Particle Filtering

Alex Pan and Alec Kosik

Reed College

2016

Hidden Markov Models (HMM)

$$X_1 \sim \mu(x_1) \text{ and } X_n|(X_{n-1} = x_{n-1}) \sim f(x_n|x_{n-1})$$

$$Y_n|(X_n = x_n) \sim g(y_n|x_n)$$

Problem Setup

Goal:

$$p(x_n|y_{1:n})$$

Have:

From Markov Property:

$$p(x_{1:n}) = \mu(x_1)\prod_{i=2}^{n} f(x_i|x_{i-1})$$

$$p(y_{1:n}|x_{1:n}) = \prod_{i=1}^{n} g(y_i|x_i)$$

Deriving our Goal $p(x_n|y_{1:n})$ from our Markovian Assumptions

$$
\begin{aligned}
p(x_{1:n}|y_{1:n}) &= \frac{p(x_{1:n}, y_{1:n})}{p(y_{1:n})} \\
&= \frac{p(y_{1:n}|x_{1:n})p(x_{1:n})}{p(y_{1:n})}
\end{aligned}
$$

Notice that $p(x_n|y_{1:n})$ is just a marginal of the above equation.

$$
p(x_n|y_{1:n}) = \int \cdots \int p(x_{1:n}|y_{1:n})\, dx_{1:n-1}
$$

Not always integrable e.g. non-linear, non-gaussian distributions

... At least we still have this ...

$$
\begin{aligned}
p(x_{1:n}|y_{1:n}) &= \frac{p(x_{1:n}, y_{1:n})}{p(y_{1:n})} \\
&= \frac{p(y_{1:n}|x_{1:n})p(x_{1:n})}{p(y_{1:n})} \\
&= \frac{\left(\prod_{i=1}^{n} g(y_i|x_i)\right)\left(\mu(x_1)\prod_{i=2}^{n} f(x_i|x_{i-1})\right)}{p(y_{1:n})} \\
&= p(x_{1:n-1}|y_{1:n-1})\frac{g(y_n|x_n))f(x_n|x_{n-1})}{p(y_n|y_{1:n-1})} \\
&= \underbrace{p(x_{1:n-1}|y_{1:n-1})}_{\text{recursion}}\underbrace{\frac{g(y_n|x_n)}{p(y_n|y_{1:n-1})}}_{\text{update/weight}}\underbrace{f(x_n|x_{n-1})}_{\text{transition probability}}
\end{aligned}
$$

Sequential Importance Sampling (SIS)

1. *Initialization:*

   For $i = 1, \dots N$:

   Sample $x_0^{(i)} \sim \mu(x_0)$

   Assign weights $\widetilde{w}_0^{(i)} = \alpha_n(x_1)$

2. *Importance Sampling:*

   For $t = 1, \dots, T$:

   Sample $x_t^{(i)} \sim f(x_t | x_{t-1}^i)$

   Assign weights

   $\widetilde{w}_t^{(i)} = w_{t-1}(x_{1:n-1})\alpha(x_{1:n}) = w_1(x_1) \prod_{k=2}^{n} \alpha_k(x_{1:k})$

3. Return $\{x_T^{(i)}, w_T^{(i)}\}_{i=1}^N$

Sequential Importance Resampling (SIR)

1. *Initialization:*

   For $i = 1, \dots N$:

   Sample $x_0^{(i)} \sim \mu(x_0)$

   Assign weights $\widetilde{w}_0^{(i)} = g(y_0 | x_0^{(i)})$

   Normalize weights $w_0^{(i)} = \frac{\widetilde{w}_0^{(i)}}{\sum_{i=1}^{n} \widetilde{w}_0^{(i)}}$

2. *Importance Sampling:*

   For $t = 1, \dots, T$:

   Sample $x_t^{(i)} \sim f(x_t | \widetilde{x}_{t-1}^{i})$

   Assign weights $\widetilde{w}_t^{(i)} = g(y_t | x_t^{(i)})$

   Normalize weights $w_t^{(i)} = \frac{\widetilde{w}_t^{(i)}}{\sum_{i=1}^{n} \widetilde{w}_t^{(i)}}$

   Resample $\widetilde{x}_t^{(i)}$ from $x_t^{(i)}$ according to the weight distribution.

3. Return $\{x_T^{(i)}, w_T^{(i)}\}_{i=1}^{N}$