Random Matrix Theory

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I demonstrate here the fundamentals of RMT, the Marchenko-Pastur theorem, and non-linear shrinkage, especially as it relates to minimizing out-of-sample portfolio variance. I will then try to sketch out how these results may be extended to producing unbiased variance forecasts.

1 Setup

To begin, assume we observe i.i.d. random values in a $T \times N$ matrix,

$$X_N = \begin{bmatrix} — & x_1 & — \\ & \vdots & \\ — & x_T & — \end{bmatrix}$$

where T is the number of observations and N is the number of variables (ie. $x_i \in \mathbb{R}^N$). I index by N since T and N will be growing to infinity. In particular, we will take $T/N \to y > 1$. Let Σ_N be the covariance matrix for X_N. Furthermore, define the empirical spectral distribution (e.s.d.) for the eigenvalues of Σ_N as,

$$H_N(\tau) = \frac{1}{N} \sum_{i=1}^{N} \mathbb{1}_{[\tau_i, +\infty)}(\tau)$$

where $\tau_1, \ldots, \tau_N$ is the system of eigenvalues. It will be implicit that this system depends on N.

A basic assumption is that $H_N(\tau)$ converges to a nonrandom limit everywhere with support on a compact interval[1].

An example of a limit law H would be a truncated exponential distribution where,

$$H_\gamma(\tau) = \frac{1 - e^{-\gamma(\tau - 1)}}{1 - e^{-\gamma}}, \quad \text{supp}(H) = [1, 2].$$

The formula for the eigenvalues would be,

$$\tau = 1 - \frac{1}{\gamma} \log\left(1 - x(1 - e^{-\gamma})\right), \quad x \in [0, 1],$$

or for some finite N,

$$\tau_i = 1 - \frac{1}{\gamma} \log\left(1 - x_i(1 - e^{-\gamma})\right), \quad x_i = \frac{i - 1}{N - 1},\ i = 1, \ldots, N$$

Figure 1.1 shows the spectrum for this function H as well as the empirical spectrum of the sample covariance matrix for different N, which we define next.

[1] We are going to assume the limit is continuous and on a compact interval to simplify this since otherwise this statement needs to be more precise.

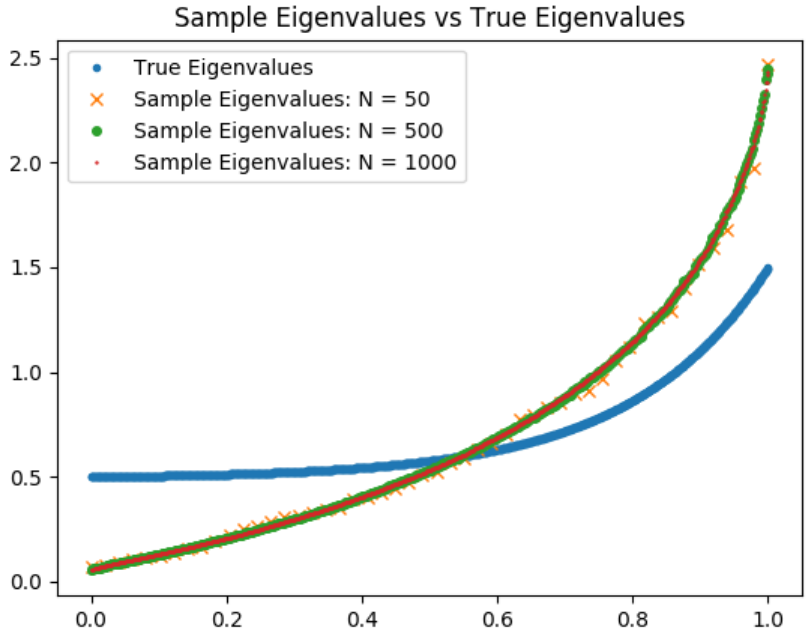

Figure 1.1: Example of sample covariance eigenvalue spectrum for various N and $y = 2$. The true eigenvalue spectrum is given by $H_\gamma(\tau) = \frac{1-e^{-\gamma(\tau-1)}}{1-e^{-\gamma}}$ with $\text{supp}(H) = [1, 2]$. The true spectrum is approximated for some N by $\tau_i = 1 - \frac{1}{\gamma}\log\left(1 - x_i(1-e^{-\gamma})\right)$ where $x_i = \frac{i-1}{N-1}$, $i = 1, \ldots, N$.

Define the sample covariance matrix as,

$$S_N = \frac{1}{T} X_N^T X_N$$

and the e.s.d. of S_N as

$$F_N(\lambda) = \frac{1}{N}\sum_{i=1}^{N} \mathbb{1}_{[\lambda_i, +\infty)}(\lambda)$$

where $\lambda_1, \ldots, \lambda_N$ is the system of eigenvalues. It is not assumed that F_N has a limit or what it is. That's where the Marchenko-Pastur theorem comes in.

2 Main Marchenko-Pastur Result

The main result of RMT and Marchenko-Pastur (MP) is convergence of F_N to a limit law F almost surely. The result is not direct but instead states convergence in terms of the *Stieltjes transform* of F_N and F.

The Stieltjes transform of a nondecreasing function G is,

$$m_G(z) = \int_{-\infty}^{\infty} \frac{dG(x)}{x - z}$$

where $z \in \mathbb{C}^+ = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$.

For an empirical distribution like F_N, this is,

$$m_{F_N}(z) = \frac{1}{T}\sum_{i=1}^{N} \frac{1}{\lambda_i - z} = \frac{1}{T}\text{Tr}\left[(S_n - zI)^{-1}\right]$$

The Marchenko-Pastur result is that $m_{F_N}(z) \to m_F(z)$ almost surely for all $z \in \mathbb{C}^+$ and $m_F(z)$ is defined through the *Marchenko-Pastur equation*,

$$m_F(z) = \int_{\text{supp}(H)} \frac{dH(\tau)}{\tau\left[1 - y^{-1} - y^{-1} z m_F(z)\right] - z}.$$

In few cases[2] can this equation be solved analytically. It can potentially be solved numerically for specified H but that is not really important. Moreover, the limit F would come from the inversion of $m_F(z)$ given by,

$$F(b) - F(a) = \lim_{\eta \to 0^+} \frac{1}{\pi} \int_a^b \text{Im}\left[m_F(\xi + i\eta)\right] d\xi$$

Figure 1.1 shows the e.s.d. F_N for various values of N. It may be difficult to see but for $N = 1000$, the e.s.d. shows convergence to the limit compared to $N = 50$.

2.1 A Generalization of M-P

Ledoit and Péché (2011) show how you can generalization to functions like

$$\Omega_N^g(\lambda) = \frac{1}{N} \sum_{i=1}^N \mathbb{1}_{[\lambda_i, +\infty)}(\lambda) \sum_{j=1}^N |u_i^T v_j|^2 g(\tau_j)$$

where

$$\begin{aligned} S_N = U\Lambda U^T, \quad U = \begin{bmatrix} u_1 & \cdots & u_N \end{bmatrix} \quad \Lambda = \text{Diag}(\lambda_1, \ldots, \lambda_N), \\ \Sigma_N = VTV^T, \quad V = \begin{bmatrix} v_1 & \cdots & v_N \end{bmatrix} \quad T = \text{Diag}(\tau_1, \ldots, \tau_N) \end{aligned}$$

The Stieltjes transform of Ω_N^g is given by,

$$\begin{aligned} \Theta_N^g(z) &= \frac{1}{N} \sum_{i=1}^N \frac{1}{\lambda_i - z} \sum_{j=1}^N |u_i^T v_j|^2 g(\tau_j) \\ &= \frac{1}{N} \text{Tr}\left[(S_N - zI)^{-1} g(\Sigma_N)\right] \end{aligned}$$

where in an abuse of notation, $g(\Sigma_N)$ is considered as a spectral function, ie. it applies g to the eigenvalues of Σ_N.

The way to see how the Trace operator relates, you just do some simple linear algebra:

$$\begin{aligned} \text{Tr}\left[(S_N - zI)^{-1} g(\Sigma_N)\right] &= \text{Tr}\left[(U\Lambda U^T - zI)^{-1} V g(T) V^T\right] \\ &= \text{Tr}\left[U^T (\Lambda - zI)^{-1} U V g(T) V^T\right] \\ &= \text{Tr}\left[(\Lambda - zI)^{-1} (UV) g(T) (UV)^T\right] \end{aligned}$$

where expanding out the last term gives $\Theta_N^g(z)$. Furthermore, if you take $g \equiv 1$, then you get back F_N and m_{F_N}.

The generalized result for $\Omega_N^g(\lambda)$ and $\Theta_N^g(z)$ is $\Theta_N^g(z)$ converges a.s. to $\Theta^g(z)$ for all $z \in \mathbb{C}^+$ where $\Theta^g(z)$ is given by,

$$\Theta^g(z) = \int_{\text{supp}(H)} \frac{g(\tau) dH(\tau)}{\tau\left[1 - y^{-1} - y^{-1} z m_F(z)\right] - z}.$$

Note how the integration kernel in the denominator stays the same and the result only differs in our weighting scheme $g(\tau)$.

[2] Perhaps only one? This would be the case when all eigenvalues are equal to a common value

2.2 Optimal Shrinkage

2.2.1 Frobenius Loss

If we consider the class of estimators

$$\hat{\Sigma}_N = UDU^T$$

where U are the eigenvectors of the sample covariance matrix, then we can try to minimize Frobenius error via,

$$\min_D \|UDU^T - \Sigma_N\|_F^2$$

The optimal eigenvalues are

$$D^* = \text{Diag}(\tilde{d}_1, \ldots, \tilde{d}_N) = \text{Diag}(u_1^T \Sigma_N u_1, \ldots, u_N^T \Sigma_N u_N) \tag{2.1}$$

From before, if we take $\Omega_N^g(\lambda)$ and $\Theta_N^g(z)$ for $g(\tau) = \tau$, then if we define,

$$\Delta_N(\lambda) = \frac{1}{N} \sum_{i=1}^{N} \tilde{d}_i \mathbb{1}_{[\lambda_i, \infty)}(\lambda) = \Omega_N^g(\lambda)$$

then based on previous stated result about the convergence of $\Theta_N^g(z)$, Ledoit and Péché (2011) show through verifying with calculus that $\Delta_N(\lambda) \to \Delta(\lambda)$ almost surely for all $\lambda \neq 0$ and

$$\Delta(\lambda) = \Omega^g(\lambda) = \int_{-\infty}^{\lambda} \delta(x) dF(x)$$

where

$$\delta(\lambda) = \begin{cases} \frac{\lambda}{|1 - y^{-1} - y^{-1}\lambda m_F(\lambda)|^2} & \lambda > 0, \\ 0 & \text{otherwise.} \end{cases}$$

The summary of this is that while the oracle estimates that are optimal, $\tilde{d}_i = u_i^T \Sigma_N u_i$, are completely infeasible, we use $\delta(\lambda_i)$ as the consistent estimate for $\tilde{d}_i$. This is seen by considering,

$$\tilde{d}_i = \lim_{\epsilon \to 0^+} \frac{\Delta_N(\lambda_i + \epsilon) - \Delta_N(\lambda_i - \epsilon)}{F_N(\lambda_i + \epsilon) - F_N(\lambda_i - \epsilon)} \approx \delta(\lambda_i)$$

where the approximation comes from the asymptotic theory. Now, it remains to actually estimate $m_F(z)$ and thus produce a consistent estimate of the shrinkage function but this has in fact been done in Ledoit and Wolf (2015).

2.2.2 Out-of-Sample Variance Loss

The goal here is to minimize

$$\mathcal{L}(\hat{\Sigma}_N, \Sigma_N, m) := \hat{w}^T \Sigma_N \hat{w} = m^T m \times \frac{m^T \hat{\Sigma}_N^{-1} \Sigma_N \hat{\Sigma}_N^{-1} m}{\left(m^T \hat{\Sigma}_N^{-1} m\right)^2}$$

for some rotation-equivariant estimator $\hat{\Sigma}_N$ given by

$$\hat{\Sigma}_N = UDU^T$$

where $D = \text{Diag}(\hat{\rho}(\lambda_1), \ldots, \hat{\rho}(\lambda_N))$ and U are the eigenvectors of S_N. Here $\hat{w}$ is a scaled form of the long-short maximum Sharpe ratio portfolio,

$$\hat{w} = \frac{\sqrt{m^T m}}{m^T \hat{\Sigma}_N^{-1} m} \times \hat{\Sigma}_N^{-1} m.$$

where m is the *return predictive signal* aka expected return and is distributed independently of S_N and its distribution is rotation invariant. The scaling $\sqrt{m^T m}$ was chosen so that the portfolio weights are invariant to m.

From Lemma 1 in Ledoit and Péché (2011) based on the properties of m,

$$\frac{1}{N} m^T \hat{\Sigma}_N^{-1} m - \frac{1}{N} \mathrm{Tr}(\hat{\Sigma}_N^{-1}) \to 0,$$

almost surely so that they converge together to,

$$\int \frac{1}{\hat{\rho}(\lambda)} dF(\lambda)$$

A similar line of reasoning shows that

$$\frac{1}{N} m^T \hat{\Sigma}_N^{-1} \Sigma_N \hat{\Sigma}_N^{-1} m - \frac{1}{N} \mathrm{Tr}(\hat{\Sigma}_N^{-1} \Sigma_N \hat{\Sigma}_N^{-1}) \to 0$$

almost surely so that they converge together. To what becomes evident by looking at the theorems for the generalization of the M-P result. Namely,

$$\frac{1}{N} \mathrm{Tr}(\hat{\Sigma}_N^{-1} \Sigma_N \hat{\Sigma}_N^{-1}) = \frac{1}{N} \mathrm{Tr}(U^T \Sigma_N U D^{-2}) = \frac{1}{N} \sum_{i=1}^{N} \frac{u_i^T \Sigma_N u_i}{\hat{\rho}(\lambda_i)^2}$$

But this is basically the same as Δ_N from the previous section, so it turns out that the limit is,

$$\int \frac{\delta(\lambda)}{\hat{\rho}(\lambda)^2} dF(\lambda)$$

These two limit results show that,

$$\mathcal{L}(\hat{\Sigma}_N, \Sigma_N, m) \to \frac{\int \frac{\delta(\lambda)}{\hat{\rho}(\lambda)^2} dF(\lambda)}{\left(\int \frac{1}{\hat{\rho}(\lambda)} dF(\lambda) \right)^2}.$$

Differentiation with respect to $\hat{\rho}(\lambda)$[3] yields the result that the first order condition is satified if and only if,

$$\frac{\hat{\rho}(\lambda)}{\delta(\lambda)} = c$$

where c is an arbitrary constant. Choosing the constant equal to 1 satisfies some consistency results with regards to Traces of the estimator and the population covariance.

The upshot here is that even for this new loss function, the shrinkage is actually the same as the one for the Frobenius norm.

2.3 Cross-Validation for Optimal Shrinkage

The optimal shrinkage is $\delta(\lambda)$ and thus is the quantity of interest.

[3] I tried replicating this result in the Goldilocks paper but I can't get it right

2.3.1 Leave-One-Out CV

Based on the suggestion of a referee, in Remark 5.2 of Ledoit and Wolf (2012), a cross-validated estimator is proposed. It is given here. Let $\left(\lambda_1^{(k)}, \ldots, \lambda_N^{(k)}\right)$ and $\left(u_1^{(k)}, \ldots, u_N^{(k)}\right)$ denote a system of eigenvalues and eigenvectors of the sample covariance matrix computed from all the observed data, except for the k-th observation x_k.

From Sections 2.2.1 and 2.2.2, the quantity to compute is,

$$\tilde{d}_i = u_i^T \Sigma_N u_i,$$

which leads to the cross-validation approximation,

$$\tilde{d}_i \approx \rho^{cv}(\lambda_i) \equiv \frac{1}{T} \sum_{k=1}^{T} ({u_i^{(k)}}^T x_k)^2$$

The motivation comes from the fact that,

$$({u_i^{(k)}}^T x_k)^2 = {u_i^{(k)}}^T x_k x_k^T u_i^{(k)}$$

where x_k is independent of $u_i^{(k)}$ and $\mathbb{E}\left[x_k x_k^T\right] = \Sigma_N$. Therefore,

$$\mathbb{E}\left[\rho^{cv}(\lambda_i)\right] = \mathbb{E}\left[{u_i^{(k)}}^T x_k x_k^T u_i^{(k)}\right] = \mathbb{E}\left[{u_i^{(k)}}^T \Sigma_N u_i^{(k)}\right]$$

The full algorithm is given by Algorithm 1.

Algorithm 1 Leave-One-Out Cross Validation for Non-Linear Shrinkage

Require: $X = [x_1, \ldots, x_T]^T \in \mathbb{R}^{T \times N}$
Require: $\Sigma_N = \mathrm{Cov}(X)$
procedure LOO-CV(X)
  $S \leftarrow \frac{1}{T} X^T X$
  $U \leftarrow [u_1, \ldots, u_N] \in \mathbb{R}^{N \times N}, \quad \Lambda \leftarrow \mathrm{Diag}(\lambda_1, \ldots, \lambda_N), \quad \text{s.t.} \quad S = U \Lambda U^T$
  $\tilde{d} \leftarrow [0, \ldots, 0]^T \in \mathbb{R}^N$
  while $k \leq T$ **do**
    $X_{-k} \leftarrow [x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_T]^T$
    $S^{(k)} \leftarrow \frac{1}{T-1} X_{-k}^T X_{-k}$
    $U^{(k)} \leftarrow [u_1^{(k)}, \ldots, u_N^{(k)}] \in \mathbb{R}^{N \times N}, \quad \Lambda^{(k)} \leftarrow \mathrm{Diag}(\lambda_1^{(k)}, \ldots, \lambda_N^{(k)}), \quad \text{s.t.} \quad S^{(k)} = U^{(k)} \Lambda^{(k)} {U^{(k)}}^T$
    $\tilde{d}_i \leftarrow \tilde{d}_i + \frac{1}{T} ({U^{(k)}}^T x_k)_i^2, \quad i = 1, \ldots, N$
    $k \leftarrow k + 1$
  end while
  $\hat{\Sigma}_N \leftarrow U D U^T, \quad \text{where} \quad D = \mathrm{Diag}(\tilde{d}_1, \ldots, \tilde{d}_N)$
  return $\hat{\Sigma}_N$ ▷ The LOO-CV non-linear shrinkage estimator for Σ_N.
end procedure

2.3.2 K-Fold CV

The standard LOO cross-validation estimator in the previous section is known to have poor properties, as discussed in Bartz (2016). One possible fix is to use K-fold sampling of the data to produce an estimate,

$$\rho^{kcv}(\lambda_i) \equiv \frac{1}{K} \sum_{k=1}^{K} {u_i^{(k)}}^T X_k X_k^T u_i^{(k)}$$

where X_k contains the observations from the k-th fold and $u_i^{(k)}$ is the i-th eigenvector of the sample covariance matrix computed on the data with the k-th fold removed.

Algorithm 2 K-Fold Cross Validation for Non-Linear Shrinkage

Require: $X = [x_1, \ldots, x_T]^T \in \mathbb{R}^{T \times N}$
Require: $\Sigma_N = \mathrm{Cov}(X)$
Require: $K \in \mathbb{Z}, \quad K > 1$
procedure K-FOLD-CV(X)
  $S \leftarrow \frac{1}{T} X^T X$
  $U \leftarrow [u_1, \ldots, u_N] \in \mathbb{R}^{N \times N}, \quad \Lambda \leftarrow \mathrm{Diag}(\lambda_1, \ldots, \lambda_N), \quad \text{s.t.} \quad S = U \Lambda U^T$
  $m \leftarrow T/K$
  $\mathcal{K} \leftarrow \{(1, \ldots m), \ldots, (T - m + 1, \ldots, T)\}$
  $\tilde{d} \leftarrow [0, \ldots, 0]^T \in \mathbb{R}^N$
  $k \leftarrow 1$
  while $k \leq K$ **do**
    $X_{-k} \leftarrow [x_{j_1}, \ldots, x_{j_{(T-m)}}]^T \in \mathbb{R}^{(T-m) \times N}, \quad j_i \notin \mathcal{K}_k$
    $X_k \leftarrow [x_{\ell_1}, \ldots, x_{\ell_m}]^T \in \mathbb{R}^{m \times N}, \quad \ell_i \in \mathcal{K}_k$
    $S^{(k)} \leftarrow \frac{1}{T-m} X_{-k}^T X_{-k}$
    $U^{(k)} \leftarrow [u_1^{(k)}, \ldots, u_N^{(k)}] \in \mathbb{R}^{N \times N}, \quad \Lambda^{(k)} \leftarrow \mathrm{Diag}(\lambda_1^{(k)}, \ldots, \lambda_N^{(k)}), \quad \text{s.t.} \quad S^{(k)} = U^{(k)} \Lambda^{(k)} {U^{(k)}}^T$
    $\tilde{d}_i \leftarrow \tilde{d}_i + \frac{1}{T} {u_i^{(k)}}^T (\frac{1}{m} X_k^T X_k) u_i^{(k)} \quad i = 1, \ldots, N$
    $k \leftarrow k + 1$
  end while
  $\hat{\Sigma}_N \leftarrow U D U^T, \quad \text{where} \quad D = \mathrm{Diag}(\tilde{d}_1, \ldots, \tilde{d}_N)$
  return $\hat{\Sigma}_N$ ▷ The K-Fold-CV non-linear shrinkage estimator for Σ_N.
end procedure

2.3.3 Isotonic-Leave-One-Out CV

Another easy and well-behaved fix is to apply isotonic regression to produce an improved estimator $\rho^{icv}(\lambda_i)$. This is given by the solution to the optimization problem,

$$\begin{aligned} \underset{a_1,\ldots,a_N}{\text{minimize}} \quad & \sum_{i=1}^{N} (a_i - \rho^{cv}(\lambda_i))^2 \\ \text{subject to} \quad & a_i > a_{i+1}, \\ & \sum_{i=1}^{N} a_i = \sum_{i=1}^{N} \rho^{cv}(\lambda_i). \end{aligned}$$

2.3.4 Isotonic-K-Fold CV

The isotonic regression can also be applied to the K-fold CV estimator that will be called $\rho^{ikcv}(\lambda_i)$.

2.3.5 Performance

Bartz (2016) shows that $\rho^{cv}(\lambda_i)$ is terrible as an estimator. The main issue is instability in the eigenvalues/eigenvectors when dropping a single observation. This is most simply mitigated by using $\rho^{kcv}(\lambda_i)$, which shows huge improvement in Figure 2.1. Applying isotonic regression, whether to the $\rho^{cv}(\lambda_i)$ or $\rho^{kcv}(\lambda_i)$ gives the best results.

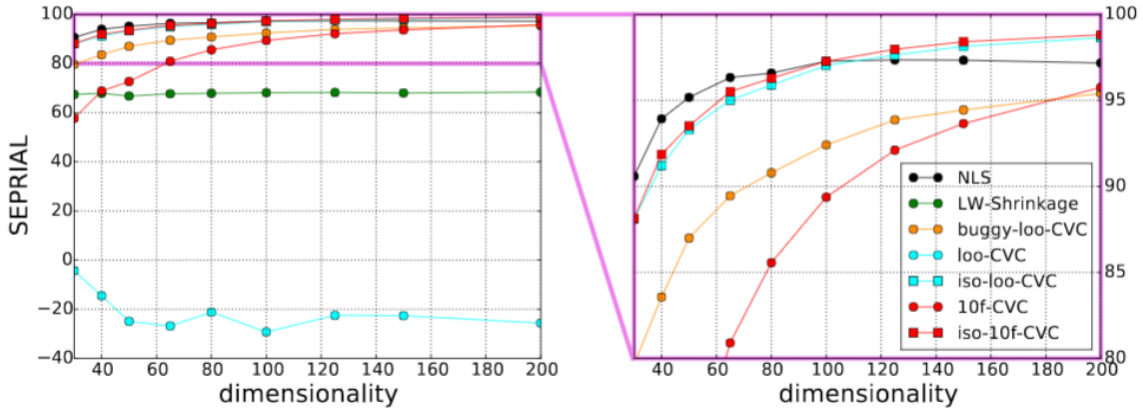

Figure 2.1: From Bartz (2016). Estimation error of cross- validated covariance estimators from Bartz (2016). Averaged over 50 repetitions. *NLS* refers to the LW optimal shrinkage. *LW-Shrinkage* refers to the linear shrinkage of Ledoit and Wolf (2004). *buggy-loo-CVC* refers to an incorrectly implemented leave-one-out cross-validated estimator that appears in Ledoit and Wolf (2012). *loo-CVC* refers to the correctly implemented leave-one-out cross-validated estimator proposed in Ledoit and Wolf (2012) and described in Section 2.3.1. *iso-loo-CVC* refers to the isotonic version of the leave-one-out cross- validated estimator described in 2.3.3. *10f-CVC* refers to the 10-fold cross-validated estimator described in 2.3.2. And *iso-10f-CVC* refers to the 10-fold, isotonic cross-validated estimator described in 2.3.4.

3 Optimal Shrinkage for Unbiased Variance Ratios

There are a few problems with the above methodologies that we have run into (and need to solidly verify numerically):

1. The above method produces a covariance matrix that will minimize out-of- sample variance for any portfolio that is colinear with that tangency portfolio. In that sense, it is optimal. But what about another portfolio? What about the minimum variance (global or long-only) portfolio? Is it still the correct shrinkage if you are interested in computing the minimum variance portfolio?
2. Numerical results (need to verify again) indicate the variance ratios for minimum variance portfolio computed from the optimal non-linear shrunk covariance matrix are suboptimal. That is, they are not unbiased around 1. As we will see, the optimal non-linear shrunk covariance matrix should produce good variance ratios for the tangency portfolios (need to verify numerically).

Consider the loss function,

$$\mathcal{R}(\hat{\Sigma}_N, \Sigma_N, \hat{w}_N) := \left(1 - \frac{\hat{w}_N^T \hat{\Sigma}_N \hat{w}_N}{\hat{w}_N^T \Sigma_N \hat{w}_N}\right)^2$$

where $\hat{w}_N$ is some portfolio rule dependent on the covariance matrix and $\hat{\Sigma}_N$ is an estimator.

3.1 Maximum Sharpe Portfolio

Theorem 3.1 (Oracle Shrinkage for Variance Ratio of Maximum Sharpe Ratio Portfolio)**.** *Under the same setup as Theorem 4.1 of Ledoit and Wolf (2017), the same oracle estimator of the covariance matrix that that minimizes out-of- sample variance also minimizes the almost sure limit of the loss function* $\mathcal{R}(\hat{\Sigma}_N, \Sigma_N, \hat{w}_N)$.

Sketch:

If we consider the same maximum Sharpe ratio portfolios as before, then,

$$\mathcal{R}(\hat{\Sigma}_N, \Sigma_N, \hat{w}_N) := \left(1 - \frac{m^T \hat{\Sigma}_N^{-1} \Sigma_N \hat{\Sigma}_N^{-1} m}{m^T \hat{\Sigma}_N^{-1} m}\right)^2.$$

From the same theory as above,

$$\mathcal{R}(\hat{\Sigma}_N, \Sigma_N, \hat{w}_N) \to \left(1 - \frac{\int \frac{\delta(\lambda)}{\hat{\rho}(\lambda)^2} dF(\lambda)}{\int \frac{1}{\hat{\rho}(\lambda)} dF(\lambda)}\right)^2,$$

which is minimized and equal to 0 if $\hat{\rho}(\lambda) = \delta(\lambda)$. Moreover, this indicates the optimal non-linear shrinkage estimator should provide good variance forecasting for the maximum Sharpe ratio portfolio.

3.2 Global Minimum Variance Portfolio

Now we consider the global minimum variance portfolio,

$$\hat{w}_N = \frac{\hat{\Sigma}_N^{-1} \mathbb{1}}{\mathbb{1}^T \hat{\Sigma}_N^{-1} \mathbb{1}}.$$

The loss function is given by,

$$\mathcal{R}(\hat{\Sigma}_N, \Sigma_N, \hat{w}_N) = \left(1 - \frac{\mathbb{1}^T \hat{\Sigma}_N^{-1} \Sigma_N \hat{\Sigma}_N^{-1} \mathbb{1}}{\mathbb{1}^T \hat{\Sigma}_N^{-1} \mathbb{1}}\right)^2.$$

Letting $\alpha = U^T \mathbb{1}$, the above loss can be written as,

$$\mathcal{R}(\hat{\Sigma}_N, \Sigma_N, \hat{w}_N) = \left(1 - \frac{\alpha^T D^{-1} U^T \Sigma_N U D^{-1} \alpha}{\alpha^T D^{-1} \alpha}\right)^2.$$

3.2.1 Oracle Optimal Shrinkage

Theorem 3.2 (Oracle Shrinkage for Variance Ratio of Minimum Variance Portfolio)**.** *Let $X \in \mathbb{R}^{T \times N}$ denote T i.i.d. draws from an N-dimensional distribution with covariance Σ_N. Define the sample covariance matrix as $S = \frac{1}{T} X^T X$ with eigendecomposition $S = U \Delta U^T$.*

Let the class of rotation invariant covariance estimators be defined by,

$$\hat{\Sigma}_N = U D U^T$$

where $D = \mathrm{Diag}(d_1, \ldots, d_N)$ are the chosen eigenvalues for the estimator.

Finally, define the plug-in estimator for global minimum variance portfolio as

$$\hat{w}_N = \frac{\hat{\Sigma}_N^{-1} \mathbb{1}}{\mathbb{1}^T \hat{\Sigma}_N^{-1} \mathbb{1}}.$$

The optimal finite sample oracle shrinkage estimator $\hat{\Sigma}_N^* = U D^* U^T$ with respect to the loss function $\mathcal{R}(\hat{\Sigma}_N, \Sigma_N, \hat{w}_N)$ is given by $D_{ii}^* = d_i^* = 1/z_i$ where $z \in \mathbb{R}^N$ is the solution to the linear system,

$$CAz = \alpha.$$

for $C = U^T \Sigma_N U$, $\alpha = U^T \mathbb{1}$, *and* $A = \text{Diag}(\alpha_1, \ldots, \alpha_N)$.

This estimator also minimizes out-of-sample variance,

$$\mathcal{L}(\hat{\Sigma}_N, \Sigma_N, \hat{w}_N) = \hat{w}_N^T \Sigma_N \hat{w}_N = \frac{\mathbb{1}^T \hat{\Sigma}_N^{-1} \Sigma_N \hat{\Sigma}_N^{-1} \mathbb{1}}{\left(\mathbb{1}^T \hat{\Sigma}_N^{-1} \mathbb{1}\right)^2}$$

Sketch:

From the preceding section, we can write,

$$\alpha^T D^{-1} \alpha = \text{Tr}(D^{-1} \alpha \alpha^T) = \sum_{i=1}^N \frac{\alpha_i^2}{d_i}$$

and,

$$\alpha^T D^{-1} U^T \Sigma_N U D^{-1} \alpha = \text{Tr}(D^{-1} U^T \Sigma_N U D^{-1} \alpha \alpha^T) = \sum_{i=1}^N \frac{\alpha_i}{d_i} \sum_{j=1}^N \frac{C_{ij} \alpha_j}{d_j}$$

where $C = U^T \Sigma_N U$ and $C_{ij} = u_i^T \Sigma_N u_j$.

If we can choose d_i to equate α_i and $\frac{C_{ji}\alpha_j}{d_j}$, then the loss function $\mathcal{R}$ will be minimized and equal to 0. This can be achieved by solving a linear system.

Let,

$$A = \text{Diag}(\alpha_1, \ldots, \alpha_N)$$

Then $d_i = 1/z_i$ where $z \in \mathbb{R}^N$ is the solution to,

$$CAz = \alpha.$$

This gives

$$z = A^{-1} U^T \Sigma_N^{-1} U \alpha = A^{-1} U^T \Sigma_N^{-1} \mathbb{1}$$

It turns out this estimator also minimizes out-of-sample variance for the global minimum variance portfolio.

The out-of-sample variance simplifies to,

$$\mathcal{L}(\hat{\Sigma}_N, \Sigma_N, \hat{w}_N) = \frac{\alpha^T D^{-1} U^T \Sigma_N U D^{-1} \alpha}{\left(\alpha^T D^{-1} \alpha\right)^2}$$

Expanding terms as before and differentiating with respect to the i-th entry in D gives,

$$\frac{\partial \mathcal{L}}{\partial d_i} = -2 \frac{\frac{\alpha_i}{d_i^2} \sum_{j=1}^N \frac{C_{ij}\alpha_j}{d_j}}{Q_2^2} + 2 \frac{Q_1}{Q_2^3} \frac{\alpha_i^2}{d_i^2}$$

where $Q_1 = \alpha^T D^{-1} U^T \Sigma_N U D^{-1} \alpha$ and $Q_2 = \alpha^T D^{-1} \alpha$.

The first-order condition $0 = \frac{\partial \mathcal{L}}{\partial d_i}$ reduces to,

$$Q_2 \sum_{j=1}^N \frac{C_{ij}\alpha_j}{d_j} = Q_1 \alpha_i.$$

If d_i derived from the solution to $CAz = \alpha$, then $Q_2 = Q_1$ and the first-order condition is satisfied.

To see this is a minimum, we rely on the chain-rule. Let $F(w) = w^T \Sigma_N w$ where w is the plug-in estimator for global minimum variance portfolio $\hat{w}_N$ and is therefore a function of D.

By the chain rule,

$$\nabla_D^2 F(w) = \nabla_D^T w \cdot \nabla_w^2 F(w) \cdot (\nabla_D^T w)^T + \nabla_D^2 w \cdot \nabla_w F(w)$$

where,

$$\begin{aligned}
(\nabla_D^2 F(w))_{ij} &= \frac{\partial^2 F(w)}{\partial d_i \partial d_j}, \\
(\nabla_w F(w)_i &= \frac{\partial F(w)}{\partial w_i}, \\
(\nabla_w^2 F(w))_{ij} &= \frac{\partial^2 F(w)}{\partial w_i \partial w_j}, \\
(\nabla_D^T w)_{ij} &= \frac{\partial w_i}{\partial d_j}, \\
(\nabla_D^2 w)_{ijk} &= \frac{\partial^2 w_i}{\partial d_j \partial d_k}.
\end{aligned}$$

The first-order condition means $\nabla_w F(w) = 0$ and thus since $\nabla_w^2 F(w) = 2\Sigma_N \succ 0$, $\nabla_D^2 F(w) \succ 0$. This verifies that the oracle estimator for D^* minimizes out-of-sample variance.

3.2.2 Asymptotically Optimal Shrinkage

From the above, it may be possible to put $\mathcal{R}(\hat{\Sigma}_N, \Sigma_N, \hat{w}_N)$ and the optimal oracle shrinkage in the RMT asymptotic framework and derive the limiting results that would indicate how one should choose D. One possible work that may have related results is Mestre (2008) which potentially derive the asymptotic results for these kinds of functionals. More needs to be done to study that work and check if it fits the framework.

3.2.3 Proposed Estimators

The solution z to the linear system,

$$CAz = \alpha,$$

has a few issues with it. First, despite being an oracle estimator like the optimal values in (2.1), the resulting values are very noisy. The optimal values in (2.1) are also noisy so this is not a problem exclusive to the new values. Second, there are no guarantees on positivity or monotonicty of the values z or d.

The following are proposed estimators that seek to remedy these problems and serve as *bona fide* estimators.

1. *Vanilla Leave-One-Out with Isotonic Regression*

   The first proposed estimator is a simple cross-validation estimator using Leave- One-Out cross-validation to generate multiple solutions to the above linear system to stabilize estimates of the eigenvalues.

   Let $C^{(k)} = {U^{(k)}}^T x_k x_k^T U^{(k)}$, $A^{(k)}$, and $\alpha^{(k)}$ be the relevant quantities computed when leaving out the k-th observation, x_k.

   Then compute a set of intermediate values

   $$\bar{d}_i = \frac{1}{T} \sum_{k=1}^{T} \frac{1}{z_i^{(k)}}$$

   where $z^{(k)}$ is the solution to,

   $$C^{(k)} A^{(k)} z^{(k)} = \alpha^{(k)}.$$

   Finally, the final estimator values are obtained following applying isotonic regression such that,

   $$\hat{d}^* = \text{Isotonic}(\bar{d}).$$

Variants could use a median computation instead of an averaging for computing $\bar{d}$. This is likely to be a better option because evidence suggests the linear system produces widely ranging values. Simple constraints like non- negativity could also be used.

2. *Vanilla K-Fold with Isotonic Regression*

   A simple of the Leave-One-Out estimator, it uses K-Fold cross-validation instead.

3. *Joint Leave-One-Out with Isotonic Regression*

   Averaging of the individual cross-validation values serves to denoise the raw values to produce better performance before applying Isotonic regression. The solution z could be obtained from solving the least squares problem,

$$\text{minimize} \quad \sum_{k=1}^{T} \|C^{(k)} A^{(k)} z - \alpha^{(k)}\|_2^2,$$

   so that $\bar{d}_i = \frac{1}{z_i}$. The final estimator is obtained via Isotonic regression,

$$\hat{d}^* = \text{Isotonic}(\bar{d}).$$

4. *Joint K-Fold with Isotonic Regression*

   Again, similar to the above but with K-Fold cross-validation instead.

5. *Nonlinear Leave-One-Out with Isotonic Regression*

   Instead of solving in the changed variables $z_i = \frac{1}{d_i}$, one can directly solve for $d_i = D_{ii}$ in,

$$\mathcal{R}(\hat{\Sigma}_N, \Sigma_N, \hat{w}_N) = \left(1 - \frac{\alpha^T D^{-1} U^T \Sigma_N U D^{-1} \alpha}{\alpha^T D^{-1} \alpha}\right)^2.$$

   Of course, to make this a *bona fide* estimator, we replace $U^T \Sigma_N U$ and α by $C^{(k)}$ and $\alpha^{(k)}$, respectively, to create the objective function,

$$f(d; C, \alpha) = \left(1 - \frac{\alpha^T D^{-1} C D^{-1} \alpha}{\alpha^T D^{-1} \alpha}\right)^2,$$

   where $D = \text{Diag}(d)$. With $f(d; C, \alpha)$, we can employ several different forms of estimator.

   This basic estimator is computed similarly to the vanilla version. Let $d^{(k)}$ denote the solution to

$$\text{minimize} \quad f(d; C^{(k)}, \alpha^{(k)})$$

   where an optional non-negativity constraint $d \geq 0$ could be added. Then compute the set of intermediate values,

$$\bar{d} = \frac{1}{T} \sum_{k=1}^{T} d^{(k)},$$

   and finally apply Isotonic regression to get,

$$\hat{d}^* = \text{Isotonic}(\bar{d}).$$

   As with the vanilla estimator, a variant could use a median computation instead of an averaging for computing $\bar{d}$.

6. *Nonlinear K-Fold with Isotonic Regression*

   A similar estimator could be generated using K-fold cross-validation instead.

7. *Joint Nonlinear Leave-One-Out with Isotonic Regression*

   Optimal values d could be obtained through joint minimization of $f(d; C^{(k)}, \alpha^{(k)})$ for $k = 1, \ldots, T$. Intermediate values are obtained from the optimization problem,

   $$\text{minimize} \quad \sum_{k=1}^{T} f(d; C^{(k)}, \alpha^{(k)}).$$

   The final estimator is obtained after Isotonic regression,

   $$\hat{d}^* = \text{Isotonic}(\bar{d}).$$

8. *Joint Nonlinear K-Fold with Isotonic Regression*

   Replace Leave-One-Out cross-validation above with K-fold cross-validation.

9. *Joint Constrained Nonlinear Leave-One-Out*

   Isotonic regression serves to create monotonicity in the estimated eigenvalues. In the nonlinear optimization of $\sum_{k=1}^{T} f(d; C^{(k)}, \alpha^{(k)})$, monotonicity constraints could be directly applied in the optimization problem. Isotonic regression also seeks to preserve the trace of the estimated eigenvalues so an additional trace constraint can be added.

   For some target trace value c and bounds $d_{\min}$, $d_{\max}$, optimal values are obtained from the optimization problem,

   $$\begin{array}{ll} \text{minimize} & \sum_{k=1}^{T} f(d; C^{(k)}, \alpha^{(k)}) \\ \text{subject to} & Gd \preceq 0, \\ & \mathbb{1}^T d = c, \\ & d_{\min} \leq d \leq d_{\max}, \end{array}$$

   where $G \in \mathbb{R}^{(N-1)\times N}$ is a differencing matrix to ensure monotonicity.

   Variants could remove the trace constraint or either or both of the bounds.

   Practical considerations for solving the optimization problem:

   - Gradient:

   $$\frac{\partial}{\partial d_i} f(d; C, \alpha) = 2\left(1 - \frac{\alpha^T D^{-1} C D^{-1} \alpha}{\alpha^T D^{-1} \alpha}\right)\left(-\frac{-2\alpha_i}{d_i^2} \sum_{j=1}^{N} \frac{C_{ij}\alpha_j}{d_j} \frac{1}{\alpha^T D^{-1} \alpha} + \frac{-\alpha_i^2}{d_i^2} \frac{\alpha^T D^{-1} C D^{-1} \alpha}{(\alpha^T D^{-1} \alpha)^2}\right)$$

   - Change of Variables: G adds $N-1$ linear inequality constraints to the problem. Augmenting a final row in G to preserve d_N creates an invertible transformation. Changing to differences gives:

     $$y = Gd, \quad g(y; C, \alpha) = f(G^{-1}y; C, \alpha), \quad \nabla_z g(y; C, \alpha) = G^{-T} \nabla_d f(G^{-1}y; C, \alpha),$$

     The new optimization problem is,

     $$\begin{array}{ll} \text{minimize} & \sum_{k=1}^{T} g(y; C^{(k)}, \alpha^{(k)}) \\ \text{subject to} & y \geq [0, \ldots, 0, d_{\min}]^T, \\ & v^T y = c, \\ & d_{\max} - \mathbb{1}^T y \geq 0, \end{array}$$

     where $v = G^{-T}\mathbb{1}$ and creates the modified trace constraint, and the final constraint represents the upper bound $d_1 \leq d_{\max}$.
     In implementation, G, G^{-1}, and $G{-}T$ can be performed without constructing matrices when inside the objective function or when transforming the gradient. As a constraint, general nonlinear optimization routines will require G in matrix form as a Jacobian of the constraint function.

   - Preconditioning: This is still an open issue but a further scaling is possible such that $z = Py$ where P is a diagonal matrix.

10. *Joint Constrained Nonlinear K-Fold*

    Replace Leave-One-Out cross-validation above with K-fold cross-validation.

4 To Address

4.1 Metrics

- Out-of-Sample Variance
- Variance forecast
- Bias statistics: $r_w/(w\Sigma w)^{1/2}$

4.2 Portfolios

- Global minimum variance
- Long-only minimum variance
- Equally weighted
- Maximum Sharpe Ratio
- Concentrated

4.3 Models

- Covariances: simple factor, broad-narrow factor, identity, known H with random U or identity U
- Returns: equivariant

4.4 Questions to Address

- What is the baseline performance of the classical sample covariance estimator?
- Does long-only outperform out-of-sample var vs global min var?
- When substituting in true eigenvectors and/or true eigenvalues, which is most important in fixing issues of estimation? What does that tell us about covariance estimation?
- How does SLR improve when model is big factor model? Where are its deficiencies? With respect to which metrics or portfolios?
- How does non-linear shrinkage do? Does the numerical verify the theory? How is the var ratio for the Max Sharpe and min var portfolios? Compare out of sample variance to portfolios along mean-variance frontier (true and estimated)
- Can we replicate the success/failure of non-linear shrinkage on the global min var and the long-only min var?
- For the oracle estimator for best var ratio performance: what portfolio does that give? Where does it fall on mean-variance frontier (true or estimated)? Does it give unbiased var ratio? Does it give good var performance too despite being designed for var ratio?
- Does the cross-validated optimal var ratio estimator work?

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