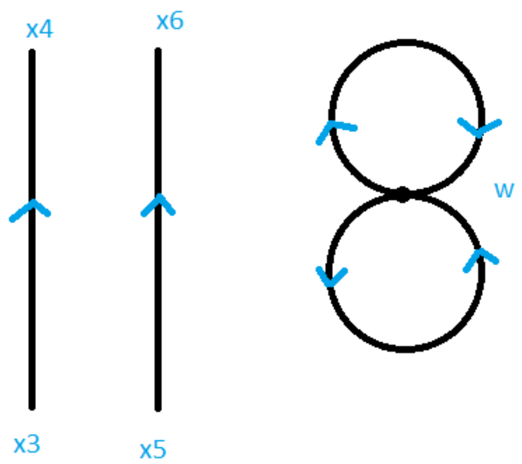
For the next step, I will need another standard mathematical trick known as differentiating under the integral sign. Notice that the inside integral (w is a dummy variable throughout the calculation, it has no physical significance unlike $x_1, x_2, ..$) is independent of w and shoving the quartic differential inside won't cause any trouble as far as the whole thing is concerned. Now I need to pick any four J's on which this can feed on and this is exactly the combinatorial game I played in the Wick picture two pages back! Picking $J(x_5), J(x_6), J(x_7), J(x_8)$ and differentiating them I end up with a bunch of ones all evaluated at w, that is, wherever you see any of x_5, x_6, x_7 or x_8 paste w. Doing this, the expression looks like

$$\mathcal{A}(J)_\lambda = \frac{i\lambda}{4!4!2^4}\int d^4w \int\int\int\int d^4x_1d^4x_2d^4x_3d^4x_4 J(x_1)J(x_2)J(x_3)J(x_4)F(x_1-w)F(x_2-w)F(x_3-w)F(x_4-w)$$

The propagators make a lot more sense now. This is nothing but our four point correlation function, where particles starting from x_1, x_2, x_3, x_4 (the sources J are still there, phew) interact at w and this point is integrated all over spacetime outside, thereby giving us the net interaction amplitude over all spacetime. The theory truly reproduces reality so far, but still, whatever would happen if my quartic differential ate up $J(x_1), J(x_2), J(x_7), J(x_8)$? Doing this instead (same idea, just paste w in the corresponding propagators, I will write F_{ab} for $F(x_a - x_b)$ to save (a lot of) ink)

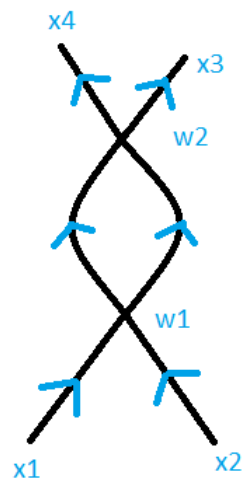
$$\mathcal{A}(J)_\lambda = \frac{i\lambda}{4!4!2^4}\int d^4w \int\int\int\int d^4x_1d^4x_2d^4x_3d^4x_4 J(x_3)J(x_4)J(x_5)J(x_6)F_{ww}F_{34}F_{56}F_{ww}$$

Whatever does this nonsense mean? Let us draw the propagators out on a Feynman diagram, it looks like the one below



This should answer the question of what would have happened if we Wick contracted the same index, we would be generating a vacuum fluctuation. This is a *disconnected* Feynman diagram for obvious reasons. We have a particle created at x_3 going to x_4, another one created at x_5 going to x_6 and then two vacuum fluctuations represented by the "8", which is actually two bubbles starting and ending at w.

It is time for some observations. Various contractions are giving us various diagrams, some connected, some disconnected. But notice that as long as you have the same power of λ, you have the same number of vertices. Also the number of propagators is (obviously) the number of edges in the Feynman graph, if you may. Thus, all in all, I can say that the Schwinger or the Wick summation actually gives me the sum total of all possible Feynman diagrams and thus all possible interactions. You need some interaction, find the number of vertices and that is the power of λ you should be looking for. Thus for the diagram given below,



I can see two vertices, that is I need to compute λ^2 terms and I have an eight order J differential ready to eat the *sixth* power of J (there are two Js raised to the sixth power giving me a total of twelve, eight of them die, four remain to be the incoming and the outgoing vertices, it all makes sense really) to give me the propagators as the six edges. You can make your own complicated Feynman diagrams and calculate their amplitudes to any order using the formalism I just developed.

The Feynman rules are nothing but the aforementioned rules stated rigidly in momentum space, instead of position space. While position space Feynman diagrams actually show us what is happening in the universe, the momentum space Feynman diagrams help us compute cross sections et cetera with a greater ease (anyway, they are Fourier transforms of each other, speak of Fourier transforming a diagram!). Also, the position space Feynman diagrams are drawn in $x - t$ space, that is one direction belongs to time (choice is yours). So, given a field theory, and a process to analyse, I draw the Feynman diagram and label all the vertices. Then I assign an arbitrary momentum to each edge (this means the particle is going with that momentum along the propagator, remember, the propagator for the Yukawa meson is not the same as that for a photon or a gluon, they are different field theories). Then assign $-i\lambda(2\pi)^4\delta^{(4)}(\sum_{in} k - \sum_{out} k)$ to each vertex, where the delta function is a mathematical way of imposing momentum conservation. Whatever edges of the graph are not external (loops et cetera), integrate those momenta with the measure $\frac{d^4k}{(2\pi)^4}$ and this essentially removes the redundancies.

It took the acumen of Freeman Dyson [FRE] to see that what Schwinger and Feynman worked out was the same thing, and it was also the same thing that Sin-Itiro Tomonaga worked out independently in Japan (before 1945). Schwinger, Feynman and Tomonaga were awarded the Nobel Prize jointly later on but Dyson never received it ([CLO] relates an anecdote about Dyson, when once the title of a talk he was to give at Princeton was misprinted to "Disturbing The University" while it was actually "Disturbing The Universe"! That later on became the award winning book by the same name [FRE1]). The weighted infinite summation of Green's functions (or correlation functions, for the condensed matter theorists) is called the Dyson series in honour of his work.

3.6 Techniques

> If you've spent a long time developing a skill and techniques, and now some 14 year-old upstart can get exactly the same result, you might feel a bit miffed I suppose, but that has happened forever.
>
> Brian Eno, Musician

Most of the current day university textbooks prescribed for a first course in quantum field theory (like [PES], or [LAN]) do not start with the path integral formalism (for no apparent reason) and instead discuss the subject in the actual way it was born, the so called *canonical formalism.* As I am talking about formalisms, I would like to suggest [EVE] for an interesting take on another formalism which did not get much attention, but is anyway interesting in its own right (see [DEW] also). I will not be deriving the entire theory in this formalism (why would I when I have the path integral working so beautifully) but I will assume that you know about the Heisenberg, Schrodinger and Interaction pictures from NRQM and quickly give the rules. You start with a vacuum ket $|0>$ in a Hilbert space and then create a harmonic oscillator (the paradigm strikes again!) using the Dirac creation operator $\hat{a}^\dagger(k)|0>$ and now my universe has a vibration with momentum k. Now to kill this vibration, I run the Dirac annihilation operator on this new state and get back the vacuum. The vacuum expectation value for this (arguably banal) process is given by

$$\mathcal{A} =< 0|\hat{a}(k)\hat{a}^\dagger(k)|0 >$$

According to my paradigm, I need to create wavepackets of harmonic oscillators to get an actual field to work with and therefore, I write my field (operator now) as a linear superposition of harmonic oscillators (it's all linear everywhere, couplings, superpositions, everything) and hence

$$\hat{\phi}(x^\mu) = \int \frac{d^D k}{(2\pi)^{D/2}} \frac{1}{(2\omega_k)^{1/2}} \hat{a}(\bar{k})e^{-ik^\mu x_\mu} + \hat{a}^\dagger(\bar{k})e^{ik^\mu x_\mu}$$

You don't need to call this déjà vu as it is the same thing I am talking about. Now, instead of just a vibration, if I need to create and (banally) destroy a field in vacuum, it will now look like

$$\mathcal{A} =< 0|\hat{\phi}(x^\mu)\hat{\phi}(0)|0 >$$

and this is the single edge (one propagation) Feynman diagram. Well, not really. We missed out the Θ functions in the oscillator expansion above. It might look like a small thing, but that encapsulates another tenet of physics called causality. Effect cannot precede the cause. So, inorder to enforce this, you can either place Θ functions directly or use the so called "Time Ordering" operator τ which tells us that whatever we are doing, if we are doing it in the presence of τ it is causal, that is earlier terms occur to the right.

$$\mathcal{A} =< 0|\tau[\hat{\phi}(x^\mu)\hat{\phi}(0)]|0 >$$

That is all there is to the canonical formalism.

3.6.1 The S Matrix

The Dyson series traces it's origins in the non relativistic quantum theory, more specifically, the time dependent (Rayleigh-Schrodinger) perturbation theory where we have a Hamiltonian $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I$ where $\mathcal{H}_0$ is the solvable Hamiltonian and $\mathcal{H}_I$ is the interaction term. As it so happens, the unitary operator for time evolution of the solvable Hamiltonian takes the simple form $e^{-i\hat{\mathcal{H}}_0 t}$ because while integrating the Schrodinger equation the Hamiltonian is time independent. However, for the interaction Hamiltonian, this is not so, and the non commutativity in time leads to the Dyson series as the direct solution of the unitary operator for time evolution [FIT]

$$U_I(t_0,t) = e^{-i\int d^4x \mathcal{H}_I} = 1 - i\int_{t_0}^{t} dt' d^3x \mathcal{H}_I(t_0,t') + (-i)^2 \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' d^3x d^3x' \mathcal{H}_I(t_0,t')\mathcal{H}_I(t_0,t'') + ...$$

The S matrix is just the time ordering operator τ applied to this operator. This is not the right way to define or introduce the S matrix (see [LAN] for a pedagogical introduction) but that is all there is to it anyway. So, there you are.

$$\hat{S} = \tau[e^{-i\int d^4x \mathcal{H}_I}]$$

My proof that this is nothing but whatever I have been doing so far is quite simple. It involves introducing another operator called the "normal ordering" operator η. In literature, sometimes people enclose a term in colons, like : $\hat{A}\hat{B}$: to tell you that the product is normal ordered. This is fancy language for asking you to shift all the operators which have daggers to the right. For instance

$$\eta[\hat{A}\hat{B}^\dagger\hat{C}^\dagger\hat{D}\hat{E}] = \hat{A}\hat{D}\hat{E}\hat{B}^\dagger\hat{C}^\dagger$$

Wick's theorem features here once again, but in a different incarnation. As stated in the introduction, any field operator can be written in terms of Dirac creators and annihilators. I write this explicitly as $\phi = \phi^c + \phi^a$. For simplicity, let me work in 1+1 spacetime. Now, if $t_1 > t_2$ then

$$\tau[\phi(x_1,t_1)\phi(x_2,t_2)] = \phi^a(x_1)\phi^a(x_2) + \phi^c(x_1)\phi^a(x_2) + \phi^c(x_1)\phi^a(x_2) + \phi^c(x_1)\phi^c(x_2) + [\phi^a(x_1), \phi^c(x_2)]$$

Except the commutator, the rest of it is properly normal ordered, and I use my new notation and write

$$\tau[\phi(x_1,t_1)\phi(x_2,t_2)] = \eta[\phi(x_1,t_1)\phi(x_2,t_2)] + [\phi^a(x_1), \phi^c(x_2)]$$

Look at the structure of the commutator. It creates a particle at x_2 and annihilates at x_1 and if $t_2 > t_1$ I have it the other way around. Essentially, it is the propagator from x_2 to x_1 or $F(x_1 - x_2)$. Now, the Wick theorem is just an extension of this idea. Note that whenever I take the expectation value of a normal ordered term, it is always zero because the dagger operators cancel out the non dagger ones (you create and annihilate ending up with vacuum, so this does not contribute to your process, it just gives you the background back). That is $< 0|\eta[(anything)]|0 >$ is zero.

$$< 0|\tau[\phi_1\phi_2...\phi_N]|0 >= \sum_{Wick} < 0|\tau[\phi_\alpha\phi_\beta]|0 >< |\tau[\phi_\gamma\phi_\delta]|0 > .... < |\tau[\phi_\epsilon\phi_\kappa]|0 >$$

where the sum is over all possible pairwise couplings or as we know them, contractions. Now I can prove that the S matrix formulation is the same as the Feynman-Schwinger-Tomonaga picture, by expanding the exponential inside the time ordering operator in the Dyson series as a power series itself. Again, I take the ϕ^4 scalar theory, and now $\mathcal{H}_I(x^\mu) = \frac{\lambda}{4!}\phi^4(x^\mu)$, and the $O(\lambda)$ term is given by

$$< q|\hat{S}_\lambda|p >= \frac{-i\lambda}{4!}\tau[\int d^4x < q|\phi^4(x^\mu)|p >]$$

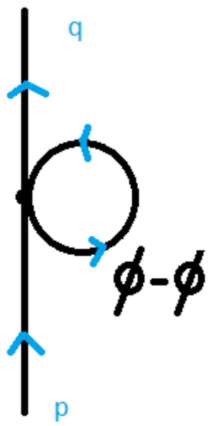
I chose p, q as arbitrary labels so that I have something going from momentum p to momentum q in my universe. Now, $|p >= \hat{a}_p^\dagger|0 >$ and $< q| =< 0|\hat{a}_q$ and plugging this I see the following

$$< q|\hat{S}_\lambda|p >= \frac{-i\lambda}{4!}\tau[\int d^4x < 0|\hat{a}_q\phi^4(x^\mu)\hat{a}_p^\dagger|0 >]$$

The time ordering has no effect on anything but operators so I can take it to the lowest level of nesting and I need to evaluate the following

$$\tau[\hat{a}_q\phi^4(x^\mu)\hat{a}_p^\dagger] = \tau[\hat{a}_q\phi(x^\mu)\phi(x^\mu)\phi(x^\mu)\phi(x^\mu)\hat{a}_p^\dagger]$$

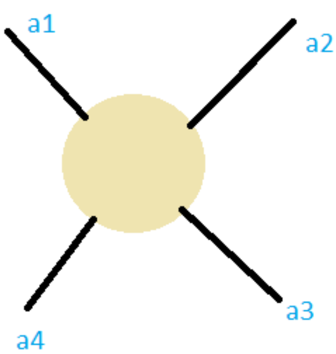
I blow the whole thing up and attack it using Wick's theorem to get contractions of various kinds. These are all the Feynman diagrams I ended up with in the last section, and I will explicitly do one to show you how it all comes about. I have six terms. I need to form pairs, and I will get three pairs anyhow. Let me pair them normally, that is the first two, then the next two, and the last two. This is one such term in the summation $\tau[\hat{a}_q \phi(x^\mu)]\tau[\phi(x^\mu)\phi(x^\mu)]\tau[\phi(x^\mu)\hat{a}_p^\dagger]$ and it happens twelve times if I count correctly. What does this mean in terms of propagators? It is the Feynman diagram shown below (which we haven't seen in the last section)



Thus, S matrix elements are Feynman diagrams themselves. I could have taken the S matrix as my axiom and indeed I can derive the entire theory again and this is how it was done historically, the S stands for scattering, yes, it's the *scattering matrix*. You have a process, you chug the components of the S matrix and you get your cross sections hassle free. According to [WEI1] it can be shown that the non analyticity of the S matrix elements gives us particles (the non analyticities are the poles, remember, if I plug $k = m$ in the propagator it blows up) and hence the entire theory of particle physics can be generated by studying the analyticity of the S matrix and this project was taken up by Geoffrey Chew, Tullio Regge, Steven Frautschi and others to create an S matrix theory for the erstwhile unknown strong interaction [CHEW].

3.6.2 The Lehmann Symanzik Zimmermann Reduction Formula

The scattering matrix idea I discussed above is discussed on a theoretical basis where I have my universe with vacuum, particles, and their interactions. The real world is not just this, and I need to find a way to think of making my formalism more robust. When I perform a real world scattering experiment, I can measure the incoming and outgoing momenta of the particles and get the S matrix elements but will this element be the same as the computation of the last section? Not exactly. Refer to the diagram below



I have introduced the term "mass shell" before, and the blob in the center of the Feynman diagram is precisely that. When my particles are "on" (near) the mass shell, if I compute the incoming and outgoing momenta and calculate the S matrix, then my answer will match (or will be off by a constant factor) the Dyson series calculation. In real world experiments, the particles start as "off" (far from) the mass shell entities, interact while they are "on" (near) the mass shell and then scurry away, again going "off" (far from) the mass shell. This important issue is addressed by the Lehmann-Symanzik-Zimmermann reduction formula and it is a reduction formula because you are reducing your real world spacetime computation to depend on the near-the-mass-shell spacetime points. The original paper is in German [LEH] and is quite readable if you have a translator near you (or you're German), and I will follow the discussion given in Coleman's magna carta [COL1], although a derivation using the Wightman axioms [WIG] is possible, see [STE].

How do I account for the "off-the-mass-shell"ness of the incoming and outgoing particles? Assuming that they are generated at some point in my universe, I will assign some linear coupling source to them and then construct a new blob which encloses the larger spacetime (until the sources) and call this the mass shell. In doing so, I have essentially added one major factor to my S matrix calculation of the previous section and that is the propagators of these initially off-the-mass-shell but now on-the-mass-shell particles. That is all there is to the derivation. Let n_1 particles enter the (older) mass shell and n_2 particles leave it. Let their off-the-mass shell momenta be $p_1, p_2, ..., p_{n_1}$ and $q_1, q_2, ..., q_{n_2}$, then from an experiment, I have the S matrix element

$$< q_1, q_2, ..., q_{n_2} | \hat{S} - \hat{1} | p_1, p_2, ..., p_{n_1} >$$

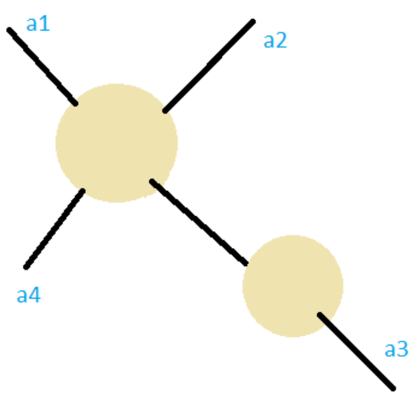
I have subtracted the unit term in the power series because these processes really don't come under the interaction paradigm, they just enter the mass shell and exit as it is ($n_1 + n_2/2$ each way in this case). As explained, I need to account for their propagation unto the mass shell, and this is done by multiplying my theoretical S matrix element with the propagator and cancel it off with its inverse when they hit the mass shell. This is simply

$$\int d^4x_1 d^4x_2.....d^4x_{n_1} d^4y_1....d^4y_{n_2} (\partial^2+m_1^2)(\partial^2+m_2^2).....(\partial^2+m_{n_2}^2) e^{+iq_1y_1} e^{+iq_2y_2}..e^{+iq_{n_2}y_{n_2}} e^{-ip_1x_1} e^{-ip_2x_2}$$

$$.....e^{-ip_{n_1}x_{n_1}} < 0|\tau[\phi(x_1)\phi(x_2)...\phi(x_{n_1})\phi(y_1)\phi(y_2).....\phi(y_{n_2})]|0 >$$

This is the LSZ reduction formula. All I have to do now is to use my formalism to calculate the vacuum expectation value inside and then plug it into this formula to see the outcome match with the experimental value registered (or the other way, usually).

The formula also tells me something else. The computation of the S matrix from the formalism is not affected by what happens off the mass shell. To understand this, I draw another Feynman diagram which basically takes one of the mass shell edges and puts it off the mass shell (it's basically on another mass shell which is far away).



For computing the vacuum amplitude for this process, I need to compute the integral for two vertices, and not for one, this time (I have introduced an internal line, whose momentum will be integrated over, causing the same effect) and I recall my complex integration math to identify that the Cauchy residue theorem here asks me to pick out the residues at points where my integral blows up. These points are precisely the mass shells or the vertices or whatever you want to call it (real particle centers). There is no restriction on the order in which I pull my residues out (and no I do not mean it that way) and doing so for the newly created mass shell I have the residue equaling a constant ($\frac{1}{2\omega_k}$) and then doing so for the older blob I get back my original S matrix element.

Therefore, if you just have the theoretical Lagrangian and the corresponding theoretical S matrix elements, you really don't have much because there could be a multitude of quantum processes happening outside your mass shell and you won't even know it because your S matrix doesn't change (this line is from [COL1]). However, if you do have some experimental data, you could check your model using the LSZ formula. That's about it.

3.6.3 Large N: First Pass

> The notion of God, as far as I have been able to make sense of it, is essentially a limiting computational conception: it is the limit as time goes to infinity of the behaviour of a complex system where the computational entities combine and grow in power into ever larger units.
>
> Ron Maimon, Physicist [MAI]

I am now ready to answer the first question posed in the Prologue using the field theory developed so far. The real universe is a system with a Lagrangian controlling the actions of a large number of fields and present day physics is far from discovering the complete Lagrangian which explains all phenomena, from the strong force to gravity, so, solving this Lagrangian using the formalism is not possible as of now. So, I will work with the scalar ϕ^4 theory for now, with a large number (N) of fields. Thence the name, the large N approximation.

If you think this is pointless because the Lagrangian is nowhere close to the real world one, then you are partly right, but you are in for a surprise. Most of the analysis done here is taken from my study of the $\frac{1}{N}$ chapter in [COL1] and I will supply references as the discussion proceeds. Before I state the Lagrangian, however, I need to tell you a bit more about some of the math involved. Firstly, the Legendre transform. This idea was introduced to me in classical mechanics, but the transform is ubiquitous in its usage, thermodynamics being the other subject where it is used mercilessly. It essentially changes the independent variables in a function of two variables using the product rule for the derivatives. Consider a function $f(x, y)$. From elementary calculus I have

$$df = (\frac{\partial f}{\partial x})_y dx + (\frac{\partial f}{\partial y})_x dy$$

Now, if I call $u = \frac{\partial f}{\partial x}$ and $w = \frac{\partial f}{\partial y}$, then I can call (u, x) and (w, y) conjugate variables, borrowing the terminology from classical mechanics. Now, I do the following

$$d(wy) = ydw + wdy$$

$$df - d(wy) = udx + wdy - ydw - wdy = udx - ydw$$

If I can call $g = f - wy$, then I have

$$dg = udx - ydw$$

Now, I can call $u = \frac{\partial g}{\partial x}$ and $y = -\frac{\partial g}{\partial w}$ and my conjugacy is invariant. This is what Legendre actually discovered. I changed the variable set from (x, y) to (x, w) and my conjugacy condition is still the same, that is, the mechanics underneath is untouched.

The next idea I will need is that of the effective action. This discussion is borrowed from the excellent book [ZUM] and is written in a very elementary way. From earlier discussions, I have $\mathcal{A} = e^{i\mathcal{W}(J)}$ as my somewhat standard way of representing the vacuum amplitude. The argument is as follows, our Feynman diagrams so generated from this expression (in whatever picture that pleases you) can be generated by piecing together *one particle irreducible vertices* with propagator lines. This is just terminology and it simply tells that there is something more topologically fundamental about the Feynman graphs.

In this instance, I would like to do a Legendre transform to convert $\mathcal{W}(J)$ into a function of the fields instead. So, my target variable is ϕ and the variable currently in the function is J. I do the same as the previous page, and I have

$$\alpha(\phi) = \mathcal{W}(J) - \int d^4x J\phi$$

The term $\alpha(\phi)$ which is the Legendre transform of $\mathcal{W}(J)$ is known as the effective action of the field. The conjugacy relations turn out to be the pair

$$\frac{\delta\mathcal{W}}{\delta J(x)} = \phi(x)$$

$$\frac{\delta\alpha}{\delta\phi(x)} = -J(x)$$

If you differentiate the $\mathcal{W}$ function(al) with J twice, you end up with the propagator sending the particle from the point where the first derivative was made to the point where the second derivative is made. That is

$$F(x_1, x_2) = \frac{\delta^2\mathcal{W}(J)}{\delta J(x_1)\delta J(x_2)} = \frac{\delta\phi(x_1)}{\delta J(x_2)}$$

Similarly, for the effective action, I can define another propagator (called the Kinetic Kernel, for fancy reasons)

$$K(x_1, x_2) = \frac{\delta^2\alpha(\phi)}{\delta\phi(x_1)\delta\phi(x_2)} = -\frac{\delta J(x_1)}{\delta\phi(x_2)}$$

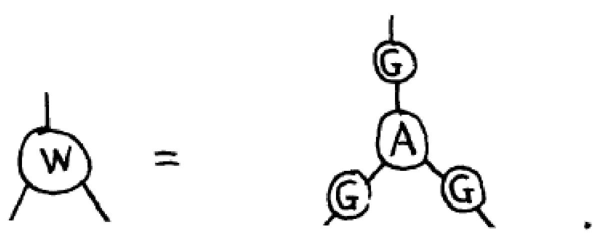
This shows the correspondence between the (let us say) original action and our new effective action term, in the sense, their propagators are reciprocal

$$F(x_1, x_2)K(x_2, x_1) = -1$$

With this correspondence, and a little more functional differentiation trickery (see [ZUM] for a full treatment), I can see the following equality

$$\frac{\delta^3\mathcal{W}}{\delta J(x_1)\delta J(x_2)\delta J(x_3)} = \int\int\int d^4u d^4v d^4w F(x_1, u)F(x_2, v)F(x_3, w)\frac{\delta^3\alpha}{\delta\phi(u)\delta\phi(v)\delta\phi(w)}$$

From my experience with drawing Feynman diagrams, I can see that the left hand side is a three particle blob (differentiating thrice) and the right hand side is another three particle blob but now I have internal edges which are being integrated over. As consistent as it may sound, the diagram below from [ZUM] should make this clear (he calls the Green's functions as G)



Now, let me differentiate the left hand side with another $J(x_4)$, that is, let me introduce another field created at x_4, and I would like to see what happens to the Feynman diagrams on the right. Firstly, my prescription of differentiating with a J can be equivalently seen as

$$\frac{\delta}{\delta J} = \frac{\delta \phi}{\delta J}\frac{\delta}{\delta \phi}$$

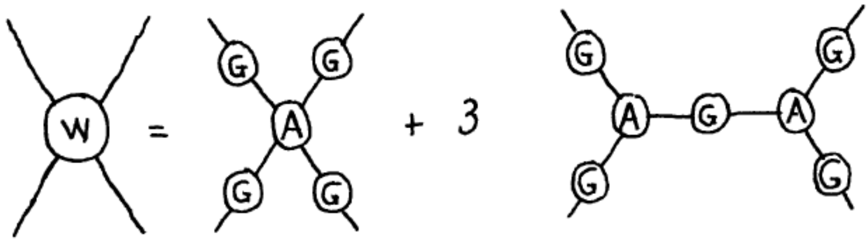
From the conjugacy relation, the first factor is nothing but a propagator and thence I have

$$\frac{\delta}{\delta J(x_a)} = F(x_a, z)\frac{\delta}{\delta \phi(z)}$$

Now, running this argument by sending the derivative inside the integral (the Leibniz rule, I think), I have the following

$$\frac{\delta^4 \mathcal{W}}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)} = \int\int\int d^4u d^4v d^4w d^4z F(x_4, z)\frac{\delta}{\delta \phi(z)}[F(x_1, u)F(x_2, v)F(x_3, w)\frac{\delta^3 \alpha}{\delta \phi(u)\delta \phi(v)\delta \phi(w)}]$$

Now it's the product rule done quickly, take the second term and you have a four point correlation function with four propagators bunching out of it, take the first term (the product of the three propagators) and you have three times two three point correlation functions with a propagator connecting the common A vertex. The diagram (again from [ZUM], he does not derive this) should make this clear



The calculations above should make it clear that the Feynman diagrams that we talked about in all of the previous sections can be decomposed into much more fundamental diagrams by prescribing an effective action functional which is the Legendre transform of $\mathcal{W}$. The concept of the effective action is quite useful in proving certain theorems and this is demonstrated in [ZUM] for proving the Ward-Takahashi identity and the Goldstone theorem.

The discussion henceforth will be based on two papers [COL2], [COL3] by Sidney Coleman, which provide lucid insight into what happens when we deal with large N fields. As promised before, the Lagrangian for a scalar ϕ^4 theory for the large N case is shown below

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi^i \partial^\mu \phi^i - \frac{1}{2}m^2 \phi^i \phi^i - \frac{\lambda}{8N}(\phi^i \phi^i)^2$$

The superscript i for the field ϕ^i denotes the ith quantum field in the universe where this parameter runs from 1 to N in our case. Questions regarding why the structure of the Lagrangian is so, and so on, can simply be attributed to the fact that it must be so for it to not violate the fundamental tenets of physics such as Lorentz invariance, et cetera, and it really isn't the problem at hand. The issue is to find a method to analyse this using our formalism, so I would ask you to refer to [WEI] or [COL1] or any other text with regard to this matter.

The group of symmetries under which our universe transforms currently is the group $O(N)$, the set of all orthogonal N dimensional matrices, the rotation matrices, if you may. That is, my universe is symmetric under rotations. This is not necessarily true for any field theory, and, to name some, electromagnetism is $U(1)$ and QCD is $SU(3)$. Now, I can see that the Lagrangian has a potential term which depends on the factor N. If I tend N to infinity, the model becomes exactly solvable (the interaction term dies) and this might not make sense, but this also is somewhat the only way I have to analyse the system, as now $\frac{1}{N}$ becomes really small, and I can do perturbation theory. Writing the potential term independently, I have

$$V = \frac{1}{2}m^2\phi^i\phi^i + \frac{\lambda}{8N}(\phi^i\phi^i)^2$$

I continue to write the fields as products as introducing exponents makes the indices and the exponent to get mixed up(I tried, it seems to be a problem with my typesetting). I would very much like to look at the extrema of V, for, since childhood, it is known that these points are probably the only points where equilibrium can be expected, and I want to see what happens to a system of a large number of fields when it gets to equilibrium (classically stable, if you want to call it). It is a quartic function of ϕ^i and it has no minima when λ is negative, so this is not of much use for me. If it is positive, then the function is strictly positive, with a minimum at $\phi^i = 0$ and this isn't a great discovery for the equilibrium is when everything is dead.

However, if my m^2 term was negative (the only other choice to see non zero minima), then I have my minimum at

$$\phi^i\phi^i = -\frac{2m^2N}{\lambda}$$

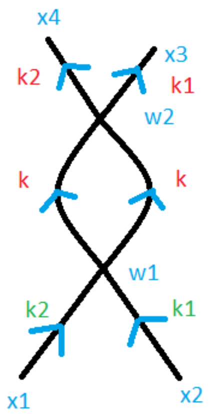
How can the mass squared be negative is the obvious issue here if my states were to get a non zero minimum. Well, only when the term is not exactly the "bare" mass of the field quantum, but is some sort of a modified (for the curious reader, "renormalized" is the term) mass. I will explain in detail how this construction is made, and this construction has a deep significance in the resurrection of quantum field theory as a sensible physical theory. For now, take it as a temporary axiom that it can be done. Thus, I have for some $i = k$ (it doesn't matter which one) $\phi^k\phi^k = -\frac{2m^2N}{\lambda}$. Something surprising has happened the moment I did this assignment, that is, the symmetry of my universe is now *broken*. I started off with $O(N)$ where all N fields were alike and now, one such field ϕ^k has gained the special status to settle at the minimum, and now my universe is $O(N-1)$.

What I could do is to call my $O(N-1)$ system as the new universe and construct a new potential which shows the separation between ϕ^k and the rest of them distinctly. I define $\phi^i = \delta_{ik}M$ and fancily call $\sigma = \phi^k - M$ and define $\pi^i = \phi^i$ for all $i \neq k$. Look at the potential below and assure yourself that it is nothing new by plugging in these terms

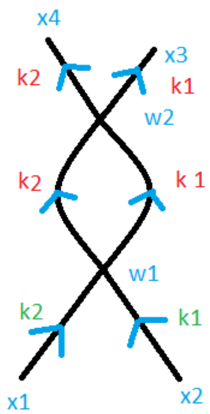
$$V' = \frac{\lambda}{8N}[\pi^i\pi^i + \sigma^2 + 2M\sigma]^2$$

Write the square in full and notice that the π^i fields are essentially massless while the one σ field has become massive with the mass $m_\sigma = -2m^2$. This is the Goldstone theorem. When symmetry breaks in your universe of N fields, it becomes equivalent to one massive field and $N-1$ massless fields whose quanta (which are massless, of course) are called as the Nambu-Goldstone bosons (Nambu got to the same conclusion independent of Jeffrey Goldstone [GOL] when he was analysing the BCS theory of superconductivity, see [NAM]). This is the first surprise one gets from the seemingly simple Lagrangian, the symmetry breaks when your system is in equilibrium and Nambu-Goldstone bosons gush into the universe (the number of these bosons is also calculable in various situations, in this case we have N-1 of them).

Another interesting aspect of the Lagrangian comes into play when you look at the topology of the Feynman diagrams and the corresponding field algebra that is computed. The Feynman formalism would be of no use if I can find two diagrams which have different field algebraic structure (I am talking about quantum fields, and not mathematical fields) but the same topological structure. Recalling the formalism, for the one vertex diagram with four propagators joining at one point, I have the amplitude for the process to be proportional to $\frac{1}{N}$ (remember that you need to look at $\frac{\lambda}{8N}$ as our new perturbation parameter λ'). The diagram



which we saw before, is also proportional to $\frac{1}{N}$ if you do the math correctly because you integrate over the internal lines (look at this as blowing up the point in the one vertex case to an internal process in the mass shell, as long as it is internal, it won't change the algebra). However, if by some divine power I knew that the internal lines were of the same momenta as the incoming fields, then I would have the diagram

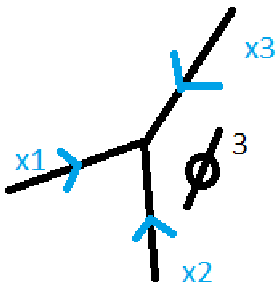


which is now proportional to $\frac{1}{N^2}$ and not $\frac{1}{N}$ because there is no integration over the internal lines (this is one specific process of the previous kind, averaging over all such processes kills of one N in the denominator). Therefore, I have two topologically equivalent Feynman graphs (say someone erased the labels et cetera) giving me different results. To counter this, Coleman suggests you fabricate a new field ξ which is fake because I define

$$\xi = \frac{1}{2}\frac{\lambda_0}{N}\phi^i\phi^i + m^2$$

$$\mathcal{L}' = \mathcal{L} + \frac{1}{2}\frac{N}{\lambda_0}(\xi - \frac{1}{2}\frac{\lambda_0}{N}\phi^i\phi^i - m^2)^2$$

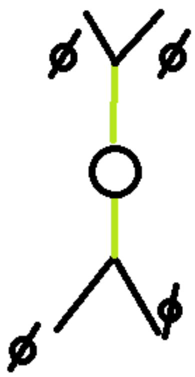
I just added a zero to the Lagrangian and made it look all complicated and scary, but then when you expand the square and actually add it, you see that the Feynman diagrams now need to account for the ξ field too. To understand this, forget about the present discussion and go back to when I started discussing interactions in my universe, and I mentioned that I don't care about cubic ϕ theories. This does not mean they do not exist or are not allowed by Physics, but if you do the computations (by all means, please do) you would get interactions with three propagators at one vertex instead of four and the Schwinger picture gives me $(\frac{\delta}{\delta J})^3$ terms in the exponent instead of $(\frac{\delta}{\delta J})^4$. That's literally all that would happen.

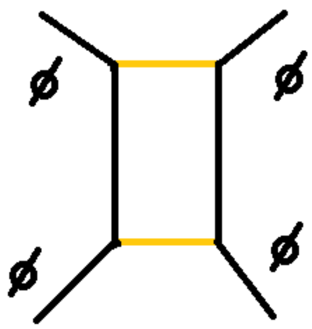


Getting that out of the way, expanding the square and cancelling terms ruthlessly I end up with the following (new look, old dynamics) Lagrangian

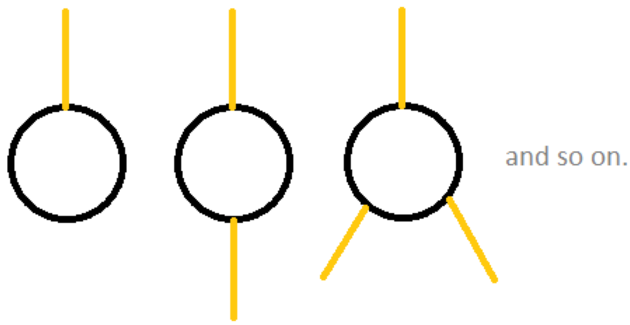
$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi^i\partial^\mu\phi^i + \frac{1}{2}\frac{N}{\lambda}\xi^2 - \frac{1}{2}\xi\phi^i\phi^i - \frac{Nm^2}{\lambda}\xi$$

The term that deserves any attention is the third summand (because the rest of them have ξ living all independently, and I know it's fake) and this is a cubic field interaction where two lines hitting the vertex are the ϕ fields and one is the ξ field. The only thing you need to know about the ξ propagator is that it is proportional to $\frac{1}{N}$ by definition. With this, you can confirm that the topologically similar diagrams are different now, as shown below





With this out of the way, I have a new Lagrangian for my N field universe and this contains a new potential term (if you're observant enough, you can see a Legendre transform sneak in somewhere) which fixed up my problem with the Feynman formalism. Well, not exactly. A closer look at the esoteric diagrams on the next page (the black lines are the ϕ propagators and the green lines are the ξ propagators) tells me that these diagrams also contribute to the $\frac{1}{N}$ order of my perturbation series.



This is not a new problem to tackle, and these one loop Feynman diagrams were analysed by Feynman and Schwinger when they were working out Quantum Electrodynamics (for them the black lines were electrons and the green lines were photons, hence the name *radiative correction* expansion is also used in the literature) and summing these diagrams was no cakewalk as the sum was visibly *divergent* (it led to the idea of "renormalization" which I will discuss in a moment) and an expression for this sum was developed by Coleman and Weinberg in their phenomenal paper [COL3] (this is also one of the end semester project problems in [PES] titled "*The Coleman-Weinberg Potential*"). I will use this sum without proof here, and my complete consistent effective potential for the universe of a large number of ϕ fields is given as

$$V = -\frac{1}{2}\frac{N}{\lambda}\xi^2 + \frac{1}{2}\xi\phi^i\phi^i + \frac{Nm^2}{\lambda}\xi + \frac{1}{2}N\int \frac{d^{D+1}k}{(2\pi)^{D+1}} ln(k^2+\xi)$$

This is startling. The answer is dependent on the dimension of the spacetime!

3.6.4 Renormalization: A Detour

> Sensible mathematics involves neglecting a quantity when it turns out to be small, not neglecting it just because it is infinitely great and you do not want it!
>
> Paul Dirac, Directions In Physics, 1978

Go back to page 33 and you see that I remarked about the self energy terms being infinite. This problem was noted long before quantum mechanics even came into existence, more precisely, when people were busy trying to make Maxwell's electrodynamics consistent. The idea that Lorentz invariance was inbuilt in the Maxwell formalism drove people to analyse the world with this formalism as the theory of everything. At the time, the electron was probably the best known elementary particle and it was essential to understand what happens to the formalism when you came really close to the electron. Abraham and Lorentz assumed that the electron was a sphere of uniform charge density and tried seeing what would happen when the electron went really fast, and you can work this out for yourself, the electron would not remain a sphere but would deform for electromagnetic signals take time to travel from one point on our spherical electron to the other (the forces that led to this deformation were called *Poincare stresses* after Henri Poincare, a hero of mine), and these forces would depend upon the *third derivative* of the position (I refuse to think the term "jerk" is standard for the third derivative, but whatever). This theory did not yet eliminate the issue, but people tried patching it up with various fancy things (the term *electromagnetic mass* also comes in here) and fancy ideas such as fields going backward and forward in time and so on. See [FEY1] for a complete description of this story.

This idea went on even after quantum electrodynamics was formulated when they noted that the radiative correction Feynman diagram sum was horribly divergent. In the time that followed Schwinger, Feynman et al swept the infinities as essentially constant terms (constantly blowing up, for more history see [CLO]) and continued to compute stuff out of quantum electrodynamics and this was met with much hostility, and even Feynman wasn't convinced that the theory was working because the infinities were neglected. It took more work and the pioneering brain of Kenneth Wilson to discover that things being divergent was because we were trying to apply the theory to energy scales where it was not applicable, and this is the core idea behind the theory of renormalization (see [KEN] for his original paper).

Even to date the subject of renormalization is presented in a way which does not sound good to the mathematician's ear, but as a pleasant surprise, it turns out that there exists something known as a *renormalization group* which when described rigorously makes complete logical sense. This idea did not grow out of quantum field theory independently, but grew in tandem with people working on condensed matter theories (such as the Ginzburg Landau theory of ferromagnetism) related to critical phenomena. I will talk about this in some detail in a while. But first, the mathematics. The discussion is based on the amazingly readable paper by Prof. Delamotte [DEL].

The one statement that needs to be stressed is the following: the infinity comes because quantum field theory is not useful for systems where we deal with length scales which are not comparable to the Compton wavelength of the quantum of the field or for energy scales that are not comparable to the mass of the quantum. You should feel cheated if I left you here and stopped this report because I promised you atleast some explanation of the complexity of the world we live in and this statement clearly tells that I cannot do this by simply drawing Feynman graphs. I need to construct some sort of a mathematical machinery which adapts this (extremely) restricted theory to the real world (of condensed matter, if you may). I will somewhat trivialize the Feynman diagram sum of the one loop processes I drew on the previous page and write it down as a function

$$A(x) = m_0 + m_0^2 A_1(x) + m_0^3 A_2(x) + ...$$

This is how the perturbative expansion looks like and (this is not how the actual sum is done, the subscripts are just to maintain sequentiality) m_0 is a quantity related to the problem I am working with. In my large N issue, it is the mass of the ϕ field quanta, in the radiative correction issue it was the charge of the electron. The problem with evaluating this sum is that the auxiliary functions A_i are not very nice

$$A_1(x) = \int_0^\infty \frac{da}{a+x}$$

This function is divergent, and more specifically, it diverges like a logarithm as it approaches the upper limit, and hence is called *logarithmically* divergent. It can diverge in other ways also, but the idea is the same (and it is logarithmic in most field theories, in our case, the propagator is a one over something similar, you see). Now, I perform experiments on the large N system at some length scale and get my results. I actually need precisely *one* measurement and I am done because I figured the value of m which is appropriate for the length scale at which I am working. Now I can tabulate the values of m that need to be used versus the length scales and I have a better description than the dead end before, and this is what I mean when I say I have renormalized the field theory. Let the length scale be some λ at which I measure the outcome of the experiment as some m_R. Mathematically

$$m_R = m_0 + m_0^2 A_1(\lambda) + m_0^3 A_2(\lambda) + ...$$

I hope that my renormalization theory converges the right hand side of the expression so that I get the finite value I hope for so that when you do the same experiment at λ', you get a convergent sensible answer and the theory has some predictive power (ignoring the divergence also does it, and this is how Hans Bethe calculated the Lamb shift).

It does not need tremendous insight to tell me that the most evident problem is with the infinity sitting on the upper bound of that integral and if I instead write the upper bound as some Λ, I can go on to other non trivial issues. This is known by the glorious name of *regularization*. It seems that I already learnt regularization even before I knew all of the Maxwell equations and this I learned in a conversation with my advisor here at PRL, Dr. Mahajan. If you remember doing electrostatics there is the problem of solving the case of the electric field of an infinite wire with uniform length distribution. It turns out that the integral there is logarithmically divergent and we then assume some cutoff radius r_0 and conclude that the answer goes like $ln(\frac{r}{r_0})$ instead of just plugging zero in the denominator. Hands are waved and we say that this cutoff really is cosmetic as the potential difference is important and not the precise potential at each point (which I never really questioned back then). What I did, by assuming a cutoff radius, was essentially regularized the electric potential. The electric field is independent of this cutoff parameter and we are all happy. Now, if I solve the same problem for a wire of finite length L, then there is no divergence really in either the potential or the field, but then the field does *not* blow up when $L \to \infty$! It has renormalized itself magically.

It turns out that if I assume my three dimensional space in which the infinite wire is actually sitting, to be a D($\neq 3$) dimensional space and then carried out all the integration and all that, even the potential doesn't blow up. This is known as *dimensional renormalization* and it is somewhat central to Wilson's paper [KEN]. Anyway, writing Λ as the upper limit I have regularized my theory and now I proceed to renormalize, now with an additional condition that if after all the calculations $\Lambda \to \infty$ then the end result converges. The idea of such summations already existed in probability theory under the name of the Borel-Cantelli theorem (after Emile Borel and F. Cantelli) which tells that if $\sum^{\infty} P(E_j)$ is bounded, then $P(\lim_{j\to\infty} E_j)$ will have to vanish so we are not really walking in the dark here.

Be what it may, it is still perturbation theory and I start by writing the constant for the theory m_0 as a series in m_R which is the renormalized constant which I need while performing my experiment at λ scale

$$m_0 = m_R + m_2 + m_3 + ..$$

where I have assumed that the theory is renormalized for the constant (therefore I wrote m_R directly instead of m_1 and then seeing that it must be m_R or else we make a mistake in the first step itself). The term m_k is proportional to m_R^k for the higher orders. To renormalize upto a quadratic term, I first write down the series $A(x)$

$$A(x) = m_0 + m_0^2 A_1(x) + ...$$

Now plug in the series for m_0 in this and this is what follows

$$A(x) = m_R + m_2 + (m_R + m_2 + ...)^2 A_1(x) + ...$$

I do not explicitly write down any term which is dimensionally greater than m^2 and expanding the square I have

$$A(x) = m_R + m_2 + m_R^2 A_1(x) + ...$$

Now, if my experiment at which m_R was determined to be right (arbitrary precision assumed) was λ then I essentially have the equality $A(\lambda) = m_R$ and combining this and the previous equation I have

$$m_R = m_R + m_2 + m_R^2 A_1(\lambda) + ...$$

$$\implies m_2 = -m_R^2 A_1(\lambda)$$

Now, all I have to do is to plug this back into my quadratic approximation for $A(x)$, blow Λ to ∞ and hope that it does not self destruct. And indeed it does not.

$$A(x) = m_R + m_R^2(A_1(x) - A_1(\lambda)) + ...$$

You need not even compute the integral because the difference of two logarithms essentially turns into a fraction and the thing that caused all the mayhem, Λ, gets killed instantly giving me a finite summation (upto the second order). You can chug and repeat the same process throughout for all orders and end up with a renormalized summation for a given experimental setup (or length scale). The process is worked out in much more detail in [DEL] and it is made very rigorous (the Lagrange-Burrmann inversion et cetera) there. One fact which might interest you is the following (proven in the appendix of [DEL]): the part of the function A_j (for any j) which blows up is always only dependent on the regulator Λ. This is obvious for the second order computation but not really so as the corrections become very weird at higher orders. [DEL] also works out the case of dimensionless m theories, such as quantum electrodynamic scattering processes, where it is the fine structure constant $\alpha = \frac{1}{137}$.

The idea of a renormalization group comes about when you look at the problem from up above and see what exactly is happening. You have the data m_R at the length scale of λ from the experiment. Someone else comes along and does the experiment at some λ_1 and ends up with m_{R1}, and then many people come and (see that it is an easy experiment) do many such runs with λ_k as the scale and end up with m_{Rk}. Wilson, Kadanoff, and many others saw that these pairs m_{Rk}, λ_k sit nicely in a group structure and it is possible to define a mathematical object τ (this is not to be confused with the time ordering operator) which takes you from one scale to another, like

$$\tau((m_1, \lambda_1)) = (m_2, \lambda_2)$$

The speciality of τ is that it has *fixed points*, that is at some pair it starts giving out the same pair over and over again. I have not seen a quantum field theoretic demonstration of this, but a statistical physics demo can be seen in the wonderful paper [KAD] by Kadanoff which concludes with the fact that these fixed points tell me that there is a critical phase transition that is happening. The idea of a phase transition is that the statistically random entities suddenly start behaving as a collective entity, which in turn implies that short range ordering has suddenly become an infinite range (or the classical range of the material) phenomenon. Transition to ferromagnetism is one such phenomenon encountered while analysing the Ginzburg Landau model with statistical field theory (another name for all that we have discussed). You can also use the renormalization group idea to solve differential equations and the group usually turns out to be the symmetry group of the solutions.

I suspected that the renormalization group for a general quantum field theory would be the Lie group for a differential equation and this equation turns out to be the Callan-Symanzik equation (due to Curtis G. Callan Jr. and Kurt Symanzik) for the n-point correlation function of the field theory according to [PET] (the paper actually establishes that the renormalization group relates to the Gell-Mann-Low functional equation, after Murray Gell-Mann and Francis Low, and the paper [HIG] relates this to the Callan-Symanzik formulation). For a philosophical take on the applicability of this approach see [KEN1], and for the paper which inspired Wilson to formulate the idea for quantum fields, see [STA]. For a more complete historical survey of the renormalization approach see [HUA].

3.6.5 Large N: Second Pass

> You are a victim of your own neural architecture which doesn't permit you to imagine anything outside of three dimensions. Even two dimensions. People know they can't visualise four or five dimensions, but they think they can close their eyes and see two dimensions. But they can't.
>
> Leonard Susskind

After the whirlwind detour of the idea of renormalization, it is time to get back to the large N potential which supposedly accounted for all the shortcomings initially. To recapitulate, I had a system of a large number of scalar ϕ^4 interacting fields, noticed that there was an issue with two topologically similar Feynman diagrams having different algebraic description, introduced a fake field ξ, this led me to discover that there are other diagrams of $\frac{1}{N}$ which I have missed and that their sum gives me a divergent solution, realized that if I can redefine the bare mass as a renormalized mass I can make the integration convergent (I will show this now), and this renormalized mass also leads to the breaking of the symmetry in my universe (the Goldstone theorem). Even without integrating, I noted that the potential depended on the dimension of spacetime I was living in. As Prof. Susskind states, the idea of a (2+1) spacetime seems easy to visualize but then if you believe that physics ultimately describes the way you see reality then (2+1) spacetime is a very weird place to live in.

To restate the effective potential I had ended up with, I have

$$V = -\frac{1}{2}\frac{N}{\lambda_0}\xi^2 + \frac{1}{2}\xi\phi^i\phi^i + \frac{Nm_0^2}{\lambda_0}\xi + \frac{1}{2}N\int \frac{d^{D+1}k}{(2\pi)^{D+1}} ln(k^2+\xi)$$

I have included the 0 subscripts on the mass and the interaction coupling to state that I am working with the bare quantities and I will use the renormalization idea to get the integral to converge. The extrema of this potential is where I am interested in for an equilibrium solution. Therefore, the field configuration for equilibrium is obtained by differentiating this with respect to both the fake and the actual field

$$\frac{\partial V}{\partial \xi} = 0, \frac{\partial V}{\partial \phi^i} = 0$$

To make things look less obfuscated, note that, at equilibrium, I have

$$\frac{dV}{d(\phi^i\phi^i)} = \frac{\partial V}{\partial(\phi^i\phi^i)} + \frac{\partial V}{\partial \xi}\frac{\partial \xi}{\partial(\phi^i\phi^i)} = \frac{\partial V}{\partial(\phi^i\phi^i)} = \frac{\xi}{2}$$

The advantage of doing this is that now I can detect for symmetry breaking directly. If ξ is zero, then I have a broken symmetry (note that this has the partial derivative of V with the square of the ϕ field). The derivative with the fake field gives me the following equality (I used the Leibniz rule to send the derivative inside the integral because ξ and k are independent)

$$\phi^i\phi^i = \frac{2N\xi}{\lambda_0} - \frac{2Nm_0^2}{\lambda_0} - N\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2+\xi} - - - - - - - - -(*)$$

It is time for some renormalization. This will feel slightly different from the mathematical approach of the last section because I have two constants which I can use to beat the divergence. I could modify λ_0 or m_0 or both. Consider the following

$$\frac{1}{k^2(k^2+M^2)} = \frac{1}{M^2}\frac{M^2+k^2-k^2}{k^2(k^2+M^2)} = \frac{1}{M^2}(\frac{1}{k^2} - \frac{1}{k^2+M^2})$$

Thus, I can see some sort of a logarithmic cancellation that I can do to kill the divergence in $(*)$ and the only constant which I can renormalize to do so has to be in the denominator of the terms, namely, λ_0 and the new term M introduced above will be of the dimensions of λ_0.

I am assuming a regulator Λ (although this is not explicitly mentioned in [COL2]) for the upper bound on the radius of the k integration shell in four dimensions (if you are using a CAS then you can skip this because it computes an indefinite integral anyway, for hand calculations the regulator prevents people from freaking out). I define the renormalized constant λ as follows

$$\frac{1}{\lambda} = \frac{1}{\lambda_0} + \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2(k^2+M^2)}$$

In doing so, looking at the partial fraction split, I see that I have essentially replaced the term $\frac{1}{k^2+\xi}$ with a similar term $\frac{1}{k^2+M^2}$ and added a $\frac{1}{k^2}$. To account for this additional k term, I have to subtract it somewhere else, and this is where I renormalize the mass instead (actually the term $\frac{m_0^2}{\lambda}$, note that in the expression *, if you need to explicitly call out the constant factors, they are $-\frac{m_0^2}{\lambda_0}$ and $\frac{1}{\lambda_0}$)

$$\frac{m^2}{\lambda} = \frac{m_0^2}{\lambda_0} + \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2}$$

Boot up your favourite CAS and plug the renormalized constants to see the integral converge as it is supposed to and the new $(*)$ expression looks like

$$\phi^i\phi^i = \frac{2N\xi}{\lambda_0} - \frac{2Nm_0^2}{\lambda_0} - \frac{N}{16\pi^2}\xi\log(\frac{\xi}{M^2}) - - - - - - - - -(a)$$

Just to confirm whether whatever we are doing is not hocus pocus, plug in $\xi = 0$ to see whether there is any symmetry breaking or not in (3+1) spacetime, and indeed, there is a vacuum state which is not zero

$$\phi^i\phi^i = \frac{-2Nm^2}{\lambda}$$

This is just the same answer I got in the first pass when I assumed that the mass squared term could be negative. Goldstone's theorem follows and my universe gushes with massless Nambu-Goldstone bosons as usual. Nothing weird happens in this spacetime. Well, again, not exactly. For a moment, if I actually think of ξ as a real field (you can think of it as a real field which coincidentally satisfies it's definition in terms of ϕ^i), then if I increase ξ it is obvious that ϕ^i increases but then it hits a maximum (you can find out where by $\frac{\partial\phi^i\phi^i}{\partial\xi} = 0$) and then goes to $-\infty$. The moment you increase ξ beyond the maximum, assume that the independent status of ξ as a field is taken away, it is fake once again, and the only thing that defines it is it's dependence on ϕ^i by definition and it so happens that it becomes *complex*.

What does this even mean? The field at a point should chug out a real value, but it gave me an imaginary term too. This is a seasonal problem in physics and this happened because we are trusting mathematics to give us insight about physical reality whereas the math does not know that we are dealing with real fields. The fact that ξ is complex also tells me (from the definition of the effective consistent potential) that the potential is complex too. If you work out the propagator fully (this is done in the last section of [COL2]) you will see that the modulus of the momentum four vector k becomes *negative*. Ah, what have we encountered? This is a particle which moves greater than the speed of light! It's a tachyon.

Where did it sneak in? There is only one place in our discussion where we assumed something will be taken care of, it is the regulator Λ. If I compute the integral in the renormalized definition of λ with the upper bound at the regulator Λ, I will get the following expression

$$\mathcal{I} = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2(k^2+M^2)} = \frac{1}{32\pi^2}ln(\frac{\Lambda^2}{M^2})$$

$$\frac{1}{\lambda} = \frac{1}{\lambda_0} + \mathcal{I}(\Lambda)$$

$$\implies \lambda_0 = \frac{\lambda}{1 - \mathcal{I}(\Lambda)\lambda}$$

The whole point of renormalization was to flip the places of λ_0 and λ and then seeing reality at the new scale, right? Look what happens when $\Lambda \to \infty$, λ_0 goes to zero indeed, but is negative at some point of it's life. This shouldn't happen because our assumption was that $\lambda_0 > 0$ for all times or else we would have $-\infty$ as the only stable field solution. And this is where tachyons come in, and this problem is called the *tachyon disaster* by Coleman. This turns out to be an inherent problem in other field theories too, and Lev Davidovich Landau found one in quantum electrodynamics (called the Landau Pole, or the Moscow Pole in literature, see [CLO]) at around 1950. This problem is deeply rooted in a fundamental property of a (gauge invariant) quantum field known as *asymptotic freedom.* As the name suggests, for systems such as electrodynamics or Yang-Mills QCD which have gauge invariance, it so happens that the interactions become weaker at short distances if we are dealing with high enough energy scales. The work of David Gross, Frank Wilczek and David Politzer in 1973 (which led to their Nobel Prize) describes this concept and eliminates the issue of the Landau Pole(see [GRO], [POL]).

The paper [CLO2] (which was published in 1974, and Politzer is one of the co-authors) recognizes that the issue with electrodynamics has been solved but for our large N field the tachyon issue persists and the only hope is to look for another vacuum state which might happen to be at a minimum at higher energies or constructing the next order theory by including $(\frac{1}{N})^2$ terms in the Lagrangian.

As simple as the scalar theory looked, it really caused some trouble, but there are some niceties too. I will not bore you with the technical details (refer [COL2] for the computation), but there seems to be no symmetry breaking for two or lower spacetime dimensions. This is the famous *Mermin-Wagner-Hohenberg Theorem* (which is sometimes esoterically referred to as the Landau-Peierls-Mermin-Wagner-Hohenberg-Coleman Theorem, see [HOH], [MER1], [COL4]) which states that two (or lower dimensional) systems cannot exist in equilibrium at non zero temperatures (it was derived in condensed matter using similar arguments, not to reiterate, but it is the same many body idea). Graphene was an interesting example of one such system which seemed to violate this theorem but it was discovered that it exists due to constant surface ripples (see [FAS]) and the structure is not really two dimensional (for a relatively advanced study of Graphene's properties using large N and renormalization, see [FOR]).

With this I end the discussion on the question regarding the complexity of matter, as for even relatively easy models there is rich physics which comes out after analysis. It has been a long trip, starting from a world where there was just vacuum to a world where there are tachyons running around and symmetries being broken. In my opinion, the relationship between the spacetime dimension and physics might be something even deeper than the above discussion. This is most certainly not the best one could do and this is barely the tip of the iceberg. As more collisions take place at the LHC, and more particles are found, it is the job of the mind which is uniquely gifted with intuition to construct more theories which form compatible explanations for regimes where current theories breakdown. Gravity, which so far has been neglected due to it's insignificance at the length scale at which calculations were being done, might have a decisive role to play in such theories, as suspected by many (if not all) top physicists of the day. The validity of quantum mechanics in explaining classical phenomena is still being probed with the present trend leading us to believe that information and computability have something to do with this(see [GER], [GER1]).

In the conclusive section I will discuss about the idea which has been pondered about since the dawn of intelligence and in much more depth since the twentieth century, the concept of time and our origins.

3.7 Time

> He lifted His hand, and from it burst a fountain-spray of fire, a million stupendous suns, which clove the blackness and soared, away and away and away, diminishing in magnitude and intensity as they pierced the far frontiers of Space, until at last they were but as diamond nailheads sparkling under the domed vast roof of the universe.
>
> Mark Twain, Letters From The Earth

In the February of 1895, Ludwig Boltzmann had an amazing insight [BOL] which would come back to bite cosmologists who ignored the issue when the concept of the cosmological constant was integrated as an essential idea to explain Edwin Hubble's observation (to be accurate, it was Georges Lemaitre's) that the galaxies were receding away from us. Instead of looking at the entropy, he looks at the time derivative $\frac{dS}{dt}$ (in the paper he refers to this as $\frac{dH}{dt}$, where H is the H function defined in the Boltzmann H theorem, which I will describe shortly). He notes the following at the end of the paper and it is a remarkable statement (known as the Stosszahlansatz)

Assuming that the universe is great enough, the probability that such a small part of it as our world should be in its present state, is no longer small. If this assumption is correct, our world would return more and more to thermal equilibrium; but because the whole universe is so great, it might be probable that at some future time some other world might deviate as far from thermal equilibrium as our world does at present. Then the aforementioned H curve would form a representation of what takes place in the universe. The summits of the curve would represent the worlds where visible motion and life exist.

It sounds really archaic but what he's trying to tell is that the idea of the irreversibility of time that is usually assigned to the Second Law of Thermodynamics is not exactly true per se with respect to statistical mechanics. The statement of the Boltzmann H theorem is that if you call $f(x, v)$ as the Maxwell-Boltzmann distribution of a gas, then the quantity

$$H = -\int f(x, v) \log(f(x, v))dxdv$$

is always increasing. This quantity is more similar to entropy in the information theoretic sense, in which you define the entropy of a source of information as

$$S = -\sum_i p_i \log(p_i)$$

where p_i is the probability that the i^{th} symbol is emitted. This definition came into existence due to Claude Shannon's landmark research on communication theory and thence the quantity H is usually referred to as the Shannon-Boltzmann entropy. Boltzmann's claim is that unlike the Second Law of Thermodynamics which states that the *derivative* of entropy of the universe is always positive, there are chances where the derivative might equal zero or even be *negative*. The so called "summits" of the H function are the stationary points where the universe is said to be in a state of some constant value for the entropy. There is no more scope of the universe being any more random and things tend to coexist.

The idea of a negative slope for entropy sounds disturbing to many because it is almost saying that in principle time can go backwards, a shattered vase can reassemble itself, a dead organism can reorganize and come back to life. The problem with this thinking is that we have gotten used to taking the Second Law of Thermodynamics as such a fundamental concept that we have related a pristine concept such as time to a seemingly arbitrary concept such as the randomness in the organization of the molecules of the universe. As a refutation to this ideology, I present two ideas, one due to Poincare and the other being the quantum mechanical analogue of the same.

For this demonstration, I have to make an ansatz which can be contested, but I need to make progress with the discussion so bear with me, because it can be justified to a certain extent. I assume that the universe is a closed volume (spherical volume, if you may) which does not interact with the external space (if there is any). The assumption is justified for we can only see as far as the light from the space which we are observing can come back to us and with this I can specify a radius for the sphere (known as the celestial sphere, I am not sure) as the ratio of the speed of light to the Hubble constant. The radius of the sphere is known to be non constant as it is observed that the Hubble constant H_0 is actually changing with time. So, this is one argument to contest the assumption, but I will take it as a constant for the discussion. Now, there is this theorem due to Poincare [WIN] which states that for a classical statistical system in a closed volume, the phase plot (position versus canonical momentum) will *repeat* in a finite interval of time. The interval of time need not be an observable duration and the time period increases rapidly as the number of entities in the closed volume increase.

To understand this, assume an isolated box in which non interacting particles are being created one after the other (these particles can be due to vacuum fluctuations but for the time being let them just be created due to some divine process). The initial configuration of vacuum is trivially periodic with a time period of 0, when one particle enters the probability of it occupying the state characterized by a certain point $\bar{x}$ is statistically proportional to $e^{-\beta\mathcal{H}(x)}$ and since the box is isolated, β is constant and the time it takes for this to happen again is simply $e^{\beta\mathcal{H}(x)}$. For two particles it is $e^{\beta(\mathcal{H}(x_0)+\mathcal{H}(x_1))}$. Thus, continuing this process, for a large number of particles the time period for a recurrence to take place is vaguely equal to $e^{\beta\sum^{N}\mathcal{H}(x_j)}$. This is a heuristic derivation of the Poincare Recurrence Theorem.

The quantum mechanical analog to this idea was published in 1957, see [BOC], and the proof is simple and direct. Working in the Schrodinger picture, let the wavefunction of the box be $\psi(t)$ at any time t, then it is sufficient if I can prove that

$$||\psi(T) - \psi(0)|| < \epsilon$$

for a really small ϵ. Now, $\psi(t) = e^{-i\mathcal{H}t}\psi(0)$ and substituting this in the inequality, I get

$$\sum_{k=0}^{\infty}(1 - \cos(E_k t)) < \epsilon$$

This inequality can be satisfied for a value of $t = \tau$ and the existence of such a τ is already well established from the theory of almost-periodic functions, refer to [JES] for a mathematical explanation. The paper [BOC] also states that this derivation does not apply if the box had a continuous spectrum, so, the assumption that I can discretize time is still questionable. Nevertheless, the authors also state that macroscopic observables will not maintain their equilibrium values once they have attained them, and this is an alarming fact. Thus, from this discussion, it is natural to ponder whether the theory of the Big Bang as the origin of the universe is correct because it looks likely to me that we are rather in another Poincare recurrence of the universe and it so happened that in this recurrence the conditions of the Earth were such that intelligent life could exist. This is also the same idea Boltzmann was pondering about because for a recurrence involving a collective state such as the Big Bang to take place the particles had to go from a random state to a collective state at some point and it is this process which shows the negative slope of the entropy. Or using our examples, if we wait for long enough the shattered vase can get back to its original shape, all of it's molecular bonds formed once again and a dead specie can get back to life with it's own structure going through a recurrence.

Of course this is all speculative, but for something precise, [WEI] section 3.6 proves that if you assume that unitarity is to be maintained then the H theorem holds true regardless, and this is in contradiction to [BOC]. To understand this better, I will sketch Weinberg's proof. This was pointed out to me by Dr. Mahajan.

Let us assume that the universe is in some state A and it is transitioning to some state B. The state A essentially encodes all the particle and anti-particle content of the universe quantum mechanically and I also take the universe to be unitary axiomatically. Now the process $B \to A$ can be seen as a forward time process if I interchange the particles in B and A with the anti-particles in B and A. It can be shown that unitarity in my universe implies that

$$\Gamma(A \to B) = \Gamma(B \to A)$$

where Γ is technically the process amplitude. Logically, the only way I can end up in the state A is either if I am already in A or the process $B \to A$ fires. Thus I can say that the probability of finding myself in A increases due to $B \to A$ and decreases due to $A \to B$.

$$\frac{dP_A}{dt} = \int dB P_B \frac{d\Gamma(B \to A)}{dA} - \int dB P_A \frac{d\Gamma(A \to B)}{dB}$$

Now, the Shannon-Boltzmann entropy can be written as (using, you guessed it, Leibniz rule, but this is subtle and it's use comes from the fact that $\int dA P_A$ is constant)

$$-\frac{d}{dt}\int dA P_A \log P_A = -\int dA(1+\log P_A)\frac{dP_A}{dt} = -\int dA \int dB(\log P_A+1)[P_B \frac{d\Gamma(B \to A)}{dA} - P_A \frac{d\Gamma(A \to B)}{dB}]$$

$$= -\int dA[\int dB(\log P_A + 1)P_B \frac{d\Gamma(B \to A)}{dA} - \int dB(\log P_A + 1)P_A \frac{d\Gamma(A \to B)}{dB}]$$

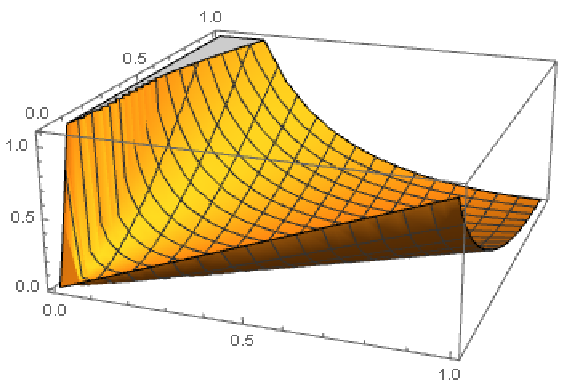
There is some subtle change of integration variable trickery which Weinberg skips but it is very easy once you write things out fully (like above). Now, if I change the variable of integration in the second term from B to A, then the non logarithmic multiplication products get cancelled because they end up to be the same term (there is actually another variable flip that happens but since the double integration variables can be flipped, this is not done explicitly ever) and I have

$$\frac{dH}{dt} = \int dA \int dB P_B \log(\frac{P_B}{P_A})\frac{d\Gamma(B \to A)}{dA}$$

Now consider the function

$$f(x, y) = y \log \frac{y}{x} - y + x$$

You could either go through the drudgery of multivariable calculus or simply (as usual) boot up your CAS and plot it. The function apparently is positive for all x, y in $[0, 1]$ and is zero at the origin.



This implies the following

$$P_B \log \frac{P_B}{P_A} \geq P_B - P_A$$

Thus, I can say

$$\frac{dH}{dt} \geq \int dA \int dB(P_B - P_A)\frac{d\Gamma(B \to A)}{dA}$$

The next step is obvious, split the integral into the difference of two terms and flip the variables on the (subtrahend? I think that's the technical term) second term and you see that the whole integrand comes down to the factor $\Gamma(B \to A) - \Gamma(A \to B)$ which is zero if you assume unitarity. Thus, Boltzmann's H theorem holds good.

That's about it, that's the answer to my second question posed in the Prologue. The box model of the universe is not exact due to another reason (this is due to Leonard Susskind, from his talk "Why Is Time A One Way Street?" at Santa Fe). The box is not really closed in the sense that there are always quantum fluctuations at the edge of the universe, or in other words, the universe is like an inside out black hole. There is yet another issue, which comes under the name of Boltzmann Brain Paradox. It is truly remarkable that an organized system such as the human body was constructed from the Big Bang, but then when compared to the vastness of the universe today, the probability is not really small as Boltzmann remarked. But if you accept this fact, then you should also accept the fact that there is a much higher probability that an organism much simpler than a human yet with the power of intuitive thinking would exist. If you can do a rough calculation, you get a whole lot of such organisms (these are the Boltzmann Brains), in fact so many that we should have observed one by now. But we haven't seen one till now and this forms the crux of the paradox (This explanation is an adaptation of the "Explain Like I'm Five" page on Reddit regarding Boltzmann Brains). People are still working this out and estimations for the lifetime of the universe are being made, see [PAG] and [ALB]. In my opinion, it is truly a miracle that it is we who are making such deep analyses and a fruitful understanding of the innards of human thought can be achieved only by cultivating such thoughts (it is in this matter that I am gravely disappointed because it turns out that because his ideas were not received well, Boltzmann grew more and more depressed and hanged himself in Trieste, Italy while on a supposed vacation with his wife and daughter, may his soul rest in peace).

3.8 Epilogue

Aim small, miss small.

Chris Kyle, American Sniper

The theoretical discussion done in the second pass of our large N discussion has been worked upon further and results have shown that there does exist a better minimum (a different ground state) but the large N approximation does not remove the tachyon disaster even at higher orders of $\frac{1}{N}$ in the effective potential (for a thorough discussion on this aspect, see [BAR]). Interesting things happen when the same model is fed to a (classical) computer, and this is done when you assume that spacetime is discrete instead of continuous and you assign a lattice spacing a. According to the report [CAL], it is said that unlike the continuum case, the discretized model of the scalar ϕ^4 theory at large N does *not* show any pathologies but it rather represents a smooth effective potential with a minimum at the origin, that is it shows a vacuum state as the stable equilibrium solution instead of the Goldstone Mexican hat model which accounts for the symmetry breaking.

The regulator Λ in the continuous case was used to set an upper limit on the energy integration and is considered to be a perturbative regulator as it enters the calculations as a renormalization by product. In the case of the lattice, the regulator is the *lattice spacing* and is essentially non perturbative. [CAL] concludes with a set of approximations which are a shade similar to what Maxwell did in thermodynamics in the case of phase transitions (and it is argued in [BAR] that the vacuum solution in the lattice case is a false vacuum and there exist two phases and not surprisingly you find tachyons in one of them). Peter LePage [LEP] describes a simple and easy to implement methodology in his paper which extends the techniques of lattice computations all the way up to QCD style lattice calculations. Present day software packages such as PyQCD also allow users to interface with Python as the base language and perform suitable calculations at chosen lattice spacings. The continuum limit is achieved when $a \to 0$ which is the same as stating that $\Lambda \to \infty$. Since we are talking about computers and simulations, it is apt to think (partly) philosophically about the idea of simulation and how well the universe can be computed. In essence, this is leading us back to Wolfram's question about the laws of physics being computed using Turing machines. Feynman gives an interesting plausibility argument regarding why a Turing machine cannot simulate reality but there are alternative ways of arguing about this ([MAI] uses Bell's theorem to refute it in a wonderful exposition of how Wolfram might have explained the origins of biology instead of physics, the present argument is from a mathematical standpoint from a conversation with one of my juniors, Yash Tibrewala).

The idea of a finite state automaton computing physical reality can invariably be seen to translate into the concept that nature is rather discrete than continuous. But if this be so, then it can be seen that I can draw a circle and try calculating the ratio of the circumference to the diameter with arbitrary accuracy and end up with a finite series of numbers after the decimal point. I can also draw a unit square and measure the length between every last elementary entity from end to end and have a finite *rational* number for the length as the base on which I have drawn my square is a lattice of non zero spacing (but of the smallest length scale possible) and the diagonal cannot occupy any point in between two rational points (for then we would have found a lattice spacing much smaller, in contradiction to the assumption). There would not be any more irrational or transcendental numbers and this really sounds chaotic (one could argue about Lindemann's exact proof of the transcendence of π but let us assume for the time being that we are working with mathematics which has not featured abstractions such as polynomials yet).

The absence of any pathologies in the discrete case for our scalar theory might be attributed to the same idea that we have missed out a lot of important physics which happens in the continuum limit and it is quite remarkable to see that all of the interesting stuff was trapped in between this small aberration which is neglected by the classical computer as an approximation.

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