S-Partitions and S-Shellings

Alexandru Papiu

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Let Δ be a simplicial complex. We define an **S-partition** to be an ordering of (not necessarily maximal) faces of Δ: $F_1, F_2, ..., F_k$ such that $F_i \cap (F_1 \cup ... \cup F_{i-1})$ is pure and $(dimF_i - 1)$-dimensional and all the facets of Δ are included in the ordering. If all the faces in the ordering are *facets* we recover the Bjorner-Wachs definition for a non-pure shelling. S-partitions were introduced by Chari in [1].

Similar to a shelling, an S - partition gives a partition of (the poset) Δ into intervals $[G_i, F_i]$ such that $\cup[G_i, F_i]$ is a simplicial complex for any i. We will call the singleton intervals of the form $[F, F]$ **critical** faces.

Chari showed in [2] that given an S-partition for Δ one can construct discrete Morse functions on Δ whose critical faces are exactly the critical faces in the S-partition.

Note that any simplicial complex admits many S-partitions. In particular we have the trivial S - partition into singelton intervals. Clearly this is not a very useful S-partition. The basic theme in this note is this: the "coarser" and S-partition is the more information it will gives us about our simplicial complex Δ both at an algebraic and homological level.

Let's also define the h^S **triangle** as follows: Let $h^S_{s,i}$ to be the number of intervals in $\mathcal{S}$ of the form $[r(F), F]$ such that $|F| = s$ and $|r(F)| = i$. Notice that

$$h_i = h_{i,i} + h_{i+1,i} + ... + h_{d,i}$$

and $c_i := h_{i,i}$ is the number of critical i-cells in $\mathcal{S}$ as well as in the corresponding Morse function $f_{\mathcal{S}}$. Denote by $c^{\mathcal{S}}(t) = \sum c_i t^i$.

Note that the h^S triangle determines the f vector by the following relation:

$$f(t) = \sum_{i,j} h_{i,j} t^j (1+t)^{i-j}$$

We can also express the h-vector in terms of the h^S triangle as follows. Using the definition of the f polynomial in terms of the h-polynomial we have

$$(1+t)^d h(\frac{t}{1+t}) = \sum_{i,j} h_{i,j} t^j (1+t)^{i-j}$$

and by doing a change of variable $\lambda = \frac{t}{1+t}$ we get

$$h(\lambda) = \sum_{i,j} h_{i,j} \lambda^j (1-\lambda)^{d-i}$$

Let's quickly introduce some algebraic notation - a good reference is [3].
The **Stanley-Reisner ring** associated to a complex Δ on vertex set $\{1, 2, ..., n\}$ is defined as the quotient ring

$$k[\Delta] = k[x_1, ..., x_n]/I_\Delta$$

where I_Δ is the ideal generated by the square-free monomials corresponding to the non-faces of Δ. An **l.s.o.p** is a collection of linear forms $\{\theta_1, ..., \theta_d\}$ in $k[x_1, ..., x_n]$, such that $k[\Delta]/(\theta_1, ..., \theta_d)$ is a finite dimensional k-vector space. We will denote $k[\Delta]/(\theta_1, ..., \theta_d)$ by $k(\Delta)$ and call it the **reduced Stanley-Resiner ring** of Δ for the specific l.s.o.p we have chosen.

We will also need the following technical definitions and result due to Stanley that characterizes l.s.o.p's in terms of a choice function. We are following the presentation in [5], section 12. For a set of linear forms $\{\theta_1, \theta_2, ..., \theta_n\}$ in $k[\Delta]$ let $M = (m_{i,j})$ be the $d \times n$ matrix defined by $\theta_i = \sum_{j=1}^{n} m_{i,j} x_j$. Let $F_1, ..., F_t$ be the *facets* of Δ and call a function $C : [t] \to 2^{[d]}$ a **nonsingular choice function** if $|C(j)| = |F_j|$ and the square submatrix with rows in $C(j)$ and columns in F_j is nonsingular, for all facets F_j.

Lemma 0.1. *(Stanley in [4], page 150) Let $\{\theta_1, \theta_2, ..., \theta_n\}$ be a set of linear forms in $k[\Delta]$. Then $\{\theta_1, \theta_2, ..., \theta_n\}$ is an l.s.o.p if and only if there exists a non-singular choice function.*

By Chari's results in [2] one can use the fundamental theorem of discrete Morse theory to show that the critical faces in an S-partitions act as a spanning set for the homology $H_*(\Delta)$. We will show next that a similar results holds true at the algebraic level in $k(\Delta)$ in the following Lemma. One can interpret as a weak version of the Klee-Kleinschmidt Lemma for shellable complexes.

Lemma 0.2. *Let $\mathcal{S}$ be an S-partition for Δ with $\Delta = \cup_{i=0}^{n} [r(F_i), F_i]$. The monomials $\{x^{r(F)} : |r(F)| = i\}$ span $k(\Delta)_i$.*

Proof. We will prove this by induction on the number of partitions. If the partition has one element, the restriction will be the empty set and $k(\Delta) = k$ as a k-vector space so the lemma is true in this case.

Now assume we have added faces $F_1, ..., F_{k-1}$ and now we are adding the interval $[r(F_k), F_k]$. Since $r(F_k)$ is the unique minimal non-face added we get that

$$k[\Delta_k]/(x^{r(F_k)}) = k[\Delta_{k-1}]$$

as rings.

Now let θ be an l.s.o.p or Δ_k. By Lemma 0.1 above this will also be an l.s.o.p for Δ_{k-1} so we get that:

$$k(\Delta_k)/(x^{r(F_k)}) = k(\Delta_{k-1})$$

.

Now by the induction hypothesis we have that $\{x^{r(F_1)}, ..., x^{r(F_{k-1})}\}$ span $k(\Delta)$ so it suffices to show that $x^{r(F_k)} x_i = 0$ in $k(\Delta)$ for any x_i.

If $i \notin F_k$ then $\{i\} \cup r(F_k)$ is not a face of Δ_k so $x^{r(F_k)}x_i = 0$ in $k[\Delta]$ and thus in $k(\Delta)$ as well.

Now let's assume $i \in F_k$ and F_k has cardinality $l < d$. By Lemma 0.1 there exits a nonsingular choice function C. Now let's select the $C(k)$ rows in the matrix $M = (m_{i,j})$ defined as above by $\theta_i = \sum_{j=1}^{n} m_{i,j}x_j$. This gives us a $l \times n$ matrix. Since the $l \times l$ restriction associated with the facet F_k is non-singular we can now use Gaussian elimination to express x_i in terms of the θ's and monomials not in F_k:

$$x_i = \sum_{j=1}^{l} \alpha_j \theta_{C(j)} + \sum_{j \notin F_k} \beta_j x_j$$

with the α's and β's in k. When we multiply by $x^{r(F_k)}$ we get both sums on the right to be zero in $k(\Delta)$. Thus $x_i x^{r(F_k)} = 0$ in $k(\Delta)$ and we are done.

□

Based on the previous lemma and Chari's result on Morse functions we get the following **Corollary**:

$$c^{\mathcal{S}}(t) \geq Hilb(H_*(\Delta), k)(t)$$
$$h^{\mathcal{S}}(t) \geq Hilb(k(\Delta), k)(t)$$

Where $Hilb(H_*(\Delta), k)(t)$ is the Betti polynomial for Δ over k counting homology ranks and $Hilb(k(\Delta), k)(t)$ is the $\mathbb{Z}$-graded Hilbert series for the reduced Stanley Reisner ring. These inequalities lead naturally to the following definitions:

Given a field k. An S-partition is **k-perfect** if the first inequality is an equality. This is equivalent to saying that the Morse function associated to the S-partition is k-perfect.

Given a field k and an l.s.o.p θ. An S-partition is an (S, θ) **- shelling** if the second inequality is an equality. This is equivalent to saying that the restriction monomials are a basis for the $\mathbb{Z}$-graded reduced Stanley-Reisner Ring.

An S-partition is **minimal** if it contains the smallest number of intervals possible.

The definition of an S-shelling, as it stands is dependent on the system of parameters we choose and this is not a desirable feat - we'd like a more combinatorial interpretation of when an S-partition is an S-shelling. It turns out that one can get such a characterization for all triangualted manifolds. This is because Schenzel's formula gives the graded dimensions of $k(\Delta)$ in terms of the h vector and homology of Δ. This allows us a clean characterization of an S-shelling without having to use the l.s.o.p directly:

Lemma 0.3. *Let Δ be a $d-1$, Buchsbaum complex (this includes all triangulated manifolds with or without boundary) and $\mathcal{S}$ and S-partition for Δ. Then $\mathcal{S}$ is an S- shelling if and only if it has length*

$$f_{d-1} + \sum_{i=1}^{d-1} \beta_{i-1}(\Delta) \binom{d-1}{i}$$

Proof. By Schenzel's Formula [3] Theorem 29 we can compute the graded dimensions of $k(\Delta)$ as follows:

$$dim_k(k(\Delta)_j) = h_j(\Delta) + \binom{d}{j} \sum_{i=1}^{j-1} (-1)^{j-1-i} \beta_{i-1}(\Delta, k)$$

.

Now adding all the graded parts we get that $k(\Delta)$ has dimension:

$$\sum_{j=0}^{d} h_j(\Delta) + \sum_{j=0}^{d} \binom{d}{j} \sum_{i=1}^{j-1} (-1)^{j-1-i} \beta_{i-1}(\Delta, k)$$

The first sum adds to f_{d-1} and the second sum is equal to

$$\sum_{i=1}^{d-1} \beta_{i-1}(\Delta) \sum_{j=i+1}^{d} \binom{d}{j} (-1)^{j-i-1} = \sum_{i=1}^{d-1} \beta_{i-1}(\Delta) \binom{d-1}{i}$$

and the result follows. □

For a 2-manifold Lemma 0.3 tells us that an S-partition is an S-shelling if and only if it has exactly $f_2 + \beta_1(\Delta, k)$ parts. This means that our S-shelling will correspond to adding the facets just like in a shelling plus critical edges, one for each basis cycle in $H_1(\Delta, k)$.

Question:A natural question to ask at this point is the following: What is the relationship between a k-perfect S-partition, minimal S-partition, and an S-shelling?

Note that since the restriction mononials span $k(\Delta)$, an S-shelling will always be minimal. However a k-perfect S-partition need not be minimal. Any collapse of a collabsible complex will give a perfect S-parition but these will usually not be minimal since they are partitions containing only intervals of size two. One could try at this point to "consolidate" the 2-partitions to create a coarser S-partitions. However as we shall see there are complexes that admit k-perfect S-partitions but are not S-shellable.

Lemma 0.4. *There exist triangulated manifolds that admit k-perfect S-partitions but do not admit S-shellings.*

Proof. Notice that if we restrict ourselves to spheres an S-partition will be an S-shelling if and only if it is a pure shelling. This follows from Lemma 0.3 since a triangulated sphere only has non-trivial homology in the top dimension.

Also Δ will admit a perfect S-partition if and only if it admits a perfect Morse function. This follows from Chari's result.

So now in order to prove our lemma we have to come up with a triangulated sphere that is perfect but not shellable. Coming up with such examples is not very easy, however in [6][Section 5.5] Benedetti an Lutz given an examples of a triangulated 3-sphere with a knotted trefoil knot on 3-edges that admits a minimal Morse vector of $(1, 0, 0, 1)$. However such a sphere cannot be shellable because of the knot.

□

Question: Will a S-shelling always be k-perfect for any k? The answer is yes for 2-manifolds.

References

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