DUALITY AND SENSITIVITY ANALYSIS:

Duality:

The Duality in linear programming problem (LPP) states that:

“For every linear programming problem there is corresponding linear programming problem called the dual”

The core linear problem is referred to as "Primal," while the derived linear problem is referred to as "Dual." The concept of duality, often known as the duality principle, is used in mathematical optimization theory to describe the possibility of viewing optimization issues from either the dual problem or the primal problem. The dual problem is a maximizing problem if the primal problem is a minimization problem (and vice versa). Any practical solution to the primary problem of minimizing is at least as big as any practical solution to the secondary problem of maximization. As a result, the dual solution is a lower bound to the primal solution and the dual solution is an upper bound to the primal solution. Weak duality is the term used to describe this fact.

In general, it's not necessary for the primal and dual problems' optimal values to be equal. The duality gap refers to their distinction. Under a constraint qualifying condition, the duality gap for convex optimization problems is equal to zero. Strong duality describes this reality.

Before attempting to solve for duality, the original linear programming problem must be expressed in its standard form. Standard form requires that all the variables in the problem be non-negative and that the "≥" sign be used for minimization and the "≤" sign for maximization.

Relationship between primal and dual problem:

There is a route or subspace of directions to move from each sub-optimal position that meets all the constraints in the linear case of the primal problem that raises the objective function. The removal of slack between the candidate solution and one or more restrictions is said to occur when moving in any of these directions. Any value of the proposed solution that exceeds one or more of the constraints is infeasible.

The dual vector in the dual problem multiplies the constraints that determine where the restrictions are in the primal problem. It is analogous to changing the upper bounds in the primal problem to change the dual vector in the dual problem. One seeks the lowest upper bound. In other words, the dual vector is minimized to close the gap between the constraints' potential places and the true optimum. A dual vector value that is too low is unfeasible. It arranges the candidates for one or more of the restrictions so that they are outside of the true optimum.

Example:

The concept of Duality can be well understood through a problem given below:

Maximize $Z = 40x_1+20x_2$

Subject to:
$2x_1 + 3x_2 \leq 200$
$3x_1 + 4x_2 \leq 350$
$X_1, x_2 \geq 0$

The duality can be applied to the above original linear programming problem as:

Minimize

$G = 200y_1+350y_2$

Subject to:

$2y_1 + 3y_1 \geq 40$
$3y_1 +4y_2 \geq 20$
$Y_1, y_2 \geq 0$

While developing the dual linear programming problem, the following observations were made:

1) The dual problem is of the minimization type, whereas the primary or original linear programming issue is of the maximizing type.

2) The variables' coefficients in the objective function of a dual problem have changed into the constraint value in the dual problem, while the constraint values 200 and 350 of the primal problem have changed into the coefficient of dual variables y1 and y2 in the dual problem.
3) The first column in the constraint inequality of a primal problem has been transformed into the first row in a dual problem, and vice versa for the second column.
4) The inequalities' orientations have also shifted, with the dual problem's sign being the opposite of a primal problem. The inequality sign in the dual problem now reads "≥" even though it read "≤" in the primary problem.

Advantages:

1) It produces a significant number of effective theorems

2) Sometimes it's simpler to solve the dual. Converting a primal problem into a dual problem can drastically shorten the computational process if the primal problem has many rows (constraints) and few columns (variables).

3) The dual solution verifies the accuracy of the primal solution by looking for computational mistakes.

4) Show that linear programming duality and the relationship between them are quite close.

5) Management will be able to make better decisions going forward by using the duals economic interpretation.

6) This is beneficial for sensitivity analysis.

7) The shadow pricing for the primal constraints is provided by the dual variables.

Sensitivity Theorem:

Sensitivity analysis is a methodical investigation into how responsive (obviously) solutions are to (minor) alterations in the data. The main goal is to be able to respond to inquiries of the following type:

1. How does the solution change, if the objective function changes?

2. How does the strategy alter, if the resources available vary?

3. How does the approach alter, if a constraint is added to the issue?

Example:

Maximize Z= $5x_1 + 4x_2$

Subject to

$$6x_1 + 4x_2 \leq 24,$$

$$x_1 + 2x_2 \leq 6,$$

$$x_2 - x_1 \leq 1,$$

$$x_2 \leq 2$$

$$x_1, x_2 \geq 0.$$

The raw material quantities 1 and 2, the market limit, and the daily demand were all depicted on the right-hand side. Now, if these right-hand sides were altered, the entire issue would be resolved. Let's say we want to know how much a specific resource is worth. More specifically, we want to know how significant it is that there are 24 units of the initial raw material accessible. Our best value shifts from 21 to 21.75 if we increase the quantity from 24 to 25. Previously, it was 21. Therefore, for every unit more of the first resource, the optimal value, which is the total profit, changes by 0.75. As a result, this might be considered the unit worth of the initial resources. This is referred to as the initial resources' shadow price in technical term.

Shadow Price:

The rate of improvement in the value of the optimal objective function, such as Z in maximizing profit or C in minimizing cost, when RHS of a particular constraint increases with all other variables held constant is referred to as the shadow price of that constraint. For a maximization model, "rate of improvement" refers to the rate of increase, whereas for a minimization model, it refers to the rate of decrease. The shadow price is the rate at which Z (or C) is affected if the RHS is reduced.

Conclusion:

The fact that the solutions to the variables of a dual problem may be used to determine the shadow pricing of the primal problem, duality aids businesses in understanding how an increase in profit can be interpreted by raising the value of the resources (constraints) by one unit. Sensitivity enables them to comprehend how changes to the objective function's parameters affect its values.

https://prinsli.com/duality-in-linear-programming/

https://en.wikipedia.org/wiki/Duality_(optimization)

https://en.wikibooks.org/wiki/Operations_Research/Sensitivity_analysis

https://econweb.ucsd.edu/~jsobel/172aw02/notes7.pdf