EE120 Course Notes

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Contents

1 Introduction to Signals and Systems

1.1 Types of Signals

Definition 1 *A signal is a function of one or more variables*

Definition 2 *A signal $x(t)$ is continuous if $x : \mathbb{R} \to \mathbb{R}$*

Definition 3 *A signal $x[n]$ is discrete if $x : \mathbb{Z} \to \mathbb{R}$*

1.1.1 Properties of the Unit Impulse

Definition 4 *The unit impulse in discrete time is defined as*

$$\delta[n] = \left\{ \begin{array}{c} 1, \textit{if } n = 0 \\ 0, \ \textit{else} \end{array} \right\}$$

- $f[n]\delta[n] = f[0]\delta[n]$
- $f[t]\delta[n - N] = f[N]\delta[n - N]$

Definition 5 *The unit impulse in continuous time is the dirac delta function*

$$\delta(t) = lim_{\Delta \to 0}\delta_{\Delta}(t)$$

$$\delta_{\Delta} = \left\{ \begin{array}{c} \frac{1}{\Delta}, \ \textit{if } \geq 0 \\ 0else \end{array} \right\}$$

- $f(t)\delta(t) = f(0)\delta(t)$
- $f(t)\delta(t - \tau) = f(\tau)\delta(t - \tau)$
- $\delta(at) = \frac{1}{|a|}\delta(t)$

Definition 6 *The unit step is defined as*

$$u[n] = \left\{ \begin{array}{c} 1, \textit{if } n \geq 0 \\ 0, \ \textit{else} \end{array} \right\}$$

1.2 Signal transformations

Signals can be transformed by modifying the variable.

- $x(t - \tau)$: Shift a signal left by τ steps.
- $x(-t)$: Rotate a signal about the $t = 0$
- $x(kt)$: Stretch a signal by a factor of k

These operations can be combined to give more complex transformations. For example, $y(t) = x(\tau - t) = x(-(t - \tau))$ flips x and shifts it right by τ timesteps. This is equivalent to shifting x left by τ timesteps and then flipping it.

1.3 Convolution

Definition 7 *The convolution of two signals in discrete time*

$$(x * h)[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Definition 8 *The convolution of two signals in continuous time*

$$(x * h)(t) = \int_{\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

While written in discrete time, these properties apply in continuous time as well.

- $(x * \delta)[n] = x[n]$
- $x[n] * \delta[n-N] = x[n-N]$
- $(x * h)[n] = (h * x)[n]$
- $x * (h_1 + h_2) = x * h_1 + x * h_2$
- $x * (h_1 * h_2) = (x * h_1) * h_2$

1.4 Systems and their properties

Definition 9 *A system is a process by which input signals are transformed to output signals*

Definition 10 *A memoryless system has output which is only determined by the input's present value*

Definition 11 *A causal system has output which only depends on input at present or past times*

Definition 12 *A stable system produces bounded output when given a bounded input. By extension, this means an unstable system is when $\exists$ a bounded input that makes the output unbounded.*

Definition 13 *A system is time-invariant if the original input $x(t)$ is transformed to $y(t)$, then $x(t-\tau)$ is transformed to $y(t-\tau)$*

Definition 14 *A system $f(x)$ is linear if and only if*

- *If $y(t) = f(x(t))$, then $f(ax(t)) = ay(t)$ (Scaling)*
- *If $y_1(t) = f(x_1(t))$ and $y_2(t) = f(x_2(t))$, then $f(x_1(t) + x_2(t)) = y_1(t) + y_2(t)$ (Superposition)*

Notice: The above conditions on linearity require that $x(0) = 0$ because if $a = 0$, then we need $y(0) = 0$ for scaling to be satisfied

Definition 15 *The impulse response of a system $f[x]$ is $h[n] = f[\delta[n]]$, which is how it response to an impulse input.*

1.5 Exponential Signals

Exponential signals are important because they can succinctly represent complicated signals using complex numbers. This makes analyzing them much easier.

$$x(t) = e^{st}, x[n] = z^n (s, z \in \mathbb{C})$$

Definition 16 *The frequency response of a system is how a system responds to a purely oscillatory signal*

2 Linear Time-Invariant Systems

Definition 17 *LTI systems are ones which are both linear and time-invariant.*

2.1 Impulse Response of LTI systems

LTI systems are special systems because their output can be determined entirely the impulse response $h[n]$.

2.1.1 The discrete case

We can think of the original signal $x[n]$ in terms of the impulse function.

$$x[n] = x[0]\delta[n] + x[1]\delta[n-1] + ... = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

This signal will be transformed in some way to get the output $y[n]$. Since the LTI system applies a functional F and the LTI is linear and time-invariant,

$$y[n] = F(\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]) = \sum_{k=-\infty}^{\infty} x[k]F(\delta[n-k]) = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Notice this operation is the convolution between the input and the impulse response.

$$y(t) = \int_{\infty}^{\infty} x(\tau)h(t-\tau)$$

2.1.2 The continuous case

We can approximate the function by breaking it into intervals of length Δ.

$$x(t) \approx \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_{\Delta}(t-k\Delta)\Delta$$

$$x(t) = lim_{\Delta \to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_{\Delta}(t-k\Delta)\Delta$$

After applying the LTI system to it,

$$y(n) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)$$

Notice this operation is the convolution between the input and the impulse response.

2.2 Determining Properties of an LTI system

Because an LTI system is determined entirely by its impulse response, we can determine its properties from the impulse response.

2.2.1 Causality

Theorem 1 *An LTI system is causal when* $h[n] = 0, \forall n < 0$

Proof 1 *Assume* $h[n] = 0, \forall n < 0$

$$y[n] = (x * h)[n] = \sum_{k=-\infty}^{\infty} x[n-k]h[k] = \sum_{k=0}^{\infty} x[n-k]h[k]$$

Notice that this does not depend on time steps prior to $n = 0$

2.2.2 Memory

Theorem 2 *An LTI system is memoryless if* $h[n] = 0, if \forall n \neq 0$

2.2.3 Stability

Theorem 3 *A system is stable if* $\sum_{n=-\infty}^{\infty} |h[n]|$ *converges.*

Proof 2
1. *Assume* $|x[n]| \leq B_x$ *to show* $|y[n]| < D$ *where* D *is some bound.*

$$|y[n]| = |\sum_{k=-\infty}^{\infty} x[n-k]h[k]| \leq \sum_{k} |x[n-k]h[k]| = \sum_{k} |x[n-k]||h[k]| \leq B_x \sum_{k} |h[k]$$

This means as long as $\sum_k |h[k]$ *converges,* $y[n]$ *will be bounded.*

2. *Assume* $\sum_n |h[n]|$ *does not converge. Show that the system is unstable. Choose* $x[n] = sgn\{h[-n]\}$

$$y[n] = \sum_{k} x[n-k]h[k]$$

so

$$y[0] = \sum_{k} x[-k]h[k] = \sum_{k} |h[k]|$$

And this is unbounded, so $y[n]$ *is unbounded.*

2.3 Frequency Response

If we pass a complex exponential into an LTI system, the output signal is the same signal but scaled. In otherwise, it is an eigenfunction of LTI systems.

$$y(t) = \int_{-\infty}^{\infty} e^{s(t-\tau)} h(\tau) d\tau = e^{st} \int_{-\infty}^{\infty} e^{-s\tau} h(\tau)$$

The integral is a constant, and the original function is unchanged. The same analysis can be done in the discrete case.

$$y[n] = \sum_{k=-\infty}^{\infty} z^{n-k} h[k] = z^n \sum_{k=-\infty}^{\infty} z^{-k} h[k]$$

Definition 18 *The frequency response of a system is the output when passed a purely oscillatory signal*

Definition 19 *The transfer function of an LTI system $H(\omega)$ is how the system scales a pure tone of frequency ω*

$$H(\omega) := \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau, H(\omega) := \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k}$$