EECS225A Course Notes

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1 Transforms

1.1 Discrete Time Fourier Transform

The Discrete Time Fourier Transform is given by

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}.$$

The Inverse Discrete Time Fourier Transform is given by

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega.$$

Since the DTFT is an infinite summation, it may or may not converge.

Definition 1 *A signal $x[n]$ belongs to the l^1 class of signals if the series converges absolutely. In other words,*

$$\sum_{k=-\infty}^{\infty} |x[k]| < \infty.$$

This class covers most real-world signals.

Theorem 1 *If $x[n]$ is a l^1 signal, then the DTFT $X(e^{j\omega})$ converges uniformly and is well-defined for every ω. $X(e^{j\omega})$ is also a continuous function.*

Definition 2 *A signal $x[n]$ belongs to the l^2 class of signals if it is square summable. In other words,*

$$\sum_{k=-\infty}^{\infty} |x[k]|^2 < \infty.$$

The l^2 class contains important functions such as sinc.

Theorem 2 *If $x[n]$ is a l^2 signal, then the DTFT $X(ej\omega)$ is defined almost everywhere and only converges in the mean-squared sense:*

$$\lim_{N\to\infty}\int_{-\pi}^{\pi}\left|\left(\sum_{k=-N}^{N} x[k]e^{-j\omega n}\right) - X(\omega)\right|^2 d\omega = 0$$

Tempered distributions like the Dirac Delta function are other functions which are important for computing the DTFT, and they arise from the theory of generalized functions.

1.2 Z-Transform

The Z-transform is a generalized version of the DTFT and is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}.$$

It is a special type of series called a **Laurent Series**.

Theorem 3 *A Laurent Series will converge absolutely on an open annulus*

$$A = \{z | r < |z| < R\}$$

for some r and R.

We can compute r and R using the signal $x[n]$.

$$r = \limsup_{n\to\infty} |x[n]|^{\frac{1}{n}}, \qquad \frac{1}{R} = \limsup_{n\to\infty} |x[-n]|^{\frac{1}{n}}.$$

2 Hilbert Space Theory

Complex random vectors form a Hilbert space with inner product $\langle X, Y\rangle = \mathbb{E}\left[XY^*\right]$. If we have a random complex vector, then we can use Hilbert Theory in a more efficient manner by looking at the matrix of inner products. For simplicity, we will call this the "inner product" of two complex vectors.

Definition 3 *Let the inner product between two random, complex vectors $\boldsymbol{Z_1}, \boldsymbol{Z_2}$*

$$\langle \boldsymbol{Z_1}, \boldsymbol{Z_2}\rangle = \mathbb{E}\left[\boldsymbol{Z_1}\boldsymbol{Z_2}^*\right]$$

The ij-th entry of the matrix is simply the scalar inner product $\mathbb{E}\left[\boldsymbol{X_i}\boldsymbol{Y_j}^*\right]$ where $\boldsymbol{X_i}$ and $\boldsymbol{Y_j}$ are the ith and jth entries of $\boldsymbol{X}$ and $\boldsymbol{Y}$ respectively. This means the matrix is equivalent to the cross correlation $\boldsymbol{R}_{XY}$ between the two vectors. We can also specify the auto-correlation $\boldsymbol{R}_X = \langle \boldsymbol{X}, \boldsymbol{X}\rangle$ and auto-covariance $\boldsymbol{\Sigma}_X = \langle \boldsymbol{X} - \mathbb{E}\left[\boldsymbol{X}\right], \boldsymbol{X} - \mathbb{E}\left[\boldsymbol{X}\right]\rangle$. One reason why we can think of this matrix as the inner product is because it also satisfies the properties of inner products. In particular, it is

1. Linear: $\langle \alpha_1\boldsymbol{V_1} + \alpha_2\boldsymbol{V_2}, \boldsymbol{u}\rangle = \alpha_1\langle \boldsymbol{V_1}, u\rangle + \alpha_2\langle \boldsymbol{V_2}, u\rangle$.
2. Reflexive: $\langle \boldsymbol{U}, \boldsymbol{V}\rangle = \langle \boldsymbol{V}, \boldsymbol{U}\rangle^*$.

3. Non-degeneracy: $\langle \boldsymbol{V}, \boldsymbol{V} \rangle = \boldsymbol{0} \Leftrightarrow \boldsymbol{V} = \boldsymbol{0}$.

Since we are thinking of the matrix as an inner product, we can also think of the norm as a matrix.

Definition 4 *The norm of a complex random vector is given by $\|\boldsymbol{Z}\|^2 = \langle \boldsymbol{Z}, \boldsymbol{Z} \rangle$.*

Since we are thinking of inner products as matrices instead of scalars, we can rewrite the Hilbert Projection Theorem to use matrices instead.

Theorem 4 (Hilbert Projection Theorem) *The minimization problem $\min_{\hat{\boldsymbol{X}}(\boldsymbol{Y})} \|\hat{\boldsymbol{X}}(\boldsymbol{Y}) - \boldsymbol{X}\|^2$ has a unique solution which is a linear function of $\boldsymbol{Y}$. The error is orthogonal to the linear subspace of $\boldsymbol{Y}$ (i.e $\langle \boldsymbol{X} - \hat{\boldsymbol{X}}, \boldsymbol{Y} \rangle = \boldsymbol{0}$)*

When we do a minimization over a matrix, we are minimizing it in a PSD sense, so for any other linear function $\boldsymbol{X}'$,

$$\|\boldsymbol{X} - \hat{\boldsymbol{X}}\|^2 \preceq \|\boldsymbol{X} - \boldsymbol{X}'\|^2.$$

3 Linear Estimation

In Linear Estimation, we are trying to estimate a random variable $\boldsymbol{X}$ using an observation $\boldsymbol{Y}$ with a llinear function of $\boldsymbol{Y}$. If $\boldsymbol{Y}$ is finite dimensional, then we can say $\hat{\boldsymbol{X}}(\boldsymbol{Y}) = \boldsymbol{W}\boldsymbol{Y}$ where $\boldsymbol{W}$ is some matrix. Using theorem 4 and the orthogonality principle, we know that

$$\langle \boldsymbol{X} - \boldsymbol{W}\boldsymbol{Y}, \boldsymbol{Y} \rangle = \boldsymbol{0} \Leftrightarrow \boldsymbol{R}_{XY} = \boldsymbol{W}\boldsymbol{R}_Y$$

This is known as the **Normal Equation**. If $\boldsymbol{R}_Y$ is invertible, then we can apply the inverse to find $\boldsymbol{W}$. Otherwise, we can apply the pseudoinverse $\boldsymbol{R}_Y^\dagger$ to find $\boldsymbol{W}$, which may not be unique. If we want to measure the quality of the estimation, since $\boldsymbol{X} = \boldsymbol{X} + (\boldsymbol{X} - \hat{\boldsymbol{X}})$,

$$\|\boldsymbol{X}\|^2 = \|\hat{\boldsymbol{X}}\|^2 + \|\boldsymbol{X} - \hat{\boldsymbol{X}}\|^2 \implies$$
$$\|\boldsymbol{X} - \hat{\boldsymbol{X}}\|^2 = \|\boldsymbol{X}\|^2 - \|\hat{\boldsymbol{X}}\|^2 = \boldsymbol{R}_X - \boldsymbol{X}_{XY}\boldsymbol{R}_Y^{-1}\boldsymbol{R}_{YX}$$