EECS225A Course Notes

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Contents

1 Transforms

1.1 Discrete Time Fourier Transform

The Discrete Time Fourier Transform is given by

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}.$$

The Inverse Discrete Time Fourier Transform is given by

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n} d\omega.$$

Since the DTFT is an infinite summation, it may or may not converge.

Definition 1 *A signal $x[n]$ belongs to the l^1 class of signals if the series converges absolutely. In other words,*

$$\sum_{k=-\infty}^{\infty} |x[k]| < \infty.$$

This class covers most real-world signals.

Theorem 1 *If $x[n]$ is a l^1 signal, then the DTFT $X(e^{j\omega})$ converges uniformly and is well-defined for every ω. $X(e^{j\omega})$ is also a continuous function.*

Definition 2 *A signal $x[n]$ belongs to the l^2 class of signals if it is square summable. In other words,*

$$\sum_{k=-\infty}^{\infty} |x[k]|^2 < \infty.$$

The l^2 class contains important functions such as sinc.

Theorem 2 *If $x[n]$ is a l^2 signal, then the DTFT $X(ej\omega)$ is defined almost everywhere and only converges in the mean-squared sense:*

$$\lim_{N\to\infty} \int_{-\pi}^{\pi} \left| \left(\sum_{k=-N}^{N} x[k]e^{-j\omega n} \right) - X(\omega) \right|^2 d\omega = 0$$

Tempered distributions like the Dirac Delta function are other functions which are important for computing the DTFT, and they arise from the theory of generalized functions.

1.2 Z-Transform

The Z-transform is a generalized version of the DTFT and is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}.$$

It is a special type of series called a **Laurent Series**.

Theorem 3 *A Laurent Series will converge absolutely on an open annulus*

$$A = \{z | r < |z| < R\}$$

for some r and R.

We can compute r and R using the signal $x[n]$.

$$r = \limsup_{n\to\infty} |x[n]|^{\frac{1}{n}}, \qquad \frac{1}{R} = \limsup_{n\to\infty} |x[-n]|^{\frac{1}{n}}.$$

2 Hilbert Space Theory

Complex random vectors form a Hilbert space with inner product $\langle X, Y \rangle = \mathbb{E}\left[XY^*\right]$. If we have a random complex vector, then we can use Hilbert Theory in a more efficient manner by looking at the matrix of inner products. For simplicity, we will call this the "inner product" of two complex vectors.

Definition 3 *Let the inner product between two random, complex vectors Z_1, Z_2*

$$\langle \boldsymbol{Z_1}, \boldsymbol{Z_2} \rangle = \mathbb{E}\left[\boldsymbol{Z_1}\boldsymbol{Z_2}^*\right]$$

The ij-th entry of the matrix is simply the scalar inner product $\mathbb{E}\left[\boldsymbol{X_i}\boldsymbol{Y_j}^*\right]$ where $\boldsymbol{X_i}$ and $\boldsymbol{Y_j}$ are the ith and jth entries of $\boldsymbol{X}$ and $\boldsymbol{Y}$ respectively. This means the matrix is equivalent to the cross correlation $\boldsymbol{R}_{XY}$ between the two vectors. We can also specify the auto-correlation $\boldsymbol{R}_X = \langle \boldsymbol{X}, \boldsymbol{X} \rangle$ and auto-covariance $\Sigma_X = \langle \boldsymbol{X} - \mathbb{E}\left[\boldsymbol{X}\right], \boldsymbol{X} - \mathbb{E}\left[\boldsymbol{X}\right] \rangle$. One reason why we can think of this matrix as the inner product is because it also satisfies the properties of inner products. In particular, it is

1. Linear: $\langle \alpha_1 \boldsymbol{V_1} + \alpha_2 \boldsymbol{V_2}, \boldsymbol{u} \rangle = \alpha_1 \langle \boldsymbol{V_1}, u \rangle + \alpha_2 \langle \boldsymbol{V_2}, u \rangle$.

2. Reflexive: $\langle \boldsymbol{U}, \boldsymbol{V} \rangle = \langle \boldsymbol{V}, \boldsymbol{U} \rangle^*$.

3. Non-degeneracy: $\langle \boldsymbol{V}, \boldsymbol{V} \rangle = \boldsymbol{0} \Leftrightarrow \boldsymbol{V} = \boldsymbol{0}$.

Since we are thinking of the matrix as an inner product, we can also think of the norm as a matrix.

Definition 4 *The norm of a complex random vector is given by* $\|\boldsymbol{Z}\|^2 = \langle \boldsymbol{Z}, \boldsymbol{Z} \rangle$.

Since we are thinking of inner products as matrices instead of scalars, we can rewrite the Hilbert Projection Theorem to use matrices instead.

Theorem 4 (Hilbert Projection Theorem) *The minimization problem* $\min_{\hat{\boldsymbol{X}}(\boldsymbol{Y})} \|\hat{\boldsymbol{X}}(\boldsymbol{Y}) - \boldsymbol{X}\|^2$ *has a unique solution which is a linear function of* $\boldsymbol{Y}$*. The error is orthogonal to the linear subspace of* $\boldsymbol{Y}$ *(i.e* $\langle \boldsymbol{X} - \hat{\boldsymbol{X}}, \boldsymbol{Y} \rangle = \boldsymbol{0}$*)*

When we do a minimization over a matrix, we are minimizing it in a PSD sense, so for any other linear function $\boldsymbol{X}'$,

$$\|\boldsymbol{X} - \hat{\boldsymbol{X}}\|^2 \preceq \|\boldsymbol{X} - \boldsymbol{X}'\|^2.$$

3 Linear Estimation

In Linear Estimation, we are trying to estimate a random variable $\boldsymbol{X}$ using an observation $\boldsymbol{Y}$ with a llinear function of $\boldsymbol{Y}$. If $\boldsymbol{Y}$ is finite dimensional, then we can say $\hat{\boldsymbol{X}}(\boldsymbol{Y}) = \boldsymbol{W}\boldsymbol{Y}$ where $\boldsymbol{W}$ is some matrix. Using theorem 4 and the orthogonality principle, we know that

$$\langle \boldsymbol{X} - \boldsymbol{W}\boldsymbol{Y}, \boldsymbol{Y} \rangle = \boldsymbol{0} \Leftrightarrow \boldsymbol{R}_{XY} = \boldsymbol{W}\boldsymbol{R}_Y$$

This is known as the **Normal Equation**. If $\boldsymbol{R}_Y$ is invertible, then we can apply the inverse to find $\boldsymbol{W}$. Otherwise, we can apply the pseudoinverse $\boldsymbol{R}_Y^\dagger$ to find $\boldsymbol{W}$, which may not be unique. If we want to measure the quality of the estimation, since $\boldsymbol{X} = \boldsymbol{X} + (\boldsymbol{X} - \hat{\boldsymbol{X}})$,

$$\|\boldsymbol{X}\|^2 = \|\hat{\boldsymbol{X}}\|^2 + \|\boldsymbol{X} - \hat{\boldsymbol{X}}\|^2 \implies$$
$$\|\boldsymbol{X} - \hat{\boldsymbol{X}}\|^2 = \|\boldsymbol{X}\|^2 - \|\hat{\boldsymbol{X}}\|^2 = \boldsymbol{R}_X - \boldsymbol{R}_{XY}\boldsymbol{R}_Y^{-1}\boldsymbol{R}_{YX}$$

3.1 Affine Estimation

If we allow ourselves to consider an affine function for estimation $\hat{\boldsymbol{X}}(\boldsymbol{Y}) = \boldsymbol{W}\boldsymbol{Y} + b$, then this is equivalent to instead finding an estimator

$$\hat{\boldsymbol{X}}(\boldsymbol{Y}') = \boldsymbol{W}\boldsymbol{Y}' \qquad \text{where } \boldsymbol{Y}' = \begin{bmatrix} \boldsymbol{Y} \\ 1 \end{bmatrix}$$

This is equivalent to the following orthogonality conditions:

1. $\langle \boldsymbol{X} - \hat{\boldsymbol{X}}, \boldsymbol{Y} \rangle$

2. $\langle \boldsymbol{X} - \hat{\boldsymbol{X}}, 1 \rangle$

Solving gives us

$$\hat{\boldsymbol{X}}(\boldsymbol{Y}) = W(\boldsymbol{Y} - \boldsymbol{\mu}_Y) + \mu_x \qquad \text{where } \boldsymbol{W}\boldsymbol{\Sigma}_Y = \boldsymbol{\Sigma}_{XY}.$$

$\boldsymbol{\Sigma}_Y$ and $\boldsymbol{\Sigma}_{XY}$ are the auto-covariance and cross-covariance respectively. Recall that if

$$\begin{bmatrix} \boldsymbol{X} \\ \boldsymbol{Y} \end{bmatrix} \sim \mathcal{N} \left(\begin{bmatrix} \boldsymbol{\mu}_X \\ \boldsymbol{\mu}_Y \end{bmatrix}, \begin{bmatrix} \boldsymbol{\Sigma}_X & \boldsymbol{\Sigma}_{XY} \\ \boldsymbol{\Sigma}_{YX} & \boldsymbol{\Sigma}_Y \end{bmatrix} \right)$$

then

$$\boldsymbol{X}|\boldsymbol{Y} \sim \mathcal{N} \left(\boldsymbol{\mu}_X + \boldsymbol{\Sigma}_{XY}\boldsymbol{\Sigma}_Y^{-1}(\boldsymbol{Y} - \boldsymbol{\mu}_Y), \boldsymbol{\Sigma}_X - \boldsymbol{\Sigma}_{XY}\boldsymbol{\Sigma}_Y^{-1}\boldsymbol{\Sigma}_{YX} \right)$$

The mean is the best affine estimator of $\boldsymbol{X}$, and the covariance is the estimation error. This has two interpretations.

1. Under the Gaussian assumption, the best nonlinear estimator $\mathbb{E}\left[\boldsymbol{X}|\boldsymbol{Y}\right]$ is affine

2. Gaussian random variables are the hardest predict because nonlinearity should improve our error, but it does not in the Gaussian case. This means if affine estimation works well, we shouldn't try and find better non-linear estimators.

3.2 Least Squares

The theory of linear estimation is very closely connected with the theory behind least squares in linear algebra. In least squares, we have a deterministic x and assume nothing else about it, meaning we are looking for an unbiased estimator. theorem 5 tells us how to find the best linear unbiased estimator in a linear setting.

Theorem 5 (Gauss Markov Theorem) *Suppose that $\boldsymbol{Y} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{Z}$ and Z is zero-mean with $\langle \boldsymbol{Z}, \boldsymbol{Z} \rangle = \boldsymbol{I}$, $\boldsymbol{H}$ is full-column rank, then $\hat{\boldsymbol{x}}_b = (\boldsymbol{H}^*\boldsymbol{H})^{-1}\boldsymbol{H}^*\boldsymbol{Y}$ is the best linear unbiased estimator.*

4 Discrete Time Random Processes

Definition 5 *A Discrete-Time Random Process is a countably infinite collection of random variables on the same probability space $\{\boldsymbol{X}_n : n \in \mathbb{Z}\}$.*

Discrete Time Random Processes have a mean function $\boldsymbol{\mu}_n = \mathbb{E}\left[\boldsymbol{X}_n\right]$ and an auto-correlation function $\boldsymbol{R}_X(n_1, n_2) = \mathbb{E}\left[\boldsymbol{X}_{n_1} \boldsymbol{X}_{n_2}^*\right]$

Definition 6 *A Wide-Sense Stationary Random Process is a disrete-time random process with constant mean, finite variance, and an autocorrelation function that can be re-written to only depend on $n_1 - n_2$.*

We call this wide-sense stationary because the mean and covariance do not change as the process evolves. In a strict-sense stationary proces, the distribution of each random variable in the process would not change.