EE222 Course Notes

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Disclaimer: These notes reflect EE222 when I took the course (Fall 2021). They may not accurately reflect current course content, so use at your own risk. If you find any typos, errors, etc, please raise an issue on the GitHub repository.

Contents

1 Real Analysis

Definition 1 *The extended real line is the set*

$$\{-\infty\} \cup \mathbb{R} \cup \{\infty\}$$

Definition 2 *The supremum of a set $S \subset \mathbb{R}$ is a value $a \in \mathbb{R}_e$ such that $\forall s \in S,\ s \leq a$ and if $b \in \mathbb{R}_e$ such that $\forall s \in S,\ s \leq b$, then $a \leq b$.*

Supremum is essentially the "least upper bound" in a set. It always exists, and is called $\sup S$. The opposite of supremum is the infinimum.

Definition 3 *The infinimum of a set $S \subset \mathbb{R}$ is a value $a \in \mathbb{R}_e$ such that $\forall s \in S,\ s \geq a$ and if $b \in \mathbb{R}_e$ such that $\forall s \in S,\ s \geq b$, then $a \geq b$.*

The infinimum is the "greatest upper bound". Like the supremum, it always exists, and it is denoted $\inf S$. Supremum and Infinimum can be applied to scalar function $f : S \to \mathbb{R}$ by letting

$$\sup_{x \in S} f(x) = \sup\{f(x) | x \in S\}.$$

1.1 Norms

Definition 4 *Let V be a vector space of $\mathbb{R}$, then $\|\cdot\| : V \to \mathbb{R}$ is a norm if $\forall \boldsymbol{x}, \boldsymbol{y} \in V, \alpha \in \mathbb{R}$,*

$$\|\boldsymbol{x}\| \geq 0 \qquad \boldsymbol{x} = 0 \Leftrightarrow \|\boldsymbol{x}\| = 0 \qquad \|\alpha \boldsymbol{x}\| = |\alpha| \|\boldsymbol{x}\| \qquad \|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$$

Definition 5 *A normed space $(V, \|\cdot\|)$ is a vector space which is equipped with a norm $\|\cdot\| : V \to \mathbb{R}$.*

If we have an operator A which takes as inputs vectors from normed space $(X, \|\cdot\|_X)$ and outputs vectors in normed space $(Y, \|\cdot\|_Y)$, then we can define another norm on the vector space of operators from $X \to Y$.

Definition 6 *Let $A : X \to Y$ be an operator between normed spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$, then the induced norm of A is*

$$\|A\|_i = \sup_{\|\boldsymbol{x} \neq 0\|_X} \frac{\|A\boldsymbol{x}\|_Y}{\|\boldsymbol{x}\|_X}$$

The induced norm can be thought of as the maximum gain of the operator.

Definition 7 *Two norms $\|\cdot\|$ and $|||\cdot|||$ on a vector space V if $\exists k_1, k_2 > 0$ suh that*

$$\forall \boldsymbol{x} \in V,\ k_1\|\boldsymbol{x}\| \leq |||\boldsymbol{x}||| \leq k_2\|\boldsymbol{x}\|$$

If V is a finite dimensional vector space if and only if all norms of V are equivalent.

1.2 Sets

Definition 8 *Let $(V, \|\cdot\|)$ be a normed space, $a \in \mathbb{R}$, $a > 0$, $\boldsymbol{x}_0 \in V$, then the open ball of radius a centered around x_0 is given by*

$$B_a(\boldsymbol{x}_0) = \{\boldsymbol{x} \in V \mid \|\boldsymbol{x} - \boldsymbol{x}_0\| < a\}$$

Definition 9 *A set $S \subset V$ is open if $\forall \boldsymbol{s}_0 \in V,\ \exists \epsilon > 0$ such that $B_\epsilon(\boldsymbol{s}_0) \subset S$.*

Open sets have a boundary which is not included in the set. By convention, we say that the empty set is open.

The opposite of an open set is a closed set.

Definition 10 *A set S is closed if $\sim S$ is open.*

Closed sets have a boundary which is included in the set.

1.3 Convergence

Definition 11 *A sequence of points $\boldsymbol{x}_k$ in normed space $(V, \|\cdot\|)$ converges to a point $\bar{\boldsymbol{x}}$ if*

$$\forall \epsilon > 0,\ \exists N < \infty,\ \text{ such that } \forall k \geq N, \|\boldsymbol{x}_k - \bar{\boldsymbol{x}}\| < \epsilon$$

Convergence means that we can always find a finite time such that after that time, all points in the sequence stay within a specicied norm ball.

Definition 12 *A sequence $\boldsymbol{x}_k$ is cauchy if*

$$\forall \epsilon > 0,\ \exists N < \infty \text{ such that } \forall n, m \geq N, \|\boldsymbol{x}_m - \boldsymbol{x}_n\| < \epsilon$$

A Cauchy sequence has a looser type of convergence than a convergent sequence since it only requires all elements to in the sequence to be part of the same norm ball after some time instead of requiring the sequence to get closed and closer to a single point.

Theorem 1 *If $\boldsymbol{x}_n$ is a convergent sequence, then $\boldsymbol{x}_n$ is a also a Cauchy sequence.*

Definition 13 *A normed space $(V, \|\cdot\|)$ is complete if every Cauchy sequence converges to a point in V*

Because a complete space requires that Cauchy sequences converge, all cauchy sequences are convergent in a complete space. Two important complete spaces are

1. Every finite dimensional vector space
2. $(C[a, b], \|\cdot\|_\infty)$, the set of continuously differentiable functions on the closed interval $[a, b]$ equipped with the infinity norm.

A complete normed space is also called a **Banach Space**.

1.4 Contractions

Definition 14 *A point x^* is a fixed point of a function $P : X \to X$ if $P(x^*) = x^*$.*

Definition 15 *A function $P : X \to X$ is a contract if there exists a constant $0 \leq c < 1$ such that*

$$\forall \boldsymbol{x}, \boldsymbol{y} \in X,\ \|P(\boldsymbol{x}) - P(\boldsymbol{y})\| \leq c\|\boldsymbol{x} - \boldsymbol{y}\|$$

Informally, a contraction is a function which makes distances smaller

Suppose we look at a sequence defined by iterates of a function

$$\boldsymbol{x}_{k+1} = P(\boldsymbol{x}_k).$$

where P is a function $P : X \to X$. When does this sequence converge, and to what point will it converge?

Theorem 2 (Contraction Mapping Theorem) *If $P : X \to X$ is a contraction on the Banach space $(X, \|\cdot\|)$, then there is a unique $\boldsymbol{x}^* \in X$ such that $P(\boldsymbol{x}^*) = \boldsymbol{x}^*$ and $\forall x_0 \in X$, the sequence $\boldsymbol{x}_{n+1} = P(\boldsymbol{x}_n)$ is converges to $\boldsymbol{x}^*$.*

The contraction mapping theorem that contractions will have a unique fixed points and that repeatedly applying the contraction will converge to the fixed point.

1.5 Continuity

Definition 16 *A function $h : V \to W$ on normed spaces $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ is continuous at a point $\boldsymbol{x}_0$ if $\forall\epsilon > 0, \exists\delta > 0$ such that*

$$\|\boldsymbol{x} - \boldsymbol{x}_0\|_V < \delta \implies \|h(\boldsymbol{x}) - h(\boldsymbol{x_0})\|_W < \epsilon$$

Continuity essentially means that given an $\epsilon-$ball in W, we can find a $\delta-$ball in V which is mapped to the ball in W. If a function is continuous at all points $\boldsymbol{x}_0$, then we say the function is continuous.

We can make the definition of continuity more restrictive by restraining the rate of growth of the function.

Definition 17 *A function $h : V \to W$ on normed spaces $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ is Lipschitz continuous at $\boldsymbol{x}_0 \in V$ if $\exists r > 0$ and $L < \infty$ such that*

$$\forall \boldsymbol{x}, \boldsymbol{y} \in B_r(\boldsymbol{x}_0),\ \|h(\boldsymbol{x}) - h(\boldsymbol{y})\|_W \leq L\|\boldsymbol{x} - \boldsymbol{y}\|_V$$

A good interpretation of Lipschitz Continuity is that given two points in a ball around $\boldsymbol{x}_0$, the slope of the line connecting those two points is less than L. It means that the function is growings slower than linear for some region around $\boldsymbol{x}_0$. Lipschitz continuity implies continuity. If a function is lipschitz continuous with respect to one norm, it is also lipschitz continuous with respect to all equivalent norms.

When the function h is a function on $\mathbb{R}^n$ and is also differentiable, then Lipschitz continuity is easy to determine.

Theorem 3 *For a differentiable function $h : \mathbb{R}^n \to \mathbb{R}^n$,*

$$\exists r > 0, L < \infty, \boldsymbol{x}_0 \in \mathbb{R}^n,\ \forall \boldsymbol{x} \in B_r(\boldsymbol{x}), \left|\left|\frac{\partial h}{\partial \boldsymbol{x}}\right|\right|_2 \leq L$$

implies Lipschitz Continuity at $\boldsymbol{x}_0$.

Definition 18 *A function $h : \mathbb{R} \to V$ is piecewise continuous if $\forall k \in \mathbb{Z}$, $h : [-k, k] \to V$ is continuous except at a possibly finite number of points, and at the points of discontinuity t_i, $\lim_{s\to 0^+} h(t_i + s)$ and $\lim_{s\to 0^-}$ exist and are finite.*

2 Nonlinear System Dynamics

Consider the non-linear system

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = f(\boldsymbol{x}, t).$$

Definition 19 *The system is autonomous if $f(x, t)$ is not explicitly dependent on time t.*

Definition 20 *A point x_0 is an equilibrium point at time t_0 if*

$$\forall t \geq t_0,\ f(x_0, t) = 0$$

Consider a single trajectory $\boldsymbol{x}(t, t_0, \boldsymbol{x}_0)$.

Definition 21 *A set S is said to be the $\omega-$limit set of ϕ if*

$$\forall y \in S, \exists t_n \to \infty, \lim_{n\to\infty} \phi(t_n, t_0, \boldsymbol{x}_0) = y$$

Whereas linear systems converge to a single point if they converge at all, nonlinear systems can converge to a set of points. Thus the $\omega-$limit set essentially generalizes the idea of a limit.

Definition 22 *A set $M \subset \mathbb{R}^n$ is said to be invariant if*

$$\forall t \geq t_0,\ \boldsymbol{y} \in M \implies \phi(t, t_0, \boldsymbol{y}) \in M$$

An invariant set is one which a trajectory of the system will never leave once it enters the set.

2.1 Solutions to Nonlinear Systems

Consider the nonlinear system

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = f(\boldsymbol{x}, t),\ \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \in \mathbb{R}^n.$$

Definition 23 *A function $\mathbf{\Phi}(t)$ is a solution to $\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = f(x,t),\ \boldsymbol{x}(t_0) = \boldsymbol{x}_0$ on the closed interval $[t_0, t]$ if $\mathbf{\Phi}(t)$ is defined on the interval $[t_0, t]$, $\frac{\mathrm{d}\mathbf{\Phi}}{\mathrm{d}t} = f(\mathbf{\Phi}(t), t)$ on the interval $[t_0, t]$, and $\mathbf{\Phi}(t_0) = \boldsymbol{x}_0$.*

We say that $\Phi(t)$ is a solution in the sense of Caratheodory if

$$\Phi(t) = \boldsymbol{x}_0 + \int_{t_0}^{t} f(\mathbf{\Phi}(\tau), \tau)d\tau.$$

Because the sytem is non-linear, it could potentially have no solution, one solution, or many solutions. These solutions could exist locally, or they could exist for all time. We might also want to know when there is a solution which depends continuously on the initial conditions.

Theorem 4 (Local Existence and Uniqueness) *Given $\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = f(\boldsymbol{x}, t),\ \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \in \mathbb{R}^n$ where f is piecewise continuous in t and $\exists T > t_0$ such that $\forall t \in [t_0, T], f$ is L-Lipschitz Continuous, then $\exists \delta > 0$ such that a solution exists and is unique $\forall t \in [t_0, t_0 + \delta]$.*

Theorem 4 can be proved using the Contraction Mapping Theorem (theorem 2) by finding δ such that the function $P : C_n[t_0, t_0 + \delta] \to C_n[t_0, t_0 + \delta]$ given by

$$P(\mathbf{\Phi})(t) = \boldsymbol{x}_0 + \int_{t_0}^{t_0+\delta} f(\mathbf{\Phi}(\tau), \tau)d\tau$$

is a contraction under the norm $\|\mathbf{\Phi}\|_\infty = \sup_{t_0 \le t \le t_0+\delta} \|\mathbf{\Phi}(t)\|$.

Theorem 5 (Global Existence and Uniqueness) *Suppose $f(\boldsymbol{x}, t)$ is piecewise continuous in t and $\forall T \in [t_0, \infty)$, $\exists L_T < \infty$ such that f is L_T Lipshitz continuous for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, then the nonlinear system has exactly one solution on $[t_0, T]$.*

Once we know that solutions to a nonlinear system exist, we can sometimes bound them.

Theorem 6 (Bellman-Gronwall Lemma) *Suppose $\lambda \in \mathbb{R}$ is a constant and $\mu : [a, b] \to \mathbb{R}$ is continuous and non-negative, then for a continuous function $y : [a, b] \to \mathbb{R}$*

$$y(t) \le \lambda + \int_a^t \mu(\tau)y(\tau)d\tau \implies y(t) \le \lambda exp\left(\int_a^t \mu(\tau)d\tau\right)$$

Another thing we might want to do is understand how the non-linear system reacts to changes in the initial condition.

Theorem 7 *Suppose the system* $\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = f(\boldsymbol{x}, t),\ \boldsymbol{x}(t_0) = \boldsymbol{x}_0$ *satisfies the conditions of global uniqueness and existence. Fix* $T \in [t_0, \infty]$ *and suppose* $\boldsymbol{x}(\cdot)$ *and* $\boldsymbol{z}(\cdot)$ *are two solutions satisfying* $\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = f(\boldsymbol{x}, t), \boldsymbol{x}(t_0) = \boldsymbol{x}_0$ *and* $\frac{\mathrm{d}\boldsymbol{z}}{\mathrm{d}t} = f(\boldsymbol{z}(t), t),\ \boldsymbol{z}(t_0) = \boldsymbol{z}_0$*, then* $\forall \epsilon > 0, \exists \delta > 0$ *such that*

$$\|\boldsymbol{x}_0 - \boldsymbol{z}_0\| < \delta \implies \|\boldsymbol{x} - \boldsymbol{z}\|_\infty < \epsilon.$$

Theorem 7 is best understood by defining a function $\Psi : \mathbb{R}^n \to C_n[t_0, t]$ where $\Psi(\boldsymbol{x}_0)(t)$ returns the solution to the system given the initial condition. If the conditions of Theorem 7 are satisfied, then the function Ψ will be continuous.

2.2 Planar Dynamical Systems

Planar dynamical systems are those with 2 state variables. Suppose we linearize the system $\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = f(\boldsymbol{x})$ at an equilibrium point.

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = D_f|_{\boldsymbol{x}=\boldsymbol{x_0}}\boldsymbol{x}$$

Depending on the eigenvalues of D_f, the Jacobian, we get several cases for how this linear system behaves. We'll let z_1 and z_2 be the eigenbasis of the *phase space*.

1. The eigenvalues are real, yielding solutions $z_1 = z_1(0)e^{\lambda_1 t}, z_2 = z_2(0)e^{\lambda_2 t}$. If we eliminate the time variable, we can plot the trajectories of the system.

$$\frac{z_1}{z_1(0)} = \left(\frac{z_2}{z_2(0)}\right)^{\frac{\lambda_1}{\lambda_2}}$$

   (a) When $\lambda_1, \lambda_2 < 0$, all trajectories converge to the origin, so we call this a **stable node**.

   (b) When $\lambda_1, \lambda_2 > 0$, all trajectories blow up, so we call this an **unstable node**.

   (c) When $\lambda_1 < 0 < \lambda_2$, the trajectories will converge to the origin along the axis corresponding to λ_1 and diverge along the axis corresponding to λ_2, so we call this a **saddle node**.

2. There is a single repeated eigenvalue with one eigenvector. As before, we can eliminate the time variable and plot the trajectories on the z_1, z_2 axes.

   (a) When $\lambda < 0$, the trajetories will converge to the origin, so we call it an **improper stable node**

(b) When $\lambda > 0$, the trajetories will diverge from the origin, so we call it an **improper unstable node**

3. When there is a complex pair of eigenvalues, the linear system will have oscillatory behavior. The Real Jordan form of D_f will look like

$$D_f = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}.$$

The parameter β will determine the direction of the trajectories (clockwise if positive).

(a) When $\alpha < 0$, the trajectories will spiral towards the origin, so we call it a **stable focus**.

(b) When $\alpha = 0$, the trajectories will remain at a constant radius from the origin, so we call it a **center**.

(c) When $\alpha > 0$, the trajectories will spiral away from the origin, so we call it an **unstable focus**.

It turns out that understanding the linear dynamics at equilibrium points can be helpful in understanding the nonlinear dynamics near equilibrium points.

Theorem 8 (Hartman-Grobman Theorem) *If the linearization of a planar dynamical system* $\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = f(\boldsymbol{x})$ *at an equilibrium point* $\boldsymbol{x_0}$ *has no zero or purely imaginary eigenvalues, then there exists a homeomorphism from a neighborhood* U *of* x_0 *into* $\mathbb{R}^2$ *which takes trajectories of the nonlinear system and maps them onto the linearization where* $h(\boldsymbol{x_0}) = 0$*, and the homeomorphism can be chosen to preserve the parameterization by time.*

Theorem 8 essentially says that the linear dynamics predict the nonlinear dynamics around equilibria, but only for a neighborhood around the equilibrium point. Outisde of this neighborhood, the linearization may be very wrong.

Non-linear systems can also have periodic solutions.

Definition 24 *A closed orbit* γ *is a trajectory of the system such that* $\gamma(0) = \gamma(T)$ *for finite* T*.*

Suppose that we have a simply connected region D (meaning D cannot be contracted to a point) and we want to know if it contains a closed orbit.

Theorem 9 (Bendixon's Theorem) *If* $\operatorname{div}(f)$ *is not identically zero in a sub-region of* D *and does not change sign in* D*, then* D *contains no closed orbits.*

Theorem 9 lets us rule out closed orbits from regions of $\mathbb{R}^2$.

Definition 25 *A region $M \subset \mathbb{R}^2$ is positively invariant for a trajectory $\phi_t(\boldsymbol{x})$ if $\forall x \in M, \forall t \geq 0, \phi_t(\boldsymbol{x}) \in M$.*

A positively invariant set essentially means that once a trajectory enters the set, it cannot leave. That means all of the vector field lines must point inside the set. If we have a positively invariant region, then we can determine whether it contains closed orbits.

Theorem 10 (Poincare-Bendixson Theorem) *If M is a compact, positively invariant set for the flow $\phi_t(\boldsymbol{x})$, then if M contains no equilibrium points, then M has a limit cycle.*

3 Stability of Nonlinear Systems

The equilibria of a system can tell us a great deal about the stability of the system. For nonlinear systems, stability is a property of the equilibrium points, and to be stable is to converge to or stay equilibrium.

Definition 26 *An equilibrium point $\boldsymbol{x}_e \in \mathbb{R}$ is a stable equilibrium point in the sense of Lyapunov if and only if $\forall \epsilon > 0, \exists \delta(t_0, \epsilon)$ such that*

$$\forall t \geq t_0,\ \|\boldsymbol{x}_0 - \boldsymbol{x}_e\| < \delta(t_0, \epsilon) \implies \|\boldsymbol{x}(t) - \boldsymbol{x}_e\| < \epsilon$$

Lyapunov Stability essentially says that a finite deviation in the initial condition from equilibrium means the resulting trajectory of the system stay close to equilibrium. Notice that this definition is nearly identical to theorem 7. That means stability of an equilibrium point is the same as saying the function which returns the solution to a system given its initial condition is continuous at the equilibrium point.

Definition 27 *An equilibrium point $\boldsymbol{x}_e \in \mathbb{R}$ is an uniformly stable equilibrium point in the sense of Lyapunov if and only if $\forall \epsilon > 0, \exists \delta(\epsilon)$ such that*

$$\forall t \geq t_0,\ \|\boldsymbol{x}_0 - \boldsymbol{x}_e\| < \delta(\epsilon) \implies \|\boldsymbol{x}(t) - \boldsymbol{x}_e\| < \epsilon$$

Uniform stability means that the δ can be chosen independently of the time the system starts at. Both stability and uniform stability do not imply convergence to the equilibrium point. They only guarantee the solution stays within a particular norm ball. Stricter notions of stabilty add this idea in.

Definition 28 *An equilibrium point $\boldsymbol{x}_e$ is attractive if $\forall t_0 > 0,\ \exists c(t_0)$ such that*

$$\boldsymbol{x}(t_0) \in B_c(\boldsymbol{x}_e) \implies \lim_{t\to\infty} \|\boldsymbol{x}(t, t_0, \boldsymbol{x}_0) - \boldsymbol{x}_e\| = 0$$

Attractive equilibria guarantee that trajectories beginning from initial conditions inside of a ball will converge to the equilibrium. However, attractivity does not imply stability since the trajectory could go arbitarily far from the equilibrium so long as it eventually returns.

Definition 29 *An equilibrium point $\boldsymbol{x}_e$ is asymptotically stable if $\boldsymbol{x}_e$ is stable in the sense of Lyapunov and attractive.*

Asymptotic stability fixes the problem of attractivity where trajectories could go far from the equilibrium, and it fixes the problem with stability where the trajectory may not converge to equilibrium. It means that trajectories starting in a ball around equilibrium will converge to equilibrium without leaving that ball. Because the constant for attractivity may depend on time, defining uniform asymptotic stability requires some modifications to the idea of attractivity.

Definition 30 *An equilibrium point is uniformly asympototically stable if $\boldsymbol{x}_e$ is uniformly stable in the sense of Lyapunov, and $\exists c$ and $\gamma : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ such that*

$$\forall \boldsymbol{x}_0 \in B_c(\boldsymbol{x}_e),\ \lim_{\tau\to\infty} \gamma(\tau, \boldsymbol{x}_0) = 0, \qquad \forall t \geq t_0,\ \|\boldsymbol{x}(t, t_0, \boldsymbol{x}_0) - \boldsymbol{x}_e\| \leq \gamma(t - t_0, \boldsymbol{x}_0)$$

The existence of the γ function helps guarantee that the rate of converges to equilibrium does not depend on t_0 since the function γ is independent of t_0. Suppose that the γ is an exponential function. Then solutions to the system will converge to the equilibrium exponentially fast.

Definition 31 *An equilibrium point $\boldsymbol{x}_e$ is locally exponentially stable if $\exists h, m, \alpha$ such that*

$$\forall \boldsymbol{x}_0 \in B_h(\boldsymbol{x}_e),\ \|\boldsymbol{x}(t, t_0, \boldsymbol{x}_0) - \boldsymbol{x}_e\| \leq m e^{-\alpha(t-t_0)} \|\boldsymbol{x}(t) - \boldsymbol{x}_e\|$$

Definitions 26, 27 and 29 to 31 are all local definitions because the only need to hold for $\boldsymbol{x}_0$ inside a ball around the equilibrium. If they hold $\forall \boldsymbol{x}_0 \in \mathbb{R}^n$, then they become global properties.

Just as we can define stability, we can also define instability.

Definition 32 *An equilibrium point* $\boldsymbol{x}_e$ *is unstable in the sense of Lyapunov if* $\exists\epsilon > 0, \forall delta > 0$ *such that*

$$\exists \boldsymbol{x}_0 \in B_\delta(\boldsymbol{x}_e) \implies \exists T \geq t_0, x(T, t_0, \boldsymbol{x}_0) \notin B_\epsilon(\boldsymbol{x}_e)$$

Instability means that for any $\delta-$ball, we can find an $\epsilon-$ball for which there is at least one initial condition whose corresponding trajectory leaves the $\epsilon-$ball.

3.1 Lyapunov Functions

In order to prove different types of stability, we will construct functions which have particular properties around equilibrium points of the system. The properties of these functions will help determine what type of stable the equilibrium point is.

Definition 33 *A class* $\mathcal{K}$ *function is a function* $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ *such that* $\alpha(0) = 0$ *and* $\alpha(s)$ *is strictly monotonically increasing in* s*.*

A subset of the class $\mathcal{K}$ functions grow unbounded as the argument approaches infinity.

Definition 34 *A class* $\mathcal{KR}$ *function is a class* $\mathcal{K}$ *function* α *where* $\lim_{s\to\infty} \alpha(s) = s$*.*

Class $\mathcal{KR}$ functions are "radially unbounded". We can use class $\mathcal{K}$ and class $\mathcal{KR}$ to bound "energy-like" functions called **Lyapunov Functions**.

Definition 35 *A function* $V(\boldsymbol{x}, t) : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ *is locally positive definite (LPDF) on a set* $G \subset \mathbb{R}^n$ *containing* $\boldsymbol{x}_e$ *if* $\exists\alpha \in \mathcal{K}$ *such that*

$$V(\boldsymbol{x}, t) \geq \alpha(\|\boldsymbol{x} - \boldsymbol{x}_e\|)$$

LPDF functions are locally "energy-like" in the sense that the equilibrium point is assigned the lowest "energy" value, and the larger the deviation from the equilibrium, the higher the value of the "energy".

Definition 36 *A function* $V(\boldsymbol{x}, t) : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ *is positive definite (PDF) if* $\exists\alpha \in \mathcal{KR}$ *such that*

$$\forall \boldsymbol{x} \in \mathbb{R}^n,\ V(\boldsymbol{x}, t) \geq \alpha(\|\boldsymbol{x} - \boldsymbol{x}_e\|)$$

Positive definite functions act like "energy functions" everywhere in $\mathbb{R}^n$.

Definition 37 *A function $V(\boldsymbol{x}, t) : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ is decrescent if $\exists \alpha \in \mathcal{K}$ such that*

$$\forall \boldsymbol{x} \in B_h(\boldsymbol{x}_e),\ V(\boldsymbol{x}, t) \leq \beta(\|\boldsymbol{x} - \boldsymbol{x}_e\|)$$

Descresence means that for a ball around the equilibrium, we can upper bound the growth of the energy.

Note that we can assume $\boldsymbol{x}_e = 0$ without loss of generality for definitions 35 to 37 since for a given system, we can always define a linear change of variables that shifts the equilibrium point to the origin.

3.2 Proving Stability

To prove the stability of an equilibrium point for a given nonlinear system, we will construct a Lyapunov function and determine stability from the properties of the Lyapunov functions which we can find.

Definition 38 *The Lie derivative of a function $V(\boldsymbol{x}, t)$ is given by*

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial \boldsymbol{x}} f(\boldsymbol{x}, t).$$

The Lie derivative is essentially the directional derivative along the trajectories of the system. Given properties of V and $\frac{\mathrm{d}V}{\mathrm{d}t}$, we can use the **Lyapunov Stability Theorems** to prove the stability of equilibria.

Theorem 11 *If $\exists V(\boldsymbol{x}, t)$ such that V is LPDF and the Lie Derivative $-\frac{\mathrm{d}V}{\mathrm{d}t} \geq 0$ locally, then $\boldsymbol{x}_e$ is stable in the sense of Lyapunov.*

Theorem 12 *If $\exists V(\boldsymbol{x}, t)$ such that V is LPDF and decrescent, and the Lie Derivative $-\frac{\mathrm{d}V}{\mathrm{d}t} \geq 0$ locally, then $\boldsymbol{x}_e$ is uniformly stable in the sense of Lyapunov.*

Theorem 13 *If $\exists V(\boldsymbol{x}, t)$ such that V is LPDF and decrescent, and the Lie Derivative $-\frac{\mathrm{d}V}{\mathrm{d}t}$ is LPDF, then $\boldsymbol{x}_e$ is uniformly asymptotically stable in the sense of Lyapunov.*

Theorem 14 *If $\exists V(\boldsymbol{x}, t)$ such that V is PDF and decrescent, and the Lie Derivative $-\frac{\mathrm{d}V}{\mathrm{d}t}$ is LPDF, then $\boldsymbol{x}_e$ is globally uniformly asymptotically stable in the sense of Lyapunov.*

Theorem 15 *If $\exists V(\boldsymbol{x}, t)$ and $h, \alpha > 0$ such that V is LPDF is decrescent, The Lie derivative $-\frac{\mathrm{d}V}{\mathrm{d}t}$ is LDPF, and*

$$\forall \boldsymbol{x} \in B_h(\boldsymbol{x}_e), \left\| \frac{\partial V}{\partial t} \right\| \leq \alpha \|\boldsymbol{x} - \boldsymbol{x}_e\|$$

The results of theorems 11 to 15 are summarized in table 1. Going down the rows

Conditions on V	Conditions on $-\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}\mathbf{t}}$	Conclusion
LPDF	≥ 0 locally	Stable
LPDF, Decrescent	≥ 0 locally	Uniformly Stable
LPDF, Decrescent	LPDF	Uniformly, Asymptotically Stable
LPDF, Decrescent	LDPF, $\exists \alpha > 0$ such that $\left\| \frac{\mathrm{d}V}{\mathrm{d}t} \right\| \leq \alpha \|\boldsymbol{x} - \boldsymbol{x}_e\|$	Exponentially Stable
PDF, Decrescent	PDF	Globally, Uniformly, Asymptotically Stable

Table 1: Summary of Lyapunov Stability Theorems

of table 1 lead to increasingly stricter forms of stability. Descresence appears to add uniformity to the stability, while $-\frac{\mathrm{d}V}{\mathrm{d}t}$ being LPDF adds asymptotic convergence. However, these conditions are only sufficient, meaning if we cannot find a suitable V, that does not mean that an equilibrium point is not stable.

One very common case where it can be difficult to find appropriate Lyapunov functions is in proving asymptotic stability since it can be hard to find V such that $-\frac{\mathrm{d}V}{\mathrm{d}t}$ is LPDF. In the case of autonomous systems, we can still prove asymptotic stability without such a V.

Theorem 16 (LaSalle's Invariance Principle) *Consider a smooth function $V : \mathbb{R}^n \to \mathbb{R}$ with bounded sub-level sets $\Omega_c = \{\boldsymbol{x} | V(\boldsymbol{x}) \leq c\}$ and $\forall \boldsymbol{x} \in \Omega_c$, the Lie derivative $-\frac{\mathrm{d}V}{\mathrm{d}t} \leq 0$. Define $S = \left\{ \boldsymbol{x} | \frac{\mathrm{d}V}{\mathrm{d}t} = 0 \right\}$ and let M be the largest invariant set in S, then*

$$\forall \boldsymbol{x}_0 \in \Omega_c, \ \boldsymbol{x}(t, t_0, x_0) \to M \text{ as } t \to \infty.$$

LaSalle's theorem helps prove general convergence to an invariant set. Since V is always decreasing in the sub-level set Ω_c, trajectories starting in Ω_c must eventually reach S. At some point, they will reach the set M in S, and then they will stay there. Thus if the set M is only the equilibrium point, or a set of equilibrium points, then we can show that the system trajectories asymptotically converges to this equilibrium or set of equilibria. Moreover, if $V(\boldsymbol{x})$ is PDF, and $\forall \boldsymbol{x} \in \mathbb{R}^n$, $\frac{\mathrm{d}V}{\mathrm{d}t} \leq 0$, then we can show global asymptotic stability as well.

LaSalle's theorem can be generalized to non-autonomous systems as well, but it is slightly more complicated since the set S may change over time.

3.2.1 Indirect Method of Lyapunov

It turns out that we can also prove the stability of systems by looking at the linearization around the equilibrium. Without loss of generality, suppose $\boldsymbol{x}_e = 0$. The linearization at the equilibrium is given by

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = f(x,t) = f(0,t) + \frac{\partial f}{\partial \boldsymbol{x}}|_{\boldsymbol{x}=0}\boldsymbol{x} + f_1(\boldsymbol{x},t) \approx A(t)\boldsymbol{x}.$$

The function $f_1(\boldsymbol{x},t)$ is the higher-order terms of the linearization. The linearization is a time-varying system. Consider the time-varying linear system

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = A(t)\boldsymbol{x},\ \boldsymbol{x}(t_0) = \boldsymbol{x}_0.$$

Definition 39 *The state transition matrix $\Phi(t,t_0)$ of a time-varying linear system is a matrix satisfying*

$$\boldsymbol{x}(t) = \Phi(t,t_0)\boldsymbol{x}_0,\ \frac{\mathrm{d}\Phi}{\mathrm{d}t} = A(t)\Phi(t,t_0),\ \Phi(t_0,t_0) = I$$

The state transition matrix is useful in determining properties of the system.

1. $\sup_{t \geq t_0} \|\Phi(t,t_0)\| = m(t_0) < \infty \implies$ the system is stable at the origin at t_0.
2. $\sup_{t_0 \geq 0} \sup_{t \geq t_0} \|\Phi(t,t_0)\| = m < \infty \implies$ the system is uniformly stable at the origin at t_0.
3. $\lim_{t \to \infty} \|\Phi(t,t_0)\| = 0 \implies$ the system is asymptotically stable.

4. $\forall t_0, \epsilon > 0, \exists T$ such that $\forall t \geq t_0+T,\ \|\Phi(t,t_0)\| < \epsilon \implies$ the system is uniformly asymptotically stable.
5. $\|\Phi(t,t_0)\| \leq Me^{-\lambda(t-t_0)} \implies$ exponential stability.

If the system was Time-Invariant, then the system would be stable so long as the eigenvalues of A were in the open left-half of the complex plane. In fact, we could use A to construct positive definite matrices.

Theorem 17 (Lyapunov Lemma) *For a matrix $A \in \mathbb{R}^{n\times n}$, its eigenvalues λ_i satisfy $\Re(\lambda_i) < 0$ if and only if $\forall Q \succ 0$, there exists a solution $P \succ 0$ to the equation*

$$A^T P + PA = -Q.$$

In general, we can use the **Lyapunov Equation** to count how many eigenvalues of A are stable.

Theorem 18 (Tausskey Lemma) *For $A \in \mathbb{R}^{n\times n}$ and given $Q \succ 0$, if there are no eigenvalues on the $j\omega$ axis, then the solution P to $A^TP + PA = -Q$ has as many positive eigenvalues as A has eigenvalues in the complex left half plane.*

The Lyapunov Lemma has extensions to the time-varying case.

Theorem 19 (Time-Varying Lyapunov Lemma) *If $A(\cdot)$ is bounded and for some $Q(t) \succeq \alpha I$, the solution $P(t)$ to $A(t)^TP(t) + P(t)A(t) = -Q(t)$ is bounded, then the origin is a asymptotically stable equilibrium point.*

It turns out that uniform asymptotic stability of the linearization of a system corresponds to uniform, asymptotic stability of the nonlinear system.

Theorem 20 (Indirect Theorem of Lyapunov) *For a nonlinear system whose higher-order terms of the linearization are given by $f(\boldsymbol{x}, t)$, if*

$$\lim_{\|\boldsymbol{x}\|\to 0} \sup_{t\geq 0} \frac{\|f_1(\boldsymbol{x},t)\|}{\|\boldsymbol{x}\|} = 0$$

and if $\boldsymbol{x}_e$ is a uniformly asymptotic stable equilibrium point of $\frac{\mathrm{d}\boldsymbol{z}}{\mathrm{d}t} = A(t)\boldsymbol{z}$ where $A(t)$ is the Jacobian at the $\boldsymbol{x}_e$, then $\boldsymbol{x}_e$ is a uniformly asymptotic stable equilibrium point of $\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = f(\boldsymbol{x}, t)$

3.3 Proving Instability

Theorem 21 *An equilibrium point $\boldsymbol{x}_{\mathrm{e}}$ is unstable in the sense of Lyapunov if $\exists V(\boldsymbol{x}, t)$ which is decrescent, the Lie derivative $\dfrac{\mathrm{d}V}{\mathrm{d}t}$ is LPDF, $V(\boldsymbol{x}_{\mathrm{e}}, t)$, and $\exists \boldsymbol{x}$ in the neighborhood of $\boldsymbol{x}_{\mathrm{e}}$ such that $V(\boldsymbol{x}_0, t) > 0$.*