Effects of Berry Curvature on Thermoelectric Transport

Archisman Panigrahi

UG 4th Year

24th June 2021

Supervisor: Prof. Subroto Mukerjee, Dept. of Physics, IISc

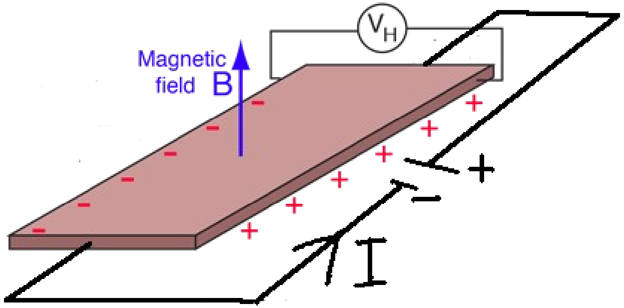
Why is this interesting?

Why is this interesting?

- There can be a transverse Hall voltage without a magnetic field

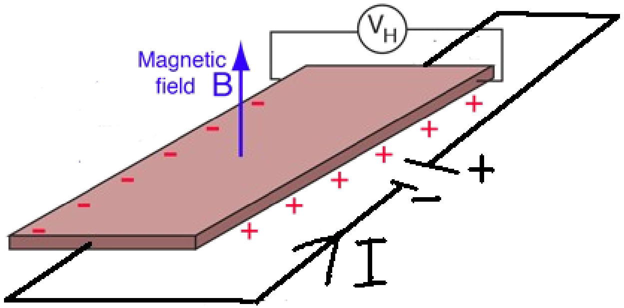
Why is this interesting?

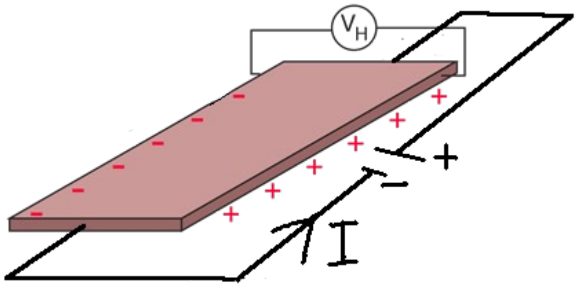
- There can be a transverse Hall voltage without a magnetic field



Why is this interesting?

- There can be a transverse Hall voltage without a magnetic field

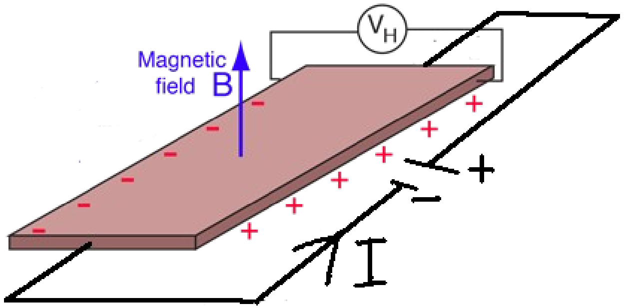


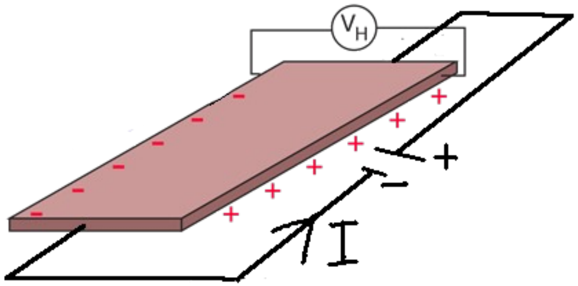


Anomalous Hall Effect

Why is this interesting?

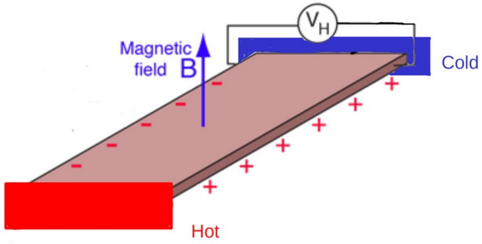
- There can be a transverse Hall voltage without a magnetic field





Anomalous Hall Effect

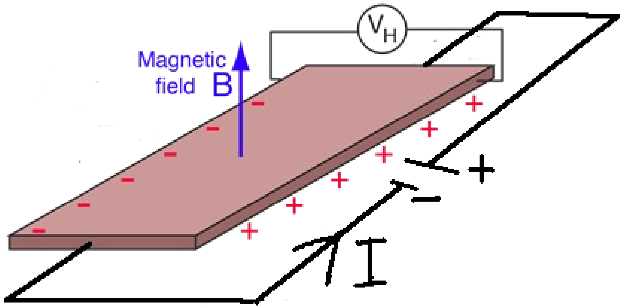
- Nernst Effect: Hall like response for a temperature gradient

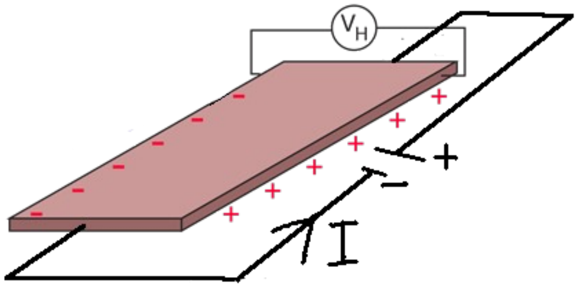


Electrically insulating temperature reservoirs

Why is this interesting?

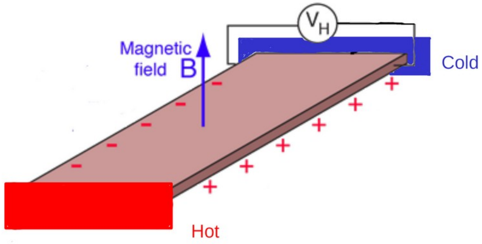
- There can be a transverse Hall voltage without a magnetic field



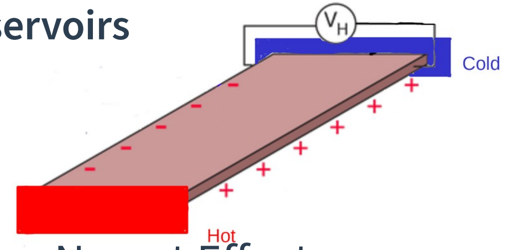


Anomalous Hall Effect

- Nernst Effect: Hall like response for a temperature gradient



Electrically insulating temperature reservoirs



Anomalous Nernst Effect

Thermoelectric transport: Einstein and Onsager relations

$$\hat{\vec{j}}_e = \overleftrightarrow{L}_{11} \cdot \left(\vec{E} + \nabla\frac{\mu}{e}\right) + \overleftrightarrow{L}_{12} \cdot (-\nabla T)$$

$$\hat{\vec{j}}_Q = \overleftrightarrow{L}_{21} \cdot \left(\vec{E} + \nabla\frac{\mu}{e}\right) + \overleftrightarrow{L}_{22} \cdot (-\nabla T)$$

Thermoelectric transport: Einstein and Onsager relations

$$\hat{\vec{j}}_e = \overleftrightarrow{L}_{11} \cdot \left(\vec{E} + \nabla \frac{\mu}{e}\right) + \overleftrightarrow{L}_{12} \cdot (-\nabla T)$$

$$\hat{\vec{j}}_Q = \overleftrightarrow{L}_{21} \cdot \left(\vec{E} + \nabla \frac{\mu}{e}\right) + \overleftrightarrow{L}_{22} \cdot (-\nabla T)$$

σ, electric conductivity

κ, thermal conductivity

Thermoelectric transport: Einstein and Onsager relations

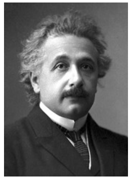
$$\hat{\vec{j}}_e = \overleftrightarrow{L}_{11} \cdot \left(\vec{E} + \nabla \frac{\mu}{e} \right) + \overleftrightarrow{L}_{12} \cdot (-\nabla T)$$

$$\hat{\vec{j}}_Q = \overleftrightarrow{L}_{21} \cdot \left(\vec{E} + \nabla \frac{\mu}{e} \right) + \overleftrightarrow{L}_{22} \cdot (-\nabla T)$$

σ, electric conductivity

κ, thermal conductivity

- **Einstein relation:** $\vec{E} \longleftrightarrow \dfrac{\nabla \mu}{e}$



Thermoelectric transport: Einstein and Onsager relations

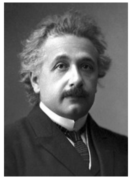
$$\hat{\vec{j}}_e = \overleftrightarrow{L}_{11} \cdot \left(\vec{E} + \nabla\frac{\mu}{e}\right) + \overleftrightarrow{L}_{12} \cdot (-\nabla T)$$

σ, electric conductivity

$$\hat{\vec{j}}_Q = \overleftrightarrow{L}_{21} \cdot \left(\vec{E} + \nabla\frac{\mu}{e}\right) + \overleftrightarrow{L}_{22} \cdot (-\nabla T)$$

κ, thermal conductivity

- **Einstein relation:** $\vec{E} \longleftrightarrow \dfrac{\nabla\mu}{e}$



- **Onsager relation:** $\overleftrightarrow{L}_{21} = T\,\overleftrightarrow{L}_{12}$

Thermoelectric transport: Einstein and Onsager relations

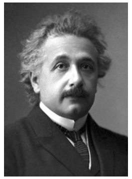
$$\hat{\vec{j}}_e = \overleftrightarrow{L}_{11} \cdot \left(\vec{E} + \nabla\frac{\mu}{e}\right) + \overleftrightarrow{L}_{12} \cdot (-\nabla T)$$

$$\hat{\vec{j}}_Q = \overleftrightarrow{L}_{21} \cdot \left(\vec{E} + \nabla\frac{\mu}{e}\right) + \overleftrightarrow{L}_{22} \cdot (-\nabla T)$$

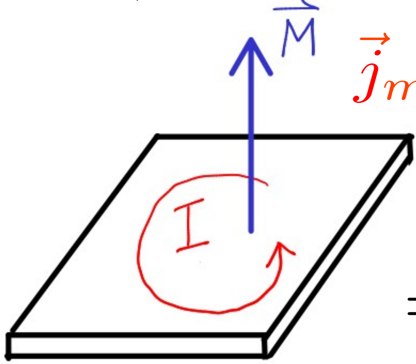
σ, electric conductivity

κ, thermal conductivity

- **Einstein relation:** $\vec{E} \longleftrightarrow \dfrac{\nabla\mu}{e}$



- **Onsager relation:** $\overleftrightarrow{L}_{21} = T\,\overleftrightarrow{L}_{12}$



$$\vec{j}_m = \nabla \times \vec{M}$$

$$\vec{j}_{\text{transport}} = \vec{j}_e - \vec{j}_M$$

N. R. Cooper, B. I. Halperin, and I. M. Ruzin. PRB **55** 2344 (1997)

Geometric phase and Berry Curvature

M. V. Berry. Proceedings of the Royal Society A. **392** 1802 (1984)

- Time dependent Hamiltonian $H(\boldsymbol{\lambda}(t))$

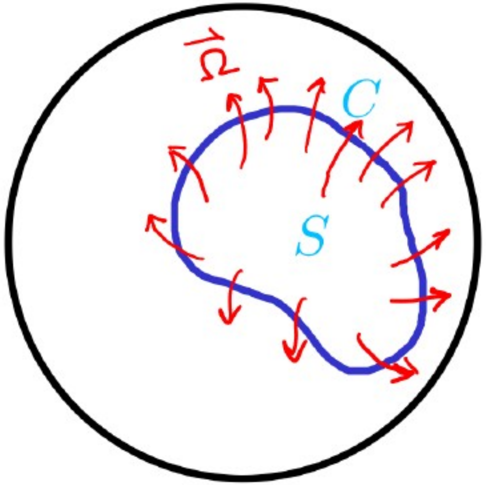
$$|\varepsilon(\boldsymbol{\lambda}(t))\rangle = e^{i\gamma(t)} e^{-\frac{i}{\hbar}\int_0^t \varepsilon(\boldsymbol{\lambda}(t'))dt'} |\varepsilon(\boldsymbol{\lambda}(0))\rangle$$

Geometric phase and Berry Curvature

M. V. Berry. Proceedings of the Royal Society A. **392** 1802 (1984)

- Time dependent Hamiltonian $H(\boldsymbol{\lambda}(t))$

$$|\varepsilon(\boldsymbol{\lambda}(t))\rangle = e^{i\gamma(t)} e^{-\frac{i}{\hbar}\int_0^t \varepsilon(\boldsymbol{\lambda}(t'))dt'} |\varepsilon(\boldsymbol{\lambda}(0))\rangle$$



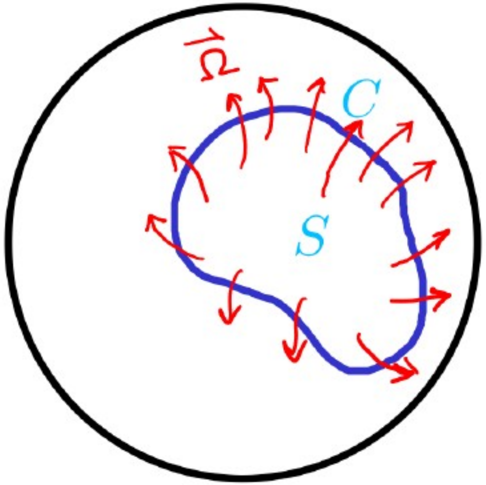
Space of $\lambda_1, \lambda_2, \ldots$

Geometric phase and Berry Curvature

M. V. Berry. Proceedings of the Royal Society A. **392** 1802 (1984)

- Time dependent Hamiltonian $H(\boldsymbol{\lambda}(t))$

$$|\varepsilon(\boldsymbol{\lambda}(t))\rangle = e^{i\gamma(t)} e^{-\frac{i}{\hbar}\int_0^t \varepsilon(\boldsymbol{\lambda}(t'))dt'} |\varepsilon(\boldsymbol{\lambda}(0))\rangle$$



$$\gamma = i \oint_C \left\langle \varepsilon(\vec{\lambda}) \middle| \nabla_{\vec{\lambda}} \middle| \varepsilon(\vec{\lambda}) \right\rangle \cdot d\vec{\lambda}$$

$$= \oint_C \vec{A}(\vec{\lambda}) \cdot d\vec{\lambda}$$

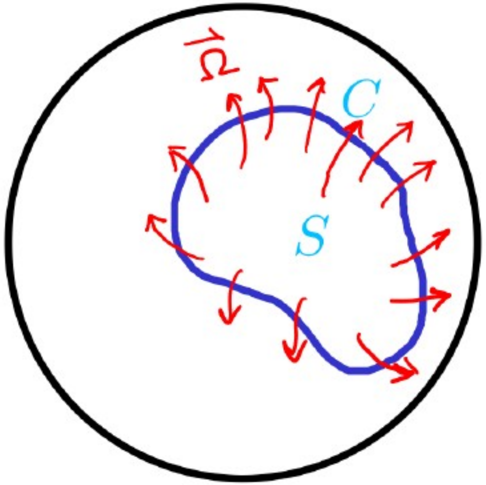
Space of $\lambda_1, \lambda_2, \ldots$

Geometric phase and Berry Curvature

M. V. Berry. Proceedings of the Royal Society A. **392** 1802 (1984)

- Time dependent Hamiltonian $H(\boldsymbol{\lambda}(t))$

$$|\varepsilon(\boldsymbol{\lambda}(t))\rangle = e^{i\gamma(t)} e^{-\frac{i}{\hbar}\int_0^t \varepsilon\left(\boldsymbol{\lambda}(t')\right)dt'} |\varepsilon(\boldsymbol{\lambda}(0))\rangle$$



$$\gamma = i\oint_C \left\langle \varepsilon(\vec{\lambda}) \middle| \nabla_{\vec{\lambda}} \middle| \varepsilon(\vec{\lambda}) \right\rangle \cdot d\vec{\lambda}$$

$$= \oint_C \vec{A}(\vec{\lambda}) \cdot d\vec{\lambda} \quad = \int_S \vec{\Omega} \cdot d\vec{a}$$

Stokes' theorem: $\vec{\Omega} = \nabla \times \vec{A}$

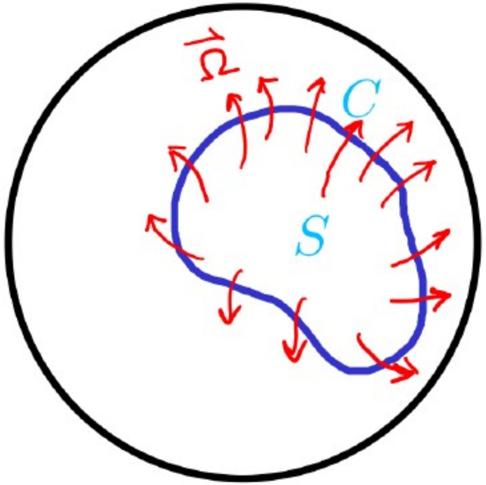
Space of $\lambda_1, \lambda_2, \ldots$

Geometric phase and Berry Curvature

M. V. Berry. Proceedings of the Royal Society A. **392** 1802 (1984)

- Time dependent Hamiltonian $H(\boldsymbol{\lambda}(t))$

$$|\varepsilon(\boldsymbol{\lambda}(t))\rangle = e^{i\gamma(t)} e^{-\frac{i}{\hbar}\int_0^t \varepsilon(\boldsymbol{\lambda}(t'))dt'} |\varepsilon(\boldsymbol{\lambda}(0))\rangle$$



Space of $\lambda_1, \lambda_2, \ldots$

$$\gamma = i \oint_C \left\langle \varepsilon(\vec{\lambda}) \middle| \nabla_{\vec{\lambda}} \middle| \varepsilon(\vec{\lambda}) \right\rangle \cdot d\vec{\lambda}$$

$$= \oint_C \vec{A}(\vec{\lambda}) \cdot d\vec{\lambda} \quad = \int_S \vec{\Omega} \cdot d\vec{a}$$

Stokes' theorem: $\vec{\Omega} = \nabla \times \vec{A}$

Like magnetic field, but in parameter space

Berry Curvature in reciprocal space

- Block wavefunctions in periodic potential

$$\psi_{n,\vec{k}}(\vec{r}) = e^{i\vec{k}\cdot\vec{r}}u_{n,\vec{k}}(\vec{r})$$

Berry Curvature in reciprocal space

- Block wavefunctions in periodic potential

$$\psi_{n,\vec{k}}(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} u_{n,\vec{k}}(\vec{r})$$

Effective Schrodinger equation

$$\left[\frac{\hbar^2}{2m}(\vec{k} - i\nabla)^2 + V(\vec{r})\right] u_{n,\vec{k}}(\vec{r}) = \varepsilon_{n,\vec{k}} u_{n,\vec{k}}(\vec{r})$$

Berry Curvature in reciprocal space

- Block wavefunctions in periodic potential

$$\psi_{n,\vec{k}}(\vec{r}) = e^{i\vec{k}\cdot\vec{r}}u_{n,\vec{k}}(\vec{r})$$

Effective Schrodinger equation

$$\left[\frac{\hbar^2}{2m}(\vec{k} - i\nabla)^2 + V(\vec{r})\right]u_{n,\vec{k}}(\vec{r}) = \varepsilon_{n,\vec{k}}u_{n,\vec{k}}(\vec{r})$$

- Crystal momentum changes with electromagneic field

$$\hbar\dot{\vec{k}} = -e\left(\vec{E} + \langle\dot{\vec{r}}\rangle \times \vec{B}\right)$$

Berry Curvature in reciprocal space

- Block wavefunctions in periodic potential

$$\psi_{n,\vec{k}}(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} u_{n,\vec{k}}(\vec{r})$$

Effective Schrodinger equation

$$\left[\frac{\hbar^2}{2m}(\vec{k} - i\nabla)^2 + V(\vec{r})\right] u_{n,\vec{k}}(\vec{r}) = \varepsilon_{n,\vec{k}} u_{n,\vec{k}}(\vec{r})$$

- Crystal momentum changes with electromagneic field

$$\hbar\dot{\vec{k}} = -e\left(\vec{E} + \langle\dot{\vec{r}}\rangle \times \vec{B}\right)$$

- Time evolution

$$u_{n,\vec{k}} \rightarrow e^{i\gamma_{\vec{k}}(\Delta t)} e^{-i\frac{\varepsilon_{\vec{k}}\Delta t}{\hbar}} u_{n,\vec{k}+\Delta\vec{k}}$$

Berry Curvature in reciprocal space

- Block wavefunctions in periodic potential

$$\psi_{n,\vec{k}}(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} u_{n,\vec{k}}(\vec{r})$$

Effective Schrodinger equation

$$\left[\frac{\hbar^2}{2m}(\vec{k} - i\nabla)^2 + V(\vec{r})\right] u_{n,\vec{k}}(\vec{r}) = \varepsilon_{n,\vec{k}} u_{n,\vec{k}}(\vec{r})$$

- Crystal momentum changes with electromagneic field

$$\hbar\dot{\vec{k}} = -e\left(\vec{E} + \langle\dot{\vec{r}}\rangle \times \vec{B}\right)$$

$$\gamma_{\vec{k}}(\Delta t) = \vec{A}(\vec{k}) \cdot \Delta\vec{k}$$

- Time evolution

$$u_{n,\vec{k}} \rightarrow e^{i\gamma_{\vec{k}}(\Delta t)} e^{-i\frac{\varepsilon_{\vec{k}}\Delta t}{\hbar}} u_{n,\vec{k}+\Delta\vec{k}}$$

$$\vec{A}(\vec{k}) = i\left\langle u_{n,\vec{k}} \middle| \nabla_{\vec{k}} \middle| u_{n,\vec{k}} \right\rangle$$

Effects of Berry curvature

Ming-Che Chang and Qian Niu PRL **75**, 1348 (1995),
Daniel C. Ralph. arXiv: 2001.04797

Effects of Berry curvature

Ming-Che Chang and Qian Niu PRL **75**, 1348 (1995),
Daniel C. Ralph. arXiv: 2001.04797

- The semiclassical equation of velocity is modified

$$\dot{\vec{r}} = \frac{1}{\hbar}\nabla_k \varepsilon_k - \dot{\vec{k}} \times \boldsymbol{\Omega}(\boldsymbol{k})$$

$$\hbar\dot{\vec{k}} = -e(\vec{E} + \dot{\vec{r}} \times \vec{B})$$

Effects of Berry curvature

Ming-Che Chang and Qian Niu PRL **75**, 1348 (1995),
Daniel C. Ralph. arXiv: 2001.04797

- The semiclassical equation of velocity is modified

$$\dot{\vec{r}} = \frac{1}{\hbar}\nabla_k \varepsilon_k - \dot{\vec{k}} \times \boldsymbol{\Omega}(\boldsymbol{k})$$

$$\hbar\dot{\vec{k}} = -e(\vec{E} + \dot{\vec{r}} \times \vec{B})$$

Anomalous velocity

Effects of Berry curvature

Ming-Che Chang and Qian Niu PRL **75**, 1348 (1995),
Daniel C. Ralph. arXiv: 2001.04797

- The semiclassical equation of velocity is modified

$$\dot{\vec{r}} = \frac{1}{\hbar}\nabla_k \varepsilon_k - \dot{\vec{k}} \times \boldsymbol{\Omega}(\boldsymbol{k})$$

$$\hbar\dot{\vec{k}} = -e(\vec{E} + \dot{\vec{r}} \times \vec{B})$$

Anomalous velocity

Decoupled

$$\dot{\vec{r}} = \frac{\frac{1}{\hbar}\frac{\partial\varepsilon}{\partial k} + \frac{e}{\hbar}(\boldsymbol{E}\times\boldsymbol{\Omega}) + \frac{e}{\hbar^2}\left(\boldsymbol{\Omega}\cdot\frac{\partial\varepsilon}{\partial k}\right)\boldsymbol{B}}{1+\frac{e}{\hbar}\boldsymbol{B}\cdot\boldsymbol{\Omega}}$$

$$\dot{\vec{k}} = -\frac{\frac{e}{\hbar}\boldsymbol{E} + \frac{e}{\hbar^2}\frac{\partial\varepsilon}{\partial k}\times\boldsymbol{B} + \frac{e^2}{\hbar^2}(\boldsymbol{E}\cdot\boldsymbol{B})\boldsymbol{\Omega}}{1+\frac{e}{\hbar}\boldsymbol{B}\cdot\boldsymbol{\Omega}}$$

Effects of Berry curvature

Ming-Che Chang and Qian Niu PRL **75**, 1348 (1995),
Daniel C. Ralph. arXiv: 2001.04797

- The semiclassical equation of velocity is modified

$$\dot{\vec{r}} = \frac{1}{\hbar}\nabla_k \varepsilon_k - \dot{\vec{k}} \times \boldsymbol{\Omega}(\boldsymbol{k})$$

$$\hbar\dot{\vec{k}} = -e(\vec{E} + \dot{\vec{r}} \times \vec{B})$$

Anomalous velocity

(2D)

Decoupled

$$\dot{\vec{r}} = \frac{\frac{1}{\hbar}\frac{\partial\varepsilon}{\partial k} + \frac{e}{\hbar}(\boldsymbol{E}\times\boldsymbol{\Omega}) + \frac{e}{\hbar^2}\left(\boldsymbol{\Omega}\cdot\frac{\partial\varepsilon}{\partial k}\right)\boldsymbol{B}}{1+\frac{e}{\hbar}\boldsymbol{B}\cdot\boldsymbol{\Omega}}$$

0

$$\dot{\vec{k}} = -\frac{\frac{e}{\hbar}\boldsymbol{E} + \frac{e}{\hbar^2}\frac{\partial\varepsilon}{\partial k}\times\boldsymbol{B} + \frac{e^2}{\hbar^2}(\boldsymbol{E}\cdot\boldsymbol{B})\boldsymbol{\Omega}}{1+\frac{e}{\hbar}\boldsymbol{B}\cdot\boldsymbol{\Omega}}$$

0

(2D)

Effects of Berry curvature

Ming-Che Chang and Qian Niu PRL **75**, 1348 (1995),
Daniel C. Ralph. arXiv: 2001.04797

- **The semiclassical equation of velocity is modified**

$$\dot{\vec{r}} = \frac{1}{\hbar}\nabla_k \varepsilon_k - \dot{\vec{k}} \times \boldsymbol{\Omega}(\boldsymbol{k})$$

$$\hbar\dot{\vec{k}} = -e(\vec{E} + \dot{\vec{r}} \times \vec{B})$$

Anomalous velocity

Decoupled

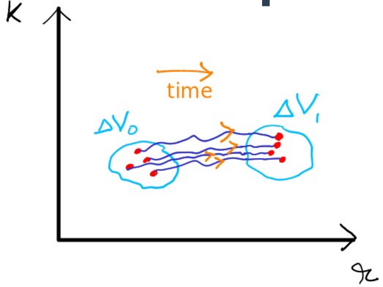
(2D)

$$\dot{\boldsymbol{r}} = \frac{\frac{1}{\hbar}\frac{\partial\varepsilon}{\partial \boldsymbol{k}} + \frac{e}{\hbar}(\boldsymbol{E}\times\boldsymbol{\Omega}) + \frac{e}{\hbar^2}\left(\boldsymbol{\Omega}\cdot\frac{\partial\varepsilon}{\partial \boldsymbol{k}}\right)\boldsymbol{B}}{1+\frac{e}{\hbar}\boldsymbol{B}\cdot\boldsymbol{\Omega}}$$

$$\dot{\boldsymbol{k}} = -\frac{\frac{e}{\hbar}\boldsymbol{E} + \frac{e}{\hbar^2}\frac{\partial\varepsilon}{\partial \boldsymbol{k}}\times\boldsymbol{B} + \frac{e^2}{\hbar^2}(\boldsymbol{E}\cdot\boldsymbol{B})\boldsymbol{\Omega}}{1+\frac{e}{\hbar}\boldsymbol{B}\cdot\boldsymbol{\Omega}}$$

(2D)

- **Phase space density is modified**



$$\Delta V\left(1 + \frac{e}{\hbar}\vec{B}\cdot\vec{\Omega}\right)$$

is a constant of motion

Di Xiao, Junren Shi, and Qian Niu. PRL 95, 137204 (2005)

Effects of Berry curvature

Ming-Che Chang and Qian Niu PRL **75**, 1348 (1995),
Daniel C. Ralph. arXiv: 2001.04797

- **The semiclassical equation of velocity is modified**

$$\dot{\vec{r}} = \frac{1}{\hbar}\nabla_k \varepsilon_k - \dot{\vec{k}} \times \boldsymbol{\Omega}(\boldsymbol{k})$$

$$\hbar\dot{\vec{k}} = -e(\vec{E} + \dot{\vec{r}} \times \vec{B})$$

Anomalous velocity

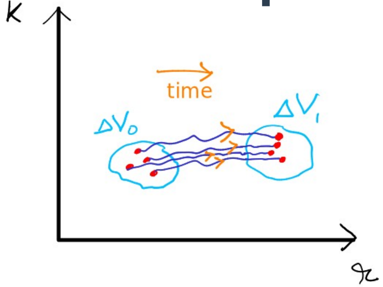
(2D)

Decoupled

$$\dot{\vec{r}} = \frac{\frac{1}{\hbar}\frac{\partial\varepsilon}{\partial k} + \frac{e}{\hbar}(\boldsymbol{E}\times\boldsymbol{\Omega}) + \frac{e}{\hbar^2}\left(\boldsymbol{\Omega}\cdot\frac{\partial\varepsilon}{\partial k}\right)\boldsymbol{B}}{1+\frac{e}{\hbar}\boldsymbol{B}\cdot\boldsymbol{\Omega}}$$

$$\dot{\vec{k}} = -\frac{\frac{e}{\hbar}\boldsymbol{E} + \frac{e}{\hbar^2}\frac{\partial\varepsilon}{\partial k}\times\boldsymbol{B} + \frac{e^2}{\hbar^2}(\boldsymbol{E}\cdot\boldsymbol{B})\boldsymbol{\Omega}}{1+\frac{e}{\hbar}\boldsymbol{B}\cdot\boldsymbol{\Omega}}$$

- **Phase space density is modified**



$$\Delta V\left(1+\frac{e}{\hbar}\vec{B}\cdot\vec{\Omega}\right)$$

is a constant of motion

Di Xiao, Junren Shi, and Qian Niu. PRL 95, 137204 (2005)

(2D)

$$\langle\mathcal{O}\rangle\,(\vec{B}=0) = 2\int\frac{d\vec{k}}{(2\pi)^d}\,\langle\mathcal{O}\rangle_{\vec{k}}\,\tilde{g}_{\vec{k}}$$

$$\langle\mathcal{O}\rangle = 2\int\frac{d\vec{k}}{(2\pi)^d}\left(1+\frac{e}{\hbar}\vec{B}\cdot\vec{\Omega}\left(\vec{k}\right)\right)\langle\mathcal{O}\rangle_{\vec{k}}\,\tilde{g}_{\vec{k}}$$

When do we get a non-zero Berry Curvature?

- If inversion symmetry holds

$$\dot{\vec{r}} = \frac{1}{\hbar}\nabla_k \varepsilon_k - \dot{\vec{k}} \times \boldsymbol{\Omega}(\boldsymbol{k})$$

$$\vec{\Omega}(\vec{k}) = \vec{\Omega}(-\vec{k})$$

- If time reversal symmetry holds

$$\vec{\Omega}(\vec{k}) = -\vec{\Omega}(-\vec{k})$$

- When both hold simultaneously, Berry curvature is identically zero.

When do we get a non-zero Berry Curvature?

$$\dot{\vec{r}} = \frac{1}{\hbar}\nabla_k \varepsilon_k - \dot{\vec{k}} \times \boldsymbol{\Omega}(\boldsymbol{k})$$

- If inversion symmetry holds

$$\vec{\Omega}(\vec{k}) = \vec{\Omega}(-\vec{k})$$

$$\vec{\Omega}(\vec{k}) = i\nabla \times \left\langle u_{n,\vec{k}} \middle| \nabla_{\vec{k}} \middle| u_{n,\vec{k}} \right\rangle$$

- If time reversal symmetry holds

$$\vec{\Omega}(\vec{k}) = -\vec{\Omega}(-\vec{k})$$

- When both hold simultaneously, Berry curvature is identically zero.

Boltzmann Transport framework

Classical charge and energy currents

$$\vec{j}_e = -nev, \vec{j}_\epsilon = n\epsilon v$$

Boltzmann Transport framework

Classical charge and energy currents

$$\vec{j}_e = -nev, \vec{j}_\epsilon = n\epsilon v$$

Semiclassical framework

$$\left\langle \hat{\vec{j}}_e \right\rangle =$$

Boltzmann Transport framework

Classical charge and energy currents

$$\vec{j}_e = -nev, \vec{j}_\epsilon = n\epsilon v$$

Semiclassical framework

$$\left\langle \hat{\vec{j}}_e \right\rangle = \qquad (-e\dot{\vec{r}})_{\vec{k}} \tilde{g}_{\vec{k}}$$

Boltzmann Transport framework

Classical charge and energy currents

$$\vec{j}_e = -nev, \vec{j}_\epsilon = n\epsilon v$$

Semiclassical framework

$$\left\langle \hat{\vec{j}}_e \right\rangle = \int \frac{d\vec{k}}{(2\pi)^d} \left(1 + \frac{e}{\hbar}\vec{B}\cdot\vec{\Omega}\left(\vec{k}\right)\right) \left(-e\dot{\vec{r}}\right)_{\vec{k}} \tilde{g}_{\vec{k}}$$

Boltzmann Transport framework

Classical charge and energy currents

$$\vec{j}_e = -nev, \vec{j}_\epsilon = n\epsilon v$$

Semiclassical framework

$$\left\langle \hat{\vec{j}}_e \right\rangle = \; 2 \int \frac{d\vec{k}}{(2\pi)^d} \left(1 + \frac{e}{\hbar} \vec{B} \cdot \vec{\Omega}\left(\vec{k}\right)\right) \left(-e\dot{\vec{r}}\right)_{\vec{k}} \tilde{g}_{\vec{k}}$$

Boltzmann Transport framework

Classical charge and energy currents

$$\vec{j}_e = -nev, \vec{j}_\epsilon = n\epsilon v$$

Semiclassical framework

$$\left\langle \hat{\vec{j}}_e \right\rangle = 2 \int \frac{d\vec{k}}{(2\pi)^d} \left(1 + \frac{e}{\hbar}\vec{B}\cdot\vec{\Omega}\left(\vec{k}\right)\right) \left(-e\dot{\vec{r}}\right)_{\vec{k}} \tilde{g}_{\vec{k}}$$

Heat current: From 1st law of thermodynamics, $dQ = dE - \mu dN$

Boltzmann Transport framework

Classical charge and energy currents

$$\vec{j}_e = -nev, \vec{j}_\epsilon = n\epsilon v$$

Semiclassical framework

$$\left\langle \hat{\vec{j}}_e \right\rangle = 2 \int \frac{d\vec{k}}{(2\pi)^d} \left(1 + \frac{e}{\hbar}\vec{B}\cdot\vec{\Omega}\left(\vec{k}\right)\right) \left(-e\dot{\vec{r}}\right)_{\vec{k}} \tilde{g}_{\vec{k}}$$

Heat current: From 1st law of thermodynamics, $dQ = dE - \mu dN$

$$\left\langle \hat{\vec{j}}_Q \right\rangle_k = \left\langle \hat{\vec{j}}_E \right\rangle_k - \mu \left\langle \hat{\vec{j}}_N \right\rangle_k$$

Boltzmann Transport framework

Classical charge and energy currents

$$\vec{j}_e = -nev, \vec{j}_\epsilon = n\epsilon v$$

Semiclassical framework

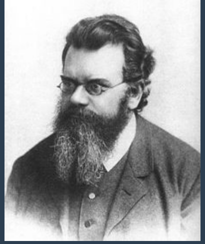
$$\left\langle \hat{\vec{j}}_e \right\rangle = 2 \int \frac{d\vec{k}}{(2\pi)^d} \left(1 + \frac{e}{\hbar}\vec{B} \cdot \vec{\Omega}\left(\vec{k}\right)\right) (-e\dot{\vec{r}})_{\vec{k}} \tilde{g}_{\vec{k}}$$

Heat current: From 1st law of thermodynamics, $dQ = dE - \mu dN$

$$\left\langle \hat{\vec{j}}_Q \right\rangle_k = \left\langle \hat{\vec{j}}_E \right\rangle_k - \mu \left\langle \hat{\vec{j}}_N \right\rangle_k$$

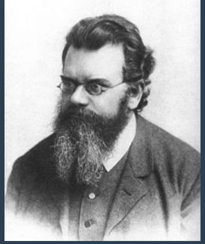
$$\left\langle \hat{\vec{j}}_Q \right\rangle = 2 \int \frac{d\vec{k}}{(2\pi)^d} \left(1 + \frac{e}{\hbar}\vec{B} \cdot \vec{\Omega}\left(\vec{k}\right)\right) (\varepsilon_{\vec{k}} - \mu)\dot{\vec{r}}_{\vec{k}} \tilde{g}_{\vec{k}}$$

Boltzmann Transport Equation



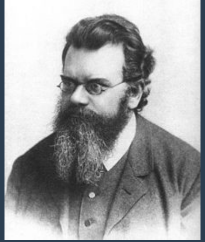
- In equilibrium, $\tilde{g}_k$ is the Fermi distribution. $f_{\vec{k}} = \dfrac{1}{e^{\beta\left(\varepsilon_{\vec{k}} - \mu\right)} + 1}$
- The external fields are small, and it deviates slightly from the Fermi distribution

Boltzmann Transport Equation



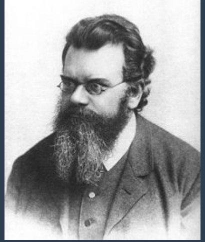
- In equilibrium, $\tilde{g}_k$ is the Fermi distribution. $f_{\vec{k}} = \dfrac{1}{e^{\beta\left(\varepsilon_{\vec{k}} - \mu\right)} + 1}$
- The external fields are small, and it deviates slightly from the Fermi distribution $\tilde{g}_k = f_k + g_k$

Boltzmann Transport Equation



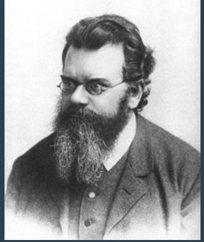
- In equilibrium, $\tilde{g}_k$ is the Fermi distribution. $f_{\vec{k}} = \dfrac{1}{e^{\beta\left(\varepsilon_{\vec{k}} - \mu\right)} + 1}$
- The external fields are small, and it deviates slightly from the Fermi distribution $\tilde{g}_k = f_k + g_k$
- The system tries to attain equilibrium, with a relaxation time $\tau \sim 10^{-14} s$

Boltzmann Transport Equation

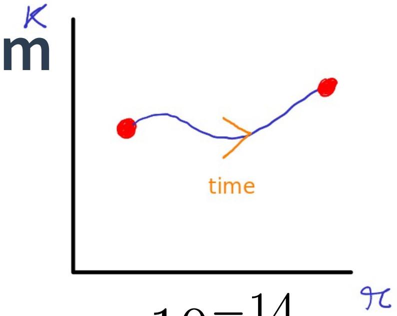


- In equilibrium, $\tilde{g}_k$ is the Fermi distribution. $f_{\vec{k}} = \dfrac{1}{e^{\beta\left(\varepsilon_{\vec{k}} - \mu\right)} + 1}$
- The external fields are small, and it deviates slightly from the Fermi distribution $\tilde{g}_k = f_k + g_k$
- The system tries to attain equilibrium, with a relaxation time $\tau \sim 10^{-14} s$
- We assume that the more the distribution deviates, the faster it would try to return to equilibrium, *relaxation time approximation.* $D_t \tilde{g}_k = -\dfrac{\tilde{g}_k - f_k}{\tau_k}$

Boltzmann Transport Equation



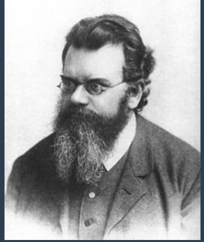
- In equilibrium, $\tilde{g}_k$ is the Fermi distribution. $f_{\vec{k}} = \dfrac{1}{e^{\beta\left(\varepsilon_{\vec{k}}-\mu\right)}+1}$
- The external fields are small, and it deviates slightly from the Fermi distribution $\tilde{g}_k = f_k + g_k$



- The system tries to attain equilibrium, with a relaxation time $\tau \sim 10^{-14}s$
- We assume that the more the distribution deviates, the faster it would try to return to equilibrium, *relaxation time approximation.* $D_t\tilde{g}_k = -\dfrac{\tilde{g}_k - f_k}{\tau_k}$

$$D_t \equiv \frac{\partial}{\partial t} + \dot{\vec{k}} \cdot \frac{\partial}{\partial \vec{k}} + \dot{\vec{r}} \cdot \frac{\partial}{\partial \vec{r}}$$

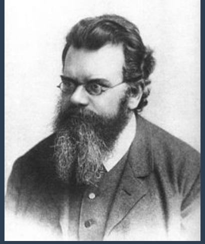
Boltzmann Transport Equation



- In steady-state, and homogeneous fields, the equation in becomes,

$$\boxed{\frac{g_k}{\tau} - \underbrace{\frac{\frac{e}{\hbar^2}\frac{\partial \varepsilon}{\partial \vec{k}} \times \vec{B}}{1+\frac{e}{\hbar}\vec{B}\cdot\vec{\Omega}} \cdot \frac{\partial}{\partial \vec{k}} g_k}_{\text{treated as a perturbation}} = \frac{\partial f}{\partial \varepsilon}\frac{1}{1+\frac{e}{\hbar}\vec{B}\cdot\vec{\Omega}}\frac{1}{\hbar}\frac{\partial\varepsilon}{\partial\vec{k}}\cdot\left[e\vec{E}+\nabla\mu+\nabla T\frac{\varepsilon-\mu}{T}\right]}$$

Boltzmann Transport Equation

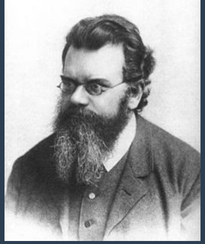


- In steady-state, and homogeneous fields, the equation in becomes,

$$\boxed{\frac{g_k}{\tau} - \underbrace{\frac{\frac{e}{\hbar^2}\frac{\partial \varepsilon}{\partial \vec{k}} \times \vec{B}}{1+\frac{e}{\hbar}\vec{B}\cdot\vec{\Omega}} \cdot \frac{\partial}{\partial \vec{k}} g_k}_{\text{treated as a perturbation}} = \frac{\partial f}{\partial \varepsilon}\frac{1}{1+\frac{e}{\hbar}\vec{B}\cdot\vec{\Omega}}\frac{1}{\hbar}\frac{\partial \varepsilon}{\partial \vec{k}} \cdot \left[e\vec{E} + \nabla\mu + \nabla T \frac{\varepsilon-\mu}{T}\right]}$$

- Why is the perturbation theory valid?

Boltzmann Transport Equation



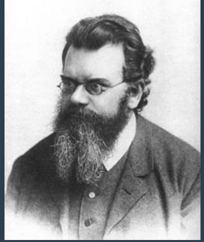
- In steady-state, and homogeneous fields, the equation in becomes,

$$\boxed{\frac{g_k}{\tau} - \underbrace{\frac{\frac{e}{\hbar^2}\frac{\partial\varepsilon}{\partial\vec{k}} \times \vec{B}}{1+\frac{e}{\hbar}\vec{B}\cdot\vec{\Omega}} \cdot \frac{\partial}{\partial\vec{k}} g_k}_{\text{treated as a perturbation}} = \frac{\partial f}{\partial\varepsilon}\frac{1}{1+\frac{e}{\hbar}\vec{B}\cdot\vec{\Omega}}\frac{1}{\hbar}\frac{\partial\varepsilon}{\partial\vec{k}} \cdot \left[e\vec{E} + \nabla\mu + \nabla T\frac{\varepsilon-\mu}{T}\right]}$$

- Why is the perturbation theory valid?

  Dimension analysis: this term = $\omega_c \tau \times$ the first term, $\omega_c = \dfrac{eB}{m^*}$ is the cyclotron frequency

Boltzmann Transport Equation



- In steady-state, and homogeneous fields, the equation in becomes,

$$\boxed{\frac{g_k}{\tau} - \underbrace{\frac{\frac{e}{\hbar^2}\frac{\partial\varepsilon}{\partial\vec{k}}\times\vec{B}}{1+\frac{e}{\hbar}\vec{B}\cdot\vec{\Omega}}\cdot\frac{\partial}{\partial\vec{k}}g_k}_{\text{treated as a perturbation}} = \frac{\partial f}{\partial\varepsilon}\frac{1}{1+\frac{e}{\hbar}\vec{B}\cdot\vec{\Omega}}\frac{1}{\hbar}\frac{\partial\varepsilon}{\partial\vec{k}}\cdot\left[e\vec{E}+\nabla\mu+\nabla T\frac{\varepsilon-\mu}{T}\right]}$$

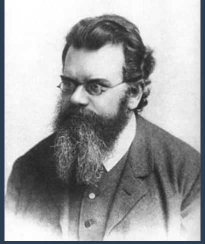
- Why is the perturbation theory valid?

  Dimension analysis: this term $\omega_c\tau\times$ the first $\omega_c = \frac{eB}{m^*}$ is the cyclotron frequency

- This is less than the first term when

$$B \ll B_{critical} = \frac{m^*}{e\tau}$$

Boltzmann Transport Equation



- In steady-state, and homogeneous fields, the equation in becomes,

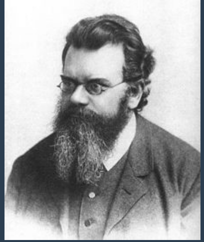
$$\boxed{\frac{g_k}{\tau} - \underbrace{\frac{\frac{e}{\hbar^2}\frac{\partial \varepsilon}{\partial \vec{k}} \times \vec{B}}{1+\frac{e}{\hbar}\vec{B}\cdot\vec{\Omega}} \cdot \frac{\partial}{\partial \vec{k}} g_k}_{\text{treated as a perturbation}} = \frac{\partial f}{\partial \varepsilon}\frac{1}{1+\frac{e}{\hbar}\vec{B}\cdot\vec{\Omega}}\frac{1}{\hbar}\frac{\partial \varepsilon}{\partial \vec{k}} \cdot \left[e\vec{E} + \nabla\mu + \nabla T\frac{\varepsilon-\mu}{T}\right]}$$

- Why is the perturbation theory valid?

  Dimension analysis: this term = $\omega_c \tau \times$ the first term, $\omega_c = \dfrac{eB}{m^*}$ is the cyclotron frequency

- This is less than the first term when $B \ll B_{critical} = \dfrac{m^*}{e\tau}$
- If $m^* \sim m, \tau \sim 10^{-14} s$ then the critical field is ~ 570 T

Boltzmann Transport Equation



- In steady-state, and homogeneous fields, the equation in becomes,

$$\boxed{\frac{g_k}{\tau} - \underbrace{\frac{\frac{e}{\hbar^2}\frac{\partial \varepsilon}{\partial \vec{k}} \times \vec{B}}{1+\frac{e}{\hbar}\vec{B}\cdot\vec{\Omega}} \cdot \frac{\partial}{\partial \vec{k}} g_k}_{\text{treated as a perturbation}} = \frac{\partial f}{\partial \varepsilon} \frac{1}{1+\frac{e}{\hbar}\vec{B}\cdot\vec{\Omega}} \frac{1}{\hbar}\frac{\partial \varepsilon}{\partial \vec{k}} \cdot \left[e\vec{E} + \nabla\mu + \nabla T \frac{\varepsilon-\mu}{T}\right]}$$

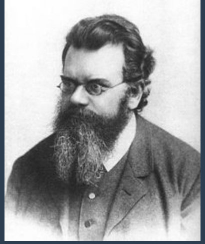
- Why is the perturbation theory valid?

  Dimension analysis: this term = $\omega_c \tau \times$ the first term, $\omega_c = \dfrac{eB}{m^*}$ is the cyclotron frequency

- This is less than the first term when $B \ll B_{critical} = \dfrac{m^*}{e\tau}$
- If $m^* \sim m, \tau \sim 10^{-14} s$ then the critical field is ~ 570 T

Why even consider this term, then??

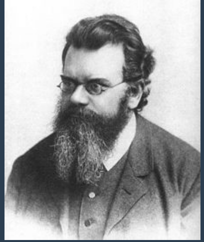
Boltzmann Transport Equation



- In steady-state, and homogeneous fields, the equation in becomes,

$$\boxed{\frac{g_k}{\tau} - \underbrace{\frac{\frac{e}{\hbar^2}\frac{\partial \varepsilon}{\partial \vec{k}} \times \vec{B}}{1+\frac{e}{\hbar}\vec{B}\cdot\vec{\Omega}} \cdot \frac{\partial}{\partial \vec{k}} g_k}_{\text{treated as a perturbation}} = \frac{\partial f}{\partial \varepsilon}\frac{1}{1+\frac{e}{\hbar}\vec{B}\cdot\vec{\Omega}}\frac{1}{\hbar}\frac{\partial \varepsilon}{\partial \vec{k}} \cdot \left[e\vec{E} + \nabla\mu + \nabla T\frac{\varepsilon-\mu}{T}\right]}$$

Boltzmann Transport Equation

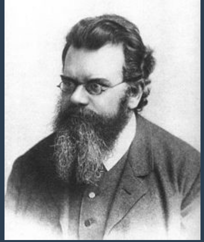


- In steady-state, and homogeneous fields, the equation in becomes,

$$\boxed{\frac{g_k}{\tau} - \underbrace{\frac{\frac{e}{\hbar^2}\frac{\partial\varepsilon}{\partial\vec{k}}\times\vec{B}}{1+\frac{e}{\hbar}\vec{B}\cdot\vec{\Omega}}\cdot\frac{\partial}{\partial\vec{k}}g_k}_{\text{treated as a perturbation}} = \frac{\partial f}{\partial\varepsilon}\frac{1}{1+\frac{e}{\hbar}\vec{B}\cdot\vec{\Omega}}\frac{1}{\hbar}\frac{\partial\varepsilon}{\partial\vec{k}}\cdot\left[e\vec{E}+\nabla\mu+\nabla T\frac{\varepsilon-\mu}{T}\right]}$$

- The solution, with $\vec{S} = e\vec{E} + \nabla\mu + \frac{\varepsilon-\mu}{T}\nabla T$

Boltzmann Transport Equation



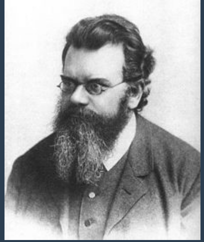
- In steady-state, and homogeneous fields, the equation in becomes,

$$\boxed{\frac{g_k}{\tau} - \underbrace{\frac{\frac{e}{\hbar^2}\frac{\partial\varepsilon}{\partial\vec{k}}\times\vec{B}}{1+\frac{e}{\hbar}\vec{B}\cdot\vec{\Omega}}\cdot\frac{\partial}{\partial\vec{k}}g_k}_{\text{treated as a perturbation}} = \frac{\partial f}{\partial\varepsilon}\frac{1}{1+\frac{e}{\hbar}\vec{B}\cdot\vec{\Omega}}\frac{1}{\hbar}\frac{\partial\varepsilon}{\partial\vec{k}}\cdot\left[e\vec{E}+\nabla\mu+\nabla T\frac{\varepsilon-\mu}{T}\right]}$$

- The solution, with $\vec{S} = e\vec{E} + \nabla\mu + \dfrac{\varepsilon-\mu}{T}\nabla T$

$$\boxed{\begin{aligned} g_k &= \frac{\partial f}{\partial\varepsilon}\frac{1}{1+\frac{e}{\hbar}\vec{B}\cdot\vec{\Omega}}\frac{\tau}{\hbar}\frac{\partial\varepsilon}{\partial\vec{k}}\cdot\vec{S} \\ &+ \frac{\frac{e\tau}{\hbar^2}}{1+\frac{e}{\hbar}\vec{B}\cdot\vec{\Omega}}\frac{\partial\varepsilon}{\partial\vec{k}}\cdot\vec{B}\times\left[\frac{\partial}{\partial\vec{k}}\left[\frac{\frac{\partial f}{\partial\varepsilon}\tau}{1+\frac{e}{\hbar}\vec{B}\cdot\vec{\Omega}}\right]\frac{\vec{S}}{\hbar}\cdot\frac{\partial\varepsilon}{\partial\vec{k}} + \frac{\frac{\partial f}{\partial\varepsilon}\tau}{1+\frac{e}{\hbar}\vec{B}\cdot\vec{\Omega}}\left[\left(\frac{1}{\hbar}\vec{S}\cdot\frac{\partial}{\partial\vec{k}}\right)\frac{\partial\varepsilon}{\partial\vec{k}}\right]\right]\end{aligned}}$$

Boltzmann Transport Equation



- In steady-state, and homogeneous fields, the equation in becomes,

$$\boxed{\frac{g_k}{\tau} - \underbrace{\frac{\frac{e}{\hbar^2}\frac{\partial\varepsilon}{\partial\vec{k}}\times\vec{B}}{1+\frac{e}{\hbar}\vec{B}\cdot\vec{\Omega}}\cdot\frac{\partial}{\partial\vec{k}}g_k}_{\text{treated as a perturbation}} = \frac{\partial f}{\partial\varepsilon}\frac{1}{1+\frac{e}{\hbar}\vec{B}\cdot\vec{\Omega}}\frac{1}{\hbar}\frac{\partial\varepsilon}{\partial\vec{k}}\cdot\left[e\vec{E}+\nabla\mu+\nabla T\frac{\varepsilon-\mu}{T}\right]}$$

Verified that it reduces to results obtained from Drude model

In free electron limit

Einstein and Onsager Relations are satisfied

- The solution, with $\vec{S} = e\vec{E} + \nabla\mu + \frac{\varepsilon-\mu}{T}\nabla T$

$$\boxed{\begin{aligned} g_k &= \frac{\partial f}{\partial\varepsilon}\frac{1}{1+\frac{e}{\hbar}\vec{B}\cdot\vec{\Omega}}\frac{\tau}{\hbar}\frac{\partial\varepsilon}{\partial\vec{k}}\cdot\vec{S} \\ &+ \frac{\frac{e\tau}{\hbar^2}}{1+\frac{e}{\hbar}\vec{B}\cdot\vec{\Omega}}\frac{\partial\varepsilon}{\partial\vec{k}}\cdot\vec{B}\times\left[\frac{\partial}{\partial\vec{k}}\left[\frac{\frac{\partial f}{\partial\varepsilon}\tau}{1+\frac{e}{\hbar}\vec{B}\cdot\vec{\Omega}}\right]\frac{\vec{S}}{\hbar}\cdot\frac{\partial\varepsilon}{\partial\vec{k}} + \frac{\frac{\partial f}{\partial\varepsilon}\tau}{1+\frac{e}{\hbar}\vec{B}\cdot\vec{\Omega}}\left[\left(\frac{1}{\hbar}\vec{S}\cdot\frac{\partial}{\partial\vec{k}}\right)\frac{\partial\varepsilon}{\partial\vec{k}}\right]\right]\end{aligned}}$$

Which all terms contribute

$$\left\langle \hat{\vec{j}}_e \right\rangle = -e \int \frac{2d\vec{k}}{(2\pi)^2} (f_{\vec{k}} + g_{\vec{k}}) \left[\frac{1}{\hbar} \frac{\partial \varepsilon_{\vec{k}}}{\partial \vec{k}} + \frac{e}{\hbar} (\vec{E} \times \vec{\Omega}(\vec{k})) \right]$$

Which all terms contribute

$$\left\langle \hat{\vec{\dot{j}}}_e \right\rangle = -e \int \frac{2d\vec{k}}{(2\pi)^2} (f_{\vec{k}} + g_{\vec{k}}) \left[\frac{1}{\hbar} \frac{\partial \varepsilon_{\vec{k}}}{\partial \vec{k}} + \frac{e}{\hbar} (\vec{E} \times \vec{\Omega}(\vec{k})) \right]$$

0, as it should be without any external field

Which all terms contribute

2nd order

$$\left\langle \hat{\vec{\dot{j}}}_e \right\rangle = -e \int \frac{2d\vec{k}}{(2\pi)^2} (f_{\vec{k}} + g_{\vec{k}}) \left[\frac{1}{\hbar} \frac{\partial \varepsilon_{\vec{k}}}{\partial \vec{k}} + \frac{e}{\hbar} (\vec{E} \times \vec{\Omega}(\vec{k})) \right]$$

0, as it should be without any external field

Which all terms contribute

2nd order

$$\left\langle \hat{\vec{j}}_e \right\rangle = -e \int \frac{2d\vec{k}}{(2\pi)^2} (f_{\vec{k}} + g_{\vec{k}}) \left[\frac{1}{\hbar} \frac{\partial \varepsilon_{\vec{k}}}{\partial \vec{k}} + \frac{e}{\hbar} (\vec{E} \times \vec{\Omega}(\vec{k})) \right]$$

0, as it should be without any external field

$$\left\langle \hat{\vec{j}}_e \right\rangle = -e \int \frac{2d\vec{k}}{(2\pi)^2} \left[f_{\vec{k}} \frac{e}{\hbar} (\vec{E} \times \vec{\Omega}(\vec{k})) + g_{\vec{k}} \frac{1}{\hbar} \frac{\partial \varepsilon}{\partial \vec{k}} \right]$$

Which all terms contribute

2nd order

$$\left\langle \hat{\vec{j}}_e \right\rangle = -e \int \frac{2d\vec{k}}{(2\pi)^2} (f_{\vec{k}} + g_{\vec{k}}) \left[\frac{1}{\hbar} \frac{\partial \varepsilon_{\vec{k}}}{\partial \vec{k}} + \frac{e}{\hbar} (\vec{E} \times \vec{\Omega}(\vec{k})) \right]$$

0, as it should be without any external field

$$\left\langle \hat{\vec{j}}_e \right\rangle = -e \int \frac{2d\vec{k}}{(2\pi)^2} \left[f_{\vec{k}} \frac{e}{\hbar} (\vec{E} \times \vec{\Omega}(\vec{k})) + g_{\vec{k}} \frac{1}{\hbar} \frac{\partial \varepsilon}{\partial \vec{k}} \right]$$

$$\left\langle \hat{\vec{j}}_e \right\rangle_{anomalous} = -2 \frac{e^2}{h} \vec{E} \times \int_{\text{filled}} \frac{d\vec{k}}{2\pi} \vec{\Omega}(\vec{k})$$

Which all terms contribute

2nd order

$$\left\langle \hat{\vec{j}}_e \right\rangle = -e \int \frac{2d\vec{k}}{(2\pi)^2} (f_{\vec{k}} + g_{\vec{k}}) \left[\frac{1}{\hbar} \frac{\partial \varepsilon_{\vec{k}}}{\partial \vec{k}} + \frac{e}{\hbar} (\vec{E} \times \vec{\Omega}(\vec{k})) \right]$$

0, as it should be without any external field

$$\left\langle \hat{\vec{j}}_e \right\rangle = -e \int \frac{2d\vec{k}}{(2\pi)^2} \left[f_{\vec{k}} \frac{e}{\hbar} (\vec{E} \times \vec{\Omega}(\vec{k})) + g_{\vec{k}} \frac{1}{\hbar} \frac{\partial \varepsilon}{\partial \vec{k}} \right]$$

$$\left\langle \hat{\vec{j}}_e \right\rangle_{anomalous} = -2 \frac{e^2}{h} \vec{E} \times \int_{\text{filled}} \frac{d\vec{k}}{2\pi} \vec{\Omega}(\vec{k})$$

For a filled band,
Chern number, integer

Which all terms contribute

2nd order

$$\left\langle \hat{\vec{j}}_e \right\rangle = -e \int \frac{2d\vec{k}}{(2\pi)^2} (f_{\vec{k}} + g_{\vec{k}}) \left[\frac{1}{\hbar} \frac{\partial \varepsilon_{\vec{k}}}{\partial \vec{k}} + \frac{e}{\hbar} (\vec{E} \times \vec{\Omega}(\vec{k})) \right]$$

0, as it should be without any external field

$$\left\langle \hat{\vec{j}}_e \right\rangle = -e \int \frac{2d\vec{k}}{(2\pi)^2} \left[f_{\vec{k}} \frac{e}{\hbar} (\vec{E} \times \vec{\Omega}(\vec{k})) + g_{\vec{k}} \frac{1}{\hbar} \frac{\partial \varepsilon}{\partial \vec{k}} \right]$$

$$\left\langle \hat{\vec{j}}_e \right\rangle_{anomalous} = -2 \frac{e^2}{h} \vec{E} \times \int_{\text{filled}} \frac{d\vec{k}}{2\pi} \vec{\Omega}(\vec{k})$$

For a filled band,

Chern number, integer

Independent of scattering!!

Which all terms contribute

- For a filled band, $\sigma_{xy} = -\sigma_{yx} = \frac{2e^2}{h}\mathcal{C}$ quantized!!

$$\left\langle \hat{\vec{j}}_e \right\rangle = -e \int \frac{2d\vec{k}}{(2\pi)^2} \left[f_{\vec{k}} \frac{e}{\hbar} (\vec{E} \times \vec{\Omega}(\vec{k})) + g_{\vec{k}} \frac{1}{\hbar} \frac{\partial \varepsilon}{\partial \vec{k}} \right]$$

$$\left\langle \hat{\vec{j}}_e \right\rangle_{anomalous} = -2\frac{e^2}{h} \vec{E} \times \int_{\text{filled}} \frac{d\vec{k}}{2\pi} \vec{\Omega}(\vec{k})$$

Independent of scattering!!

Which all terms contribute

- For a filled band, $\sigma_{xy} = -\sigma_{yx} = \dfrac{2e^2}{h}\mathcal{C}$ quantized!!
- This is not the (usual) quantum Hall effect

$$\left\langle \hat{\vec{j}}_e \right\rangle = -e \int \frac{2d\vec{k}}{(2\pi)^2}\left[f_{\vec{k}} \frac{e}{\hbar}(\vec{E}\times\vec{\Omega}(\vec{k})) + g_{\vec{k}}\frac{1}{\hbar}\frac{\partial \varepsilon}{\partial \vec{k}} \right]$$

$$\left\langle \hat{\vec{j}}_e \right\rangle_{anomalous} = -2\frac{e^2}{h}\vec{E}\times \int_{\text{filled}} \frac{d\vec{k}}{2\pi}\vec{\Omega}(\vec{k})$$

Independent of scattering!!

Which all terms contribute

- For a filled band, $\sigma_{xy} = -\sigma_{yx} = \frac{2e^2}{h}\mathcal{C}$ quantized!!
- This is not the (usual) quantum Hall effect
- But we have an issue here

$$\left\langle \hat{\vec{j}}_e \right\rangle = -e \int \frac{2d\vec{k}}{(2\pi)^2} \left[f_{\vec{k}} \frac{e}{\hbar} (\vec{E} \times \vec{\Omega}(\vec{k})) + g_{\vec{k}} \frac{1}{\hbar} \frac{\partial \varepsilon}{\partial \vec{k}} \right]$$

$$\left\langle \hat{\vec{j}}_e \right\rangle_{anomalous} = -2\frac{e^2}{h}\vec{E} \times \int_{\text{filled}} \frac{d\vec{k}}{2\pi} \vec{\Omega}(\vec{k})$$

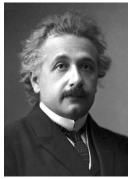
Independent of scattering!!

Issue

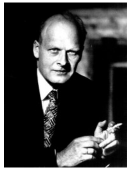
- Apparently, a temp gradient or
  a chemical potential gradient
  cannot give rise to anomalous Hall effect
- No term like $\nabla\mu \times \vec{\Omega}$ and $\nabla T \times \vec{\Omega}$

Issue

- Apparently, a temp gradient or a chemical potential gradient cannot give rise to anomalous Hall effect
- No term like $\nabla\mu \times \vec{\Omega}$ and $\nabla T \times \vec{\Omega}$



☹

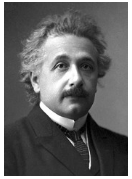
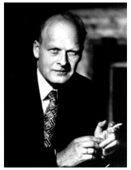


☹

☹

(Nernst)

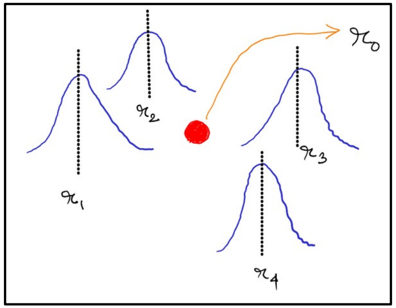
Issue


☹ ☹ ☹

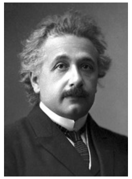
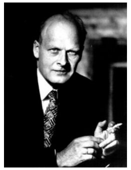
(Nernst)

- Apparently, a temp gradient or
  a chemical potential gradient
  cannot give rise to anomalous Hall effect
- No term like $\nabla \mu \times \vec{\Omega}$ and $\nabla T \times \vec{\Omega}$
- **What we missed:**
- The wavepackets are not localized



Issue

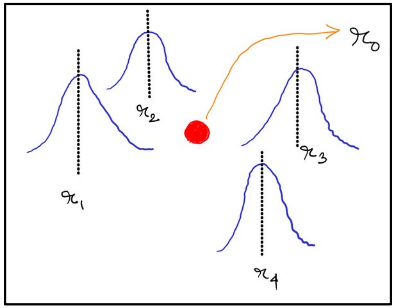
- **Apparently, a temp gradient or a chemical potential gradient cannot give rise to anomalous Hall effect**
- **No term like** $\nabla \mu \times \vec{\Omega}$ **and** $\nabla T \times \vec{\Omega}$

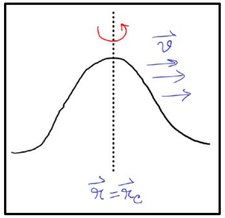




☹ ☹ ☹

(Nernst)

- **What we missed:**
- The wavepackets are not localized
- Circulating magnetization currents



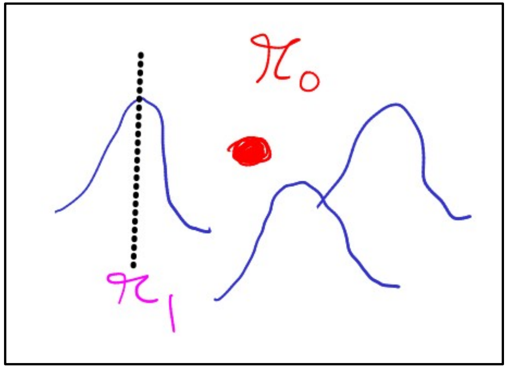


Orbital magnetic moment

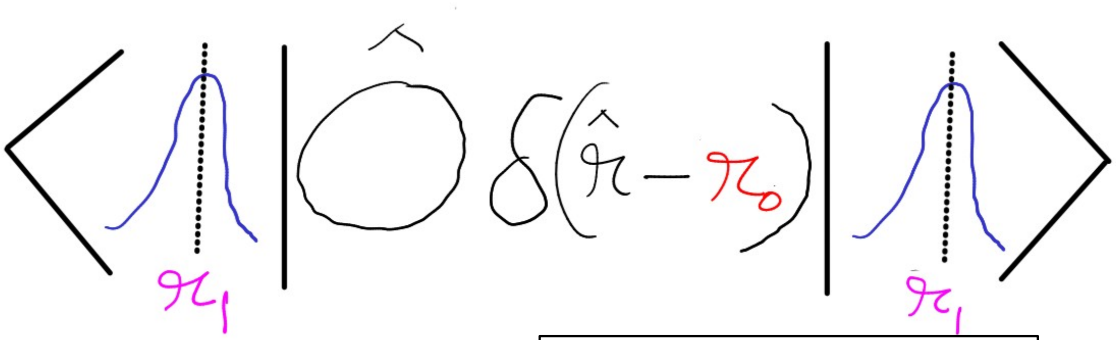
$$\vec{m}_k = -\frac{e}{2m} \langle \psi_{k,r_0} | (\hat{\vec{r}} - \vec{r}_0) \times \hat{\vec{p}} | \psi_{k,r_0} \rangle$$

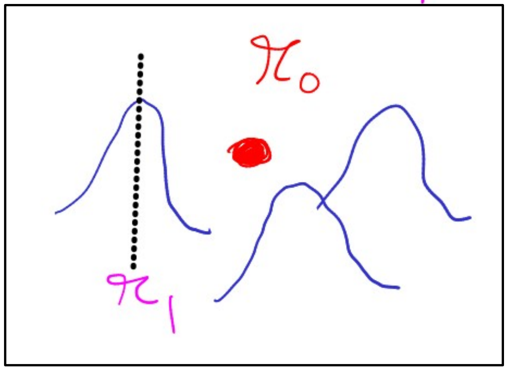
Di Xiao, Yugui Yao, Zhong Fang, and Qian Niu. PRL **97**, 026603 (2006)

Wavepackets are not localized



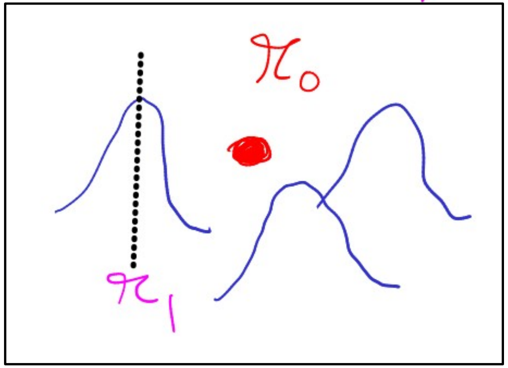
Wavepackets are not localized





Wavepackets are not localized

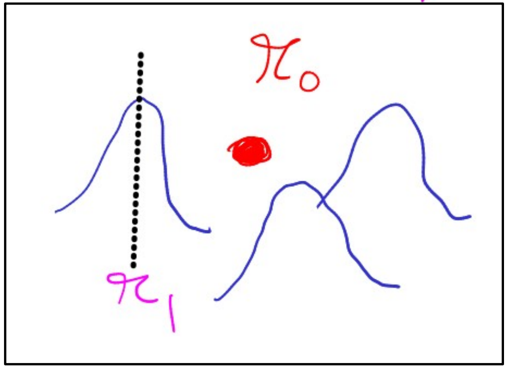
$$\langle \hat{O} \rangle (r_0) = \int_{r_1} \int_{k} \left\langle r_1 \right| \hat{O}\, \delta(\hat{r} - r_0) \left| r_1 \right\rangle$$



Wavepackets are not localized

$$\left\langle \hat{O} \right\rangle (r_0) = \int_{r_1} \int_{k} \left\langle \, r_1 \, \middle| \, \hat{O}\, \delta(\hat{r} - r_0) \, \middle| \, r_1 \, \right\rangle$$

- To be technically correct, we should use the operator $\frac{\hat{O}\delta(\hat{r}-\vec{r}_0)+\delta(\hat{r}-\vec{r}_0)\hat{O}}{2}$ in case $\hat{O}$ does not commute with $\hat{r}$



- To calculate electric current, we use $\hat{O} = \frac{-e\hat{p}}{m}$

$$\left\langle \hat{\vec{j}}_e \right\rangle\Big|_{\vec{B}=0} = -e \int \frac{2d\vec{k}}{(2\pi)^d} g_{\vec{k}} \frac{1}{\hbar} \frac{\partial \varepsilon_0(\vec{k})}{\partial \vec{k}} + f_{\vec{k}} \frac{e}{\hbar} (\vec{E} \times \vec{\Omega}(\vec{k})) + \nabla \times \int \frac{2d\vec{k}}{(2\pi)^d} f_{\vec{k}} \vec{m}_{\vec{k}}$$

- Similarly, for heat current

$$\left\langle \hat{\vec{j}}_Q \right\rangle = \int \frac{2d\vec{k}}{(2\pi)^d} (\varepsilon_0(\vec{k}) - \mu) \left[g_{\vec{k}} \frac{1}{\hbar} \frac{\partial \varepsilon_0(\vec{k})}{\partial \vec{k}} + \frac{e}{\hbar} f_{\vec{k}} (\vec{E} \times \vec{\Omega}(\vec{k})) \right] - \vec{E} \times \int \frac{2d\vec{k}}{(2\pi)^d} f_{\vec{k}} \vec{m}_{\vec{k}}$$

Effects of orbital magnetization

- Energy eigenvalues are modified $\varepsilon_{\vec{k}} = \varepsilon_0(\vec{k}) - \vec{m}_{\vec{k}} \cdot \vec{B}$

Effects of orbital magnetization

- Energy eigenvalues are modified $\varepsilon_{\vec{k}} = \varepsilon_0(\vec{k}) - \vec{m}_{\vec{k}} \cdot \vec{B}$
- Net magnetization can be obtained from the Free energy

Effects of orbital magnetization

- Energy eigenvalues are modified $\varepsilon_{\vec{k}} = \varepsilon_0(\vec{k}) - \vec{m}_{\vec{k}} \cdot \vec{B}$
- Net magnetization can be obtained from the Free energy

$$G = -\frac{1}{\beta} \sum_{\vec{k}} \log\left(1 + e^{-\beta(\varepsilon_{\vec{k}} - \vec{m}_{\vec{k}} \cdot \vec{B} - \mu)}\right)$$

$$= -\frac{1}{\beta} \int \frac{d\vec{k}}{(2\pi)^d} \left(1 + \frac{e}{\hbar}\vec{B} \cdot \vec{\Omega}\right) \log\left(1 + e^{-\beta(\varepsilon_{\vec{k}} - \vec{m}_{\vec{k}} \cdot \vec{B} - \mu)}\right)$$

Effects of orbital magnetization

- Energy eigenvalues are modified $\varepsilon_{\vec{k}} = \varepsilon_0(\vec{k}) - \vec{m}_{\vec{k}} \cdot \vec{B}$
- Net magnetization can be obtained from the Free energy

$$G = -\frac{1}{\beta}\sum_{\vec{k}} \log\left(1 + e^{-\beta(\varepsilon_{\vec{k}} - \vec{m}_{\vec{k}}\cdot\vec{B} - \mu)}\right)$$

$$= -\frac{1}{\beta}\int \frac{d\vec{k}}{(2\pi)^d}\left(1 + \frac{e}{\hbar}\vec{B}\cdot\vec{\Omega}\right)\log\left(1 + e^{-\beta(\varepsilon_{\vec{k}} - \vec{m}_{\vec{k}}\cdot\vec{B} - \mu)}\right)$$

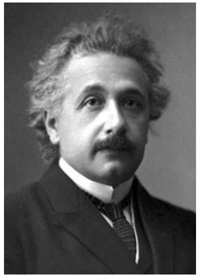
$$\vec{M}^e = -\left.\frac{\partial G}{\partial \vec{B}}\right|_{\vec{B}=0} = \frac{1}{\beta}\int \frac{2d\vec{k}}{(2\pi)^d}\frac{e}{\hbar}\vec{\Omega}(\vec{k})\log\left(1 + e^{-\beta(\varepsilon_0(\vec{k}) - \mu)}\right) + \int \frac{2d\vec{k}}{(2\pi)^d} f \vec{m}_{\vec{k}}$$

Phase space density correction is important

Transport current

Now we are ready to calculate the transport electric current

$$
\begin{aligned}
\vec{j}^{e}_{\text{transport}} &= \left\langle \hat{\vec{j}}_e \right\rangle - \nabla \times \vec{M}^e \\
&= \int \frac{2d\vec{k}}{(2\pi)^d}(-e)g_{\vec{k}}\frac{1}{\hbar}\frac{\partial \varepsilon_0(\vec{k})}{\partial \vec{k}} - f_{\vec{k}}\frac{e^2}{\hbar}\left(\left[\vec{E} + \frac{\nabla \mu}{e}\right] \times \vec{\Omega}(\vec{k})\right) \\
&- \frac{\nabla T}{T} \times \int \frac{2d\vec{k}}{(2\pi)^d}\frac{e}{\hbar}\vec{\Omega}(\vec{k})\left[f_{\vec{k}}(\varepsilon_0(\vec{k}) - \mu) + k_B T \log\left(1 + e^{-\beta(\varepsilon_0(\vec{k})-\mu)}\right)\right]
\end{aligned}
$$



How about Onsager relation?

- We can define energy magnetization, $\vec{M}^E = \vec{M}_0^E + \underbrace{\phi(\vec{r})}_{\text{electric potential energy}} \vec{M}^N$

N. R. Cooper, B. I. Halperin, and I. M. Ruzin. PRB **55** 2344 (1997)

How about Onsager relation?

- We can define energy magnetization, $\vec{M}^E = \vec{M}_0^E + \underbrace{\phi(\vec{r})}_{\text{electric potential energy}} \vec{M}^N$

$$
\begin{aligned}
\vec{j}^Q_{\text{transport}} &= \vec{j}^E_{\text{transport}} - \mu \vec{j}^N_{\text{transport}} = \vec{j}^E_{\text{transport}} - \frac{\mu}{-e} \vec{j}^e_{\text{transport}} \\
&= \int \frac{2d\vec{k}}{(2\pi)^d} (\varepsilon_0(\vec{k}) - \mu) \frac{1}{\hbar} \frac{\partial \varepsilon_0(\vec{k})}{\partial \vec{k}} g_k \\
&+ \vec{E} \times \int \frac{2d\vec{k}}{(2\pi)^d} \left[(\varepsilon_0(\vec{k}) - \mu) \frac{e}{\hbar} \vec{\Omega} f_k + k_B T \frac{e\vec{\Omega}}{\hbar} \log\left(1 + e^{-\beta(\varepsilon_0(\vec{k}) - \mu)}\right) \right] \\
&+ \int \frac{2d\vec{k}}{(2\pi)^d} (-\mu) \frac{1}{\hbar} \nabla \mu \times \vec{\Omega} f_k \\
&- \frac{\mu}{\hbar} \int \frac{2d\vec{k}}{(2\pi)^d} \frac{\nabla T}{T} \times \vec{\Omega} \left[f_k \left(\varepsilon_0(\vec{k})_k - \mu \right) + k_B T \log\left(1 + e^{-\beta(\varepsilon_0(\vec{k}) - \mu)}\right) \right] \\
&- \nabla \mu \times \frac{\partial \vec{M}_0^E}{\partial \mu} - \nabla T \times \frac{\partial \vec{M}_0^E}{\partial T}
\end{aligned}
$$

N. R. Cooper, B. I. Halperin, and I. M. Ruzin. PRB **55** 2344 (1997)

How about Onsager relation?

- We can define energy magnetization, $\vec{M}^E = \vec{M}_0^E + \underbrace{\phi(\vec{r})}_{\text{electric potential energy}} \vec{M}^N$

$$
\begin{aligned}
\vec{j}^{Q}_{\text{transport}} &= \vec{j}^{E}_{\text{transport}} - \mu \vec{j}^{N}_{\text{transport}} = \vec{j}^{E}_{\text{transport}} - \frac{\mu}{-e}\vec{j}^{e}_{\text{transport}} \\
&= \int \frac{2d\vec{k}}{(2\pi)^d}(\varepsilon_0(\vec{k}) - \mu)\frac{1}{\hbar}\frac{\partial \varepsilon_0(\vec{k})}{\partial \vec{k}} g_k \\
&+ \vec{E} \times \int \frac{2d\vec{k}}{(2\pi)^d}\left[(\varepsilon_0(\vec{k}) - \mu)\frac{e}{\hbar}\vec{\Omega} f_k + k_B T \frac{e\vec{\Omega}}{\hbar}\log\left(1 + e^{-\beta(\varepsilon_0(\vec{k})-\mu)}\right)\right] \\
&+ \int \frac{2d\vec{k}}{(2\pi)^d}(-\mu)\frac{1}{\hbar}\nabla\mu \times \vec{\Omega} f_k \\
&- \frac{\mu}{\hbar}\int \frac{2d\vec{k}}{(2\pi)^d}\frac{\nabla T}{T} \times \vec{\Omega}\left[f_k\left(\varepsilon_0(\vec{k})_k - \mu\right) + k_B T \log\left(1 + e^{-\beta(\varepsilon_0(\vec{k})-\mu)}\right)\right] \\
&- \nabla\mu \times \frac{\partial \vec{M}_0^E}{\partial \mu} - \nabla T \times \frac{\partial \vec{M}_0^E}{\partial T}
\end{aligned}
$$

Di Xiao, Yugui Yao, Zhong Fang, and Qian Niu. PRL **97**, 026603 (2006)

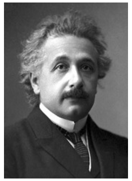
N. R. Cooper, B. I. Halperin, and I. M. Ruzin. PRB **55** 2344 (1997)

How about Onsager relation?

- We can define energy magnetization, $\vec{M}^E = \vec{M}_0^E + \underbrace{\phi(\vec{r})}_{\text{electric potential energy}} \vec{M}^N$

$$
\begin{aligned}
\vec{j}^{Q}_{\mathrm{transport}} &= \vec{j}^{E}_{\mathrm{transport}} - \mu \vec{j}^{N}_{\mathrm{transport}} = \vec{j}^{E}_{\mathrm{transport}} - \frac{\mu}{-e}\vec{j}^{e}_{\mathrm{transport}} \\
&= \int \frac{2d\vec{k}}{(2\pi)^d}(\varepsilon_0(\vec{k}) - \mu)\frac{1}{\hbar}\frac{\partial \varepsilon_0(\vec{k})}{\partial \vec{k}} g_k \\
&+ \vec{E} \times \int \frac{2d\vec{k}}{(2\pi)^d}\left[(\varepsilon_0(\vec{k}) - \mu)\frac{e}{\hbar}\vec{\Omega} f_k + k_B T \frac{e\vec{\Omega}}{\hbar}\log\left(1 + e^{-\beta(\varepsilon_0(\vec{k})-\mu)}\right)\right] \\
&+ \int \frac{2d\vec{k}}{(2\pi)^d}(-\mu)\frac{1}{\hbar}\nabla\mu \times \vec{\Omega} f_k \\
&- \frac{\mu}{\hbar}\int \frac{2d\vec{k}}{(2\pi)^d}\frac{\nabla T}{T} \times \vec{\Omega}\left[f_k\left(\varepsilon_0(\vec{k})_k - \mu\right) + k_B T \log\left(1 + e^{-\beta(\varepsilon_0(\vec{k})-\mu)}\right)\right] \\
&- \nabla\mu \times \frac{\partial \vec{M}_0^E}{\partial \mu} - \nabla T \times \frac{\partial \vec{M}_0^E}{\partial T}
\end{aligned}
$$

Di Xiao, Yugui Yao, Zhong Fang, and Qian Niu. PRL **97**, 026603 (2006)



N. R. Cooper, B. I. Halperin, and I. M. Ruzin. PRB **55** 2344 (1997)

Possible resolution for the Einstein relation to hold

$$\boxed{-\frac{\partial \vec{M}_0^E}{\partial \mu} = \int \frac{2d\vec{k}}{(2\pi)^d}\frac{e\vec{\Omega}}{\hbar}\left[\varepsilon_0(\vec{k})f_k + k_B T\log\left(1+e^{-\beta(\varepsilon_0(\vec{k})-\mu)}\right)\right]}$$

$$\approx \int \frac{2d\vec{k}}{(2\pi)^d}\frac{e\vec{\Omega}}{\hbar}\mu f_k \underbrace{=}_{2D} \frac{2e\mu}{h}\left(\sum_{fully filled} \mathcal{C}_n + \int_{partially filled}\frac{\Omega_z}{2\pi}\right)$$

- Interpretation ?? Along the lines of Raffaele Resta 2010 J. Phys.: Condens. Matter 22 123201
- Zero for trivial states, and bound energy current is due to topological states, as a response to change in chemical potential

Possible resolution for the Einstein relation to hold

$$\boxed{-\frac{\partial \vec{M}_0^E}{\partial \mu} = \int \frac{2d\vec{k}}{(2\pi)^d}\frac{e\vec{\Omega}}{\hbar}\left[\varepsilon_0(\vec{k})f_k + k_B T \log\left(1 + e^{-\beta(\varepsilon_0(\vec{k})-\mu)}\right)\right]}$$

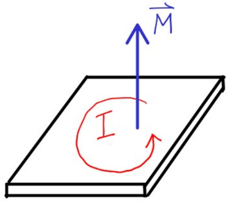
$$\approx \int \frac{2d\vec{k}}{(2\pi)^d}\frac{e\vec{\Omega}}{\hbar}\mu f_k \underbrace{=}_{2D} \frac{2e\mu}{h}\Big(\sum_{fullyfilled} \mathcal{C}_n + \int_{partiallyfilled} \frac{\Omega_z}{2\pi}\Big)$$

- Interpretation ?? Along the lines of Raffaele Resta 2010 J. Phys.: Condens. Matter 22 123201
- Zero for trivial states, and bound energy current is due to topological states, as a response to change in chemical potential

$$\frac{\Delta I_0^E}{\Delta \mu} = \frac{\mu}{e}\left(2\frac{e^2}{h}\right)\#\text{Topological states}$$

$$I_0^E = \left|\vec{M}_0^E \times \hat{n}\right|$$

Edge and bulk



Berry curvature and Chern numbers of some microscopic models

- In a two band system with Hamiltonian $H = d_x\sigma_x + d_y\sigma_y + d_z\sigma_z$

Berry curvature and Chern numbers of some microscopic models

- In a two band system with Hamiltonian $H = d_x\sigma_x + d_y\sigma_y + d_z\sigma_z$

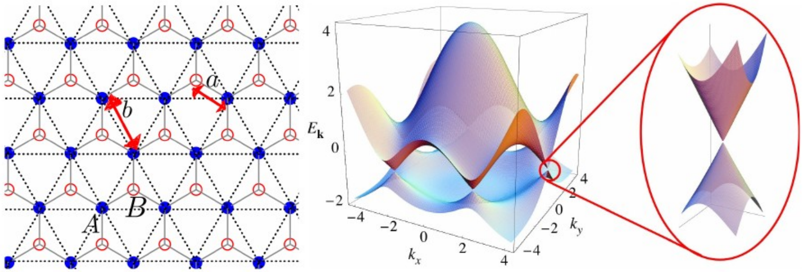
$$\Omega_z = i\int d\vec{r}\nabla_{\vec{k}_x} u^*_{\vec{k}} \times \nabla_{\vec{k}_y} u_{\vec{k}} = \frac{1}{2}\hat{d}\cdot\left(\frac{\partial\hat{d}}{\partial k_x}\times\frac{\partial\hat{d}}{\partial k_y}\right)$$

Berry curvature and Chern numbers of some microscopic models

- In a two band system with Hamiltonian $H = d_x\sigma_x + d_y\sigma_y + d_z\sigma_z$

$$\Omega_z = i\int d\vec{r}\nabla_{\vec{k}_x} u^*_{\vec{k}} \times \nabla_{\vec{k}_y} u_{\vec{k}} = \frac{1}{2}\hat{d}\cdot\left(\frac{\partial\hat{d}}{\partial k_x}\times\frac{\partial\hat{d}}{\partial k_y}\right)$$

- Monolayer graphene

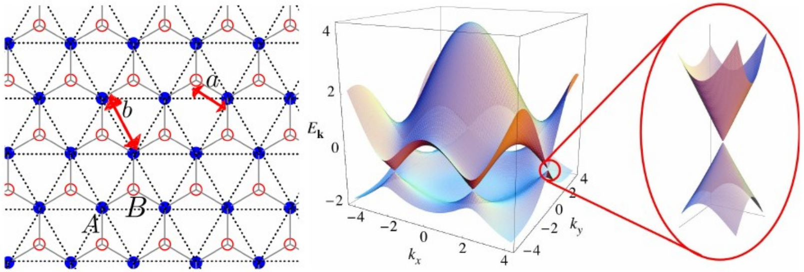


Berry curvature and Chern numbers of some microscopic models

- In a two band system with Hamiltonian $H = d_x\sigma_x + d_y\sigma_y + d_z\sigma_z$

$$\Omega_z = i\int d\vec{r}\nabla_{\vec{k}_x} u^*_{\vec{k}} \times \nabla_{\vec{k}_y} u_{\vec{k}} = \frac{1}{2}\hat{d}\cdot\left(\frac{\partial\hat{d}}{\partial k_x}\times\frac{\partial\hat{d}}{\partial k_y}\right)$$

- Monolayer graphene



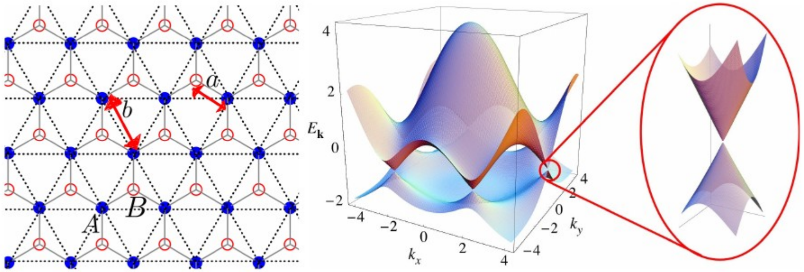
Non-zero Berry curvature when there is a finite band gap : growing on BN substrate

Berry curvature and Chern numbers of some microscopic models

- In a two band system with Hamiltonian $H = d_x\sigma_x + d_y\sigma_y + d_z\sigma_z$

$$\Omega_z = i\int d\vec{r}\nabla_{\vec{k}_x} u^*_{\vec{k}} \times \nabla_{\vec{k}_y} u_{\vec{k}} = \frac{1}{2}\hat{d}\cdot\left(\frac{\partial\hat{d}}{\partial k_x}\times\frac{\partial\hat{d}}{\partial k_y}\right)$$

- Monolayer graphene



Non-zero Berry curvature when there is a finite band gap : growing on BN substrate

However, there is Time Reversal Symmetry, so $\vec{\Omega}(\vec{k}) = -\vec{\Omega}(-\vec{k}), \mathcal{C} = 0$

Valley Polarization

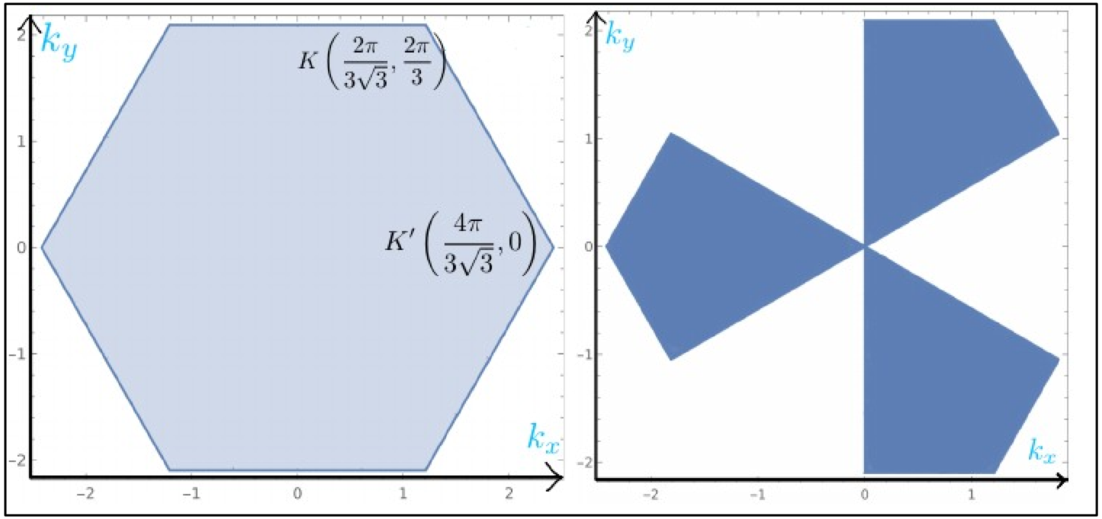
- **Electrons can be made to selectively occupy valleys with circularly polarized light**

A. Friedlan and M. M. Dignam PRB **103**, 075414 (2021)

Valley Polarization

- **Electrons can be made to selectively occupy valleys with circularly polarized light**

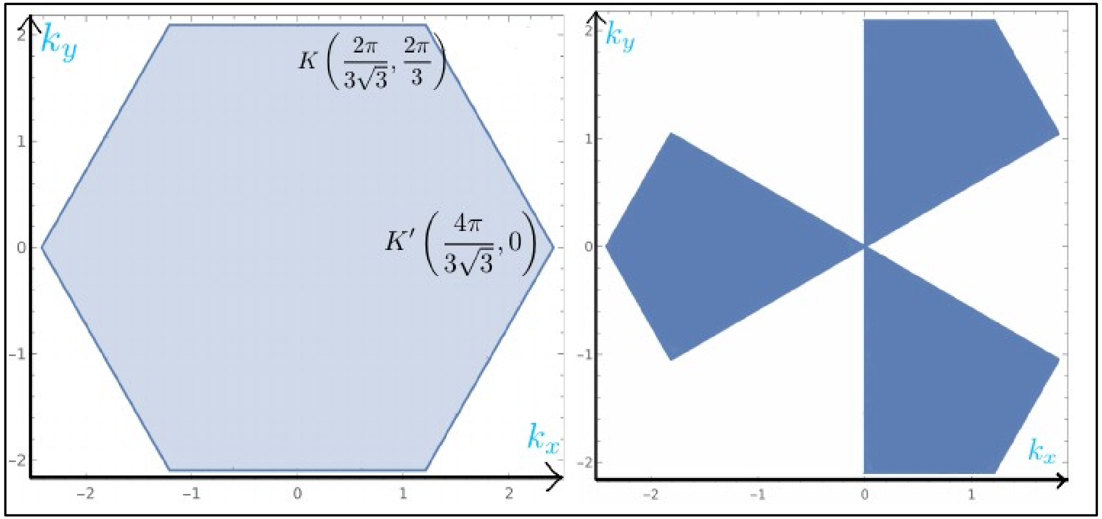
A. Friedlan and M. M. Dignam PRB **103**, 075414 (2021)

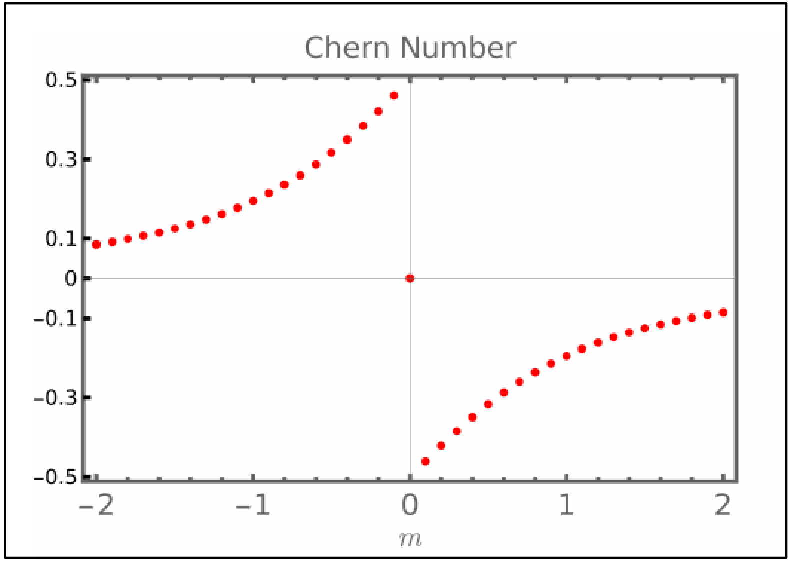


Valley Polarization

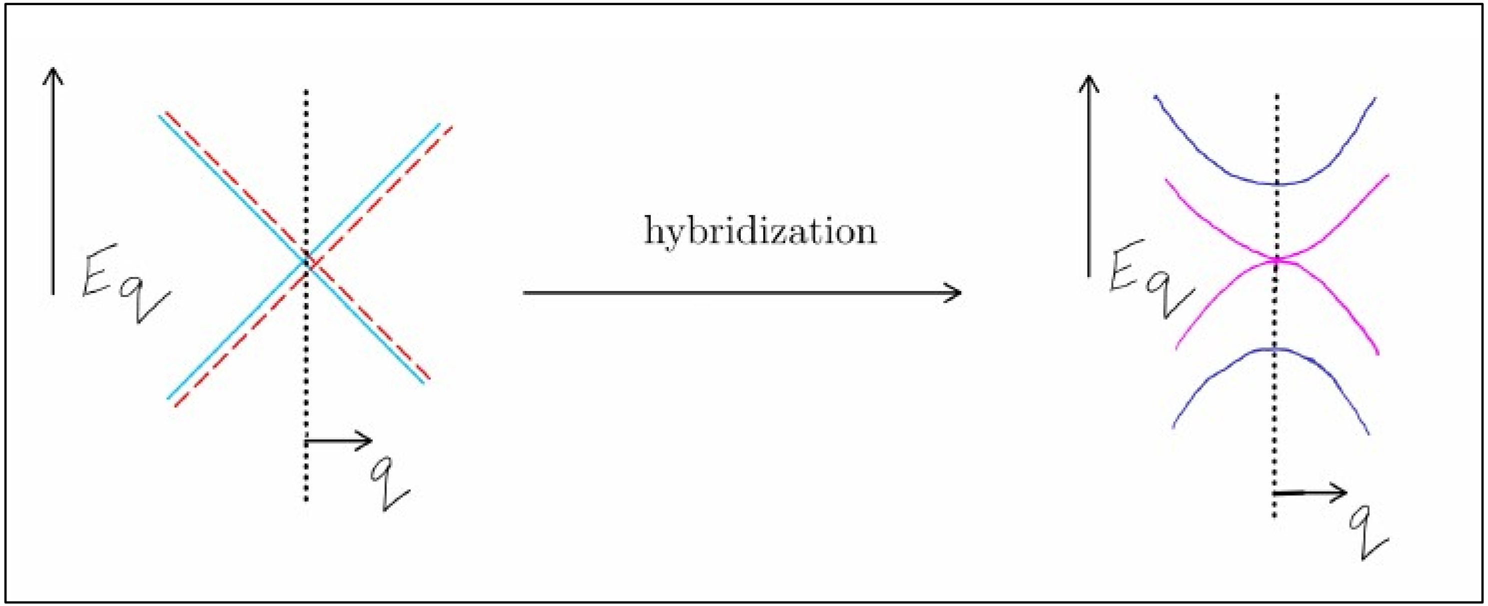
- **Electrons can be made to selectively occupy valleys with circularly polarized light**

A. Friedlan and M. M. Dignam PRB **103**, 075414 (2021)

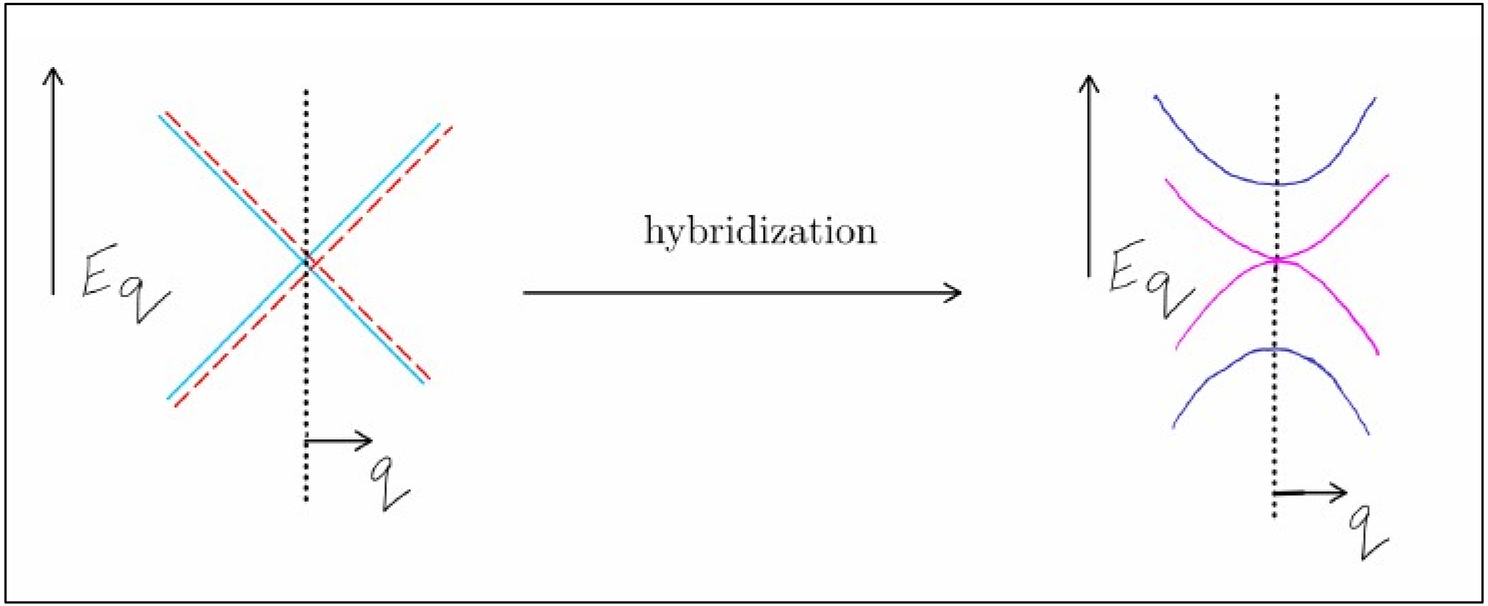




Bilayer Graphene

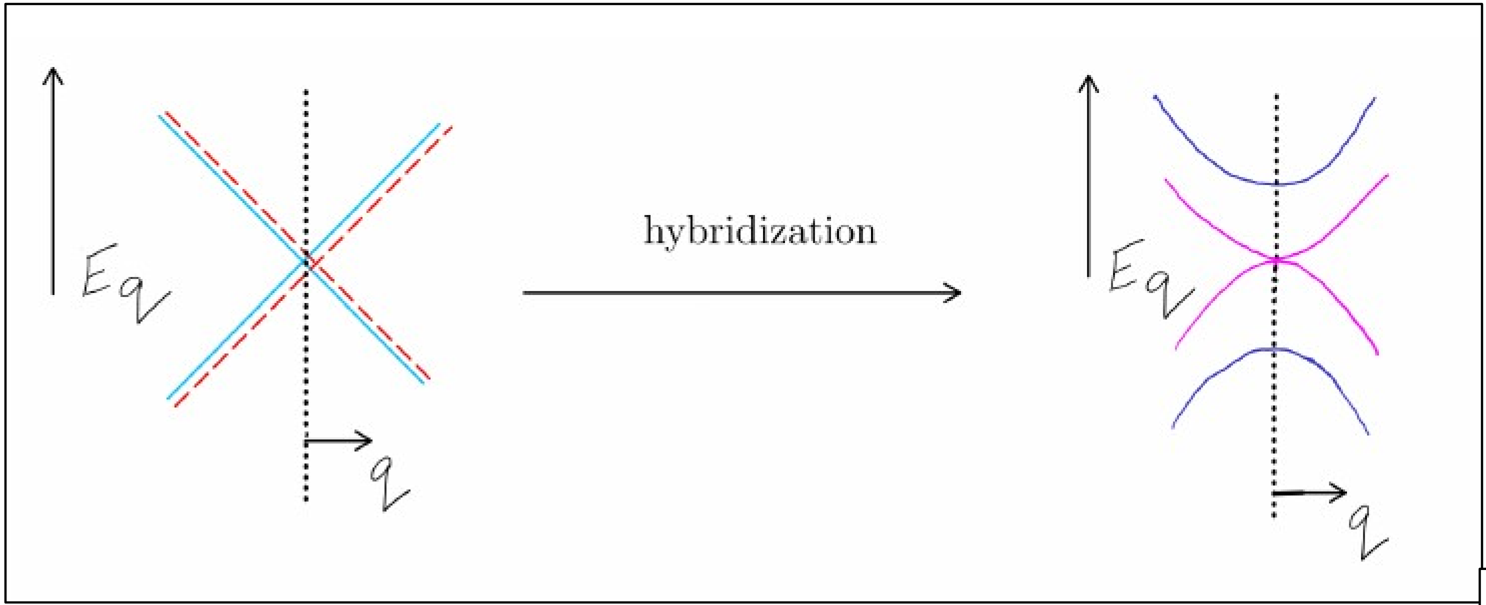


Bilayer Graphene

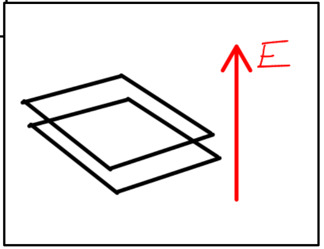


- We can generate a band gap by applying an electric field

Bilayer Graphene



- We can generate a band gap by applying an electric field



Results for valley Chern number

		$\vec{\Omega}$	C
	$E=K$	0	0
	$E=\sqrt{K^2+m^2}$	$\neq 0$	$\pm\frac{1}{2}$
	$E=K^2$	0	0
	$E=\sqrt{K^4+\Delta^2}$	$\neq 0$	± 1
	$E=\sqrt{k^2+\alpha^2 k^{4n}}$	$\neq 0$	$-\frac{1}{2},\ n>\frac{1}{2}$ $+\frac{1}{2},\ n<\frac{1}{2}$

Results for valley Chern number

		$\vec{\Omega}$	C
	$E=k$	0	0
	$E=\sqrt{k^2+m^2}$	$\neq 0$	$\pm\frac{1}{2}$
	$E=k^2$	0	0
	$E=\sqrt{k^4+\Delta^2}$	$\neq 0$	± 1
	$E=\sqrt{k^2+\alpha^2 k^{4n}}$	$\neq 0$	$-\frac{1}{2},\ n>\frac{1}{2}$ $+\frac{1}{2},\ n<\frac{1}{2}$

$$H(\vec{k}) = k_x\sigma_x + k_y\sigma_y + \alpha(k_x^2+k_y^2)^n\sigma_z$$

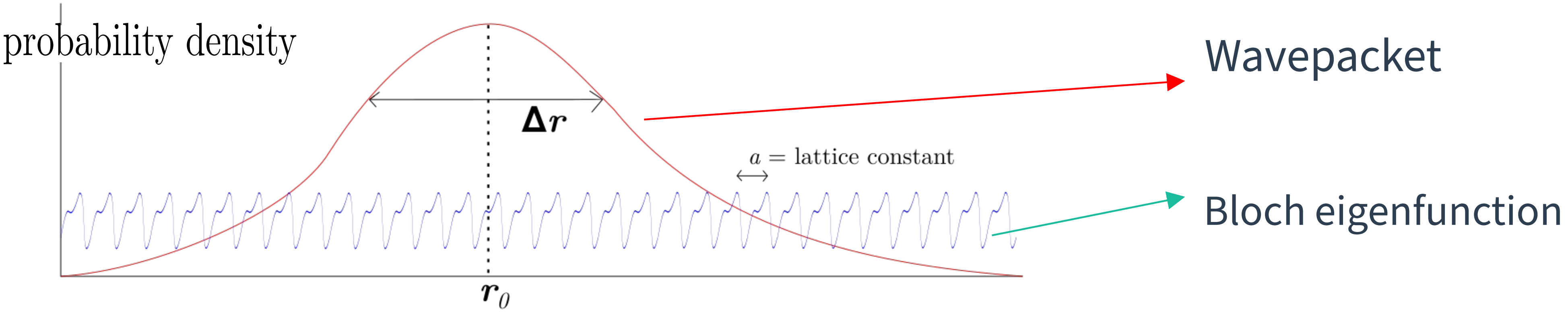
New results

- The condition on the energy magnetization for Einstein relation to hold for anomalous Heat current

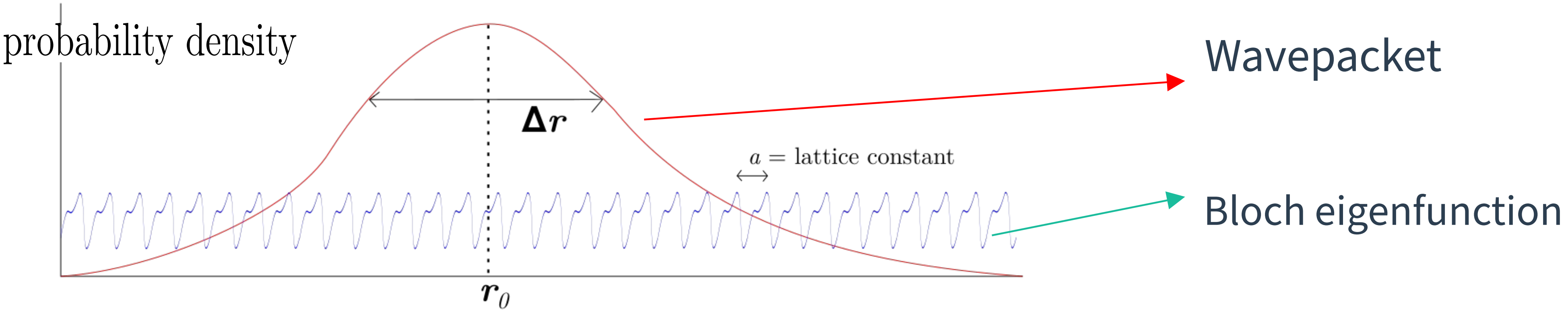
- Solution of Boltzmann transport equation upto linear order, that captures the effects of Berry curvature in regular Ohmic, regular Hall and regular Nernst effects

Thank you

Appendix: Construction of Bloch wavepacket, and its evolution

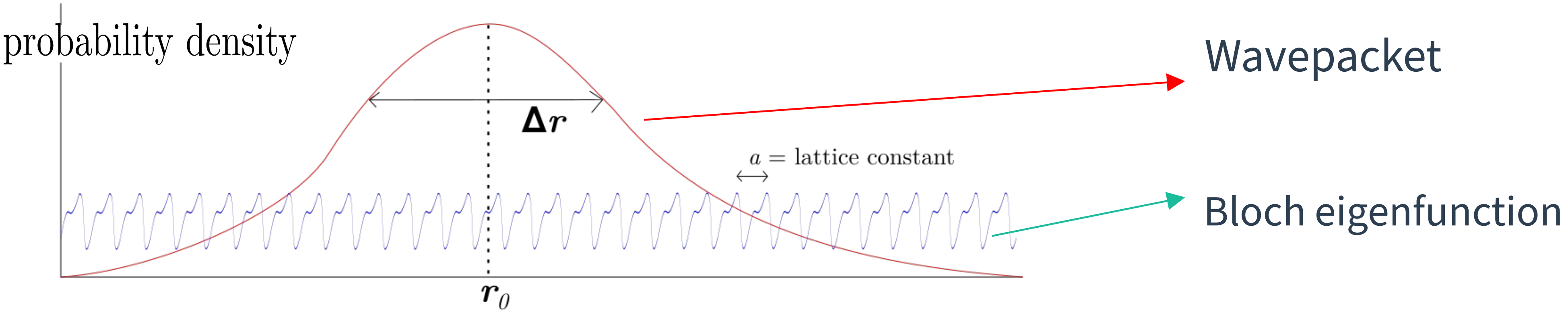


Appendix: Construction of Bloch wavepacket, and its evolution



$$\psi(\vec{r}, t=0) \underbrace{=}_{?} \int d\vec{k} w(\vec{k}-\vec{k}_0) e^{-i\vec{k}\cdot\vec{r}_0} \left(e^{i\vec{k}\cdot\vec{r}} u_{n,\vec{k}}(\vec{r}) \right)$$

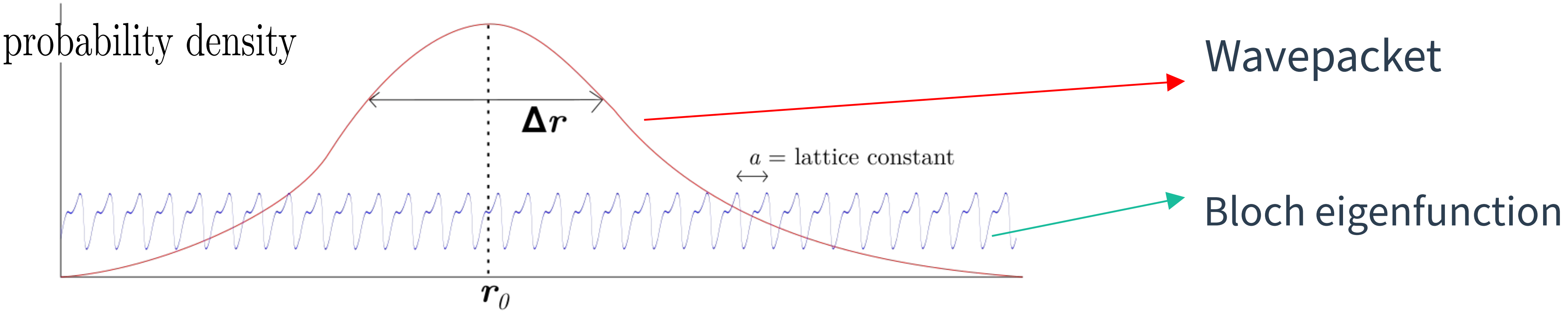
Appendix: Construction of Bloch wavepacket, and its evolution



$$\psi(\vec{r}, t=0) \underbrace{=}_{?} \int d\vec{k} w(\vec{k}-\vec{k}_0) e^{-i\vec{k}\cdot\vec{r}_0} \left(e^{i\vec{k}\cdot\vec{r}} u_{n,\vec{k}}(\vec{r}) \right)$$

$$\psi(\vec{r}, t=0) = \int d\vec{k} w(\vec{k}-\vec{k}_0) e^{i\vec{A}(\vec{k_0})\cdot(\vec{k}-\vec{k}_0)} e^{-i\vec{k}\cdot\vec{r}_0} \left(e^{i\vec{k}\cdot\vec{r}} u_{n,\vec{k}}(\vec{r}) \right)$$

Appendix: Construction of Bloch wavepacket, and its evolution



$$\psi(\vec{r}, t=0) \underbrace{=}_{?} \int d\vec{k}\, w(\vec{k}-\vec{k}_0)\, e^{-i\vec{k}\cdot\vec{r}_0} \left(e^{i\vec{k}\cdot\vec{r}} u_{n,\vec{k}}(\vec{r})\right)$$

$$\psi(\vec{r}, t=0) = \int d\vec{k}\, w(\vec{k}-\vec{k}_0)\, e^{i\vec{A}(\vec{k}_0)\cdot(\vec{k}-\vec{k}_0)} e^{-i\vec{k}\cdot\vec{r}_0} \left(e^{i\vec{k}\cdot\vec{r}} u_{n,\vec{k}}(\vec{r})\right)$$

$$\psi(\vec{r}, t=\Delta t)$$

$$\left(e^{i(\vec{k}+\Delta\vec{k})\cdot\vec{r}} e^{i\vec{A}\cdot\Delta\vec{k}} e^{-i\frac{\varepsilon_{\vec{k}}\Delta t}{\hbar}} u_{n,\vec{k}+\Delta\vec{k}}\right)$$