EE120 Course Notes

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Fall 2019 - Professor Murat Arcak

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1 Introduction to Signals and Systems

1.1 Types of Signals

Definition 1 *A signal is a function of one or more variables*

Definition 2 *A continuous signal* $x(t)$ *maps* $\mathbb{R} \rightarrow \mathbb{R}$

Definition 3 *A discrete signal* $x[n]$ *maps* $\mathbb{Z} \rightarrow \mathbb{R}$

1.1.1 Properties of the Unit Impulse

Definition 4 *The unit impulse in discrete time is defined as*

$$\delta[n] = \left\{ \begin{array}{c} 1, \textit{if } n = 0 \\ 0, \ \textit{else} \end{array} \right\}$$

- $f[n]\delta[n] = f[0]\delta[n]$
- $f[t]\delta[n - N] = f[N]\delta[n - N]$

Definition 5 *The unit impulse in continuous time is the dirac delta function*

$$\delta(t) = lim_{\Delta \rightarrow 0}\delta_{\Delta}(t)$$

$$\delta_{\Delta} = \left\{ \begin{array}{c} \frac{1}{\Delta}, \ \textit{if } \geq 0 \\ 0else \end{array} \right\}$$

- $f(t)\delta(t) = f(0)\delta(t)$
- $f(t)\delta(t - \tau) = f(\tau)\delta(t - \tau)$
- $\delta(at) = \frac{1}{|a|}\delta(t)$

Definition 6 *The unit step is defined as*

$$u[n] = \left\{ \begin{array}{c} 1, \textit{if } n \geq 0 \\ 0, \ \textit{else} \end{array} \right\}$$

1.2 Signal transformations

Signals can be transformed by modifying the variable.

- $x(t - \tau)$: Shift a signal left by τ steps.
- $x(-t)$: Rotate a signal about the $t = 0$
- $x(kt)$: Stretch a signal by a factor of k

These operations can be combined to give more complex transformations. For example, $y(t) = x(\tau - t) = x(-(t - \tau))$ flips x and shifts it right by τ timesteps. This is equivalent to shifting x left by τ timesteps and then flipping it.

1.3 Convolution

Definition 7 *The convolution of two signals in discrete time*

$$(x * h)[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Definition 8 *The convolution of two signals in continuous time*

$$(x * h)(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

While written in discrete time, these properties apply in continuous time as well.

- $(x * \delta)[n] = x[n]$
- $x[n] * \delta[n-N] = x[n-N]$
- $(x * h)[n] = (h * x)[n]$
- $x * (h_1 + h_2) = x * h_1 + x * h_2$
- $x * (h_1 * h_2) = (x * h_1) * h_2$

1.4 Systems and their properties

Definition 9 *A system is a process by which input signals are transformed to output signals*

Definition 10 *A memoryless system has output which is only determined by the input's present value*

Definition 11 *A causal system has output which only depends on input at present or past times*

Definition 12 *A stable system produces bounded output when given a bounded input. By extension, this means an unstable system is when $\exists$ a bounded input that makes the output unbounded.*

Definition 13 *A system is time-invariant if the original input $x(t)$ is transformed to $y(t)$, then $x(t-\tau)$ is transformed to $y(t-\tau)$*

Definition 14 *A system $f(x)$ is linear if and only if*

- *If $y(t) = f(x(t))$, then $f(ax(t)) = ay(t)$ (Scaling)*
- *If $y_1(t) = f(x_1(t))$ and $y_2(t) = f(x_2(t))$, then $f(x_1(t) + x_2(t)) = y_1(t) + y_2(t)$ (Superposition)*

Notice: The above conditions on linearity require that $x(0) = 0$ because if $a = 0$, then we need $y(0) = 0$ for scaling to be satisfied

Definition 15 *The impulse response of a system $f[x]$ is $h[n] = f[\delta[n]]$, which is how it response to an impulse input.*

Definition 16 *A system has a Finite Impulse Response (FIR) if $h[n]$ decays to zero in a finite amount of time*

Definition 17 *A system has an Infinite Impulse Response (IIR) if $h[n]$ does not decay to zero in a finite amount of time*

1.5 Exponential Signals

Exponential signals are important because they can succinctly represent complicated signals using complex numbers. This makes analyzing them much easier.

$$x(t) = e^{st}, x[n] = z^n (s, z \in \mathbb{C})$$

Definition 18 *The frequency response of a system is how a system responds to a purely oscillatory signal*

2 The Fourier Series

2.1 Continuous Time

Definition 19 *A function $x(t)$ is periodic if $\exists T$ such that $\forall t, x(t - T) = x(t)$.*

The smalled such T which satisfies the periodicity property is known as the **Fundamental Period**.

Theorem 1 *If $x(t)$ and $y(t)$ are functions with period T_1 and T_2 respectively, then $x(t) + y(t)$ is periodic if $\exists m, n \in \mathbb{Z}$ such that $mT_1 = nT_2$.*

Definition 20 *Given a periodic function $x(t)$ with fundamental period T and fundamental frequency $\omega_0 = \frac{2\pi}{T}$, the Fourier Series of x is a weighted sum of the harmonic functions.*

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

To find the coefficients a_k:

$$x(t) \cdot e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 t(k-n)}$$

$$\int_T x(t) \cdot e^{-jn\omega_0 t} dt = \sum_{k=-\infty}^{\infty} a_k \int_T e^{j\omega_0 t(k-n)} dt = \left\{ \begin{array}{cc} Ta_k & \text{if k=n} \\ 0 & \text{else} \end{array} \right\}$$

Rearranging this, we can see that

$$a_n = \frac{1}{T} \int_T x(t) e^{-jn\omega_0 t} dt$$

. For a_0, the DC offset term, this formula makes a lot of sense because it is just the average value of the function over one period.

$$a_0 = \frac{1}{T} \int_T x(t) dt$$

Because the Fourier Series is an infinite sum, there is a worry that for some functions $x(t)$, it will not converge. The **Dirichlet Convergent Requirements** tell us when the Fourier Series converges. More specificially, they tell us when

$$\lim_{M \to \infty} x_M(\tau) = x(\tau) \forall \tau, x_M(t) = \sum_{k=-M}^{M} a_k e^{jk\omega_0 t}$$

will converge.

Theorem 2 *The Fourier Series of a continuous time periodic function $x(t)$ will converge when x is piecewise continuous and $\frac{d}{dt}x$ is piecewise continuous.*

- *If x is continuous at τ,* $\lim_{M \to \infty} x_M(\tau) = x(\tau)$
- *If x is discontinuous at τ, then* $\lim_{M \to \infty} x_M(\tau) = \frac{1}{2}(x(\tau^- + x(\tau^+)))$

These convergence requirements are for pointwise convergence only. They do not necessarily imply that the graphs of the Fourier Series and the original function will look the same.

2.2 Discrete Time

The definition for periodicity in discrete time is the exact same as the definition in continuous time.

Definition 21 *A function $x[n]$ is periodic with period $N \in \mathbb{Z}$ if $\forall n, x[n+N] = x[n]$*

However, there are some differences. For example, $x[n] = cos(\omega_0 n)$ is only periodic in discrete time if $\exists N, M \in \mathbb{Z}, \omega_0 N = 2\pi M$.

Theorem 3 *The sum of two discrete periodic signals is periodic*

The above statement is not always true in continuous time but it is in discrete time.

The Fourier Series in discrete time is the same idea as the Fourier series in continuous time: to express every signal as a linear combination of complex exponentials. The discrete time basis that we use are the Nth roots of unity.

$$\phi_k[n] = e^{jk\frac{2\pi}{N}n}$$

- $\phi_k[n]$ is perioidic in n (i.e $\phi_k[n+N] = \phi_k[n]$)
- $\phi_k[n]$ is periodic in k (i.e $\phi_{k+N}[n] = \phi_k[n]$)
- $\phi_k[n] \cdot \phi_m[n] = \phi_{k+m}[n]$

Notice that with this basis, there are only N unique functions that we can use. An additional property of the $\phi_k[n]$ is that

$$\sum_{n=<N>} \phi_k[n] = \left\{ \begin{array}{cc} N & \text{if } k = 0, \pm N, \pm 2N \\ 0 & \text{otherwise} \end{array} \right\}$$

Definition 22 *Given a periodic discrete-time function $x[n]$ with period N, the Fourier series of the function is a weighted sum of the roots of unity basis functions.*

$$x[n] = \sum_{k=0}^{N-1} a_k \phi_k[n]$$

In order to find the values of a_k, we can perform a similar process as in continuous time.

$$x[n] = \sum_{k=0}^{N-1} a_k \phi_k[n]$$

$$x[n]\phi_{-M}[n] = \sum_{k=0}^{N-1} a_k \phi_k[n]\phi_{-M}[n]$$

$$\sum_{n=<N>} x[n]\phi_{-M}[n] = \sum_{n=<N>} \sum_{k=<N>} a_k \phi_{k-M}[n] = \sum_{k=<N>} a_k \sum_{n=<N>} \phi_{k-M}[n]$$

$$\sum_{n=<N>} x[n]\phi_{-M}[n] = a_M N$$

$$a_M = \frac{1}{N} \sum_{n=<N>} x[n]\phi_{-M}[n]$$

2.3 Properties of the Fourier Series

Linearity: If a_k and b_k are the coefficients of the Fourier Series of $x(t)$ and $y(t)$ respectively, then $Aa_k + Bb_k$ are the coefficients of the Fourier series of $Ax(t) + By(t)$

Time Shift: If a_k are the coefficients of the Fourier Series of $x(t)$, then $b_k = e^{-jk\frac{2\pi}{T}t_0}a_k$ are the coefficients of the Fourier Series of $\hat{x}(t) = x(t - t_0)$

Time Reversal: If a_k are the coefficients of the Fourier Series of $x(t)$, then $b_k = a_{-k}$ are the coefficients of the Fourier Series of $x(-t)$

Conjugate Symmetry: If a_k are the coefficients of the Fourier Series of $x(t)$, then a_k^* are the coefficients of the Fourier Series of $x^*(t)$. This means that $x(t)$ is a real valued signal, then $a_k = a_{-k}^*$

Theorem 4 (Parseval's Theorem)

$$\frac{1}{T}\int |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2 (ContinousTime)$$

$$\frac{1}{N}\sum_{n=<N>} |x[n]|^2 = \sum_{k=<N>} |a_k|^2 (DiscreteTime)$$

2.4 Interpreting the Fourier Series

A good way to interpret the Fourier Series is as a change of basis. In both the continuous and discrete case, we are projecting our signal x onto a set of basis functions, and the coefficients a_k are the coordinates of our signal in the new space.

2.4.1 Discrete Time

Since in discrete time, signal is periodic in N, we can turn any it into a vector $\vec{x} \in \mathbb{C}^N$.

$$\vec{x} = \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix} \in \mathbb{C}^N$$

We can use this to show that ϕ_k form an orthogonal basis. If we take two of them $\phi_k[n]$ and $\phi_M[n]$ ($k \neq M$) and compute their dot product of their vector forms, then

$$\phi_k[n] \cdot \phi_M[n] = \phi_M^* \phi_k = \sum_{<n>} \phi_{k-M}[n] = 0$$

That means that ϕ_k and ϕ_M are orthogonal, and they are N of them, therefore they are a basis. If we compute their magnitudes, we see

$$\phi_k \cdot \phi_k = ||\phi_k||^2 = N, \therefore ||\phi_k|| = \sqrt{N}$$

Finally, if we compute $\vec{x}\phi_M$ where $\vec{x}$ is the vector form of an N-periodic signal,

$$\vec{x} \cdot \vec{\phi_M} = (\sum_{i=0}^{N-1} a_i\phi_i) \cdot \phi_M = Na_m$$

$$a_m = \frac{1}{N}\vec{x} \cdot \phi_M$$

This is exactly the equation we use for finding the Fourier Series coefficients, and notice that it is a projection since $N = ||\phi_m||^2$. This gives a nice geometric intution for Parseval's theorem.

$$\frac{1}{N}\sum |x[n]|^2 = \frac{1}{N}||\vec{x}||^2 = \sum |a_k|^2$$

because we know the norms of two vectors in different bases must be equal.

2.4.2 Continuous Time

In continuous time, our bases functions are $\phi_k(t) = e^{jk\frac{2\pi}{T}t}$ for $k \in (-\infty, \infty)$ Since we can't convert continuous functions into vectors, these ϕ_k are really a basis for the vector space of square integrable functions on the interval $[0, T]$. The inner product for this vector space is

$$< x, y >= \int_0^T x(t)y^*(t)$$

We can use this inner product to conduct the same proof as we did in discrete time.

3 The Fourier Transform

3.1 Continuous Time Fourier Transform

Definition 23 *The Continuous Time Fourier Transform converts an aperiodic signal into the frequency domain.*

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

The intuition for this transform comes from the Fourier Series. Only periodic signals can be represented by the Fourier Series. If we start with a finite signal $x(t)$, then we can just make it periodic by copying the domain over which it is nonzero so it repeats over a period T. Call this signal $\tilde{x}(t)$. Since $\tilde{x}$ is perioidc, we can find its fourier series coefficients.

$$a_k = \frac{1}{T}\int_T \tilde{x}(t)e^{-jn\frac{2\pi}{T}t} = \frac{1}{T}\int_T x(t)e^{-jn\frac{2\pi}{T}t} = \frac{1}{T}\int_{-\infty}^{\infty} x(t)e^{-jn\frac{2\pi}{T}t}$$

These steps are possible because $\tilde{x}(t) = x(t)$ over a single period, and $x(t)$ is zero outside that period.

$$Ta_k = \int_{-\infty}^{\infty} x(t)e^{-jn\frac{2\pi}{T}t}$$

notice that if we let T approach infinity, then $\omega_0 = \frac{2\pi}{T}$ becomes very small, so the Ta_k can almost be thought of as samples of some continuous time function. What this means is for a general aperiodic signal, regardless of if it is finite or not, we can think of it as having "infinite period" and thus made up of a continuous set of frequencies. This is what motivates the continuous time fourier transform.

Definition 24 *The Inverse Continuous Time Fourier Transform takes us from the frequency domain reprsentation of a function $X(\omega)$ to its time domain representation $x(t)$*

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

We can arrive at this equation by starting from the Fourier series again Our faux signal $\tilde{x}(t)$ which was the periodic function we constructed out of our aperiod one is represented by its Fourier Series

$$\tilde{x}(t) = \sum_{k-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} (\frac{1}{T}X(\omega))e^{j\omega t}|_{w=k\omega_0}$$

Notice this is just rewrite a_k as the samples of the Fourier Transform $X(\omega)$. $T = \frac{2\pi}{\omega_0}$ so

$$\tilde{x}(t) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} \omega_0 X(\omega)e^{j\omega t}|_{w=k\omega_0}$$

$$x(t) = \lim_{T->\infty}\tilde{x}(t) = lim_{\omega_0->0}\tilde{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

3.1.1 Properties of the CTFT

For all these properties, assume that $x(t) \leftrightarrow X(\omega)$ and $y(t) \leftrightarrow Y(\omega)$
Linearity:

$$ax(t) + by(t) \leftrightarrow aX(\omega) + bY(\omega)$$

Time Shift:

$$x(t - t_0) \leftrightarrow e^{-j\omega t_0}X(\omega)$$

Time/Frequency Scaling:

$$x(at) \leftrightarrow \frac{1}{|a|}X(\frac{\omega}{a})$$

Conjugation:

$$x^*(t) \leftrightarrow X^*(-\omega)$$

Derivative:

$$\frac{d}{dt}x(t) \leftrightarrow j\omega X(\omega), \frac{d}{d\omega}X(\omega) \leftrightarrow -jtx(t)$$

Convolution/Multiplication:

$$(x * y)(t) \leftrightarrow X(\omega)Y(\omega), x(t)y(t) \leftrightarrow \frac{1}{2\pi}(X * Y)(\omega)$$

Frequency Shift:

$$e^{j\omega_0 t}x(t) \leftrightarrow X(\omega - \omega_0)$$

Parsevals Theorem:

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

3.1.2 Convergence of the CTFT

A big question that arises when thinking about the Fourier Transform is whether or not the integral $\int x(t)e^{-j\omega t}$ actually converges.

Theorem 5 *If $\int_{-\infty}^{\infty} |x(t)|dt$ converges, then $X(\omega)$ exists and is continuous. In addition, $X(\omega)$ approaches 0 as $|\omega|$ approaches ∞*

Conceptually, this theorem makes sense because

$$|x(t)e{-}j\omega t| = |x(t)||e^{-j\omega t}| = |x(t)|$$

So if one converges, the other must converge. However, this means that $x(t) = 1$, $x(t) = sin(\omega t)$, $x(t) = cos(\omega t)$ don't have a "strict" Fourier Series because the integral doesn't converge for these periodic signals. In order to get around this, we can define a "generalized" Fourier Transform which operates on periodic signals.

Starting with $x(t) = 1$, we know that in the frequency domain, the only consituent frequency is $\omega = 0$. This means that $X(\omega) = k\delta(\omega)$ where k is some scalar. Using the Inverse Fourier Transform,

$$x(t) = \frac{1}{2\pi}\int k\delta(\omega)e^{j\omega t}d\omega = \frac{k}{2\pi}$$

That means $k = 2\pi$, so

$$x(t) = 1 \leftrightarrow X(\omega) = 2\pi\delta(\omega)$$

Now if we apply the frequency shift property, we see that

$$x(t) = e^{j\omega_0 t} = 2\pi\delta(\omega - \omega_0)$$

With this, we can define our generalized Fourier Transform for periodic signals.

Definition 25 *The generalized Fourier Transform for a periodic signal $x(t)$ is*

$$X(\omega) = \sum_{-\infty}^{\infty} a_k \cdot 2\pi\delta(\omega - \omega_0)$$

where a_k are the coefficients of the Fourier Series of $x(t)$

This definition works because any periodic signal can be represented by its Fourier Series. The rational behind using the Dirac Delta in this generalized Fourier Transform is exlained by the Theory of Distributions which can be found in the Appendix.

3.2 Discrete Time Fourier Transform

Definition 26 *The Discrete Time Fourier Transform converts aperiodic discrete signals into the frequency domain.*

$$X(\omega) = \sum_{-\infty}^{\infty} x[n]e^{-j\omega n}$$

The intution for the discrete time fourier transform is more or less the same as that of the Continuous Time Fourier Transform.

Definition 27 *The Inverse Discrete Time Fourier Transform converts the frequency domain representation of a signal back into its time domain representation.*

$$x[n] = \frac{1}{2\pi}\int_{<2\pi>} X(\omega)e^{j\omega n}d\omega$$

3.2.1 Properties of the DTFT

For all these properties, assume that $x[n] \leftrightarrow X(\omega)$ and $y[n] \leftrightarrow (\omega)$
Time Shift:

$$x[n-n_0] \leftrightarrow e^{-j\omega n_0}X(\omega)$$

Frequency Shift:

$$X(\omega-\omega_0) \leftrightarrow e^{j\omega_0 n}x[n]$$

Time Reversal:

$$x[-n] \leftrightarrow X(-\omega)$$

Conjugation:

$$x^*[n] = X^*(-\omega)$$

Time Expansion:
In discrete time, compression of a signal doesn't make sense because we can't have partial steps (i.e n must be an integer). However, we can stretch a signal.

$$x_M[n] \leftrightarrow X(M\omega), x_M[n] = \left\{ \begin{array}{cc} x[\frac{n}{M}] & \text{when } M|n \\ 0 & \text{else} \end{array} \right\}$$

Derivative Property:

$$nx[n] \leftrightarrow j\frac{d}{d\omega}X(\omega)$$

Multiplication Property:

$$x[n]y[n] \leftrightarrow \frac{1}{2\pi}\int_{2\pi} X(\theta)Y(\omega-\theta)d\theta$$

Convolution Property:

$$(x * y)[n] = X(\omega)Y(\omega)$$

3.2.2 Convergence of the DTFT

Just like in continuous time, it was unclear whether or not the integral would converge, in discrete time, it is unclear if the infinite sum will converge. The convergence theorem for both are essentially the same.

Theorem 6 *If $\sum_{-\infty}^{\infty} |x[n]|$ converges, then $X(\omega)$ exists and is continuous.*

Just like in continuous time, periodic signals like $x[n] = 1, x[n] = sin(\omega_0 t), x[n] = cos(\omega_0 t)$... are problematic because they don't converge under the "strict" transform, so they require a generalied transform. In the frequency domain, a constant signal like $x[n] = 1$ will be the sum of all frequencies. This will look like a sum of Dirac Deltas.

$$X(\omega) = k \sum_{l=-\infty}^{\infty} \delta(\omega - 2\pi l)$$

Applying the synthesis equation to this, we get

$$x(t) = \frac{1}{2\pi} \int_{2\pi} k \sum_{l=-\infty}^{\infty} \delta(\omega - 2\pi l) = \frac{k}{2\pi} \sum_{l=-\infty}^{\infty} \int_{2\pi} \delta(\omega - 2\pi l) = \frac{k}{2\pi} \int_{2\pi} \delta(\omega - 2\pi \cdot 0) = \frac{k}{2\pi}$$

Therefore $k = 2\pi$, so

$$x[n] = 1 \leftrightarrow X(\omega) = 2\pi \sum_{l=-\infty}^{\infty} \delta(\omega - 2\pi l)$$

and we can apply the frequency shift property to get

$$x[n] = e^{j\omega_0 n} \leftrightarrow X(\omega) = 2\pi \sum_{l=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi l)$$

Once again using the Fourier Series representation of $x[n]$, we can define the generalized Discrete Time Fourier Transform.

Definition 28 *For a perioidic signal $x[n]$, the generalized Discrete Fourier Transform is*

$$x[n] \leftrightarrow 2\pi \sum_{-\infty}^{\infty} a_k \delta(\omega - \frac{2\pi}{N} k)$$

where a_k are the Fourier Series coefficients of $x[n]$

4 Linear Time-Invariant Systems

Definition 29 *LTI systems are ones which are both linear and time-invariant.*

4.1 Impulse Response of LTI systems

LTI systems are special systems because their output can be determined entirely the impulse response $h[n]$.

4.1.1 The Discrete Case

We can think of the original signal $x[n]$ in terms of the impulse function.

$$x[n] = x[0]\delta[n] + x[1]\delta[n-1] + ... = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

This signal will be transformed in some way to get the output $y[n]$. Since the LTI system applies a functional F and the LTI is linear and time-invariant,

$$y[n] = F(\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]) = \sum_{k=-\infty}^{\infty} x[k]F(\delta[n-k]) = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Notice this operation is the convolution between the input and the impulse response.

4.1.2 The Continuous Case

We can approximate the function by breaking it into intervals of length Δ.

$$x(t) \approx \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_{\Delta}(t-k\Delta)\Delta$$

$$x(t) = lim_{\Delta\to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_{\Delta}(t-k\Delta)\Delta$$

After applying the LTI system to it,

$$y(n) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)$$

Notice this operation is the convolution between the input and the impulse response.

4.2 Determining Properties of an LTI system

Because an LTI system is determined entirely by its impulse response, we can determine its properties from the impulse response.

4.2.1 Causality

Theorem 7 *An LTI system is causal when $h[n] = 0, \forall n < 0$*

Proof 1 *Assume $h[n] = 0, \forall n < 0$*

$$y[n] = (x * h)[n] = \sum_{k=-\infty}^{\infty} x[n-k]h[k] = \sum_{k=0}^{\infty} x[n-k]h[k]$$

Notice that this does not depend on time steps prior to $n = 0$

4.2.2 Memory

Theorem 8 *An LTI system is memoryless if $h[n] = 0, \forall n \neq 0$*

Memoryless means that the system doesn't depend on past values, so its impulse response should just be a scaled version of δ.

4.2.3 Stability

Theorem 9 *A system is stable if $\sum_{n=-\infty}^{\infty} |h[n]|$ converges.*

Proof 2
1. *Assume $|x[n]| \leq B_x$ to show $|y[n]| < D$ where D is some bound.*

$$|y[n]| = |\sum_{k=-\infty}^{\infty} x[n-k]h[k]| \leq \sum_{k} |x[n-k]h[k]| = \sum_{k} |x[n-k]||h[k]| \leq B_x \sum_{k} |h[k]$$

This means as long as $\sum_k |h[k]$ converges, $y[n]$ will be bounded.

2. *Assume $\sum_n |h[n]|$ does not converge. Show that the system is unstable. Choose $x[n] = sgn\{h[-n]\}$*

$$y[n] = \sum_{k} x[n-k]h[k]$$

so

$$y[0] = \sum_{k} x[-k]h[k] = \sum_{k} |h[k]|$$

And this is unbounded, so $y[n]$ is unbounded.

4.3 Frequency Response

Definition 30 *The frequency response of a system is the output when passed a purely oscillatory signal*

If we pass a complex exponential into an LTI system, the output signal is the same signal but scaled. In otherwise, it is an eigenfunction of LTI systems.

$$y(t) = \int_{-\infty}^{\infty} e^{s(t-\tau)}h(\tau)d\tau = e^{st}\int_{-\infty}^{\infty} e^{-s\tau}h(\tau)$$

The integral is a constant, and the original function is unchanged. The same analysis can be done in the discrete case.

$$y[n] = \sum_{k=-\infty}^{\infty} z^{n-k}h[k] = z^n \sum_{k=-\infty}^{\infty} z^{-k}h[k]$$

We give these constant terms a special name called the transfer function.

Definition 31 *The transfer function of an LTI system $H(\omega)$ is how the system scales a pure tone of frequency ω*

$$H(\omega) := \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau, H(\omega) := \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$

Notice: The transfer function is the fourier transform of the impulse response! This means the Fourier Transform takes us from the impulse response of the system to the frequency response.

4.4 Special LTI Systems

4.4.1 Linear Constant Coefficient Difference/Differential Equations

Definition 32 *A linear constant coefficient difference equation is a system of one of the following forms*

$$\text{Discrete: } \sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$\text{Continuous: } \sum_{k=0}^{N} a_k \frac{d^k y}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x}{dt^k}$$

Theorem 10 *Systems described by a linear constant coefficient difference equation are causal LTI iff $a_0 \neq 0$ and the system is initially at rest ($y[n] = 0$ for $n < n_0$ where n_0 is the first instant $x[n] \neq 0$)*

Notice that if $a_1..a_n = 0$, then the system will have a finite impulse response because eventually the signal will die out. It turns out that all causal FIR systems can be written as a linear constant coefficient difference equation.

Theorem 11 *Systems of the form*

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

are causal, FIR LTI systems and their impulse response is

$$h[n] = \sum_{k=0}^{M} b_k \delta[n-k]$$

Theorem 12 *Given a constant coefficient difference/differential equation, the transfer function $H(\omega)$ is*

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k} \text{ [Continuous Case]}$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{\sum_{k=0}^{N} a_k e^{-j\omega k}} \text{ [Discrete Case]}$$

Proof 3
The Continuous Case

$$\sum_{k=0}^{N} a_k \frac{d^k y}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x}{dt^k}$$

Taking the Fourier Transform,

$$\sum_{k=0}^{N} a_k (j\omega)^k Y(\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(\omega)$$

$$\frac{Y(\omega)}{X(\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$$

$$y(t) = (h * x)(t) \leftrightarrow H(\omega)X(\omega)$$

$$\therefore H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$$

The Discrete Case

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

Remember the frequecy response is the impulse response, so let $x[n] = \delta[n]$

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k \delta[n-k]$$

Take the DTFT

$$\sum_{k=0}^{N} a_k e^{-j\omega k} H(\omega) = \sum_{k=0}^{M} b_k e^{-j\omega k}$$

$$H(\omega) = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{\sum_{k=0}^{N} a_k e^{-j\omega k}}$$

5 Appendix

5.1 Theory of Distributions

The Theory of Distributions is the mathematical framework which underlies the generalized Fourier Transforms.

Definition 33 *Given a test function x, a distribution T operates on x to produce a number $< T, x >$.*

Definition 34 *The distrbution induced by a function g is defined as*

$$< T_g, x >= \int_{-\infty}^{\infty} g(t)^* x(t) dt$$

Notice two things:

- $< T_g, x >$ is linear
- $< \alpha T_g, x >= \alpha^* < T_g, x >$

With these definitions, we can now define the Dirac delta in terms of distrubions. Let g be any function such that

$$\int_{\infty}^{\infty} g(t) dt = 1$$

Define g_ϵ to be

$$g_\epsilon = \frac{1}{\epsilon} g(\frac{t}{\epsilon})$$

Now we can defined $\delta(t) = \lim_{\epsilon \to 0} T_{g_\epsilon}$

$$< \delta, x >= \int_{-\infty}^{\infty} \delta(t) x(t) dt = x(0)$$

which is the property of the Dirac Delta we want. Now we can define the generalized Fourier Transform in terms of distributions.

Definition 35 *The generalized Continuous Time Fourier Transform of a distrubtion T is*

$$< FT, X >= 2\pi < T, x >$$

for test function x whose Fourier Transform is X