Effects of Berry Curvature on Thermoelectric Transport of Bilayer Graphene and Other Materials

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INTRODUCTION

intro

BERRY CURVATURE IN RECIPROCAL SPACE

Berry Curvature

When the Hamiltonian of a system has a paramater $\boldsymbol{\lambda}$ which evolves slowly (compared to the timescale $\frac{\hbar}{\varepsilon}$ of a state $|\varepsilon\rangle$), an eigenstate of the Hamiltonian evolves as,[1]

$$|\varepsilon(\boldsymbol{\lambda}(t))\rangle = e^{i\gamma(t)} e^{-\frac{i}{\hbar}\int_0^t \varepsilon(\boldsymbol{\lambda}(t'))dt'} |\varepsilon(\boldsymbol{\lambda}(0))\rangle \tag{1}$$

where the geometric phase $\gamma(t)$ is given by $i\int_{\boldsymbol{\lambda}_i}^{\boldsymbol{\lambda}_f} \langle\varepsilon(\boldsymbol{\lambda})|\,\nabla_{\boldsymbol{\lambda}}\,|\varepsilon(\boldsymbol{\lambda})\rangle \cdot d\boldsymbol{\lambda}$. It depends on the trajectory in the parameter space $\boldsymbol{\lambda}$, not just the endpoints. The real quantity $i\langle\varepsilon(\boldsymbol{\lambda})|\,\nabla_{\boldsymbol{\lambda}}\,|\varepsilon(\boldsymbol{\lambda})\rangle = \boldsymbol{A}(\boldsymbol{\lambda})$ is known as the Berry connection, and its curl, $\boldsymbol{\Omega} = \nabla_{\boldsymbol{\lambda}} \times \boldsymbol{A}$ is known as the Berry curvature.

Note: $\langle\varepsilon(\boldsymbol{\lambda})|\,\nabla_{\boldsymbol{\lambda}}\,|\varepsilon(\boldsymbol{\lambda})\rangle$ stands for $\int d\boldsymbol{r}\varepsilon^*_{\boldsymbol{\lambda}}(\boldsymbol{r})\nabla_{\boldsymbol{\lambda}}\varepsilon_{\boldsymbol{\lambda}}(\boldsymbol{r})$, where $\varepsilon_{\boldsymbol{\lambda}}(\boldsymbol{r})$ is the normalized position space wavefunction of $|\varepsilon(\boldsymbol{\lambda})\rangle$.

While the overall phase of the state does not affect its physical observables, if we take a linear combination of such eigenstates, each of them would evolve with a different geometric phase, and that would indeed affect the physical properties of the system. We would now see how such a geometric phase manifests in electrons moving in a lattice, when an external electromagnetic field is applied.

Electrons in a lattice

Now, let us consider an electron in a periodic potential $V(\boldsymbol{r})$ (e.g. the potential due to a lattice), without any external fields. The Hamiltonian describing it is given by $\hat{H} = \frac{\hat{\boldsymbol{p}}^2}{2m} + V(\hat{\boldsymbol{r}})$.

The eigenfunctions of this Hamiltonian are of the form,

$$\psi_{n,\boldsymbol{k}}(\boldsymbol{r}) = e^{i\boldsymbol{k}\cdot\boldsymbol{r}} u_{n,\boldsymbol{k}}(\boldsymbol{r})$$

Here $u_{n,\boldsymbol{k}}(\boldsymbol{r})$ is a function with the same periodicity as $V(\boldsymbol{r})$ (e.g. in a lattice, it would be periodic in every unit cell). The quantity $\hbar\mathbf{k}$ is known as the crystal momentum and n is the band index. This result is known as Bloch's theorem (See chapter 8 of [2]). The crystal momentum is a good quantum number for an electron in a lattice, and it is conserved upto a reciprocal lattice vector (times $\hbar$). Now, we can rewrite the eigenvalue equation $\hat{H}\psi_{n,\boldsymbol{k}}(\boldsymbol{r}) = \varepsilon_{n,\boldsymbol{k}}\psi_{n,\boldsymbol{k}}(\boldsymbol{r})$ as,

$$\left[\frac{\hbar^2}{2m}(\boldsymbol{k} - i\nabla)^2 + V(\boldsymbol{r})\right] u_{n,\boldsymbol{k}}(\boldsymbol{r}) = \varepsilon_{n,\boldsymbol{k}} u_{n,\boldsymbol{k}}(\boldsymbol{r}) \tag{2}$$

In this equation, $\boldsymbol{k}$ is just a parameter in the effective Hamiltonian. When we apply an external electromagnetic field, the crystal momentum is not anymore a good quantum number. Its evolution is governed by the Lorenz force, (See Appendix)

$$\hbar\dot{\boldsymbol{k}} = -e\left(\boldsymbol{E} + \langle\dot{\boldsymbol{r}}\rangle \times \boldsymbol{B}\right) \tag{3}$$

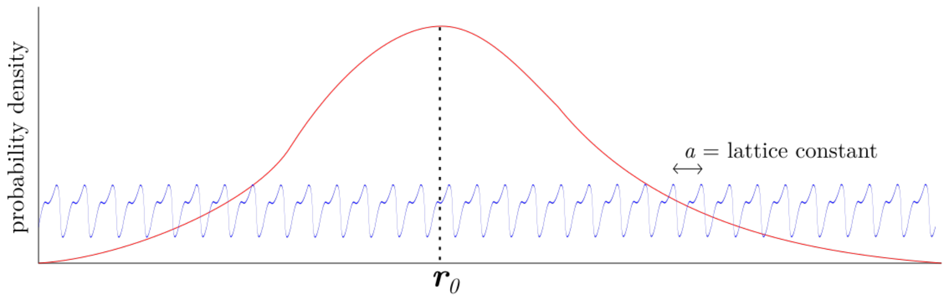

FIG. 1. The red

As $\boldsymbol{k}$ changes, it would give rise to a geometric phase. After a small time interval Δt, $\boldsymbol{k}$ would evolve to $\boldsymbol{k} + \Delta\boldsymbol{k}$ (with $\Delta\boldsymbol{k} = \dot{\boldsymbol{k}}\Delta t$), and the wavefunction of a state labeled with $\boldsymbol{k}$ would evolve as $u_{n,\boldsymbol{k}} \to e^{i\boldsymbol{A}\cdot\boldsymbol{\Delta k}} e^{-i\frac{\varepsilon_{\boldsymbol{k}}\Delta t}{\hbar}} u_{n,\boldsymbol{k}+\Delta\boldsymbol{k}}$, where $\boldsymbol{A}(\boldsymbol{k}) = i\left\langle u_{n,\boldsymbol{k}}\right| \nabla_{\boldsymbol{k}} \left|u_{n,\boldsymbol{k}}\right\rangle$ is the Berry connection in the reciprocal space. Again, for a state labeled with a single $\boldsymbol{k}$, this geometric phase is an overall global phase, which would not show up in physical observables. However, if we take a linear superposition of Bloch wavefunctions with different values of $\boldsymbol{k}$, each of them would evolve with a different geometric phase $\gamma_n(\boldsymbol{k}) = \int_0^t dt' \boldsymbol{A(k)} \cdot \frac{d\boldsymbol{k}}{dt'}$, and we would soon see that the interference of such phase factors would lead to many interesting effects.

EFFECTS OF BERRY CURVATURE

Modification of semiclassical equations of motion of an wavepacket

The probability density of the wavefunction $\psi_{n,\boldsymbol{k}}(\boldsymbol{r}) = e^{i\boldsymbol{k}\cdot\boldsymbol{r}} u_{n,\boldsymbol{k}}(\boldsymbol{r})$ is not localized anywhere, it is periodic over all unit cells (The blue curve in Fig. 1). We can construct a wave packet by forming a linear superposition of many such states, such that the wavepacket is localized at some point $\boldsymbol{r}_0$ in the real space, and its crystal momentum is also localized around a value of $\hbar\boldsymbol{k}_0$.

Violation of Liouville's theorem and Modification of phase space density

Derivation from the semiclassical equations of Motion

DECOUPLING OF THE EQUATIONS OF MOTION AND THEIR SIMPLIFICATION

$$\dot{\mathbf{r}} = \frac{\frac{1}{\hbar}\frac{\partial\varepsilon}{\partial\mathbf{k}} + \frac{e}{\hbar}(\mathbf{E}\times\boldsymbol{\Omega}) + \frac{e}{\hbar^2}(\Omega\cdot\frac{\partial\varepsilon}{\partial\mathbf{k}})\mathbf{B}}{1+\frac{e}{\hbar}\mathbf{B}\cdot\boldsymbol{\Omega}} \tag{4}$$

$$\dot{\mathbf{k}} = -\frac{\frac{e}{\hbar}\mathbf{E} + \frac{e}{\hbar^2}\frac{\partial\varepsilon}{\partial\mathbf{k}}\times\mathbf{B} + \frac{e^2}{\hbar^2}(\mathbf{E}\cdot\mathbf{B})\Omega}{1+\frac{e}{\hbar}\mathbf{B}\cdot\boldsymbol{\Omega}} \tag{5}$$

In a 2D sample, $\boldsymbol{\Omega}$ is along the z axis, while $\frac{\partial\varepsilon}{\partial\mathbf{k}}$ is in the xy plane, then $\boldsymbol{\Omega}\cdot\frac{\partial\varepsilon}{\partial\mathbf{k}} = 0$.

Also, if we take perpendicular electric and magnetic fields, then $\mathbf{E}\cdot\mathbf{B} = 0$.

Then the equations (4) and (5) simplify to,

$$\dot{\mathbf{r}} = \frac{\frac{1}{\hbar}\frac{\partial\varepsilon}{\partial\mathbf{k}} + \frac{e}{\hbar}(\mathbf{E}\times\boldsymbol{\Omega})}{1+\frac{e}{\hbar}\mathbf{B}\cdot\boldsymbol{\Omega}} \tag{6}$$

$$\dot{\mathbf{k}} = -\frac{\frac{e}{\hbar}\mathbf{E} + \frac{e}{\hbar^2}\frac{\partial\varepsilon}{\partial\mathbf{k}}\times\mathbf{B}}{1+\frac{e}{\hbar}\mathbf{B}\cdot\boldsymbol{\Omega}} \tag{7}$$

STRATEGY OF CALCULATION OF THERMAL AND HEAT CURRENTS

Equilibrium and Non-Equilibrium distribution functions

BOLTZMANN TRANSPORT EQUATION AND ITS SOLUTION

Some discussions about the relaxation time approximation....

Under the relaxation time approximation, the Boltzmann Transport Equation (BTE) takes the form

$$\dot{\mathbf{k}} \cdot \frac{\partial}{\partial \mathbf{k}}(f+g) + \dot{\mathbf{r}} \cdot \frac{\partial}{\partial \mathbf{r}}(f+g) = -\frac{g}{\tau} \tag{8}$$

,

which can be cast into the form

$$\frac{g}{\tau} + \dot{\mathbf{k}} \cdot \frac{\partial}{\partial \mathbf{k}} g + \dot{\mathbf{r}} \cdot \frac{\partial}{\partial \mathbf{r}} g = -\dot{\mathbf{k}} \cdot \frac{\partial}{\partial \mathbf{k}} f - \dot{\mathbf{r}} \cdot \frac{\partial}{\partial \mathbf{r}} f \tag{9}$$

.

Here $f = \frac{1}{e^{\beta(\varepsilon_{\mathbf{k}}-\mu)}+1}$ is the equilibrium Fermi distribution, and g is the non-equilibrium part. Using the decoupled equations of motion, right hand side of Eq. (9) simplifies to (*Need to show the steps*),

$$\frac{\partial f}{\partial \varepsilon} \frac{1}{1+\frac{e}{\hbar}\mathbf{B}\cdot\mathbf{\Omega}} \frac{1}{\hbar} \frac{\partial \varepsilon}{\partial \mathbf{k}} \cdot \left[e\mathbf{E} + \nabla\mu + \nabla T \frac{\varepsilon-\mu}{T} \right]$$

.

When the fields are constant in space and time, we assume that $\frac{\partial g}{\partial \mathbf{r}} = 0$.

Validity of the assumption: When we solve the equation after discarding this term, we would find that (See Eq. (11)) this term is second order in the applied fields. Then upto linear order, we can discard this term.

Then, the BTE becomes, $\frac{g}{\tau} - \frac{\frac{e}{\hbar}\mathbf{E}+\frac{e}{\hbar^2}\frac{\partial\varepsilon}{\partial\mathbf{k}}\times\mathbf{B}}{1+\frac{e}{\hbar}\mathbf{B}\cdot\mathbf{\Omega}} \cdot \frac{\partial}{\partial \mathbf{k}} g = \frac{\partial f}{\partial \varepsilon} \frac{1}{1+\frac{e}{\hbar}\mathbf{B}\cdot\mathbf{\Omega}} \frac{1}{\hbar} \frac{\partial \varepsilon}{\partial \mathbf{k}} \cdot \left[e\mathbf{E} + \nabla\mu + \nabla T \frac{\varepsilon-\mu}{T} \right]$

Solution for $\mathbf{B} \neq 0$, $\mathbf{\Omega} \neq 0$, $\mathbf{E} \neq 0$, $\nabla\mu = \nabla T = 0$

In this limit, the equation becomes,

$$\frac{g}{\tau} - \frac{\frac{e}{\hbar}\mathbf{E}+\frac{e}{\hbar^2}\frac{\partial\varepsilon}{\partial\mathbf{k}}\times\mathbf{B}}{1+\frac{e}{\hbar}\mathbf{B}\cdot\mathbf{\Omega}} \cdot \frac{\partial}{\partial \mathbf{k}} g = \frac{\partial f}{\partial \varepsilon} \frac{1}{1+\frac{e}{\hbar}\mathbf{B}\cdot\mathbf{\Omega}} \frac{1}{\hbar} \frac{\partial \varepsilon}{\partial \mathbf{k}} \cdot e\mathbf{E}$$

If we discard the term $\mathbf{E} \cdot \frac{\partial \varepsilon}{\partial \mathbf{k}}$ in the LHS, we would find that g is a linear function of $\mathbf{E}$. Then, $\mathbf{E} \cdot \frac{\partial \varepsilon}{\partial \mathbf{k}}$ would be quadratic in $\mathbf{E}$. This is why, we can discard this term. (This is not a circular argument. If we do this, everything remains self consistent upto linear order)

Then, the equation takes the form,

$$\frac{g}{\tau} - \frac{\frac{e}{\hbar^2}}{1+\frac{e}{\hbar}\mathbf{B}\cdot\mathbf{\Omega}} \frac{\partial\varepsilon}{\partial\mathbf{k}} \times \mathbf{B} \cdot \frac{\partial}{\partial \mathbf{k}} g = \frac{\partial f}{\partial \varepsilon} \frac{1}{1+\frac{e}{\hbar}\mathbf{B}\cdot\mathbf{\Omega}} \frac{1}{\hbar} \frac{\partial \varepsilon}{\partial \mathbf{k}} \cdot e\mathbf{E} \tag{10}$$

We solve this equation, treating the second term in LHS as a perturbation (See Appendix). We write $g = g_0 + g_1$, such that $\frac{g_0}{\tau} = \frac{\partial f}{\partial \varepsilon} \frac{1}{1+\frac{e}{\hbar}\mathbf{B}\cdot\mathbf{\Omega}} \frac{1}{\hbar} \frac{\partial \varepsilon}{\partial \mathbf{k}} \cdot e\mathbf{E}$, and $\frac{g_1}{\tau} - \frac{\frac{e}{\hbar^2}}{1+\frac{e}{\hbar}\mathbf{B}\cdot\mathbf{\Omega}} \frac{\partial\varepsilon}{\partial\mathbf{k}} \times \mathbf{B} \cdot \frac{\partial}{\partial \mathbf{k}} g_0 = 0$. Then, g_1 is a linear function of $\mathbf{B}$, and it is justified to discard the term $\frac{\frac{e}{\hbar^2}}{1+\frac{e}{\hbar}\mathbf{B}\cdot\mathbf{\Omega}} \frac{\partial\varepsilon}{\partial\mathbf{k}} \times \mathbf{B} \cdot \frac{\partial}{\partial \mathbf{k}} g_1$, as it would be quadratic in $\mathbf{B}$

Then, $g_0 = \frac{\partial f}{\partial \varepsilon} \frac{1}{1+\frac{e}{\hbar}\mathbf{B}\cdot\mathbf{\Omega}} \frac{e\tau}{\hbar} \frac{\partial \varepsilon}{\partial \mathbf{k}} \cdot \mathbf{E}$. To find g_1, we need to calculate $\frac{\partial g_0}{\partial \mathbf{k}}$, i.e., the quantity $\frac{\partial}{\partial \mathbf{k}} \left[\frac{\partial f}{\partial \varepsilon} \frac{1}{1+\frac{e}{\hbar}\mathbf{B}\cdot\mathbf{\Omega}} \frac{e\tau}{\hbar} \frac{\partial \varepsilon}{\partial \mathbf{k}} \cdot \mathbf{E} \right]$.

To calculate it, let us first calculate $\nabla[\phi\mathbf{A}\cdot\mathbf{C}]$, where ϕ is a scalar function, $\mathbf{A}$ is a vector function, and $\mathbf{C}$ is a constant vector.

$\nabla[\phi\mathbf{A}\cdot\mathbf{C}] = (\nabla\phi)(\mathbf{A}\cdot\mathbf{C}) + \phi(\mathbf{C}\cdot\nabla)A + \phi(\mathbf{C}\times(\nabla\times\mathbf{A}))$. In our calculation, $\phi \sim \frac{\partial f}{\partial \varepsilon} \frac{1}{1+\frac{e}{\hbar}\mathbf{B}\cdot\mathbf{\Omega}} \frac{e\tau}{\hbar}$, $\mathbf{A} \sim \frac{\partial \varepsilon}{\partial \mathbf{k}}$, and $\mathbf{C} \sim \mathbf{E}$.

Then,

$$\frac{\partial}{\partial \mathbf{k}}\left[\frac{\partial f}{\partial \varepsilon}\frac{1}{1+\frac{e}{\hbar}\mathbf{B}\cdot\mathbf{\Omega}}\frac{e\tau}{\hbar}\frac{\partial \varepsilon}{\partial \mathbf{k}}\cdot\mathbf{E}\right] = \frac{\partial}{\partial \mathbf{k}}\left[\frac{\frac{\partial f}{\partial \varepsilon}\tau}{1+\frac{e}{\hbar}\mathbf{B}\cdot\mathbf{\Omega}}\right]\frac{e\mathbf{E}}{\hbar}\cdot\frac{\partial \varepsilon}{\partial \mathbf{k}} + \frac{\frac{\partial f}{\partial \varepsilon}\tau}{1+\frac{e}{\hbar}\mathbf{B}\cdot\mathbf{\Omega}}\left[\left(\frac{e}{\hbar}\mathbf{E}\cdot\frac{\partial}{\partial \mathbf{k}}\right)\frac{\partial \varepsilon}{\partial \mathbf{k}} + \frac{e}{\hbar}\mathbf{E}\times\left(\frac{\partial}{\partial \mathbf{k}}\times\frac{\partial \varepsilon}{\partial \mathbf{k}}\right)\right]$$

.

The last term is zero because it is the curl of a gradient, $\frac{\partial}{\partial \mathbf{k}}\times\frac{\partial \varepsilon}{\partial \mathbf{k}} = \nabla_{\mathbf{k}}\times(\nabla_{\mathbf{k}}\varepsilon) = 0$.
Finally,

$$g = \frac{\partial f}{\partial \varepsilon}\frac{1}{1+\frac{e}{\hbar}\mathbf{B}\cdot\mathbf{\Omega}}\frac{\tau}{\hbar}\frac{\partial \varepsilon}{\partial \mathbf{k}}\cdot e\mathbf{E} + \frac{\frac{e\tau}{\hbar^2}}{1+\frac{e}{\hbar}\mathbf{B}\cdot\mathbf{\Omega}}\frac{\partial \varepsilon}{\partial \mathbf{k}}\cdot\mathbf{B}\times\left[\frac{\partial}{\partial \mathbf{k}}\left[\frac{\frac{\partial f}{\partial \varepsilon}\tau}{1+\frac{e}{\hbar}\mathbf{B}\cdot\mathbf{\Omega}}\right]\frac{e\mathbf{E}}{\hbar}\cdot\frac{\partial \varepsilon}{\partial \mathbf{k}} + \frac{\frac{\partial f}{\partial \varepsilon}\tau}{1+\frac{e}{\hbar}\mathbf{B}\cdot\mathbf{\Omega}}\left[\left(\frac{e}{\hbar}\mathbf{E}\cdot\frac{\partial}{\partial \mathbf{k}}\right)\frac{\partial \varepsilon}{\partial \mathbf{k}}\right]\right] \quad (11)$$

Verification: When $\mathbf{\Omega} = 0$, and $\varepsilon = \frac{\hbar^2 k^2}{2m^*}$, this term produces $\sigma_{xy} = -\omega_c \tau \frac{ne^2\tau}{m^*}$. (*Need to show the steps*)

Solution for $\mathbf{B} \neq 0$, $\mathbf{\Omega} \neq 0$, $\mathbf{E} = 0$, $\nabla\mu \neq 0$, $\nabla T \neq 0$

Here

$$\dot{\mathbf{r}} = \frac{\frac{1}{\hbar}\frac{\partial \varepsilon}{\partial \mathbf{k}}}{1+\frac{e}{\hbar}\mathbf{B}\cdot\mathbf{\Omega}} \quad (12)$$

$$\dot{\mathbf{k}} = -\frac{\frac{e}{\hbar^2}\frac{\partial \varepsilon}{\partial \mathbf{k}}\times\mathbf{B}}{1+\frac{e}{\hbar}\mathbf{B}\cdot\mathbf{\Omega}} \quad (13)$$

We again take $\frac{\partial g}{\partial \mathbf{k}} \sim 0$.
Under these circumstances, the equation becomes,

$$\frac{g}{\tau} - \frac{\frac{e}{\hbar^2}\frac{\partial \varepsilon}{\partial \mathbf{k}}\times\mathbf{B}}{1+\frac{e}{\hbar}\mathbf{B}\cdot\mathbf{\Omega}}\cdot\frac{\partial}{\partial \mathbf{k}}g = \frac{\partial f}{\partial \varepsilon}\frac{1}{1+\frac{e}{\hbar}\mathbf{B}\cdot\mathbf{\Omega}}\frac{1}{\hbar}\frac{\partial \varepsilon}{\partial \mathbf{k}}\cdot\left[\nabla\mu + \nabla T\frac{\varepsilon-\mu}{T}\right]$$

Let $g = g_0 + g_1$, with $g_0 = \frac{\partial f}{\partial \varepsilon}\frac{1}{1+\frac{e}{\hbar}\mathbf{B}\cdot\mathbf{\Omega}}\frac{\tau}{\hbar}\frac{\partial \varepsilon}{\partial \mathbf{k}}\cdot\left[\nabla\mu + \nabla T\frac{\varepsilon-\mu}{T}\right]$ and $\frac{g_1}{\tau} - \frac{\frac{e}{\hbar^2}\frac{\partial \varepsilon}{\partial \mathbf{k}}\times\mathbf{B}}{1+\frac{e}{\hbar}\mathbf{B}\cdot\mathbf{\Omega}}\cdot\frac{\partial}{\partial \mathbf{k}}g_0 = 0$
Then, *(Need to show the steps)*

$$\frac{\partial g_0}{\partial \mathbf{k}} = \frac{\partial}{\partial \mathbf{k}}\left[\frac{\frac{\partial f}{\partial \varepsilon}\tau}{1+\frac{e}{\hbar}\mathbf{B}\cdot\mathbf{\Omega}}\right]\frac{\nabla\mu + \frac{\varepsilon-\mu}{T}}{\hbar}\cdot\frac{\partial \varepsilon}{\partial \mathbf{k}} + \frac{\frac{\partial f}{\partial \varepsilon}\tau}{1+\frac{e}{\hbar}\mathbf{B}\cdot\mathbf{\Omega}}\left[\left(\frac{\nabla\mu + \frac{\varepsilon-\mu}{T}}{\hbar}\cdot\frac{\partial}{\partial \mathbf{k}}\right)\frac{\partial \varepsilon}{\partial \mathbf{k}} + \frac{1}{\hbar}\left(\nabla\mu + \frac{\varepsilon-\mu}{T}\right)\times\underbrace{\left(\frac{\partial}{\partial \mathbf{k}}\times\frac{\partial \varepsilon}{\partial \mathbf{k}}\right)}_{0}\right]$$

$$+\frac{\frac{\partial f}{\partial \varepsilon}\tau}{1+\frac{e}{\hbar}\mathbf{B}\cdot\mathbf{\Omega}}\left[\frac{1}{\hbar}\left(\frac{\partial \varepsilon}{\partial \mathbf{k}}\cdot\frac{\partial}{\partial \mathbf{k}}\right)\left(\frac{\varepsilon\nabla T}{T}\right) + \frac{1}{\hbar}\frac{\partial \varepsilon}{\partial \mathbf{k}}\times\left(\frac{\partial}{\partial \mathbf{k}}\times\left(\frac{\varepsilon\nabla T}{T}\right)\right)\right]$$

The term inside the final third bracket can be further simplified.

$$\left(\frac{\partial \varepsilon}{\partial \mathbf{k}}\cdot\frac{\partial}{\partial \mathbf{k}}\right)\left(\frac{\varepsilon\nabla T}{T}\right) + \frac{\partial \varepsilon}{\partial \mathbf{k}}\times\left(\frac{\partial}{\partial \mathbf{k}}\times\left(\frac{\varepsilon\nabla T}{T}\right)\right) = \left(\frac{\partial \varepsilon}{\partial \mathbf{k}}\cdot\frac{\partial \varepsilon}{\partial \mathbf{k}}\right)\left(\frac{\nabla T}{T}\right) + \frac{\partial \varepsilon}{\partial \mathbf{k}}\times\left(\frac{\partial \varepsilon}{\partial \mathbf{k}}\times\left(\frac{\nabla T}{T}\right)\right)$$

$$= \left(\frac{\partial \varepsilon}{\partial \mathbf{k}}\cdot\frac{\partial \varepsilon}{\partial \mathbf{k}}\right)\left(\frac{\nabla T}{T}\right) + \left(\frac{\partial \varepsilon}{\partial \mathbf{k}}\cdot\frac{\nabla T}{T}\right)\frac{\partial \varepsilon}{\partial \mathbf{k}} - \left(\frac{\partial \varepsilon}{\partial \mathbf{k}}\cdot\frac{\partial \varepsilon}{\partial \mathbf{k}}\right)\left(\frac{\nabla T}{T}\right)$$

$$= \left(\frac{\partial \varepsilon}{\partial \mathbf{k}}\cdot\frac{\nabla T}{T}\right)\frac{\partial \varepsilon}{\partial \mathbf{k}}$$

.

When we calculate $g_1 = \tau \frac{\frac{e}{\hbar^2}\frac{\partial \varepsilon}{\partial \mathbf{k}} \times \mathbf{B}}{1+\frac{e}{\hbar}\mathbf{B}\cdot\mathbf{\Omega}} \cdot \frac{\partial}{\partial \mathbf{k}} g_0$, this term cancels because $\frac{\partial \varepsilon}{\partial \mathbf{k}} \times \mathbf{B} \frac{\partial \varepsilon}{\partial \mathbf{k}} = 0$.

Finally we get,

$$g = \frac{\partial f}{\partial \varepsilon} \frac{1}{1+\frac{e}{\hbar}\mathbf{B}\cdot\mathbf{\Omega}} \frac{\tau}{\hbar} \frac{\partial \varepsilon}{\partial \mathbf{k}} \cdot \mathbf{S} + \frac{\frac{e\tau}{\hbar^2}}{1+\frac{e}{\hbar}\mathbf{B}\cdot\mathbf{\Omega}} \frac{\partial \varepsilon}{\partial \mathbf{k}} \cdot \mathbf{B} \times \left[\frac{\partial}{\partial \mathbf{k}} \left[\frac{\frac{\partial f}{\partial \varepsilon}\tau}{1+\frac{e}{\hbar}\mathbf{B}\cdot\mathbf{\Omega}} \right] \frac{\mathbf{S}}{\hbar} \cdot \frac{\partial \varepsilon}{\partial \mathbf{k}} + \frac{\frac{\partial f}{\partial \varepsilon}\tau}{1+\frac{e}{\hbar}\mathbf{B}\cdot\mathbf{\Omega}} \left[\left(\frac{1}{\hbar}\mathbf{S} \cdot \frac{\partial}{\partial \mathbf{k}}\right) \frac{\partial \varepsilon}{\partial \mathbf{k}} \right] \right] \tag{14}$$

with $\mathbf{S} = \nabla\mu + \frac{\varepsilon-\mu}{T}$.

Einstein and Onsager relations are perfectly valid.

EINSTEIN AND ONSAGER RELATIONS

Their apparent violations (when $\mathbf{\Omega} \neq 0$) and the resolution with magnetization currents.

CHERN NUMBER IN DIFFERENT MODELS

Monolayer and Bilayer Graphene

Selective Valley Population

Valley Chern Number

Calculation of Effective Hamiltonian and its Chern Number

Calculation of Berry Phase

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Time evolution of Crystal Momentum

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Validity of this perturbation theory

Classically, $\dot{\mathbf{k}} \cdot \frac{\partial}{\partial \mathbf{k}} g \sim \dot{\mathbf{p}} \cdot \frac{\partial}{\partial \mathbf{p}} g \sim e(\mathbf{v} \times B) \cdot \frac{\partial}{m \partial \mathbf{v}} g \sim \boldsymbol{\omega} \times \mathbf{v} \cdot \frac{\partial g}{\partial \mathbf{v}} \sim \omega g$.

When $\omega \ll \frac{1}{\tau}$ (in the limit of low magnetic field), it is justified to treat ωg as a perturbation over $\frac{g}{\tau}$, in the LHS of Eq. (9).

How low magnetic is 'low'?

[1] M. V. Berry, Quantal phase factors accompanying adiabatic changes, Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences **392**, 45 (1984).
[2] N. W. Ashcroft and N. D. Mermin, *Solid State Physics* (Holt-Saunders, 1976).