1 Differentiating

To compute the complex derivatives use the Total Derivative theorem:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t}\frac{\mathrm{d}t}{\mathrm{d}t} + \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \dots \tag{1}$$

1.1 Differentiating atan2

$$\begin{aligned}\frac{\mathrm{d}\,\mathrm{atan2}\,(y,x)}{\mathrm{d}\theta} &= \frac{\partial\,\mathrm{atan2}\,(y,x)}{\partial\theta}\frac{\mathrm{d}\theta}{\mathrm{d}\theta} + \frac{\partial\,\mathrm{atan2}\,(y,x)}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}\theta} + \frac{\partial\,\mathrm{atan2}\,(y,x)}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}\theta} \\ &= -\frac{y}{x^2+y^2}\frac{\mathrm{d}x}{\mathrm{d}\theta} + \frac{x}{x^2+y^2}\frac{\mathrm{d}y}{\mathrm{d}\theta}\end{aligned} \tag{2}$$

2 3D Transformations

The vector ${}^{\{A\}}\boldsymbol{t}_{\{B\}} = [x\ y\ z\ \psi\ \theta\ \phi]$ represents a rigid transformation between two frames $\{A\}$ and $\{B\}$, specifically, how $\{B\}$ is seen from $\{A\}$.

2.1 Inverse compounding

$$\boldsymbol{t}' = \ominus\boldsymbol{t} = \begin{bmatrix} x' \\ y' \\ z' \\ \psi' \\ \theta' \\ \phi' \end{bmatrix} = \begin{bmatrix} -\boldsymbol{R}^T \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\ \mathrm{atan2}(o'_z, a'_z) \\ \mathrm{atan2}(-n'_z, n'_x\cos(\phi') + n'_y\sin(\phi')) \\ \mathrm{atan2}(n'_y, n'_x) \end{bmatrix} \tag{3}$$

where:

$$\boldsymbol{R} = \begin{bmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{bmatrix}; \boldsymbol{R}' = \boldsymbol{R}^T; \boldsymbol{R}' = \begin{bmatrix} n'_x & o'_x & a'_x \\ n'_y & o'_y & a'_y \\ n'_z & o'_z & a'_z \end{bmatrix} \tag{4}$$

$$\boldsymbol{J}_\ominus = \begin{bmatrix} -\boldsymbol{R}^T & \boldsymbol{N} \\ \boldsymbol{0}_{3\times 3} & \boldsymbol{Q} \end{bmatrix} \tag{5}$$

with:

$$\boldsymbol{N} = \begin{bmatrix} 0 & -n_z x\cos(\phi) - n_z y\sin(\phi) + z\cos(\theta) & n_y x - n_x y \\ z' & -o_z x\cos(\phi) - o_z y\sin(\phi) + z\sin(\theta)\sin(\psi) & o_y x - o_x y \\ -y' & a_z x\cos(\phi) - a_z y\sin(\phi) + z\sin(\theta)\cos(\psi) & a_y x - a_x y \end{bmatrix} \tag{6}$$

$$\boldsymbol{Q} = \begin{bmatrix} -n_x/(1-a_x^2) & -o_x\cos(\psi)/(1-a_x^2) & a_z a_x/(1-a_x^2) \\ o_x/(1-a_x^2)^{1/2} & -a_z\cos(\phi)/(1-a_x^2)^{1/2} & a_y/(1-a_x^2)^{1/2} \\ n_x a_x/(1-a_x^2) & -a_y\cos(\phi)/(1-a_x^2) & -a_z/(1-a_x^2) \end{bmatrix} \tag{7}$$

The inverted rotation matrix is:

$$\boldsymbol{R}' = \begin{bmatrix} \cos(\phi)\cos(\theta) & \cos(\phi)\sin(\theta)\sin(\psi) - \sin(\phi)\cos(\psi) & \cos(\phi)\sin(\theta)\cos(\psi) + \sin(\phi)\sin(\psi) \\ \sin(\phi)\cos(\theta) & \sin(\phi)\sin(\theta)\sin(\psi) + \cos(\phi)\cos(\psi) & \sin(\phi)\sin(\theta)\cos(\psi) - \cos(\phi)\sin(\psi) \\ -\sin(\theta) & \cos(\theta)\sin(\psi) & \cos(\theta)\cos(\psi) \end{bmatrix}^T \tag{8}$$

$$\boldsymbol{R}' = \begin{bmatrix} \cos(\phi)\cos(\theta) & \sin(\phi)\cos(\theta) & -\sin(\theta) \\ \cos(\phi)\sin(\theta)\sin(\psi) - \sin(\phi)\cos(\psi) & \sin(\phi)\sin(\theta)\sin(\psi) + \cos(\phi)\cos(\psi) & \cos(\theta)\sin(\psi) \\ \cos(\phi)\sin(\theta)\cos(\psi) + \sin(\phi)\sin(\psi) & \sin(\phi)\sin(\theta)\cos(\psi) - \cos(\phi)\sin(\psi) & \cos(\theta)\cos(\psi) \end{bmatrix} \tag{9}$$

$$\boldsymbol{R}' = \begin{bmatrix} \cos(\phi')\cos(\theta') & \cos(\phi')\sin(\theta')\sin(\psi') - \sin(\phi')\cos(\psi') & \cos(\phi')\sin(\theta')\cos(\psi') + \sin(\phi')\sin(\psi') \\ \sin(\phi')\cos(\theta') & \sin(\phi')\sin(\theta')\sin(\psi') + \cos(\phi')\cos(\psi') & \sin(\phi')\sin(\theta')\cos(\psi') - \cos(\phi')\sin(\psi') \\ -\sin(\theta') & \cos(\theta')\sin(\psi') & \cos(\theta')\cos(\psi') \end{bmatrix} \tag{10}$$

First line of $\boldsymbol{Q}$:

$$Q_{1,1} = \frac{\mathrm{d}\psi'}{\mathrm{d}\psi} = \frac{\mathrm{d}\,\mathrm{atan2}(o'_z, a'_z)}{\mathrm{d}\psi} = -\frac{o'_z}{{a'_z}^2 + {o'_z}^2}\frac{\mathrm{d}a'_z}{\mathrm{d}\psi} + \frac{a'_z}{{a'_z}^2 + {o'_z}^2}\frac{\mathrm{d}o'_z}{\mathrm{d}\psi} \tag{11}$$

$$Q_{1,2} = \frac{\mathrm{d}\psi'}{\mathrm{d}\theta} = \frac{\mathrm{d}\,\mathrm{atan2}(o'_z, a'_z)}{\mathrm{d}\theta} = -\frac{o'_z}{{a'_z}^2 + {o'_z}^2}\frac{\mathrm{d}a'_z}{\mathrm{d}\theta} + \frac{a'_z}{{a'_z}^2 + {o'_z}^2}\frac{\mathrm{d}o'_z}{\mathrm{d}\theta} \tag{12}$$

$$Q_{1,3} = \frac{\mathrm{d}\psi'}{\mathrm{d}\phi} = \frac{\mathrm{d}\,\mathrm{atan2}(o'_z, a'_z)}{\mathrm{d}\phi} = -\frac{o'_z}{{a'_z}^2 + {o'_z}^2}\frac{\mathrm{d}a'_z}{\mathrm{d}\phi} + \frac{a'_z}{{a'_z}^2 + {o'_z}^2}\frac{\mathrm{d}o'_z}{\mathrm{d}\phi} \tag{13}$$

$$a'_z = \cos(\theta)\cos(\psi) \tag{14}$$

$$\frac{\mathrm{d}a'_z}{\mathrm{d}\psi} = -\cos(\theta)\sin(\psi) \tag{15}$$

$$\frac{\mathrm{d}a'_z}{\mathrm{d}\theta} = -\sin(\theta)\cos(\psi) \tag{16}$$

$$\frac{\mathrm{d}a'_z}{\mathrm{d}\phi} = 0 \tag{17}$$

$$o'_z = \sin(\phi)\sin(\theta)\cos(\psi) - \cos(\phi)\sin(\psi) \tag{18}$$

$$\frac{\mathrm{d}o'_z}{\mathrm{d}\psi} = -\left(\sin(\phi)\sin(\theta)\sin(\psi) + \cos(\phi)\cos(\psi)\right) = -o'_y \tag{19}$$

$$\frac{\mathrm{d}o'_z}{\mathrm{d}\theta} = \sin(\phi)\cos(\theta)\cos(\psi) \tag{20}$$

$$\frac{\mathrm{d}o'_z}{\mathrm{d}\phi} = \cos(\phi)\sin(\theta)\cos(\psi) + \sin(\phi)\sin(\psi) = n'_z \tag{21}$$

$$\tag{22}$$

Third line of $\boldsymbol{Q}$:

$$Q_{3,1} = \frac{\mathrm{d}\phi'}{\mathrm{d}\psi} = \frac{\mathrm{d}\,\mathrm{atan2}(n'_y, n'_x)}{\mathrm{d}\psi} = -\frac{n'_y}{{n'_x}^2 + {n'_y}^2}\frac{\mathrm{d}n'_x}{\mathrm{d}\psi} + \frac{n'_x}{{n'_x}^2 + {n'_y}^2}\frac{\mathrm{d}n'_y}{\mathrm{d}\psi} \tag{23}$$

$$Q_{3,2} = \frac{\mathrm{d}\phi'}{\mathrm{d}\theta} = \frac{\mathrm{d}\,\mathrm{atan2}(n'_y, n'_x)}{\mathrm{d}\theta} = -\frac{n'_y}{{n'_x}^2 + {n'_y}^2}\frac{\mathrm{d}n'_x}{\mathrm{d}\theta} + \frac{n'_x}{{n'_x}^2 + {n'_y}^2}\frac{\mathrm{d}n'_y}{\mathrm{d}\theta} \tag{24}$$

$$Q_{3,3} = \frac{\mathrm{d}\phi'}{\mathrm{d}\phi} = \frac{\mathrm{d}\,\mathrm{atan2}(n'_y, n'_x)}{\mathrm{d}\phi} = -\frac{n'_y}{{n'_x}^2 + {n'_y}^2}\frac{\mathrm{d}n'_x}{\mathrm{d}\phi} + \frac{n'_x}{{n'_x}^2 + {n'_y}^2}\frac{\mathrm{d}n'_y}{\mathrm{d}\phi} \tag{25}$$

$$n'_x = \cos(\phi)\cos(\theta) \tag{26}$$

$$\frac{\mathrm{d}n'_x}{\mathrm{d}\psi} = 0 \tag{27}$$

$$\frac{\mathrm{d}n'_x}{\mathrm{d}\theta} = -\cos(\phi)\sin(\theta) \tag{28}$$

$$\frac{\mathrm{d}n'_x}{\mathrm{d}\phi} = -\sin(\phi)\cos(\theta) \tag{29}$$

$$n'_y = \cos(\phi)\sin(\theta)\sin(\psi) - \sin(\phi)\cos(\psi) \tag{30}$$

$$\frac{\mathrm{d}n'_y}{\mathrm{d}\psi} = \cos(\phi)\sin(\theta)\cos(\psi) + \sin(\phi)\sin(\psi) = n'_z = \frac{\mathrm{d}o'_z}{\mathrm{d}\phi} \tag{31}$$

$$\frac{\mathrm{d}n'_y}{\mathrm{d}\theta} = \cos(\phi)\cos(\theta)\sin(\psi) \tag{32}$$

$$\frac{\mathrm{d}n'_y}{\mathrm{d}\phi} = -\left(\sin(\phi)\sin(\theta)\sin(\psi) + \cos(\phi)\cos(\psi)\right) = -o'_y = \frac{\mathrm{d}o'_z}{\mathrm{d}\psi} \tag{33}$$

$$\tag{34}$$

Second line of $\boldsymbol{Q}$:

$$Q_{2,1} = \frac{\mathrm{d}\theta'}{\mathrm{d}\psi} = \frac{\mathrm{d}\,\mathrm{atan2}(y,x)}{\mathrm{d}\psi} = -\frac{y}{x^2+y^2}\frac{\mathrm{d}x}{\mathrm{d}\psi} + \frac{x}{x^2+y^2}\frac{\mathrm{d}y}{\mathrm{d}\psi} \tag{35}$$

$$Q_{2,2} = \frac{\mathrm{d}\theta'}{\mathrm{d}\theta} = \frac{\mathrm{d}\,\mathrm{atan2}(y,x)}{\mathrm{d}\theta} = -\frac{y}{x^2+y^2}\frac{\mathrm{d}x}{\mathrm{d}\theta} + \frac{x}{x^2+y^2}\frac{\mathrm{d}y}{\mathrm{d}\theta} \tag{36}$$

$$Q_{2,3} = \frac{\mathrm{d}\theta'}{\mathrm{d}\phi} = \frac{\mathrm{d}\,\mathrm{atan2}(y,x)}{\mathrm{d}\phi} = -\frac{y}{x^2+y^2}\frac{\mathrm{d}x}{\mathrm{d}\phi} + \frac{x}{x^2+y^2}\frac{\mathrm{d}y}{\mathrm{d}\phi} \tag{37}$$

$$\begin{aligned} x = n'_x \cos(\phi') + n'_y \sin(\phi') = \cos(\phi)\cos(\theta)\cos(\phi') + \\ (\cos(\phi)\sin(\theta)\sin(\psi) - \sin(\phi)\cos(\psi))\sin(\phi') \end{aligned} \tag{38}$$

$$\frac{\mathrm{d}x}{\mathrm{d}\psi} = \frac{\partial x}{\partial \psi}\frac{\mathrm{d}\psi}{\mathrm{d}\psi} + \frac{\partial x}{\partial \phi'}\frac{\mathrm{d}\phi'}{\mathrm{d}\psi} \tag{39}$$

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = \frac{\partial x}{\partial \theta}\frac{\mathrm{d}\theta}{\mathrm{d}\theta} + \frac{\partial x}{\partial \phi'}\frac{\mathrm{d}\phi'}{\mathrm{d}\theta} \tag{40}$$

$$\frac{\mathrm{d}x}{\mathrm{d}\phi} = \frac{\partial x}{\partial \phi}\frac{\mathrm{d}\phi}{\mathrm{d}\phi} + \frac{\partial x}{\partial \phi'}\frac{\mathrm{d}\phi'}{\mathrm{d}\phi} \tag{41}$$

$$\frac{\partial x}{\partial \psi} = (\cos(\phi)\sin(\theta)\cos(\psi) + \sin(\phi)\sin(\psi))\sin(\phi') = n'_z \sin(\phi') \tag{42}$$

$$\frac{\partial x}{\partial \theta} = \cos(\phi)\cos(\theta)\sin(\psi)\sin(\phi') - \cos(\phi)\sin(\theta)\cos(\phi') \tag{43}$$

$$\begin{aligned} \frac{\partial x}{\partial \phi} &= -\left[(\sin(\phi)\sin(\theta)\sin(\psi) + \cos(\phi)\cos(\psi))\sin(\phi') + \sin(\phi)\cos(\theta)\cos(\phi')\right] \\ &= -\left(o_y \sin(\phi') + \sin(\phi)\cos(\theta)\cos(\phi')\right) \end{aligned} \tag{44}$$

$$\begin{aligned} \frac{\partial x}{\partial \phi'} &= (\cos(\phi)\sin(\theta)\sin(\psi) - \sin(\phi)\cos(\psi))\cos(\phi') - \cos(\phi)\cos(\theta)\sin(\phi') \\ &= n'_y \cos(\phi') - \cos(\phi)\cos(\theta)\sin(\phi') \end{aligned} \tag{45}$$

$$y = -n'_z = -\cos(\phi)\sin(\theta)\cos(\psi) - \sin(\phi)\sin(\psi) \tag{46}$$

$$\frac{\mathrm{d}y}{\mathrm{d}\psi} = \cos(\phi)\sin(\theta)\sin(\psi) - \sin(\phi)\cos(\psi) \tag{47}$$

$$\frac{\mathrm{d}y}{\mathrm{d}\theta} = -\cos(\phi)\cos(\theta)\cos(\psi) \tag{48}$$

$$\frac{\mathrm{d}y}{\mathrm{d}\phi} = \sin(\phi)\sin(\theta)\cos(\psi) - \cos(\phi)\sin(\psi) \tag{49}$$

$$\tag{50}$$