Connectivity of Pseudomanifolds

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1 Preliminaries and History

Given a polytope or a triangulated manifold it is natural to try to understand the structure of its 1-dimensional skeleton. This was first done by Steinitz in 1922 where he solved the problem in the 3-dimensional case: the graphs of 3-polytopes are exactly the 3-connected planar graphs. Balinksi extended these results to any dimension by proving that the graphs of d-polytopes are d-connected. This was generalized further by Barnette who showed that the graph of every d-dimensional polyhedral pseudomanifold is (d + 1)-connected.

More recently these results have been sharpened in cases where more is known about the structure of the simplical complex: Athanasiadis [1] proved better connectivity bounds for flag pseudomanifolds and Bjorner and Vorwerk [3] and Adiprasito, Goodarzi and Verbaro [4] exteneted the results for banner complexes.

Here we present a straightforward approach rooted in combinatorial topology that extends and simplifies previous approaches. Here is the basic idea: Let Δ be a triangulation of a (pseudo)manifold from which we remove a subset of vertices W together with the induced subcomplex Δ_W. We will analyze how the number of connected components of the 1-skeleton $G(\Delta)$ on $V \setminus W$ vertices relates to $H_{d-1}(\Delta_W)$. In some cases one completely determines the other, in other cases the way $H_{d-1}(\Delta_W)$ "sits" inside $H_{d-1}(\Delta)$ matters. For pseudomanifolds we obtain that if $H_{d-1}(\Delta_W)$ is trivial removing W does not disconnect the graph and use this to prove lower bounds on connectivity of different classes of complexes.

?maybe more background on simplicial complexes here?

Let Δ be a simplicial complex. The underlying graph (or 1-skeleton) $G(\Delta)$ of Δ is the graph obtained by restricting Δ to faces of cardinality at most two. A graph G is **k-connected** if it has at least k-vertices and removing any $k-1$ vertices does not disconnect G. All simplicial complexes we consider will be pure. We will work with homology over $\mathbb{Z}_2$.

A d-dimensional simplicial complex Δ is a **weak pseudomanifold** if every $d-1$ face is contained in exactly two d faces. If in addition the link of each face is connected we call Δ a **normal pseudomanifold**. A **pseudomanifold** is a weak pseudomanifold in which the facet graph is connected. It's not hard to show that every normal pseudomanifold is indeed a pseudomanifold.

? maybe more background on pseudomanifolds - they're minimal cycles ?

2 Main Connectivity Bound

We will introduce the notion of strong connected components and use it to prove the main theorem on the connectivity of manifolds. Let W be a subset of the vertex set of Δ. We define the following relation on the facets of Δ not contained in W: $F_1 \sim F_n$ if there is a sequence of facets $F_1, F_2, ..., F_n$ such that $F_i \cap F_{i+1}$ has co-dimension 1 and is not in W. It's easy to see that this is an equivalence relation and we will call the equivalence classes thus obtained **strong components of** Δ/W. We will denote the number of such classes by $S(\Delta/W)$.

Theorem 2.1. *Let Δ be a d-pseudomanifold and W a subset of vertices. If $H_{d-1}(\Delta_W) = 0$ then removing W does not disconnect $G(\Delta)$.*

Proof. We will in fact prove the stronger statement:

$$dimH_0(\Delta_{V-W}) \leq S(\Delta/W) \leq dimH_{d-1}(\Delta_W) + 1 \tag{1}$$

Let K be a strong component in Δ/W. Let v_1, v_2 be two vertices contained in $v_i \in F_i$ for $i = 1, 2$. We can then build a path between v_1, v_2 in Δ_{V-W} by following the sequence of facets connecting F_1 and F_2 and at each step choosing a point in $F_i \cap F_{i+1}$ which is not in W. This proves the first inequality.

For the second inequality let $K_1, K_2, ..., K_n$ be the strong components of Δ/W. These are elements in $C_d(\Delta)$ and $\partial(K_i) \in C_{d-1}(\Delta_W)$ since otherwise one could extend the strong component K_i over a $d-2$ face not in contained in W. Since $\partial(\partial(K_i) = 0$ we can think of $[\partial(K_i)]$ as elements in $H_{d-1}(\Delta_W)$. Now assume some of the strong components satisfy a linear relation in $H_{d_1}(\Delta_W)$ so their sum will be equal to $\partial(\sigma)$ with σ in $C_d(\Delta)$. This implies that

$$\partial(\sum K_i + \sigma) = 0$$

However since Δ is a pseudomanifold, the top homology of Δ must be supported on the entire complex and this can only happen if all the K_i are in the sum. Thus we get that any $n-1$ strong components are linearly independent so $S(\Delta/W) = n \leq dimH_{d-1}(\Delta_W) + 1$ □

3 Stronger Results for Normal Pseudomanifolds

One would expect better estimates on the connectivity of $G(\Delta)$ if we restrict ourselves to spaces without singularities, say triangulated manifolds. In fact a much more lax condition is necessary, namely requiring that the links be connected, to force the first inequality in (1) to become an equality.

Lemma 3.1. *If Δ is a normal pseudomanifold then $dim(H_d(\Delta, \Delta_W)) = dimH_0(\Delta_{V-W}) = S(\Delta/W)$*

Proof. For the first inequality: Let v, w be in the same connected component of G/W. We now need to show is that any two facets F and G with the first containing v and the second w are in the same strong component of Δ/W. The way we will build the facet sequence is by following the path between v and w and taking advantage of the facet connectivity of the links. Say v_i, v_{i+1} are two consecutive vertices in the path from v to

w and F_i, F_{i+1} facets with $v_i \in F_i$ and $v_{i+1} \in F_{i+1}$ Now the link of v_i is easily checked to also be a normal peseudomanifold and thus facet connected. Let $F_{i,i+1}$ be a face in the link of $\{v_i, v_{i+1}\}$. We can find a sequence of facets in Δ going from F_i to $F_{i,1+1}$ using the facet connectivity of lkv_i. Furthermore any two such facets have v_i in common so this is a strong sequence in Δ/W. Analogously we can find a sequence from $F_{i,1+1}$ to F_i so F_i and $F_{i+1}$are in the same strong component of Δ/W Following the path between v to w and applying the procedure above one gets that F and G are in the same strong component of Δ/W.

For the second inequality: Let σ be a relative cycle in ker∂_d, F be a facet of Δ contained in σ and K_F the corresponding strong component. Asume there is a facet F' in K_F that is not in σ. There will thus exist F_i, F_{i+1} in the facet sequence connecting F and F' in such that $F_i \cap F_{i+1} = G \not\subset W$ with $F_i \in \sigma$ and $F_{i+1} \notin \sigma$ .But since G is contained in exactly two facets we get that G is in $\partial_d(\sigma) \subset W$ - a contradiction. So every relative cycle is a sum of the strong compoenents. It follows that the strong components of Δ/W are a basis for the kernel and the result follows.

□

Theorem 3.2. *Let Δ be a normal pseudomanifold and let i be the inclusion map i : $H_{d-1}(\Delta_W) \to H_{d-1}(\Delta)$. Then dim$H_0(\Delta_{V-W})$= dim (ker i)-1.*

Proof. We will be using the fact that $H_d(\Delta, \Delta_W)$ fits into the exact long sequence

$$0 \longrightarrow H_d(\Delta) \longrightarrow H_d(\Delta, \Delta_W) \longrightarrow H_{d-1}(\Delta_W) \longrightarrow H_{d-1}(\Delta) \longrightarrow ... \tag{2}$$

By using the exact sequence above we get $S(\Delta/W) = dim(H_d(\Delta, \Delta_W)$ =dim(ker i)-1 and combined with Lemma 3.1 the result follows. □

Remarks: For normal pseudomanifolds we get that $H_0(\Delta_{V-W}) \cong H_d(\Delta, \Delta_W)$ which can be intepreted as a weak form of Poinacere-Lefschetz Duality for normal pseudomanifolds.

Corollary: If Δ is a normal d-pseudomanifold with $H_{d-1}(\Delta) = 0$ then $v(\Delta/W) =$ dim$H_{d-1}(\Delta_W) - 1$ and $k(G(\Delta))$ =min$\{|W| : H_{d-1}(\Delta_W) \neq 0\}$

4 Applications

Let $C(i, d)$ the class of simplicial complexes with $H_d(\Delta) \neq 0$ and no missing faces of dimension greater than i. These complexes were introduced in [2] by Nevo For given i, d there exist unique integers $0 \leq r$ and $1 \leq r \leq i$ such that $d + 1 = qi + r$. Define

$$S(i, d) = \partial\sigma^i \star ... \star \partial\sigma^i \star \partial\sigma^r$$

where $\partial\sigma^i$, the boundary of the i-simplex, appears q-times in the join. We have the following lemma.

Lemma 4.1. *(Nevo [2]) If Δ is in $C(i, d)$ then $f_0(\Delta) \geq f_0(C(i, d))$= q(i+1)+(r+1) = d +1 + q + 1*

Theorem 4.2. *Let Δ be a d - pseudomanifold with no missing faces of dimension higher than i then $G(\Delta)$ is $d+q-1$ - connected.*

Proof. Assume removing the subset of vertices W disconnects $G(\Delta)$. By Theorem 1 we get that $H_{d-1}(\Delta_W) \neq 0$ and since the complex Δ_W is induced it cannot have any missing faces of dimension higher than i. It follows that $\Delta_W \in C(i, d-1)$ so $f_0(\Delta_W) \geq f_0(S(i, d-1) \geq d+q-1$.

$\square$

Note that the two previous results of Barnette (in the simplicial case) and Athanasiadis follow from Theorem 4.2:

Corollary: Let Δ be a pseudomanifold then $G(\Delta)$ is $d+1$ connected. Furthermore if Δ is a flag complex then $G(\Delta)$ is $2d$ connected.

Remark: Results similar to Theorem 2 have been published in [4] using methods from algebraic topology similar to ours but also commutative algebra. The results there are for a class of complexes called banner complexes which are a generalization of flag complexes. We will show however that the connectivity bounds we get are tighter.

We illustrate this by an example: Let $\Delta = C_3 * C_4$ be the join of 3-cycle and a 4-cycle. This is a 3-sphere. Δ is not flag since it contains an empty triangle, however it is easily seen that $\Delta \in C(2,3)$ since no empty tetrahedra are introduced by the join construction. Thus by Theorem 4.2 we get that $G(\Delta)$ is 5-connected and notice that this is the best bound possible - it is easily checked that the connectivity of $G(\Delta)$ is 5. However $b(\Delta)$ as defined in [4] page 3 will be 2. This is because the link of any vertex in the 4-cycle C_4 is the boundary of a triangular bipyramid which is not banner. Thus by Theorem 12 in [4] we only get that $G(\Delta)$ is 4 - connected which is true but weaker than our result.

References

[1] Christos A. Athanasiadis, *Some Combinatorial Properties Of Flag Simplicial Pseudomanifolds And Spheres*, Ark. Mat., 00 (2008), 112

[2] Eran Nevo, *Remarks on Missing Faces and Generalized Lower Bounds on Face Numbers*

[3] Anders Bjorner, Kathrin Vorwerk *On The Connectivity of Manifold Graphs*, Proceedings Of The American Mathematical Society Volume 143, Number 10, October 2015

[4] Karim A. Adiprasito, Afshin Goodarzi, Matteo Varbaro *Connectivity Of Pseudomanifold Graphs From An Algebraic Point Of View*