EE222 Course Notes

Anmol Parande

Spring 2022 - Professors Shankar Shastry and Koushil Srinath

Disclaimer: These notes reflect EE222 when I took the course (Fall 2021). They may not accurately reflect current course content, so use at your own risk. If you find any typos, errors, etc, please raise an issue on the GitHub repository.

Contents

1 Nonlinear System Dynamics

Consider the non-linear system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, t).$$

Definition 1 *The system is autonomous if $f(x,t)$ is not explicitly dependent on time t.*

Definition 2 *A point x_0 is an equilibrium point at time t_0 if*

$$\forall t \geq t_0,\ f(x_0, t) = 0$$

1.1 Planar Dynamical Systems

Planar dynamical systems are those with 2 state variables. Suppose we linearize the system $\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = f(\boldsymbol{x})$ at an equilibrium point.

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = D_f|_{\boldsymbol{x}=\boldsymbol{x_0}}\boldsymbol{x}$$

Depending on the eigenvalues of D_f, the Jacobian, we get several cases for how this linear system behaves. We'll let z_1 and z_2 be the eigenbasis of the *phase space*.

1. The eigenvalues are real, yielding solutions $z_1 = z_1(0)e^{\lambda_1 t}$, $z_2 = z_2(0)e^{\lambda_2 t}$. If we eliminate the time variable, we can plot the trajectories of the system.

   $$\frac{z_1}{z_1(0)} = \left(\frac{z_2}{z_2(0)}\right)^{\frac{\lambda_1}{\lambda_2}}$$

   (a) When $\lambda_1, \lambda_2 < 0$, all trajectories converge to the origin, so we call this a **stable node**.

   (b) When $\lambda_1, \lambda_2 > 0$, all trajectories blow up, so we call this an **unstable node**.

   (c) When $\lambda_1 < 0 < \lambda_2$, the trajectories will converge to the origin along the axis corresponding to λ_1 and diverge along the axis corresponding to λ_2, so we call this a **saddle node**.

2. There is a single repeated eigenvalue with one eigenvector. As before, we can eliminate the time variable and plot the trajectories on the z_1, z_2 axes.

(a) When $\lambda < 0$, the trajetories will converge to the origin, so we call it an **improper stable node**

(b) When $\lambda > 0$, the trajetories will diverge from the origin, so we call it an **improper unstable node**

3. When there is a complex pair of eigenvalues, the linear system will have oscillatory behavior. The Real Jordan form of D_f will look like

$$D_f = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}.$$

The parameter β will determine the direction of the trajectories (clockwise if positive).

(a) When $\alpha < 0$, the trajectories will spiral towards the origin, so we call it a **stable focus**.

(b) When $\alpha = 0$, the trajectories will remain at a constant radius from the origin, so we call it a **center**.

(c) When $\alpha > 0$, the trajectories will spiral away from the origin, so we call it an **unstable focus**.

It turns out that understanding the linear dynamics at equilibrium points can be helpful in understanding the nonlinear dynamics near equilibrium points.

Theorem 1 (Hartman-Grobman Theorem) *If the linearization of a planar dynamical system* $\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = f(\boldsymbol{x})$ *at an equilibrium point* $\boldsymbol{x_0}$ *has no zero or purely imaginary eigenvalues, then there exists a homeomorphism from a neighborhood* U *of* x_0 *into* $\mathbb{R}^2$ *which takes trajectories of the nonlinear system and maps them onto the linearization where* $h(\boldsymbol{x_0}) = 0$*, and the homeomorphism can be chosen to preserve the parameterization by time.*

Theorem 1 essentially says that the linear dynamics predict the nonlinear dynamics around equilibria, but only for a neighborhood around the equilibrium point. Outisde of this neighborhood, the linearization may be very wrong.

Non-linear systems can also have periodic solutions.

Definition 3 *A closed orbit* γ *is a trajectory of the system such that* $\gamma(0) = \gamma(T)$ *for finite* T*.*

Suppose that we have a simply connected region D (meaning D cannot be contracted to a point) and we want to know if it contains a closed orbit.

Theorem 2 (Bendixon's Theorem) *If* $\mathrm{div}(f)$ *is not identically zero in a sub-region of* D *and does not change sign in* D*, then* D *contains no closed orbits.*

Theorem 2 lets us rule out closed orbits from regions of $\mathbb{R}^2$.

Definition 4 *A region $M \subset \mathbb{R}^2$ is positively invariant for a trajectory $\phi_t(\boldsymbol{x})$ if $\forall x \in M, \forall t \geq 0, \phi_t(\boldsymbol{x}) \in M$.*

A positively invariant set essentially means that once a trajectory enters the set, it cannot leave. That means all of the vector field lines must point inside the set. If we have a positively invariant region, then we can determine whether it contains closed orbits.

Theorem 3 (Poincare-Bendixson Theorem) *If M is a compact, positively invariant set for the flow $\phi_t(\boldsymbol{x})$, then if M contains no equilibrium points, then M has a limit cycle.*