Picking the Best Players for Fantasy Football

The basic problem of picking a fantasy football team is roughly as follows:

- You must pick 16 players from given lists of players of various types, Quarterbacks (QB), Running backs (RB), Wide receivers (WR), Tight ends (TE), Defense/Special teams (D/ST) and Place kickers (K).
- Each player makes points throughout the season for successful plays, such as touch downs, rushing yards and receiving yards. Players also lose points for interceptions thrown or fumbles. Thus each player has a value or a potential value.
- Players are recruited with a draft system where teams make sequential choices from the roster. Trading players during the season is sometimes allowed.
- Each player has a cost.
- As a team manager, you have a maximum amount to spend, so the total cost of your players must be less than that specific amount.
- Thus you must maximize the number of points your team will earn, subject to constraints on budget and on the number and availability of players.

How Valuable is a player?

We will start with the problem of determining how valuable a player is.

- There are two ways of doing this:
- We might just be interested in determining whether one player is more valuable than another, in which case we would be satisfied with a ranking or an ordering of players according to their value without actually stating a specific value for the players.
- On the other hand, we might want to come up with a specific numerical value for each player such as a formula that gives us a prediction of the number of points each player will earn next season.
- We will first look at some different ways to amalgamate existing rankings from voting theory and then we will look at how to use simple linear regression to make predictions.
- Lastly, we will look take a quick look at the optimization problem of maximizing the number of points your team will earn subject to budget constraints. This problem is known as the knapsack problem in optimization theory.

Ranking Players

Ranking players is useful when making choices among the available players and also may help to identify undervalued players.

- There are a number of methods of amalgamating statistics to produce a ranking for players which are beyond the scope of course. Many are discussed in the book *"Who's Number 1"* by *Amy N. Langville and Carl D. Meyer* which is among our list of references for the course.
- We will satisfy ourselves with using rankings based on individual statistics or rankings published by experts with a good reputation.
- One of the first things we will will notice when we use rankings by trusted experts to pick our players is that the rankings often do not agree.
- We will first look at some different ways to amalgamate these rankings from voting theory.
- We will see that different ways of amalgamating rankings also often leads to different rankings or different choices for the best player. In fact Arrow's Theorem says that there is no perfect way to amalgamate rankings.

Rankings

A ranking refers to a rank-ordered list of competitors and a rating gives us a list of numerical scores. Every rating gives us a ranking for the competitors.

- Example We can rank Quarterbacks according to the number of completed passes in the previous season. The number of completed passes for the 2014 season (COMP) gives us a rating. We can convert this to a ranking, the top ranked (# 1 player) being the one with the most completed passes. The top ten quarterbacks for the 2014 season according to this ranking are shown below.

Completions Leaders - All Players

RK	PLAYER	TEAM	COMP
1	Drew Brees, QB	NO	456
2	Matt Ryan, QB	ATL	415
3	Ben Roethlisberger, QB	PIT	408
4	Peyton Manning, QB	DEN	395
5	Ryan Tannehill, QB	MIA	392
6	Andrew Luck, QB	IND	380
7	Eli Manning, QB	NYG	379
	Philip Rivers, QB	SD	379
9	Tom Brady, QB	NE	373
10	Jay Cutler, QB	CHI	370

-

Rankings

On the other hand, we could rank Quartebacks according to the number of passing yards they have acquired in in the 2014 season. The number of passing yards (YDS) gives us a rating as shown below for the 2014 season. We can convert this to a ranking, the top ranked (# 1 player) being the one with the most passing yards. The top ten quarterbacks for the 2014 season according to this ranking are shown the right above. Note that it gives a different ranking than the one we got using completions.

Passing Yards Leaders - All Players

RK	PLAYER	TEAM	COMP	ATT	PCT	YDS
1	Drew Brees, QB	NO	456	659	69.2	**4,952**
	Ben Roethlisberger, QB	PIT	408	608	67.1	**4,952**
3	Andrew Luck, QB	IND	380	616	61.7	**4,761**
4	Peyton Manning, QB	DEN	395	597	66.2	**4,727**
5	Matt Ryan, QB	ATL	415	628	66.1	**4,694**
6	Eli Manning, QB	NYG	379	601	63.1	**4,410**
7	Aaron Rodgers, QB	GB	341	520	65.6	**4,381**
8	Philip Rivers, QB	SD	379	570	66.5	**4,286**
9	Matthew Stafford, QB	DET	363	602	60.3	**4,257**
10	Tom Brady, QB	NE	373	582	64.1	**4,109**
RK	PLAYER	TEAM	COMP	ATT	PCT	YDS

Rankings

There are also a number of combined statistical ratings to choose from, for example the **Expected Points Added (EPA)** from Football Analytics.com

Rank	Player	Team	G	WPA	EPA	W
1	12-A.Rodgers	GB	18	5.87	223.8	
2	12-T.Brady	NE	19	4.16	176.8	
3	9-T.Romo	DAL	17	4.94	166.5	
4	7-B.Roethlisberger	PIT	17	4.03	153.1	
5	5-J.Flacco	BLT	18	3.07	148.8	
6	9-D.Brees	NO	16	2.81	137.2	
7	18-P.Manning	DEN	17	3.69	126.7	
8	12-A.Luck	IND	19	1.86	126.4	
9	3-R.Wilson	SEA	19	3.38	121.6	
10	17-P.Rivers	SD	16	3.25	109.0	
11	2-M.Ryan	ATL	16	2.92	107.6	
12	9-M.Stafford	DET	17	3.72	76.0	
13	17-R.Tannehill	MIA	16	2.53	70.7	
14	11-A.Smith	KC	15	1.33	63.4	
15	10-E.Manning	NYG	16	3.17	62.6	
16	1-C.Newton	CAR	16	2.12	53.1	
17	7-C.Kaepernick	SF	16	1.36	51.0	
18	3-C.Palmer	ARZ	6	1.96	48.2	
19	14-R.Fitzpatrick	HST	12	0.22	41.0	
20	5-T.Bridgewater	MIN	13	1.33	38.4	

Rankings

or ESPN's Total Quarterback rating (Total QBR) as shown below.

Season Leaders | Weekly Leaders | Best Games | Best Teams | All-Time Best Seasons* | All-Time Best Games*

2014 Regular Season NFL Leaders

RK	PLAYER	PASS EPA	RUN EPA	SACK EPA	PEN EPA	TOTAL EPA	ACT PLAYS	QB PAR	QB PAA	TOTAL QBR
1	Tony Romo, DAL	86.2	3.3	-18.8	5.1	75.7	537	103.9	61.1	**82.7**
2	Aaron Rodgers, GB	91.9	11.7	-17.0	6.6	93.2	648	125.0	73.4	**82.6**
3	Peyton Manning, DEN	91.3	-3.3	-10.8	5.4	82.5	704	118.6	62.4	**77.2**
4	Tom Brady, NE	81.6	3.6	-12.5	6.2	78.9	699	109.6	53.9	**74.3**
5	Ben Roethlisberger, PIT	91.8	-2.1	-19.2	5.1	75.5	713	107.0	50.1	**72.5**
6	Drew Brees, NO	89.7	2.4	-17.6	7.8	82.4	765	112.2	51.2	**71.6**
7	Eli Manning, NYG	84.4	1.8	-16.4	4.6	74.4	701	101.3	45.4	**70.9**
8	Joe Flacco, BAL	65.5	-0.9	-13.9	12.9	63.7	659	87.2	34.6	**67.3**
9	Matt Ryan, ATL	81.6	2.9	-18.6	1.5	67.3	741	97.1	38.0	**67.0**
10	Philip Rivers, SD	81.0	-0.1	-20.0	7.2	68.1	707	92.3	35.9	**66.8**

Rankings

In addition to rankings created from ratings derived from statistics, many sports analysts publish rankings created from algorithms (which produce ratings and in turn rankings) and opinions such as those shown below:

Jamey Eisenberg
Senior Fantasy Writer
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Updated: Mon, Mar 16, 2015 12:01 am

Quarterbacks	
1	Aaron Rodgers GB
2	Andrew Luck IND
3	Peyton Manning DEN
4	Tom Brady NE
5	Matt Ryan ATL
6	Cam Newton CAR
7	Ben Roethlisberger PIT
8	Russell Wilson SEA
9	Drew Brees NO
10	Matthew Stafford DET
11	Tony Romo DAL
12	Eli Manning NYG
13	Ryan Tannehill MIA

Dave Richard
Senior Fantasy Writer
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Updated: Fri, Mar 20, 2015 2:51 pm

Quarterbacks	
1	Aaron Rodgers GB
2	Andrew Luck IND
3	Tom Brady NE
4	Peyton Manning DEN
5	Matt Ryan ATL
6	Cam Newton CAR
7	Russell Wilson SEA
8	Tony Romo DAL
9	Drew Brees NO
10	Eli Manning NYG
11	Matthew Stafford DET
12	Ryan Tannehill MIA
13	Philip Rivers SD

Combining Rankings

None of the above ranking systems will be 100% accurate in predicting points earned per player in the upcoming season. Some sites keep track of accuracy levels of the various rankings such as “Fantasy Pros”. We might therefore consider amalgamating some of the top rankings to make an overall ranking. We will go through some of the most common methods below. We will use some terminology from voting theory so that we will not confuse the overall ranking we wish to achieve with the rankings we are combining. We will refer to

- the ranking systems or statistics we use for a ranking as voters,
- each individual ranking will be referred to as a ballot
- and the rank given to each player as a vote.

Plurality

This is a method for choosing a single "winning" candidate from a a number of rankings or voters who have ranked the candidates. The winning candidate is the one with the most number 1 votes. In the case of a tie, some back up plan is needed to decide between the winners.

- Suppose for example that we want to choose a wide receiver from the following 5 for our fantasy football team. We have ranked the players using EPA/P (Expected points added per play), CR(Catch Rate), YPR(Yards Per reception), RecTD(Received Touch Downs) and Rec(Number of receptions) for the 2014 season from the site "Advanced Football Analytics".
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Rankings $->$	EPA	CR	YPR	RecTD	Rec
E. Sanders	1	1	4	3	1
T. Williams	4	5	1	1	5
S. Smith	2	2	3	4	2
M. Floyd	3	4	2	5	4
M. Wallace	5	3	5	2	3

- According to the plurality method Sanders would be our top choice, since he has 3 number one votes.

Splitting The Vote

In addition to the problem with ties, the plurality method only takes into account the number 1 votes. This can lead to a situation where the number 1 votes are split among athletes of similar similar high ability and it is conceivable that an athlete who is least preferred by a majority of the voters could be the winner of the plurality vote as in the following example:

- Lets assume we are considering picking one of the five wide receivers listed below as W.R. 1- W.R. 5. Lets also assume that we are using the plurality method with eight good ranking systems RA- RG to make our choice. We have the following results:
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Rankings $->$	RA	RB	RC	RD	RE	RF	RG	RF
W.R. 1	1	1	2	3	2	2	2	2
W.R. 2	3	4	3	2	4	4	5	5
W.R. 3	2	2	1	1	3	3	3	3
W.R. 4	4	3	4	4	1	5	4	4
W.R. 5	5	5	5	5	5	1	1	1

- If one uses the Plurality method to put together these rankings, the winner is W.R. 5 despite the fact that W.R. 5 is the lowest ranked athlete in a majority of the rankings

Plurality with Runoff

One could get around this problem by having a runoff between the top two athletes. (If there is a tie for second place, we eliminate all but those in first and second place or use tie breaking rules if we must).

- We redistribute the number 1 votes for the eliminated athletes by assigning them to the athlete ranked highest among the remaining candidates.
- We continue the process until we have a winner.
- In the above example, this would lead to the following result

Rankings	RA	RB	RC	RD	RE	RF	RG	RH	#1 Votes in Round 1	#1 Votes in Round 2	#1 Votes in Rd 3
W.R. 1	1	1	2	3	2	2	2	2	2	3	5(winner)
W.R. 2	3	4	3	2	4	4	5	5	0*		
W.R. 3	2	2	1	1	3	3	3	3	2	2*	
W.R. 4	4	3	4	4	1	5	4	4	1*		
W.R. 5	5	5	5	5	5	1	1	1	3	3	3

Instant Runoff

This is the same as plurality with runoff described above, where we eliminate the athlete with the least number of number one votes in each round.
To apply the Instant Runoff method:

- Each voter ranks the list of athletes in order of preference; the athletes are ranked in ascending order with a “1” next to the most preferred candidate and so forth. (In some cases, the voter ranks as many or as few choices as they wish while in others, they are required to rank all of the athletes.)
- In the initial count, the first preference of each voter is counted and used to order the athletes. Each first preference counts as one vote for the appropriate athlete
- Once all the first preferences are counted, if one athlete holds a majority (more than 50% of votes cast), that athlete wins. Otherwise the athlete who holds the fewest first preferences is eliminated (If there is an exact tie for last place in numbers of votes, tie-breaking rules determine which is eliminated.)
- Ballots assigned to eliminated athletes are recounted and assigned to one of the remaining athletes based on the next preference on each ballot.
- The process repeats until one candidate achieves a majority (more than 50%) of votes cast for continuing candidates.

Instant Runoff

Example Suppose we wish to choose one of the following Quarterbacks who are ranked by ranking systems A through E as follows:

Rankings	RA	RB	RC	RD	RE	RF	#1's in R 1	#1's in R 2	#1's in R 3
C. Newton	1	1	2	3	5	6	2	2	2
D. Brees	2	2	1	1	2	4	2	2	4 (wins)
B. Roethlisberger	3	3	5	5	4	3	0*		
M. Ryan	4	4	4	2	1	5	1	1	
E. Manning	5	6	3	4	6	2	0*		
M. Stafford	6	5	6	6	3	1	1	1	

Simple Borda Count

Another method of amalgamating votes is to simply to obtain a rating by averaging the ranks for any given athlete and use the resulting ranking.
If voters rank the entire list of candidates or choices in order of preference from the first choice to the last choice then using Borda's method the votes are tallied as follows:

- On a particular ballot, the lowest ranking candidate is given 1 point, the second lowest is given 2 points, and so on, the top candidate receiving points equal to the number of candidates .
- The number of points given to each candidate is summed across all ballots. This is called the Borda Count for the candidate.
- The winner is the candidate with the highest Borda count.
- A Simple Borda Count results in the same ranking as that derived from average rankings: As it turns out the ranking we get for the candidates from a Borda Count is the same as the ranking we get from the ratings calculated by averaging the votes, where a lower rating is given a better(lower) ranking.

Example: Borda Count

Example Let us consider the rankings from the above example and suppose that we wish to choose two of the following Quarterbacks who are ranked by ranking systems A through E by using the Borda Method.

Rankings − >	RA	RB	RC	RD	RE	RF	Ave. Rank	Borda Rank
Cam Newton	1	1	2	3	5	6	3	2
Drew Brees	2	2	1	1	2	4	2	1
Ben Roethlisberger	3	3	5	5	4	3	3.83	4
Matt Ryan	4	4	4	2	1	5	3.33	3
Eli Manning	5	6	3	4	6	2	4.33	5
Matthew Stafford	6	5	6	6	3	1	4.5	6

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Example: Borda Count

The number 1 athlete from the Borda Method does not necessarily always coincide with the number 1 player that we get from the instant runoff method. Consider the following example:

Borda Method

Rankings	RA	RB	RC	RD	RE	RF	RG	RH	Ave.	Borda Rank
Player 1	1	1	1	3	2	6	5	6	3.125	2
Player 2	2	2	2	2	1	2	2	3	2	1
Player 3	5	6	3	6	3	1	1	2	3.375	4
Player 4	3	4	5	1	4	5	3	1	3.25	3
Player 5	4	3	4	5	5	4	4	5	4.25	5
Player 6	6	5	6	4	6	3	6	4	5	6

Instant Runoff

We break ties by eliminating the person with the largest vote sum.

	RA	RB	RC	RD	RE	RF	RG	RH	#1's R 1	# 1's R 2	# 1's R 3	# 1's R 4
P 1	1	1	1	3	2	6	5	6	3	3	4	4(wins)
P 2	2	2	2	2	1	2	2	3	1	1*		
P 3	5	6	3	6	3	1	1	2	2	2	2*	
P 4	3	4	5	1	4	5	3	1	2	2	2	4
P 5	4	3	4	5	5	4	4	5	0*			
P 6	6	5	6	4	6	3	6	4	0*			

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Modified Borda Count

If there are N candidates and some voters do not rank the entire list of candidates, instead ranking their top M candidates, where $M < N$ we can modify the Borda count in the following way:

- On a particular ballot where the candidates are ranked 1 through M, the lowest ranking candidate is given 1 point, the second lowest is given 2 points, and so on, the top candidate receiving points equal to M . The unranked candidates are given 0 points.
- The number of points given to each caniddate is summed across all ballots. This is called the **Borda Count** for the candidate.
- The winner is the candidate with the highest Borda count.
- The **ranking resulting from a modified Borda Count** for an election with N candidates is equivalent to the **ranking resulting from an averaging process** modified as follows:
- On a particular ballot where the candidates are ranked 1 through M and $M < N$, the unranked individuals all receive a rank of $\mathrm{M} + 1$.
- the average across all ballots is then taken for each candidate.

Example: Modified Borda Count

To find a ranking for the top quarterbacks, I might choose the top twenty quarterbacks from some of the most accurate ranking methods as measured by "Fantasy Pros". Below I show some preseason rankings for 2015, showing the top twenty players for each ranking if available.

Rankings −>	A	B	C	D	E	F
Aaron Rodgers	1	2	1	1	1	1
Andrew Luck	2	1	16	13	3	2
Russell Wilson	3	7	7	4	6	6
Peyton Manning	4	5	4	3	2	3
Tom Brady	5	3	20	19	5	4
Cam Newton	6	8	5	6	12	7
Drew Brees	7	9	3	2	4	5
Ben Roethlisberger	8	10	10	8	11	8
Matt Ryan	9	13	9	5	7	10
Eli Manning	10	17	6	7	14	15
Matthew Stafford	11	15	11	11	9	11
Tony Romo	12	4	2	9	8	9
Ryan Tannehill	13	12	12	10	13	13
Philip Rivers	14	11	15	16	10	12
Carson Palmer	15	19	8		18	19
Colin Kaepernick	16	16	19	20	16	14
Sam Bradford	17	20		12		
Jay Cutler	18	14	14	15	17	16
Robert Griffin III	19		17	17		20
Nick Foles	20	18			19	
Joe Flacco		6				17
Andy Dalton			18	18	20	
Teddy Bridgewater			13	14	15	18

We see here that the top 20 quarterbacks are not the same group for each ranking system. One method of dealing with this is to use the modified Borda Method above with $N = 23$ and $M = 20$. We can fill in the missing rankings with 21 and calculate the average

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Example: Modified Borda Count

Rankings − >	A	B	C	D	E	F	Average	Borda Method Rank
Aaron Rodgers	1	2	1	1	1	1	1.17	1
Andrew Luck	2	1	16	13	3	2	6.17	5
Russell Wilson	3	7	7	4	6	6	5.5	4
Peyton Manning	4	5	4	3	2	3	3.5	2
Tom Brady	5	3	20	19	5	4	9.3	10
Cam Newton	6	8	5	6	12	7	7.3	6
Drew Brees	7	9	3	2	4	5	5	3
Ben Roethlisberger	8	10	10	8	11	8	9.17	9
Matt Ryan	9	13	9	5	7	10	8.83	8
Eli Manning	10	17	6	7	14	15	11.5	12
Matthew Stafford	11	15	11	11	9	11	11.33	11
Tony Romo	12	4	2	9	8	9	7.33	7
Ryan Tannehill	13	12	12	10	13	13	12.17	13
Philip Rivers	14	11	15	16	10	12	13	14
Carson Palmer	15	19	8	21	18	19	16.67	16
Colin Kaepernick	16	16	19	20	16	14	16.83	17
Sam Bradford	17	20	21	12	21	21	18.67	20
Jay Cutler	18	14	14	15	17	16	15.67	15
Robert Griffin III	19	21	17	17	21	20	19.17	21
Nick Foles	20	18	21	21	19	21	20	23
Joe Flacco	21	6	21	21	21	17	17.83	19
Andy Dalton	21	21	18	18	20	21	19.83	22
Teddy Bridgewater	21	21	13	14	15	18	17	18

Head-To-Head Comparisons

Another, more time consuming, method of amalgamating rankings is by running a head-to-head comparison between each pair of athletes. If we know the preference rankings of the voters, we can compare the votes of any two athletes to see which one would win in a plurality election in the absence of the other candidates. Such a comparison is called a Head-To-Head comparison of the two candidates.

- For example if I run a head-to-head comparison between Russell Wilson and Peyton Manning from the above rankings;

Rankings $->$	A	B	C	D	E	F
Russell Wilson	3	7	7	4	6	6
Peyton Manning	4	5	4	3	2	3

- I see that Voter A ranks Russell Wilson higher than Peyton Manning and all other voters rank Payton Manning higher than Russell Wilson.
- Thus Payton Manning is the winner of this head-to head comparison.
- In the absence of all other candidates, Peyton Manning would win in a plurality contest with 5 number one votes versus one number one vote for Russell Wilson.
- Peyton Manning (5) vs. Russell Wilson (1) $->$ Peyton Manning will be used as a shorthand method of representing the result of head-to-yead comparisons such as this in later examples.

Head-To-Head Comparisons

If I run a head-to-head comparison between each pair of athletes, then each matchup gives either a winner and loser or a draw between the two athletes compared.

- Thus if we compare every possible pair of candidates in this way, we can treat the results as we would those of a round robin tournament.
- We could rank candidates by wins minus losses (see **Copeland's method below**) or if we keep track of the magnitude of the differences, we could use the equivalent of the point differential (voters in favor minus voters against).

Condorcet Winner

An athlete who is the winner of a head-to-head comparison with every other athlete is called a Condorcet winner.

- If an athlete beats or ties with every other athlete in a head-to-head comparison, that athlete is called a weak Condorcet winner.
- For any set of rankings (assuming all athletes are ranked), there may or may not be a Condorcet winner or a weak Condorcet winner.
- If a Condorcet winner exists, he/she is unique whereas there may be more than one weak Condorcet winner.
- If a Condorcet winner exists it is a popularly held belief that they should be the winner of the election or in this case, they should be our top choice for a position on our fantasy team.
- It is certainly true that if a competitor beats all other competitors in a round robin tournament then they should be the winner.

Copeland's Method

In the absence of a Condorcet winner one needs a backup method to decide on a winner. A simple method widely used to decide the winner of round robin tournaments is to calculate wins minus losses. In voting theory the analog (using head-to-head comparisons as the analog of matches between players) is Copeland's Method.

- With Copeland's method, athletes are ordered by the number of pairwise victories (from head-to-head comparisons) minus the number of pairwise defeats.
- This method is easily understood and easy to calculate, however it often leads to ties and puts more emphasis on the number of victories and defeats rather than their magnitude.
- Note that a Condorcet winner will be the highest ranked athlete when we apply Copeland's method.

Comparing Methods of Amalgamating Rankings

We will see in the next example that if we wish to amalgamate a number of rankings(or choose a best player), the resulting ranking(or best player) will depend on which method we choose to amalgamate the rankings(or decide on who is the best).

Example: Comparing Methods.

Consider the following players ranked according to rankings $A - E$.

Rankings$->$	**A**	**B**	**C**	**D**	**E**
Player 1	3	2	5	5	5
Player 2	1	1	2	2	2
Player 3	5	5	1	1	1
Player 4	4	3	3	3	4
Player 5	2	4	4	4	3

- (a) Rank the players using Borda's method
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Rankings$->$	**A**	**B**	**C**	**D**	**E**	Average	Borda Rank
Player 1	3	2	5	5	5	4	5
Player 2	1	1	2	2	2	1.6	1
Player 3	5	5	1	1	1	2.6	2
Player 4	4	3	3	3	4	3.4	3
Player 5	2	4	4	4	3	3.4	3

- We see that Player 2 is the number one player using Borda's method.

Example: Comparing Methods.

(b) Rank the Players using Copeland's method.

Rankings$->$	**A**	**B**	**C**	**D**	**E**
Player 1	3	2	5	5	5
Player 2	1	1	2	2	2
Player 3	5	5	1	1	1
Player 4	4	3	3	3	4
Player 5	2	4	4	4	3

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	Player 1	Player 2	Player 3	Player 4	Player 5	W - L	Copeland's Rank
Player 1	–	0 – 5	2 – 3	2 – 3	1 – 4	−4	5
Player 2	5 – 0	–	2 – 3	5 – 0	5 – 0	2	2
Player 3	3 – 2	3 – 2	–	3 – 2	3 – 2	4	1
Player 4	3 – 2	0 – 5	2 – 3	–	3 – 2	0	3
Player 5	4 – 1	0 – 5	2 – 3	2 – 3	–	−2	4

- The table shows head to head comparisons between each pair of players. We calculate wins minus losses (W-L) as if we were dealing with the results of a round robin tournament and this gives us a rating for the players which in turn gives a ranking.
- The number one player using Copeland's method is Player 3.

Example: Comparing Methods.

(c) Who would be the "best player" if we used the plurality method with a runoff between the top two candidates to decide?

Rankings $->$	**A**	**B**	**C**	**D**	**E**
Player 1	3	2	5	5	5
Player 2	1	1	2	2	2
Player 3	5	5	1	1	1
Player 4	4	3	3	3	4
Player 5	2	4	4	4	3

- We see that Player 3 already has more than 50% of the number one votes, so Player 3 would be considered the best player if we use the plurality method with a runoff between the top two candidates to decide

Example: Comparing Methods.

(d) Is there a Condorcet winner?

Rankings $->$	**A**	**B**	**C**	**D**	**E**
Player 1	3	2	5	5	5
Player 2	1	1	2	2	2
Player 3	5	5	1	1	1
Player 4	4	3	3	3	4
Player 5	2	4	4	4	3

- To answer this, we go back to our table showing the results of all head to head comparisons between pairs of candidates.
-

	Player 1	Player 2	Player 3	Player 4	Player 5	W - L
Player 1	–	$0-5$	$2-3$	$2-3$	$1-4$	-4
Player 2	$5-0$	–	$2-3$	$5-0$	$5-0$	2
Player 3	$3-2$	$3-2$	–	$3-2$	$3-2$	4
Player 4	$3-2$	$0-5$	$2-3$	–	$3-2$	0
Player 5	$4-1$	$0-5$	$2-3$	$2-3$	–	-2

- We see that player 3 beats everyone else in a head-to-head comparison and is therefore the Condorcet winner.

A Best Method

Now that we have seen that different methods of amalgamating rankings lead to different results, it is natural to ask if there is a "best" method of amalgamating rankings. We saw that there are drawbacks to each type of voting system we studied. The plurality method with three or more candidates may lead to a winner who is undesirable to the majority of voters. We saw that a Condorcet winner may not be the winner using the Borda method and in fact may not even make it into the runoff in a Plurality voting system with runoff. Furthermore a Condorcet winner may not even exist.

- Of course the answer depends on what qualities we expect the best method to have.
- There are a number of qualities that we would want an amalgamation of rankings to have, some of which which are listed on the next page.
- We will look at Arrow's Theorem which essentialy says that there is no perfect way of amalgamating rankings (or votes).

Properties of a Good Voting System

In our discussion below, the term voting system will refer to a method of amalgamating rankings such as the ones we have discussed already, the Borda method, the instant runoff method or Copeland's method. To define what we mean by a good voting system, we follow the ideas of Nobel Laureate Kenneth Arrow who, beginning in the 1940's explored methods of ordering choices among public policies. We first begin with a list of properties that most people would consider desirable in a voting system.

- Universal Domain Any ordering of the candidates/players is allowed, that is, there are no restrictions placed on the ranking of the candidates/players a voter may choose.
- Pareto optimality If all voters prefer candidate/player A to candidate/player B, then the group choice should not prefer candidate/player B to candidate/player A.
- Non-Dictatorship No one individual voter preference totally determines the group choice.
- Independence from irrelevant alternatives If a group of voters choose candidate/player A over candidate/player B, then the addition or subtraction of other candidates/players should not change the group choice to B.

Properties of a Good Voting System

That our voting system be Independence from irrelevant alternatives (If a group of voters choose candidate/player A over candidate/player B, then the addition or subtraction of other candidates/players should not change the group choice to B.) is the most debatable especially in sport.

- On the one hand a choice between A and B should not depend on what other choices are available,
- on the other hand however, it is only by comparison with other possibilities that voters' perception of differences between candidates/players can be brought to light.

Example: Independence From Irrelevant Alternatives

Let's see how the independence from irrelevant alternatives criterion breaks down for the method of Plurality with runoff. Suppose that you and your friends are discussing two players, Player1 and Player 2 and everyone ranks them as shown in the table below:

Rankings $->$	**A**	**B**	**C**	**D**	**E**	**F**	**G**
Player 1	1	1	1	1	2	2	2
Player 2	2	2	2	2	1	1	1

Clearly Player 1 has the most number 1's and wins the vote.

- Now suppose that someone says you should also consider Player 3 and everybody ranks all three players:

Rankings $->$	**A**	**B**	**C**	**D**	**E**	**F**	**G**
Player 1	2	2	2	1	2	2	2
Player 2	3	3	3	2	1	1	1
Player 3	1	1	1	3	3	3	3

- Three of those who initially ranked Player 1 first, now rank Player 3 as number 1, but still prefer Player 1 to Player 2. We see that Player 2 now wins(getting 4 number one votes after Player 1 is eliminated) , which means that the introduction of the irrelevant alternative Player 3 reversed the outcome.

Arrow's Impossibility Theorem

Arrow's Impossibility Theorem There is no voting system based on rankings that satisfies the properties of universal domain, Pareto optimality, non-dictatorship and independence from irrelevant alternatives.

- It is important to note here that Arrows theorem applies in the context of rankings, where voters give an ordering of their preferences. It does not apply to the situation where voters give a measure of the worth or utility or strength of performance of each candidate.
- Basically it says that it is impossible to find a function or rule that will amalgamate a sequence of individual rankings or ballots (represented by lists or lists with ties) in a reasonable way.
- We also note that there are many other desirable properties of voting systems which are not listed above and should be considered when choosing a way to amalgamate ballots.

Example: Independence From Irrelevant Alternatives

Copeland's method violates the independence from irrelevant alternatives criterion.

Suppose you wish to amalgamate the given rankings for the following players using Copeland's method in order to rank players from this category for your draft picks:

#**Voters**	**A**	**B**	**C**	**D**	**E**	**F**	**G**	**H**	**I**	**J**
Player 1	1	1	2	2	2	2	3	3	2	5
Player 2	4	4	1	1	1	1	2	4	5	4
Player 3	3	3	3	3	5	5	1	1	4	2
Player 4	2	2	4	4	3	3	4	2	1	3
Player 5	5	5	5	5	4	4	5	5	3	1

- We get

Rankings$- >$	Player 1	Player 2	Player 3	Player 4	Player 5	W - L	Copeland's Copeland's
Player 1	—	4 — 6	7 — 3	7 — 3	9 — 1	2	1*
Player 2	6 — 4	—	4 — 6	5 — 5	8 — 2	1	2
Player 3	3 — 7	6 — 4	—	5 — 5	6 — 4	1	2
Player 4	3 — 7	5 — 5	5 — 5	—	9 — 1	0	4
Player 5	1 — 9	2 — 8	4 — 6	1 — 9	—	-4	5

Example: Independence From Irrelevant Alternatives

Now right before the draft, one of the players (Player 3) gets injured and is out for the season. Since Player 3 was not the top choice, this fact should have no effect on the choice of Player 1 as the first draft, or should it ?
After rewriting the rankings as rankings 1 through 4 for the 4 remaining players, we get

#Voters	**A**	**B**	**C**	**D**	**E**	**F**	**G**	**H**	**I**	**J**
Player 1	1	1	2	2	2	2	2	2	2	4
Player 2	3	3	1	1	1	1	1	3	4	3
Player 4	2	2	3	3	3	3	3	1	1	2
Player 5	4	4	4	4	4	4	4	6	3	1

- Now applying Copeland's method, we get:

Rankings− >	Player 1	Player 2	Player 4	Player 5	W - L	Copeland's Rank
Player 1	—	4 — 6	7 — 3	9 — 1	1	2
Player 2	6 — 4	—	5 — 5	8 — 2	2	1*
Player 4	3 — 7	5 — 5	—	9 — 1	0	3
Player 5	1 — 9	2 — 8	1 — 9	—	−3	4

- We see that Player 2 is the new Number 1 player and that the removal of Player 3 from the pool, switches the ranks of Player 1 and Player 2 even though the preferences of the voters have not changed.

Assigning a Value To a Player

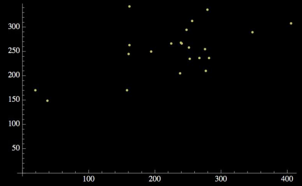
Ideally we would like to know the points that each player will score throughout the upcoming season and use it as the value of the player. Unfortunately, these statistics are not available to us before the season begins, so we must try to predict the number of points that the players will score in the upcoming season.

- It is natural to expect that there might be a relationship between the points scored by a player last season and the points scored in the upcoming season.
- We don't expect that the number of points in the upcoming season will be exactly the same as those in last season.
- To find an approximate relationship between points in consecutive seasons, we will try to find a relationship between points scored last season and points scored in the previous season.

Relationship between points last year and this year

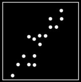
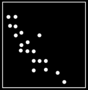
In this table we show the scores for two consecutive seasons for 22 quarterbacks.

	2013	2014
Aaron Rodgers	162	342
Andrew Luck	279	336
Russell Wilson	256	312
Peyton Manning	406	307
Ben Roethlisberger	248	295
Drew Brees	348	290
Matt Ryan	239	268
Tom Brady	241	267
Ryan Tannehill	225	266
Eli Manning	162	263
Tony Romo	252	258
Philip Rivers	276	254
Joe Flacco	194	249
Jay Cutler	160	244
Matthew Stafford	267	237
Cam Newton	282	237
Colin Kaepernick	253	234
Andy Dalton	277	210
Alex Smith	238	205
Kyle Orton	19	171
Ryan Fitzpatrick	158	171
Brian Hoyer	38	149

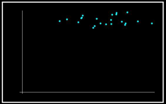
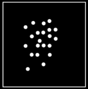
- We can make a picture called a scatterplot from this data, which helps us determine if there is a strong relationship between them. We plot the points from the 2013 season on the horizontal axis and those from the 2014 season on the vertical axis.
- 

Fitting a line to a scatterplot

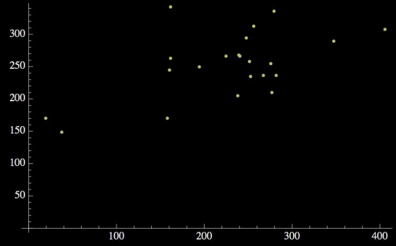
Using the method of least squares, we can find a line that best fits the points on the scatterplot. How well the line fits the data will be discussed in the next section. For now, we just focus on finding a linear fit.

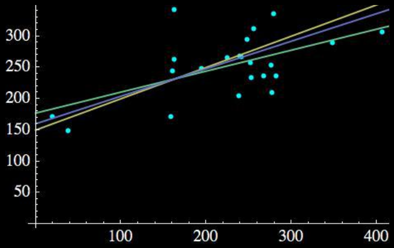
-  If the dots (roughly) lie along a line sloping upwards then there is a positive relationship between the two variables where higher values of the horizontal variable are associated to higher values of the vertical variable and vice versa. A steeper slope to the line indicates a stronger relationship between the two variables
-  IIf the points on the scatterplot lie along a line sloping downwards there is a negative relationship between the 2 variables and higher values of the vertical variable are related to lower values of the horizontal variable.

Fitting a line to a scatterplot

-  If the points lie along a horizontal line then the values of the vertical variable are roughly the same no matter what the values of the horizontal variable are.
-  If the points form a disc like shape or a nonlinear shape one can fit a line to the data, but visually one can see that it is not appropriate. In the next section, measures of the goodness of fit of a model for the data will be discussed.

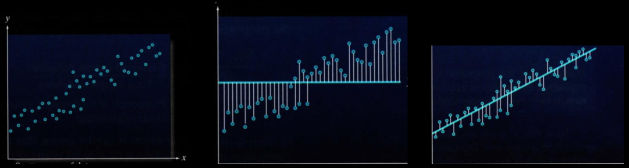
Least Squares Line for the 2013/2014 data





- Visually, we can see that the data points (very) roughly lie along a line sloping upwards indicating that there is some positive and roughly linear relationship between points scored in 2013 and points scored in 2104.
- There are however many lines with positive slopes which fit the data reasonably well and we need a method to choose between them.
- In the method outlined below, we find the line which minimized the sum of the squared vertical distances of the points from the line.

The least Squares Line



- Given a set of data points in the xy-plane, $\{(x_1, y_1), (x_2, y_2), (x_3, y_3), \quad \ldots \quad, (x_n, y_n)\}$ such as those shown above, we can find an equation for the line which best fits the data using the method of least squares.
- This line minimizes the squares of the difference between the y values on the line and the y values for the points in the data. Clearly for the line on the right above the sum of the squared (vertical) distances from the data points to the line is smaller than that for the line on the left. We look for the line which among all possible lines, minimizes that sum.
- When we find the equation of the least squares line it gives us a linear formula which estimates the relationship between the variable x and the variable y which we can use for predictions.

The least Squares Line

Recall that the equation of a line is of the form $y = \beta_0 + \beta_1 x$ where β_0 and β_1 are constants.

- The idea is to find values of β_0 and β_1 so that the sum
$$\text{SSE} = ((y_1 - y(x_1))^2 + (y_2 - y(x_2))^2 + \cdots + (y_n - y(x_n))^2)$$
is minimal where $y(x_i)$ is tha value corresponding to x_i from the formula $y = \beta_0 + \beta_1 x$ and y_i is the value corresponding to x_i in the datapoint (x_i, y_i). (SSE stands for the sum of the squared errors.)

Calculating the sum of the squared error (SSE).

Consider the data for our quarterbacks above. Let x_i be the number of fantasy points scored by quarterback i in 2013 and let y_i denote the number of fantasy points scored by quarterback i is 2014. Lets calculate the sum of the squared error (SSE) for a particular line $y = 150 + (0.5)x$.

points 2013	points 2014	predicted values	Errors	Squared Errors
x_i	y_i	$y(x_i) = 150 + (0.5)x_i$	$y_i - y(x_i)$	$(y_i - y(x_i))^2$
162	342	231	111	12321
279	336	289.5	46.5	2162.25
256	312	278	34	1156
406	307	353	-46	2116
248	295	274	21	441
348	290	324	-34	1156
239	268	269.5	-1.5	2.25
241	267	270.5	-3.5	12.25
225	266	262.5	3.5	12.25
162	263	231	32	1024
252	258	276	-18	324
276	254	288	-34	1156
194	249	247	2	4
160	244	230	14	196
267	237	283.5	-46.5	2162.25
282	237	291	-54	2916
253	234	276.5	-42.5	1806.25
277	210	288.5	-78.5	6162.25
238	205	269	-64	4096
19	171	159.5	11.5	132.25
158	171	229	-58	3364
38	149	169	-20	400
			$SE = -225$	$SSE = 43122.25$

The least Squares Line

From the table above , we see that for the line $y = 150 + (0.5)x$, the sum of the errors (SE = - 225) is an unreliable statistic in measuring how well the line fits the data due to cancellation. We avoid this problem by squaring the error and use the Sum of Squares of The Error (SSE) to measure how well the line fits the data. Naturally a smaller SSE will indicate that a line is a better fit for the data.

- There is a unique line for which SSE is at a minimum. This line is called the Least Squares Line. The methodology used to obtain the equation of this line is called the method of least squares.
- We can solve for the coefficients β_0 and β_1 of such a line, for a particular set of data, by using calculus to find the minimum of the function

$$SSE = \Sigma[y_i - (\beta_0 + \beta_1 x_i)]^2,$$

for the variable β_0 and β_1.

The least Squares Line

Definition The least squares line $y = \beta_0 + \beta_1 x$, for a set of data, is the unique line with the following properties:

- The sum of errors equals 0; $\mathrm{SE} = 0$
- The sum of squared errors (SSE) is smaller than that for any other straight line model.
- The values of β_0 and β_1 for the least squares line are given by the following formulas (where $\bar{x}$ and $\bar{y}$ demote the means of the data sets $\{x_1, x_2, \ldots, x_n\}$ and $\{y_1, y_2, \ldots, y_n\}$ respectively):

$$\beta_1 = \frac{SS_{xy}}{SS_{xx}}, \qquad \beta_0 = \bar{y} - \beta_1 \bar{x}$$

where

$$SS_{xy} = \sum (x_i - \bar{x})(y_i - \bar{y}) = \sum x_i y_i - \frac{(\sum x_i)(\sum y_i)}{n}$$

and

$$SS_{xx} = \sum (x_i - \bar{x})^2 = \sum x_i^2 - \frac{(\sum x_i)^2}{n}$$

We have

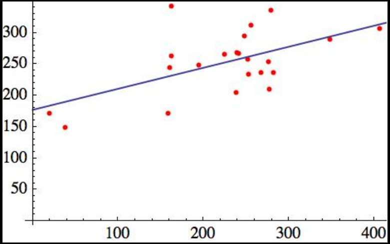
$$SSE = \sum (y_i - (\beta_0 + \beta_1 x_i))^2 = SS_{yy} - \beta_1 SS_{xy}$$

where

$$SS_{yy} = \sum y_i^2 - \frac{\sum (y_i)^2}{n}.$$

The least Squares Line for data on points in 2013/2014

For the above data, we can apply the formulas to see that the least squares line is given by $y = 177.06 + 0.33x$ and the sum of the squared errors for this line is $\text{SSE} = 36487.3$. This is less than the sum of the squared errors for the line shown above, in fact it is the minimum such sum possible for any line that we might fit to the data.

- 

- We could use this line to estimate the fantasy points that a quarterback will score id 2015 given the number of points he scored in 2014. For example if a quarterback scored 250 points in fantasy football in 2014, we might expect the number of points he scores in 2015 to be roughly $177.06 + 0.33(250) \approx 259.56$.

The least Squares Line for data on points in 2013/2014

For the above data, we can apply the formulas to see that the least squares line is given by $y = 177.06 + 0.33x$ and the sum of the squared errors for this line is SSE $= 36487.3$. This is less than the sum of the squared errors for the line shown above, in fact it is the minimum such sum possible for any line that we might fit to the data.

- There is quite a bit of calculation involved in finding the least squares line and for a large number of data points one can use statistical software. Also Wolfram Alpha allows you to calculate the least squares line for a limited number of data points as does a TI (83 and up) calculator. I have included directions for calculating the least squares line on a TI83 in the notes. You can find directions for TI84's and up online.

Several Variables

The process discussed above is called Linear Regression.

- It is likely that a number of variables will have a positive or negative influence the number of points scored in the upcoming season. I have listed a few candidates below:
  - H = Height of player
  - A = Age of player
  - GP = Games Played (career)
  - C = Completion percentage (previous season)
  - AY = yards per pass attempted (previous season)
  - QBR = Total Quarterback rating (ESPN Previous season)
  - INT = Interception percentage (previous season)
  - FUM = Fumbles (previous season)
  - LNG = Longest Pass Play (career)
  - PP = Fantasy points previous season.

  We can collect data on these variables and consider their relative weight or influence on the points scored in the 2014 season using a generalization of the method of linear regression described above. The method is relatively easy to apply if you have some statistical software. In Chapter 3 his book *Mathletics*, Winston describes how to run such a regression in `Excel` and gives a compact explanation of how to interpret the results.

The Optimal Combination of Players

Although we cannot explore the (knapsack) problem of maximizing expected points scored subject to budget constraints without some techniques from linear programming or some computer programming skills, we can have a quick look at how we might go about setting up the problem in a way that could be solved with the help of a computer. If you can program a little, you will find some suggestions for computer solutions online by googling "knapsack problem in fantasy football".

The Optimal Combination of Players

Lets suppose

- you want to choose 3 players from a set of
  3 quarterbacks,
  2 Running Backs,
  3 wide receivers
- You must choose at least one quarterback
- You do not wish to choose more than 1 quarterback
- You must not spend more than $45
- The value of each player (the number of points you expect them to earn or last year's points) and their cost is shown in the table below. (Quarterbacks are named Q1-Q3, Running Backs are named RB1, RB2, Wide Receivers are named WR1-WR3)

Player	Points Exp.	Cost
QB1	250	$20
QB2	220	$18
QB3	170	$15
RB1	270	$20
RB2	225	$16
WR1	230	$14
WR2	210	$12
WR3	150	$5

The Optimal Combination of Players

For each player we create a variable which can take one of the values 0 or 1. These variables will represent the number of each player that we will purchase.

- We call the variables q_1, q_2, q_3 for the quarterbacks, r_1, r_2 for the running backs and w_1, w_2, w_3 for the wide receivers. Let $P(q_1)$ denote the points we expect quarterback 1 to earn next season and let $C(q_1)$ denote their cost etc...
- Our problem with our constraints can be summarized as follows: We wish to find the values of $q_1, q_2, q_3, r_1, r_2, w_1, w_2, w_3$ such that :
- the sum
  $q_1 P(q_1) + q_2 P(q_2) + q_3 P(q_3) + r_1 P(r_1) + r_2 P(r_2) + w_1 P(w_1) + w_2 P(w_2) + w_3 P(w_3)$
  (sum of the quantity of each player times the number of points) is maximized subject to the constraints;
- $q_1 C(q_1) + q_2 C(q_2) + q_3 C(q_3) + r_1 C(r_1) + r_2 C(r_2) + w_1 C(w_1) + w_2 C(w_2) + w_3 C(w_3) \leq 45$
  (sum of quantity of players times costs ≤ 45)
- We must also have that $1 = q_1 + q_2 + q_3$ (exactly one quarterback), and $q_1 + q_2 + q_3 + r_1 + r_2 + w_1 + w_2 + w_3 = 3$ (a total of 3 players).

The Optimal Combination of Players

A solution to the above problem is a sequence of 0's and 1's of length 8 $(q_1, q_2, q_3, r_1, r_2, w_1, w_2, w_3)$ which satisfies all of the constraints.

- There are 256 possible sequences of 0's and 1's of length 8. However not all of the 256 such sequences will satisfy our constraints.
- The constraint $1 = q_1 + q_2 + q_3$ means that exactly one of the first three variables must be equal to 1.
- Putting this together with $q_1 + q_2 + q_3 + r_1 + r_2 + w_1 + w_2 + w_3 = 3$, we gat that exactly two of the 5 variables r_1, r_2, w_1, w_2, w_3 must equal 1.
- There are exactly 10 such combinations of these 5 variables. We can put any of these 10 combinations together with the 3 choices of quarterback to make 30 possible combinations which fit the last two constraints on the list. On the next page, we calculate the cost for each of these to check if it is a feasible combination satisfying the budget and then we choose the one which maximizes points.

The Optimal Combination of Players

q_1	q_2	q_3	r_1	r_2	w_1	w_2	w_3	Cost	Points
1	0	0	1	1	0	0	0	56	
1	0	0	1	0	1	0	0	54	
1	0	0	1	0	0	1	0	52	
1	0	0	1	0	0	0	1	45	670 (Max)
1	0	0	0	1	1	0	0	50	
1	0	0	0	1	0	1	0	48	
1	0	0	0	1	0	0	1	41	625
1	0	0	0	0	1	1	0	46	
1	0	0	0	0	0	1	1	37	610
1	0	0	0	0	1	0	1	39	630
0	1	0	1	1	0	0	0	54	
0	1	0	1	0	1	0	0	52	
0	1	0	1	0	0	1	0	50	
0	1	0	1	0	0	0	1	43	640
0	1	0	0	1	1	0	0	48	
0	1	0	0	1	0	1	0	46	
0	1	0	0	1	0	0	1	39	595
0	1	0	0	0	1	1	0	44	660
0	1	0	0	0	0	1	1	35	580
0	1	0	0	0	1	0	1	37	600
0	0	1	1	1	0	0	0	51	
0	0	1	1	0	1	0	0	49	
0	0	1	1	0	0	1	0	47	
0	0	1	1	0	0	0	1	40	
0	0	1	0	1	1	0	0	51	
0	0	1	0	1	0	1	0	47	
0	0	1	0	1	0	0	1	40	545
0	0	1	0	0	1	1	0	41	610
0	0	1	0	0	0	1	1	32	530
0	0	1	0	0	1	0	1	34	550

The Optimal Combination of Players

There are algorithmic ways of tackling this problem using the Branch and Bound Method for Integer Programming along with Techniques from Linear Programming along with some dynamic programming methods. Computers and computer programming are obviously necessary for finding an optimal solution when larger numbers of players are involved.