DTFT

$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$ $\qquad$ $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$

Symmetry Properties

$x[n]$	$X(e^{j\omega})$
$x^*[n]$	$X^*(e^{-j\omega})$
$x^*[-n]$	$X^*(e^{j\omega})$
$x[-n] = x^\star[n]$	$X(e^{j\omega}) = X^\star(e^{j\omega})$ (Real)
$x[-n] = -x^\star[n]$	$X^\star(e^{j\omega}) = -X(e^{j\omega})$ (Imaginary)
$x[n] = x_e[n] + x_o[n]$	$Re\{X(e^{j\omega})\} + jIm\{X(e^{j\omega})\}$
Real x[n]	Conjugate Symmetric Real part is even Imaginary part is odd Magnitude is even Phase is odd

Theorems

$x[n]$	$X(e^{j\omega})$
$\alpha x[n] + \beta y[n]$	$\alpha X(e^{j\omega}) + \beta Y(e^{j\omega})$
$x[n-n_d]$	$e^{-j\omega n_d}X(e^{j\omega})$
$x[-n]$	$X(e^{-j\omega})$
$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$
$x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
$x[n]y[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
$\sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \lvert X(e^{j\omega})\rvert^2 d\omega$	
$\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega$	
Converges if $\sum_{n=-\infty}^{\infty} \lvert x[n]\rvert < \infty$	

Transform Pairs

Signal	Fourier Transform
$\delta[n]$	1
$\delta[n-n_0]$	$e^{-j\omega n_0}$
1	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega+2\pi k)$
$a^n u[n] (\alpha < 1)$	$\frac{1}{1-\alpha e^{-j\omega}}$
$u[n]$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega+2\pi k)$
$(n+1)a^n u[n]$	$\frac{1}{(1-\alpha e^{-j\omega})^2}$
$\frac{\sin\omega_c n}{\pi n}$	$\left\{\begin{array}{ll} 1 & \lvert\omega\rvert < \omega_c, \\ 0 & \omega_c < \lvert\omega\rvert \le \pi \end{array}\right\}$
$\left\{\begin{array}{ll} 1 & 0 \le n \le M, \\ 0 & else \end{array}\right\}$	$\frac{sin(\omega(M+1)/2)}{sin[\omega]/2}e^{-j\omega M/2}$
$e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega-\omega_0+2\pi k)$
$cos(\omega_0 n + \phi)$	$\sum_{k=-\infty}^{\infty} \pi e^{j\phi}\delta(\omega-\omega_0+2\pi k) + \pi e^{-j\phi}\delta(\omega+\omega_0+2\pi k)$

Z - Transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

Theorems

Sequence	Transform	ROC
$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	$R_1 \bigcap R_2$
$x[n-n_0]$	$z^{-n_0}X(z)$	R
$z_0^n x[n]$	$X(\frac{z}{z_0})$	$\|z_0\|R$
$nx[n]$	$-z\frac{dX(z)}{dz}$	R
$x^*[n]$	$X^*(z^*)$	R
$Re\{x[n]\}$	$\frac{1}{2}\left[X(z) + X^*(z^*)\right]$	Contains R
$Im\{x[n]\}$	$\frac{1}{2j}\left[X(z) - X^*(z^*)\right]$	Contains R
$x^*[-n]$	$X^*\left(\frac{1}{z^*}\right)$	$\frac{1}{R}$
$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	$R_1 \bigcap R_2$

Transforms

Sequence	Transform	ROC
$\delta[n]$	1	$\mathbb{C}$
$u[n]$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\|z\| < 1$
$\delta[n-m]$	z^{-m}	$\mathbb{C}$ except 0 or ∞
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
$r^n cos(\omega_0 n)u[n]$	$\frac{1-rcos(\omega_0)z^{-1}}{1-2rcos(\omega_0)z^{-1}+r^2z^{-2}}$	$\|z\| > r$
$r^n sin(\omega_0 n)u[n]$	$\frac{rsin(\omega_0)z^{-1}}{1-2rcos(\omega_0)z^{-1}+r^2z^{-2}}$	$\|z\| > r$
$\left\{ \begin{array}{ll} a^n & 0 \leq n \leq N-1 \\ 0 & \end{array} \right\}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\|z\| > 0$

ROC properties

1. ROC is either $|z| \geq r$, $|z| \leq r$, or $r_1 \leq |z| \leq r_2$
2. Fourier transform exists iff ROC includes unit circle
3. ROC contains no poles
4. If $x[n]$ is finite, ROC is entire plane
5. If $x[n]$ is right sided, ROC extends from outermost pole to ∞
6. If $x[n]$ is left sided, ROC extends from innermost pole to 0
7. If $x[n]$ is two sided, then ROC is an annulus
8. ROC is connected region
9. $\sum_{n=-\infty}^{\infty} |x[n]| \implies$ ROC includes unit circle
   For LTI Systems
10. Stable $\implies$ Causal if and only if right-sided ROC
11. Causal $\implies$ Stable if and only if poles inside unit circle

Discrete Fourier Series

$$\tilde{x}[n] = \frac{1}{N}\sum_k \tilde{X}[k]e^{j\frac{2\pi}{N}kn} \qquad \tilde{X}[n] = \sum_n \tilde{x}[n]e^{-j\frac{2\pi}{n}kn}$$

Properties

Periodic sequence $\tilde{x}[n]$	Periodic coefficients $\tilde{X}[k]$
$a\tilde{x}_1[n] + b\tilde{x}_2[n]$	$a\tilde{X}_1[k] + b\tilde{X}_2[k]$
$\tilde{X}[n]$	$N\tilde{x}[-k]$
$\tilde{x}[n-m]$	$e^{-j\frac{2\pi}{N}km}\tilde{X}[k]$
$e^{j\frac{2\pi}{N}mn}\tilde{x}[n]$	$\tilde{X}[k-m]$
$\sum_{m=0}^{N-1}\tilde{x_1}[m]\tilde{x}_2[n-m]$	$\tilde{X}_1[k]\tilde{X}_2[k]$
$\tilde{x}^*[n]$	$\tilde{X}^*[-k]$
$\tilde{x}^*[-n]$	$\tilde{X}^*[k]$
$Re\{\tilde{x}[n]\}$	$\tilde{X}_e[k] = \frac{1}{2}\left(\tilde{X}[k] + \tilde{X}^*[-k]\right)$
$jIm\{\tilde{x}[n]\}$	$\tilde{X}_o[k] = \frac{1}{2}\left(\tilde{X}[k] - \tilde{X}^*[-k]\right)$
$\tilde{x}_e[n] = \frac{1}{2}\left(\tilde{x}[n] + \tilde{x}^*[-n]\right)$	$Re\{\tilde{X}[k]\}$
$\tilde{x}_o[n] = \frac{1}{2}\left(\tilde{x}[n] - \tilde{x}^*[-n]\right)$	$jIm\{\tilde{X}[k]\}$
	Conjugate Symmetric
	Real Part Even
Real $\tilde{x}[n]$	Imaginary Part Odd
	Magnitude is even
	Phase is odd

Discrete Fourier Transform

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j\frac{2\pi}{N}kn} \qquad X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{n}kn}$$

Properties

Finite Length sequence $x[n]$	N Point DFT $X[k]$
$ax_1[n] + bx_2[n]$	$aX_1[k] + bX_2[k]$
$X[n]$	$Nx[((-k))_N]$
$x[((n-m))_N]$	$e^{-j\frac{2\pi}{N}km}X[k]$
$e^{j\frac{2\pi}{N}mn}x[n]$	$X[((k-m))_N]$
$\sum_{m=0}^{N-1}x_1[m]x_2[((n-m))_N]$	$X_1[k]X_2[k]$
$x^*[n]$	$X^*[((-k]))_N$
$x^*[((-n))_N]$	$X^*[k]$
$Re\{x[n]\}$	$X_e[k] = \frac{1}{2}\left(X[k] + X^*[-k]\right)$
$jIm\{x[n]\}$	$X_o[k] = \frac{1}{2}\left(X[k] - X^*[-k]\right)$
$x_e[n] = \frac{1}{2}\left(x[n] + x^*[((-n))_N]\right)$	$Re\{X[k]\}$
$x_o[n] = \frac{1}{2}\left(x[n] - x^*[((-n))_N]\right)$	$jIm\{X[k]\}$
	Conjugate Symmetric
	Real Part Even
Real $x[n]$	Imaginary Part Odd
	Magnitude is even
	Phase is odd

$$\sum_{n=0}^{N-1}|x[n]|^2 = \frac{1}{N}\sum_{k=0}^{N-1}|X[k]|^2 \qquad x[n] = \frac{1}{N}DFT\{X^*[k]\}^*$$

Definitions

Causality: $y[n]$ only depends on $x[n]$

Memoryless: $y[n]$ depends only on $x[n]$

Time Invariance: $y[n] = T\{x[n]\} \Rightarrow y[n-n_0] = T\{x[n-n_0]\}$

Bibo Stability: $|x[n]| \le B_x < \infty \ \forall n \Rightarrow |y[n]| \le B_y < \infty \ \forall n$

Stable Sequence: $\sum_{n=-\infty}^{\infty} |x[n]| < \infty$

LTI Systems

- Causal: $h[n<0] = 0$
- Stable: $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$
- Stable $\Rightarrow$ Causal $\Leftrightarrow$ ROC of $H(z)$ right sided
- Causal $\Rightarrow$ Stable $\Leftrightarrow$ poles of $H(z)$ inside unit circle

Linear Convolution via DFT

$x[n]$ $0 \le n \le L-1$ & $h[n]$ $0 \le n \le P-1$

1. Pad vectors to length $N \ge L+P-1$
2. Compute $Y[k] = H[k]X[k]$
3. Take the inverse DFT

Overlap-Add Linear Convolution

1. $x_r[n] = \begin{cases} x[n] & rL \le n < (r+1)L \\ 0 & \text{else} \end{cases}$
2. Zero pad $x_r[n]$ & $h[n]$ to $N \ge L+P-1$
3. Linearly convolve $x_r[n]$ & $h[n]$
4. $y[n] = \sum_r x_r[n] * h[n]$

* Neighboring outputs share $P-1$ points

Overlap-Save Linear Convolution

1. $x_r[n] = x[n + r(L-P+1) - P + 1]$ $(0 \le n \le L-1)$
2. $y_r[n] = \begin{cases} x_r[n] * h[n] & P-1 \le n \le L-1 \\ 0 & \text{else} \end{cases}$
3. $y[n] = \sum_r y_r[n - r(L-P+1) + P - 1]$

* First $P-1$ samples of each y_r are bad

FFT

$W_N = e^{-j\frac{2\pi}{N}}$

$W_N^{k(N-n)} = W_N^{-n} = (W_N^{kn})^*$

$W_N^{kn} = W_N^{k(n+N)}$

$W_N^2 = W_{N/2}$

Decimation in Time

$X[k] = \sum x[2r] W_N^{2rk} + \sum x[2r+1] W_N^{(2r+1)k}$

$\hookrightarrow X[k] = G[k] + W_N^k H[k]$

$X[k + \frac{N}{2}] = G[k] - W_N^k H[k]$

Decimation in Frequency

$X[2r] = \sum x[n] W_N^{2rn} + \sum x[n+\frac{N}{2}] W_N^{2r(n+\frac{N}{2})}$

$= \sum (x[n] + x[n+\frac{N}{2}]) W_{N/2}^{rn}$

$X[2r+1] = \sum (x[n] - x[n+\frac{N}{2}]) W_{N/2}^{rn}$

Windowing

- Larger window → smaller main lobe → better frequency resolution
- Smaller main lobe → larger side lobes
- More zero-padding → DFT better samples DTFT
- Smaller window → larger main lobe → better time resolution

STFT & DTDFT

STFT: $X[n, w] = \sum_m x[n+m] w[m] e^{-jwm}$

DTDFT: $X_r[k] = \sum_{m=0}^{L-1} x[rR+m] w[m] e^{-j\frac{2\pi}{N}km}$

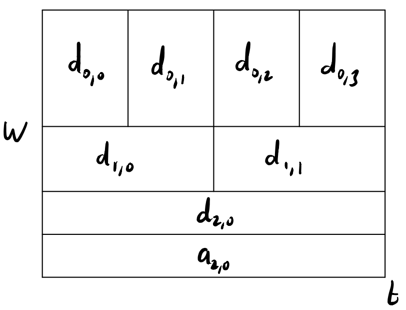
L: window length, R: Sample jump, $N \ge L$: DFT

In time frequency tiling, $\Delta w = \frac{2\pi}{L}$, $\Delta t = L$

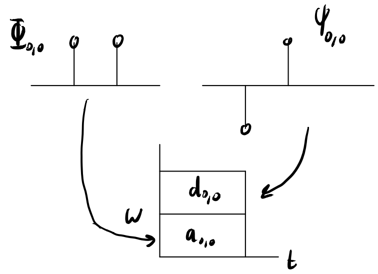
R, zero pad blends tiling together

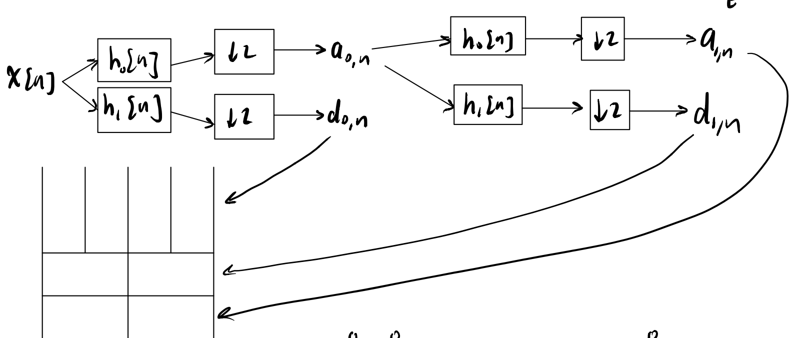
Discrete Wavelet Transform

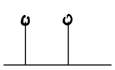
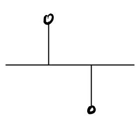
$d_{s,u} = \sum_{n=0}^{N-1} x[n] \psi_{s,u}[n]$, $a_{s,u} = \sum_{n=0}^{N-1} x[n] \phi_{s,u}[n]$



Haar For $n=2$





Normalize! $h_0[n] =$  $h_1[n]$ 

Wavelet Transform

$Wf(u,s) = \int_{-\infty}^{\infty} f(t) \cdot \frac{1}{\sqrt{s}} \psi^*\left(\frac{t-u}{s}\right) dt = \{f(t) * \bar{\psi}_s(t)\}(u)$

$\int_{-\infty}^{\infty} |\psi(t)|^2 dt = 1$ $\quad \int_{-\infty}^{\infty} \psi(t) dt = 0$

Haar Mother: $\psi(t) = \begin{cases} -1 & 0 \le t < 1/2 \\ 1 & 1/2 \le t < 1 \end{cases}$

Haar Father: $\Phi(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{else} \end{cases}$