Unilateral Laplace Transform

$$X(s) = \int_{0^-}^{\infty} x(t)e^{-st}dt$$

Theorems

$x(t)$	$X(s)$	ROC
$x(t-t_0)$	$e^{-st_0}X(s)$	R
$e^{s_0 t}x(t)$	$X(s-s_0)$	$R + Re(s_0)$
$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{s}{a}\right)$	aR
$x^*(t)$	$X(s^*)^*$	R
$(x_1 * x_2)(t)$	$X_1(s)X_2(s)$	$R_1 \bigcap R_2$
$-tx(t)$	$\dfrac{\mathrm{d}X}{\mathrm{d}s}$	R
$\dfrac{\mathrm{d}^n x}{\mathrm{d}t^n}$	$s^n X(s) - \sum_{i=0}^{n-1} s^{n-i-1}\dfrac{\mathrm{d}^i x}{\mathrm{d}t^i}\vert_{t=0^-}$	R

Transforms

Signal	Transform	ROC
$\delta(t-T)$	e^{-sT}	$\mathbb{C}$
$\frac{t^{n-1}}{(n-1)!}u(t)$	$\frac{1}{s^n}$	$Re(s) > 0$
$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$	$\frac{1}{(s+a)^n}$	$Re(s) > a$
$e^{-at}\cos(\omega_0 t)u(t)$	$\frac{s+a}{(s+a)^2+\omega_0^2}$	$Re(s) > a$
$e^{-at}\sin(\omega_0 t)u(t)$	$\frac{\omega_0}{(s+a)^2+\omega_0}$	$Re(s) > a$

Electro-Mechanical Equivalence

Equivalent Quantities

Translational Mechanical System	Rotational Mechanical System	Electrical System
Force (F)	Torque	Voltage (V)
Mass (M)	Moment of Inertia (J)	Inductance (L)
Damping Coefficient (B)	Rotational Damping Coefficient (B)	Resistance (R)
Spring Constant (K)	Torsional Spring Constant (K)	Reciprocal of Capacitance $\left(\frac{1}{C}\right)$
Displacement (x)	Angular Displacement (θ)	Charge (Q)
Velocity (v)	Angular Velocity (ω)	Current (I)

Equation Equivalence

Translational Mechanical System	Rotational Mechanical System	Electrical System
$Ms^2X(s)$	$Js^2\Theta(s)$	$LsI(s)$
$BsX(s)$	$Bs\Theta(s)$	$RI(s)$
$KX(s)$	$K\Theta(s)$	$\frac{1}{Cs}I(s)$
-	$\frac{T_2(s)}{T_1(s)} = \frac{\Theta_1(s)}{\Theta_2(s)} = \frac{N_2}{N_1}$	$\frac{N_p}{N_s} = \frac{V_p(s)}{V_s(s)} = \frac{I_s(s)}{I_p(s)}$

Conversion Rules

1. The force at two ends of a damper (or spring) must be equal $\Leftrightarrow$ the voltage across the resistor (or capacitor) must be equal
2. Parallel in one domain $\implies$ Series in the other domain
3. $\sum F = 0$ at a massless node $\Leftrightarrow$ $\sum V = 0$ at an electrical node
4. Rotational impedances are reflected through gear trains by multiplying by $\left(\frac{N_{dest}^2}{N_{source}^2}\right)$

Conversion Procedure

Electrical to Mechanical

1. Label all currents such that only one current flows through inductors
2. Write loop equations for each loop
3. Re-write equations using the analogous quantities. Each loop is replaced by a position
4. Draw mechanical system corresponding to equations

Mechanical to Electrical

1. Write force equations for each position
2. Re-write equations using analogous quantities. Each equation becomes a loop
3. Draw loops such that only one current flows through each inductor

Second-Order System

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2}$$

Poles are at $s = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$.

Cases of Interest

$\zeta = 0$: $s = \pm\omega_n j$. The system is marginally stable.

$0 < \zeta < 1$: $s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$. The system is underdamped.

$\zeta = 1$: $s = -\omega_n$. The system is critically damped.

$\zeta > 1$: $s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$. The system is overdamped

Underdamped systems

Time to Peak: $T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$

% Overshoot: $e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}$

Settling Time: $T_s = \frac{4}{\zeta\omega_n}$

Rule of Thumb: If two poles are at $s = -a \pm bj$, then if $Re\{c\} \leq 5 \cdot Re\{a\}$, the system is approximately second order.

Steady-State Errors

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} \frac{sR(s)}{1+G(s)}$$

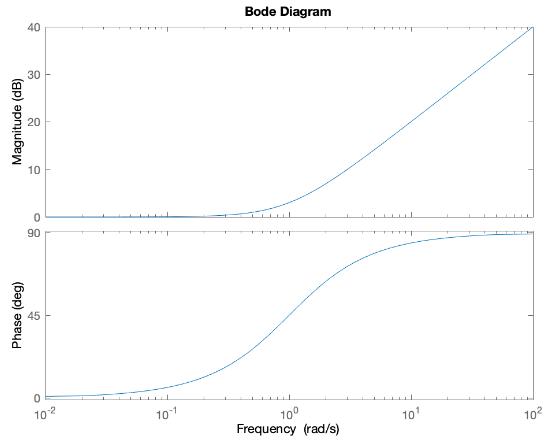
$$K_p = \lim_{s\to 0} G(s) \quad K_v = \lim_{s\to 0} sG(s) \quad K_a = \lim_{s\to 0} s^2 G(s)$$

The **type** of a system is how many poles at 0 the open loop system has.

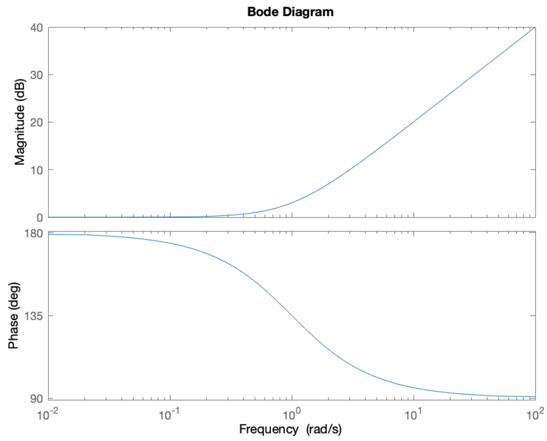
		Type 0		**Type 1**		**Type 2**	
Input	**SSE**	**Constant**	**Error**	**Constant**	**Error**	**Constant**	**Error**
$u(t)$	$\frac{1}{1+K_p}$	$K_p =$ Constant	$\frac{1}{1+K_p}$	$K_p = \infty$	0	$K_p = \infty$	0
$tu(t)$	$\frac{1}{K_v}$	$K_v = 0$	∞	$K_v =$ Constant	$\frac{1}{K_v}$	$K_v = \infty$	0
$\frac{1}{2}t^2u(t)$	$\frac{1}{K_a}$	$K_a = 0$	∞	$K_a = 0$	∞	$K_a =$ Constant	$\frac{1}{K_a}$

Bode Diagrams

First-Order Zeros

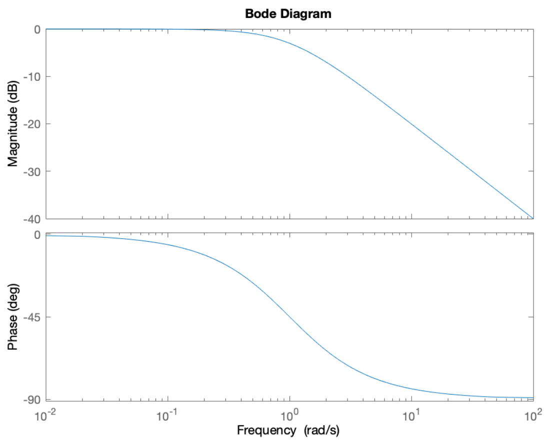


(a) LHP Zero

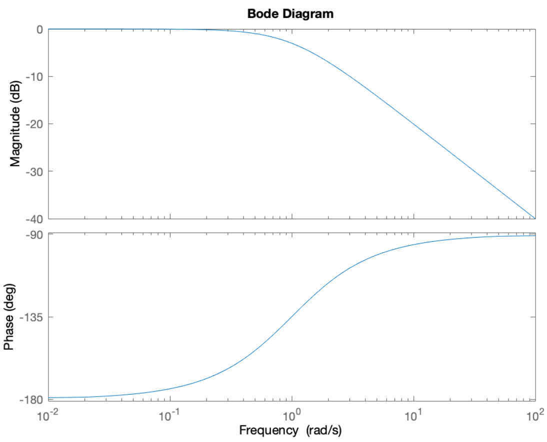


(b) RHP Zero

First-Order Poles

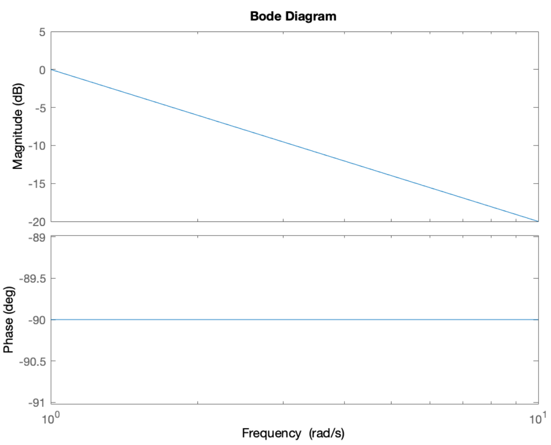


(a) LHP pole

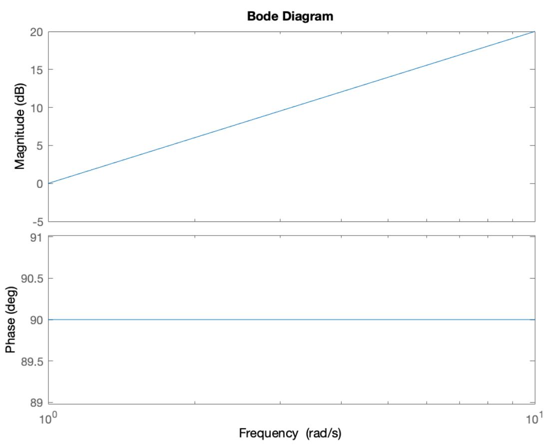


(b) RHP Pole

Pole/Zero at $\omega - 0$

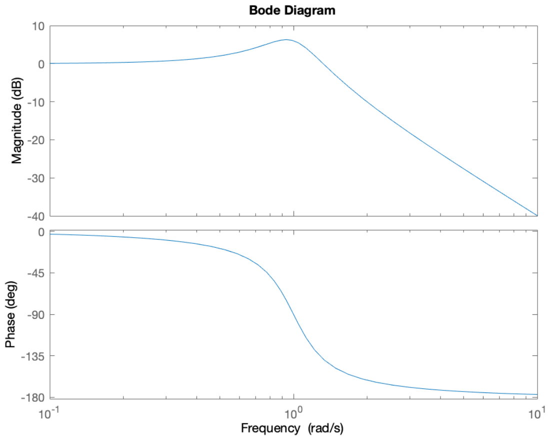


(a) Pole at zero

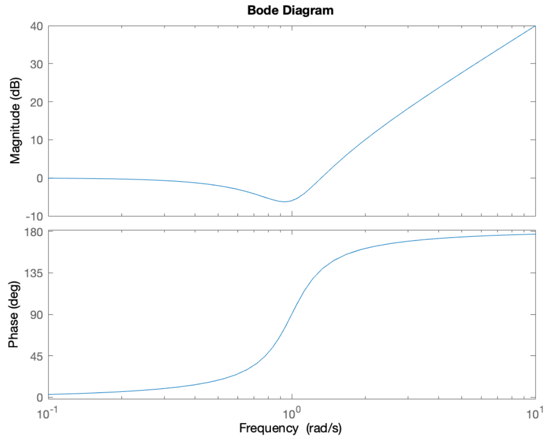


(b) Zero at zero

Second-Order behavior



(a) Second-Order poles



(b) Second-Order zeros

Nyquist Diagrams

In the standard negative feedback setup with plant G and controller H,

1. Poles of $1 + GH$ are the poles of the open loop system.
2. Zeros of $1 + GH$ are the poles of the closed loop system.

Where N is the number of CCW encirclements of -1, P is the number of RHP poles of GH, and Z is number of RHP zeros of $1 + GH$

$$N = P - Z$$

If $Z = 0$, then the system is stable.

Design by Frequency Response

Gain Margin: The change in open loop gain that will make the closed loop system unstable.

$$\angle G(j\omega_{GM}) = (2k+1)\pi \implies G_m = \frac{1}{G(j\omega_{GM})}$$

Phase Margin: The change in open loop phase to make the closed loop system unstable.

$$|G(j\omega_{PM})| = 1 \implies \phi_m = (2k+1)\pi + \angle G(j\omega_{PM})$$

For a second-order system,

$$\omega_{PM} = \omega_n\sqrt{-2\zeta^2 + \sqrt{1+4\zeta^4}}$$
$$\phi_M = \tan^{-1}\frac{2\zeta}{\sqrt{-2\zeta^2+\sqrt{1+4\zeta^4}}}$$

Lag Controller

$$G_c(s) = \frac{s+\frac{1}{T}}{s+\frac{1}{\alpha T}}, \quad \alpha > 1$$

Purpose: To reduce static error constant by increasing low frequency gain and to increase the phase margin of the system.

Design Procedure

1. Set gain K to the value that satisfies the SSE specification and plot the Bode diagram at that gain.
2. Find ω_{PM} such that ϕ_M is 5°to 12°larger than required.
3. Let the high frequency asymptote be $-20\log K_{PM}$dB at ω_{PM} where $K_{PM} = |G(j\omega_{PM})|$.
4. Choose the upper break frequency to be $\frac{\omega_{PM}}{10}$
5. Set the low frequency asymptote to be 0 dB and locate the lower break frequency.
6. Reset the system gain K to compensate for attenuation.

Lead Controller

$$G_c(s) = \frac{1}{\beta}\frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}}, \quad \beta < 1$$

Purpose: To change the phase margin and decrease percent overshoot and reduce rise/settling time.
The lead controller has a peak phase ϕ_{max}.

$$\omega_{max} = \frac{1}{T\sqrt{\beta}} \quad, \phi_{max} = \sin^{-1}\frac{1-\beta}{1+\beta}, \quad |G_c(j\omega_{max})| = \frac{1}{\sqrt{\beta}}$$

Design Procedure

1. Set gain K of the uncompensated system to a value satisfying SSE requirement.
2. Plot bode diagram for system with gain K and determine ϕ_M.
3. Find ϕ_M needed to meet requirements and evaluate additional phase contribution from compenstor.
4. Determine β.
5. Determine $|G_c(j\omega_{max})|$.
6. Determine ω_{PM} where $|G(j\omega)| = -20\log|G_c(j\omega_{max})|$
7. Find the break frequencies.
8. Reset the gain
9. Simulate and tweak.