EE120 Course Notes

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Disclaimer: These notes reflect 120 when I took the course (Fall 2019). They may not accurately reflect current course content, so use at your own risk. If you find any typos, errors, etc, please raise an issue on the GitHub repository.

Contents

1 Introduction to Signals and Systems

1.1 Types of Signals

Definition 1 *A signal is a function of one or more variables*

Definition 2 *A continuous signal $x(t)$ maps $\mathbb{R} \to \mathbb{R}$*

Definition 3 *A discrete signal $x[n]$ maps $\mathbb{Z} \to \mathbb{R}$*

1.1.1 Properties of the Unit Impulse

Definition 4 *The unit impulse in discrete time is defined as*

$$\delta[n] = \left\{ \begin{array}{c} 1, if\ n = 0 \\ 0,\ \ else \end{array} \right\}$$

- $f[n]\delta[n] = f[0]\delta[n]$
- $f[t]\delta[n-N] = f[N]\delta[n-N]$

Definition 5 *The unit impulse in continuous time is the dirac delta function*

$$\delta(t) = lim_{\Delta \to 0}\delta_{\Delta}(t)$$

$$\delta_{\Delta} = \left\{ \begin{array}{c} \frac{1}{\Delta},\ \ if\ \ \geq 0 \\ 0else \end{array} \right\}$$

- $f(t)\delta(t) = f(0)\delta(t)$
- $f(t)\delta(t-\tau) = f(\tau)\delta(t-\tau)$
- $\delta(at) = \frac{1}{|a|}\delta(t)$

Definition 6 *The unit step is defined as*

$$u[n] = \left\{ \begin{array}{c} 1, if\ n \geq 0 \\ 0,\ \ else \end{array} \right\}$$

1.2 Signal transformations

Signals can be transformed by modifying the variable.

- $x(t-\tau)$: Shift a signal left by τ steps.
- $x(-t)$: Rotate a signal about the $t = 0$
- $x(kt)$: Stretch a signal by a factor of k

These operations can be combined to give more complex transformations. For example, $y(t) = x(\tau - t) = x(-(t-\tau))$ flips x and shifts it right by τ timesteps. This is equivalent to shifting x left by τ timesteps and then flipping it.

1.3 Convolution

Definition 7 *The convolution of two signals in discrete time*

$$(x * h)[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Definition 8 *The convolution of two signals in continuous time*

$$(x * h)(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

While written in discrete time, these properties apply in continuous time as well.

- $(x * \delta)[n] = x[n]$
- $x[n] * \delta[n-N] = x[n-N]$
- $(x * h)[n] = (h * x)[n]$
- $x * (h_1 + h_2) = x * h_1 + x * h_2$
- $x * (h_1 * h_2) = (x * h_1) * h_2$

1.4 Systems and their properties

Definition 9 *A system is a process by which input signals are transformed to output signals*

Definition 10 *A memoryless system has output which is only determined by the input's present value*

Definition 11 *A causal system has output which only depends on input at present or past times*

Definition 12 *A stable system produces bounded output when given a bounded input. By extension, this means an unstable system is when $\exists$ a bounded input that makes the output unbounded.*

Definition 13 *A system is time-invariant if the original input $x(t)$ is transformed to $y(t)$, then $x(t-\tau)$ is transformed to $y(t-\tau)$*

Definition 14 *A system $f(x)$ is linear if and only if*

- *If $y(t) = f(x(t))$, then $f(ax(t)) = ay(t)$ (Scaling)*
- *If $y_1(t) = f(x_1(t))$ and $y_2(t) = f(x_2(t))$, then $f(x_1(t) + x_2(t)) = y_1(t) + y_2(t)$ (Superposition)*

Notice: The above conditions on linearity require that $x(0) = 0$ because if $a = 0$, then we need $y(0) = 0$ for scaling to be satisfied

Definition 15 *The impulse response of a system $f[x]$ is $h[n] = f[\delta[n]]$, which is how it response to an impulse input.*

Definition 16 *A system has a Finite Impulse Response (FIR) if $h[n]$ decays to zero in a finite amount of time*

Definition 17 *A system has an Infinite Impulse Response (IIR) if $h[n]$ does not decay to zero in a finite amount of time*

1.5 Exponential Signals

Exponential signals are important because they can succinctly represent complicated signals using complex numbers. This makes analyzing them much easier.

$$x(t) = e^{st}, x[n] = z^n (s, z \in \mathbb{C})$$

Definition 18 *The frequency response of a system is how a system responds to a purely oscillatory signal*

2 The Fourier Series

2.1 Continuous Time

Definition 19 *A function $x(t)$ is periodic if $\exists T$ such that $\forall t, x(t-T) = x(t)$.*

The smalled such T which satisfies the periodicity property is known as the **Fundamental Period**.

Theorem 1 *If $x(t)$ and $y(t)$ are functions with period T_1 and T_2 respectively, then $x(t) + y(t)$ is periodic if $\exists m, n \in \mathbb{Z}$ such that $mT_1 = nT_2$.*

Definition 20 *Given a periodic function $x(t)$ with fundamental period T and fundamental frequency $\omega_0 = \frac{2\pi}{T}$, the Fourier Series of x is a weighted sum of the harmonic functions.*

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

To find the coefficients a_k:

$$x(t) \cdot e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 t(k-n)}$$

$$\int_T x(t) \cdot e^{-jn\omega_0 t} dt = \sum_{k=-\infty}^{\infty} a_k \int_T e^{j\omega_0 t(k-n)} dt = \left\{ \begin{array}{cc} Ta_k & \text{if k=n} \\ 0 & \text{else} \end{array} \right\}$$

Rearranging this, we can see that

$$a_n = \frac{1}{T} \int_T x(t) e^{-jn\omega_0 t} dt$$

. For a_0, the DC offset term, this formula makes a lot of sense because it is just the average value of the function over one period.

$$a_0 = \frac{1}{T} \int_T x(t) dt$$

Because the Fourier Series is an infinite sum, there is a worry that for some functions $x(t)$, it will not converge. The **Dirichlet Convergent Requirements** tell us when the Fourier Series converges. More specificially, they tell us when

$$\lim_{M \to \infty} x_M(\tau) = x(\tau) \forall \tau, x_M(t) = \sum_{k=-M}^{M} a_k e^{jk\omega_0 t}$$

will converge.

Theorem 2 *The Fourier Series of a continuous time periodic function $x(t)$ will converge when x is piecewise continuous and $\frac{d}{dt}x$ is piecewise continuous.*

- *If x is continuous at τ, $\lim_{M \to \infty} x_M(\tau) = x(\tau)$*
- *If x is discontinuous at τ, then $\lim_{M \to \infty} x_M(\tau) = \frac{1}{2}(x(\tau^- + x(\tau^+)))$*

These convergence requirements are for pointwise convergence only. They do not necessarily imply that the graphs of the Fourier Series and the original function will look the same.

2.2 Discrete Time

The definition for periodicity in discrete time is the exact same as the definition in continuous time.

Definition 21 *A function $x[n]$ is periodic with period $N \in \mathbb{Z}$ if $\forall n, x[n+N] = x[n]$*

However, there are some differences. For example, $x[n] = cos(\omega_0 n)$ is only periodic in discrete time if $\exists N, M \in \mathbb{Z}, \omega_0 N = 2\pi M$.

Theorem 3 *The sum of two discrete periodic signals is periodic*

The above statement is not always true in continuous time but it is in discrete time.

The Fourier Series in discrete time is the same idea as the Fourier series in continuous time: to express every signal as a linear combination of complex exponentials. The discrete time basis that we use are the Nth roots of unity.

$$\phi_k[n] = e^{jk\frac{2\pi}{N}n}$$

- $\phi_k[n]$ is perioidic in n (i.e $\phi_k[n+N] = \phi_k[n]$)
- $\phi_k[n]$ is periodic in k (i.e $\phi_{k+N}[n] = \phi_k[n]$)
- $\phi_k[n] \cdot \phi_m[n] = \phi_{k+m}[n]$

Notice that with this basis, there are only N unique functions that we can use. An additional property of the $\phi_k[n]$ is that

$$\sum_{n=<N>} \phi_k[n] = \left\{ \begin{array}{cc} N & \text{if } k = 0, \pm N, \pm 2N \\ 0 & \text{otherwise} \end{array} \right\}$$

Definition 22 *Given a periodic discrete-time function $x[n]$ with period N, the Fourier series of the function is a weighted sum of the roots of unity basis functions.*

$$x[n] = \sum_{k=0}^{N-1} a_k \phi_k[n]$$

In order to find the values of a_k, we can perform a similar process as in continuous time.

$$x[n] = \sum_{k=0}^{N-1} a_k \phi_k[n]$$

$$x[n]\phi_{-M}[n] = \sum_{k=0}^{N-1} a_k \phi_k[n]\phi_{-M}[n]$$

$$\sum_{n=<N>} x[n]\phi_{-M}[n] = \sum_{n=<N>} \sum_{k=<N>} a_k \phi_{k-M}[n] = \sum_{k=<N>} a_k \sum_{n=<N>} \phi_{k-M}[n]$$

$$\sum_{n=<N>} x[n]\phi_{-M}[n] = a_M N$$

$$a_M = \frac{1}{N} \sum_{n=<N>} x[n]\phi_{-M}[n]$$

2.3 Properties of the Fourier Series

Linearity: If a_k and b_k are the coefficients of the Fourier Series of $x(t)$ and $y(t)$ respectively, then $Aa_k + Bb_k$ are the coefficients of the Fourier series of $Ax(t) + By(t)$

Time Shift: If a_k are the coefficients of the Fourier Series of $x(t)$, then $b_k = e^{-jk\frac{2\pi}{T}t_0}a_k$ are the coefficients of the Fourier Series of $\hat{x}(t) = x(t - t_0)$

Time Reversal: If a_k are the coefficients of the Fourier Series of $x(t)$, then $b_k = a_{-k}$ are the coefficients of the Fourier Series of $x(-t)$

Conjugate Symmetry: If a_k are the coefficients of the Fourier Series of $x(t)$, then a_k^* are the coefficients of the Fourier Series of $x^*(t)$. This means that $x(t)$ is a real valued signal, then $a_k = a_{-k}^*$

Theorem 4 (Parseval's Theorem)

$$\frac{1}{T}\int |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2 (ContinousTime)$$

$$\frac{1}{N}\sum_{n=<N>} |x[n]|^2 = \sum_{k=<N>} |a_k|^2 (DiscreteTime)$$

2.4 Interpreting the Fourier Series

A good way to interpret the Fourier Series is as a change of basis. In both the continuous and discrete case, we are projecting our signal x onto a set of basis functions, and the coefficients a_k are the coordinates of our signal in the new space.

2.4.1 Discrete Time

Since in discrete time, signal is periodic in N, we can turn any it into a vector $\vec{x} \in \mathbb{C}^N$.

$$\vec{x} = \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix} \in \mathbb{C}^N$$

We can use this to show that ϕ_k form an orthogonal basis. If we take two of them $\phi_k[n]$ and $\phi_M[n]$ ($k \neq M$) and compute their dot product of their vector forms, then

$$\phi_k[n] \cdot \phi_M[n] = \phi_M^* \phi_k = \sum_{<n>} \phi_{k-M}[n] = 0$$

That means that ϕ_k and ϕ_M are orthogonal, and they are N of them, therefore they are a basis. If we compute their magnitudes, we see

$$\phi_k \cdot \phi_k = ||\phi_k||^2 = N, \therefore ||\phi_k|| = \sqrt{N}$$

Finally, if we compute $\vec{x}\phi_M$ where $\vec{x}$ is the vector form of an N-periodic signal,

$$\vec{x} \cdot \vec{\phi_M} = (\sum_{i=0}^{N-1} a_i \phi_i) \cdot \phi_M = Na_m$$

$$a_m = \frac{1}{N}\vec{x} \cdot \phi_M$$

This is exactly the equation we use for finding the Fourier Series coefficients, and notice that it is a projection since $N = ||\phi_m||^2$. This gives a nice geometric intution for Parseval's theorem.

$$\frac{1}{N}\sum |x[n]|^2 = \frac{1}{N}||\vec{x}||^2 = \sum |a_k|^2$$

because we know the norms of two vectors in different bases must be equal.

2.4.2 Continuous Time

In continuous time, our bases functions are $\phi_k(t) = e^{jk\frac{2\pi}{T}t}$ for $k \in (-\infty, \infty)$ Since we can't convert continuous functions into vectors, these ϕ_k are really a basis for the vector space of square integrable functions on the interval $[0, T]$. The inner product for this vector space is

$$< x, y >= \int_0^T x(t)y^*(t)$$

We can use this inner product to conduct the same proof as we did in discrete time.

3 The Fourier Transform

3.1 Continuous Time Fourier Transform

Definition 23 *The Continuous Time Fourier Transform converts an aperiodic signal into the frequency domain.*

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

The intuition for this transform comes from the Fourier Series. Only periodic signals can be represented by the Fourier Series. If we start with a finite signal $x(t)$, then we can just make it periodic by copying the domain over which it is nonzero so it repeats over a period T. Call this signal $\tilde{x}(t)$. Since $\tilde{x}$ is perioidc, we can find its fourier series coefficients.

$$a_k = \frac{1}{T}\int_T \tilde{x}(t)e^{-jn\frac{2\pi}{T}t} = \frac{1}{T}\int_T x(t)e^{-jn\frac{2\pi}{T}t} = \frac{1}{T}\int_{-\infty}^{\infty} x(t)e^{-jn\frac{2\pi}{T}t}$$

These steps are possible because $\tilde{x}(t) = x(t)$ over a single period, and $x(t)$ is zero outside that period.

$$Ta_k = \int_{-\infty}^{\infty} x(t)e^{-jn\frac{2\pi}{T}t}$$

notice that if we let T approach infinity, then $\omega_0 = \frac{2\pi}{T}$ becomes very small, so the Ta_k can almost be thought of as samples of some continuous time function. What this means is for a general aperiodic signal, regardless of if it is finite or not, we can think of it as having "infinite period" and thus made up of a continuous set of frequencies. This is what motivates the continuous time fourier transform.

Definition 24 *The Inverse Continuous Time Fourier Transform takes us from the frequency domain reprsentation of a function $X(\omega)$ to its time domain representation $x(t)$*

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

We can arrive at this equation by starting from the Fourier series again Our faux signal $\tilde{x}(t)$ which was the periodic function we constructed out of our aperiod one is represented by its Fourier Series

$$\tilde{x}(t) = \sum_{k-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} (\frac{1}{T}X(\omega))e^{j\omega t}|_{w=k\omega_0}$$

Notice this is just rewrite a_k as the samples of the Fourier Transform $X(\omega)$. $T = \frac{2\pi}{\omega_0}$ so

$$\tilde{x}(t) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} \omega_0 X(\omega)e^{j\omega t}|_{w=k\omega_0}$$

$$x(t) = \lim_{T->\infty}\tilde{x}(t) = lim_{\omega_0->0}\tilde{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

3.1.1 Properties of the CTFT

For all these properties, assume that $x(t) \leftrightarrow X(\omega)$ and $y(t) \leftrightarrow Y(\omega)$
Linearity:

$$ax(t) + by(t) \leftrightarrow aX(\omega) + bY(\omega)$$

Time Shift:

$$x(t - t_0) \leftrightarrow e^{-j\omega t_0}X(\omega)$$

Time/Frequency Scaling:

$$x(at) \leftrightarrow \frac{1}{|a|}X(\frac{\omega}{a})$$

Conjugation:

$$x^*(t) \leftrightarrow X^*(-\omega)$$

Derivative:

$$\frac{d}{dt}x(t) \leftrightarrow j\omega X(\omega), \frac{d}{d\omega}X(\omega) \leftrightarrow -jtx(t)$$

Convolution/Multiplication:

$$(x * y)(t) \leftrightarrow X(\omega)Y(\omega), x(t)y(t) \leftrightarrow \frac{1}{2\pi}(X * Y)(\omega)$$

Frequency Shift:

$$e^{j\omega_0 t}x(t) \leftrightarrow X(\omega - \omega_0)$$

Parsevals Theorem:

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

3.1.2 Convergence of the CTFT

A big question that arises when thinking about the Fourier Transform is whether or not the integral $\int x(t)e^{-j\omega t}$ actually converges.

Theorem 5 *If $\int_{-\infty}^{\infty} |x(t)|dt$ converges, then $X(\omega)$ exists and is continuous. In addition, $X(\omega)$ approaches 0 as $|\omega|$ approaches ∞*

Conceptually, this theorem makes sense because

$$|x(t)e{-}j\omega t| = |x(t)||e^{-j\omega t}| = |x(t)|$$

So if one converges, the other must converge. However, this means that $x(t) = 1$, $x(t) = sin(\omega t)$, $x(t) = cos(\omega t)$ don't have a "strict" Fourier Series because the integral doesn't converge for these periodic signals. In order to get around this, we can define a "generalized" Fourier Transform which operates on periodic signals.

Starting with $x(t) = 1$, we know that in the frequency domain, the only consituent frequency is $\omega = 0$. This means that $X(\omega) = k\delta(\omega)$ where k is some scalar. Using the Inverse Fourier Transform,

$$x(t) = \frac{1}{2\pi}\int k\delta(\omega)e^{j\omega t}d\omega = \frac{k}{2\pi}$$

That means $k = 2\pi$, so

$$x(t) = 1 \leftrightarrow X(\omega) = 2\pi\delta(\omega)$$

Now if we apply the frequency shift property, we see that

$$x(t) = e^{j\omega_0 t} = 2\pi\delta(\omega - \omega_0)$$

With this, we can define our generalized Fourier Transform for periodic signals.

Definition 25 *The generalized Fourier Transform for a periodic signal $x(t)$ is*

$$X(\omega) = \sum_{-\infty}^{\infty} a_k \cdot 2\pi\delta(\omega - \omega_0)$$

where a_k are the coefficients of the Fourier Series of $x(t)$

This definition works because any periodic signal can be represented by its Fourier Series. The rational behind using the Dirac Delta in this generalized Fourier Transform is exlained by the Theory of Distributions which can be found in the Appendix.

3.2 Discrete Time Fourier Transform

Definition 26 *The Discrete Time Fourier Transform converts aperiodic discrete signals into the frequency domain.*

$$X(\omega) = \sum_{-\infty}^{\infty} x[n] e^{-j\omega n}$$

The intution for the discrete time fourier transform is more or less the same as that of the Continuous Time Fourier Transform.

Definition 27 *The Inverse Discrete Time Fourier Transform converts the frequency domain representation of a signal back into its time domain representation.*

$$x[n] = \frac{1}{2\pi} \int_{<2\pi>} X(\omega) e^{j\omega n} d\omega$$

3.2.1 Properties of the DTFT

For all these properties, assume that $x[n] \leftrightarrow X(\omega)$ and $y[n] \leftrightarrow (\omega)$
Time Shift:

$$x[n - n_0] \leftrightarrow e^{-j\omega n_0} X(\omega)$$

Frequency Shift:

$$X(\omega - \omega_0) \leftrightarrow e^{j\omega_0 n} x[n]$$

Time Reversal:

$$x[-n] \leftrightarrow X(-\omega)$$

Conjugation:

$$x^*[n] = X^*(-\omega)$$

Time Expansion:
In discrete time, compression of a signal doesn't make sense because we can't have partial steps (i.e n must be an integer). However, we can stretch a signal.

$$x_M[n] \leftrightarrow X(M\omega), x_M[n] = \left\{ \begin{array}{cc} x[\frac{n}{M}] & \text{when } M|n \\ 0 & \text{else} \end{array} \right\}$$

Derivative Property:

$$nx[n] \leftrightarrow j\frac{d}{d\omega} X(\omega)$$

Multiplication Property:

$$x[n]y[n] \leftrightarrow \frac{1}{2\pi} \int_{2\pi} X(\theta) Y(\omega - \theta) d\theta$$

Convolution Property:

$$(x * y)[n] = X(\omega)Y(\omega)$$

3.2.2 Convergence of the DTFT

Just like in continuous time, it was unclear whether or not the integral would converge, in discrete time, it is unclear if the infinite sum will converge. The convergence theorem for both are essentially the same.

Theorem 6 *If $\sum_{-\infty}^{\infty} |x[n]|$ converges, then $X(\omega)$ exists and is continuous.*

Just like in continuous time, periodic signals like $x[n] = 1, x[n] = sin(\omega_0 t), x[n] = cos(\omega_0 t)$... are problematic because they don't converge under the "strict" transform, so they require a generalied transform. In the frequency domain, a constant signal like $x[n] = 1$ will be the sum of all frequencies. This will look like a sum of Dirac Deltas.

$$X(\omega) = k \sum_{l=-\infty}^{\infty} \delta(\omega - 2\pi l)$$

Applying the synthesis equation to this, we get

$$x(t) = \frac{1}{2\pi} \int_{2\pi} k \sum_{l=-\infty}^{\infty} \delta(\omega - 2\pi l) = \frac{k}{2\pi} \sum_{l=-\infty}^{\infty} \int_{2\pi} \delta(\omega - 2\pi l) = \frac{k}{2\pi} \int_{2\pi} \delta(\omega - 2\pi \cdot 0) = \frac{k}{2\pi}$$

Therefore $k = 2\pi$, so

$$x[n] = 1 \leftrightarrow X(\omega) = 2\pi \sum_{l=-\infty}^{\infty} \delta(\omega - 2\pi l)$$

and we can apply the frequency shift property to get

$$x[n] = e^{j\omega_0 n} \leftrightarrow X(\omega) = 2\pi \sum_{l=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi l)$$

Once again using the Fourier Series representation of $x[n]$, we can define the generalized Discrete Time Fourier Transform.

Definition 28 *For a perioidic signal $x[n]$, the generalized Discrete Time Fourier Transform is*

$$x[n] \leftrightarrow 2\pi \sum_{-\infty}^{\infty} a_k \delta(\omega - \frac{2\pi}{N} k)$$

where a_k are the Fourier Series coefficients of $x[n]$

3.3 Discrete Fourier Transform

Whereas the CTFT takes a continuous signal and outputs a continuous frequency spectrum and the DTFT takes a discrete signal and outputs a continuous, periodic frequecy spectrum, the Discrete Fourier Transform takes a discrete periodic signal and outputs a discrete frequency spectrum.

Definition 29 *For a length N finite sequence $\{x[n]\}_0^{n-1}$, the Discrete Fourier Transform of the signal is a length N finite sequence $\{X[k]\}_0^{n-1}$ where*

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$$

One way to interpret the DFT is in terms of the Fourier series for a disrete periodic signal $\tilde{x}[n] = x[n \bmod \mathrm{N}]$. Recall that the coefficient of the kth term of the Fourier Series

$$a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$$

Notice that the a_k of the Fourier Series are the DFT values except scaled by a factor of N. This gives an intuitive inverse DFT.

Definition 30 *For a length N finite sequence $\{X[k]\}_0^{N-1}$ representing the DFT of a finite perioidc signal $\{x[n]\}_0^{N-1}$, the inverse DFT is given by*

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}kn}$$

One important property of the DFT is its complex conjugacy. When $x[n]$ is a real valued signal, then $X[n-k] = X[k]^*$. This can easily be shown by substituting $N-k$ into the DFT formula. Further intuition for the DFT comes from relating it to the DTFT. Suppose we have a finite signal $x[n]$ which is 0 for $n < 0$ and $n > N-1$. The DTFT of this signal is

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = \sum_{n=0}^{N-1} x[n] e^{-j\omega n}$$

Suppose we sample the DTFT at intervals of $\frac{2\pi}{N}k$, then

$$X[k] = X\left(\frac{2\pi}{N}k\right) = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$$

Thus we can think of the DFT as a N point sample of the DTFT.

3.4 2D Fourier Transforms

So far, our Fourier Transforms have been limited to signals of a single dimension. However, in the real world, signals might be multidimensional (think images). Thankfully, each of the Fourier Transforms generalizes easily into higher dimensions.

$$\textbf{2D CTFT: } X(\omega_1, \omega_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t_1, t_2)e^{-j\omega_0 t}e^{-j\omega_1 t}dt_1 dt_2$$

$$\textbf{Inverse 2D CTFT: } x(t_1, t_2) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(\omega_1, \omega_2)e^{j\omega_1 t_1}e^{j\omega_2 t_2}d\omega_1 d\omega_2$$

$$\textbf{2D DTFT: } X(\omega_1, \omega_2) = \sum_{n_2=-\infty}^{\infty}\sum_{n_1=-\infty}^{\infty} x[n_1, n_2]e^{-j\omega_1 n_1}e^{-j\omega_2 n_2}$$

$$\textbf{Inverse 2D DTFT: } x[n_1, n_2] = \sum_{\omega_2=-\infty}^{\infty}\sum_{\omega_1=-\infty}^{\infty} X(\omega_1, \omega_2)e^{j\omega_1 n_1}e^{j\omega_2 n_2}$$

$$\textbf{2D DFT: } X[k_1, k_2] = \sum_{n_2=0}^{N_2-1}\sum_{n_1=0}^{N_1-1} x[n_1, n_2]e^{-j\frac{2\pi}{N_1}k_1 n_1}e^{-j\frac{2\pi}{N_2}k_2 n_2}$$

$$\textbf{2D DFT: } x[n_1, n_2] = \sum_{k_2=0}^{N_2-1}\sum_{k_1=0}^{N_1-1} X[k_1, k_2]e^{j\frac{2\pi}{N_1}k_1 n_1}e^{j\frac{2\pi}{N_2}k_2 n_2}$$

Just like in 1 dimension, absolute summability/integrability guarantee the convergence of these transforms. It turns out that when a 2D signal is simply a multiplication of two 1D signals, the Fourier Transforms are very easy to compute.

Theorem 7 *If $x(t_1, t_2) = x(t_1)x(t_2)$, then $X(\omega_1, \omega_2) = X(\omega_1)X(\omega_2)$*

4 Laplace Transform

It turns out the CTFT is just a specific case of a more general transform called the Laplace transform.

Definition 31

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt \text{ for } s \in \mathbb{C}$$

Notice that the Fourier Transform is merely $X(j\omega)$. Like the Fourier Transform, the Laplace Transform does not always converge. The values of s for which the transform does converge is known as the region of convergence (ROC). If the ROC contains the imaginary axis, then the signal has a well-defined Fourier Transform.

Unlike the Fourier Transform, there is no easy way to take the Inverse Laplace Transform. However, for many LTI systems/signals, we can use partial fraction expansion and then use known Laplace Transform pairs in order to compute the inverse.

Two useful features of laplace transforms are their poles and zeros. Suppose

$$X(s) = \frac{N(s)}{D(s)}$$

The poles of the system are $\{s|D(s) = 0\}$ and the zeros are $\{s|N(s) = 0\}$

4.1 Properties of the Laplace Transform

The properties of the Laplace transform are largely the same as the properties of the Fourier Transform. One must just be careful about the Region of Convergence. For all these properties, assume that $x(t) \leftrightarrow X(s)$ and $y(t) \leftrightarrow Y(s)$ with original region of convergence R.

Linearity:

$$ax(t) + by(t) \leftrightarrow aX(s) + bY(s)$$

Time Shift:

$$x(t - t_0) \leftrightarrow e^{-st_0}X(s)$$

Time/Frequency Scaling:

$$x(at) \leftrightarrow \frac{1}{|a|}X(\frac{s}{a}), \text{ ROC: } s \in aR$$

Conjugation:

$$x^*(t) \leftrightarrow X^*(s^*)$$

Derivative:

$$\frac{d}{dt}x(t) \leftrightarrow sX(s), \frac{d}{ds}X(s) \leftrightarrow -tx(t)$$

Convolution/Multiplication:

$$(x * y)(t) \leftrightarrow X(s)Y(s)$$

Frequency Shift:

$$e^{s_0 t}x(t) \leftrightarrow X(s - s_0), \text{ ROC: } s \in R + s_0$$

5 Linear Time-Invariant Systems

Definition 32 *LTI systems are ones which are both linear and time-invariant.*

5.1 Impulse Response of LTI systems

LTI systems are special systems because their output can be determined entirely the impulse response $h[n]$.

5.1.1 The Discrete Case

We can think of the original signal $x[n]$ in terms of the impulse function.

$$x[n] = x[0]\delta[n] + x[1]\delta[n-1] + ... = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

This signal will be transformed in some way to get the output $y[n]$. Since the LTI system applies a functional F and the LTI is linear and time-invariant,

$$y[n] = F(\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]) = \sum_{k=-\infty}^{\infty} x[k]F(\delta[n-k]) = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Notice this operation is the convolution between the input and the impulse response.

5.1.2 The Continuous Case

We can approximate the function by breaking it into intervals of length Δ.

$$x(t) \approx \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_{\Delta}(t-k\Delta)\Delta$$

$$x(t) = lim_{\Delta\to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_{\Delta}(t-k\Delta)\Delta$$

After applying the LTI system to it,

$$y(n) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)$$

Notice this operation is the convolution between the input and the impulse response.

5.2 Determining Properties of an LTI system

Because an LTI system is determined entirely by its impulse response, we can determine its properties from the impulse response.

5.2.1 Causality

Theorem 8 *An LTI system is causal when $h[n] = 0, \forall n < 0$*

Proof 1 *Assume $h[n] = 0, \forall n < 0$*

$$y[n] = (x * h)[n] = \sum_{k=-\infty}^{\infty} x[n-k]h[k] = \sum_{k=0}^{\infty} x[n-k]h[k]$$

Notice that this does not depend on time steps prior to $n = 0$

5.2.2 Memory

Theorem 9 *An LTI system is memoryless if $h[n] = 0, \forall n \neq 0$*

Memoryless means that the system doesn't depend on past values, so its impulse response should just be a scaled version of δ.

5.2.3 Stability

Theorem 10 *A system is stable if $\sum_{n=-\infty}^{\infty} |h[n]|$ converges.*

Proof 2
1. *Assume $|x[n]| \leq B_x$ to show $|y[n]| < D$ where D is some bound.*

$$|y[n]| = |\sum_{k=-\infty}^{\infty} x[n-k]h[k]| \leq \sum_k |x[n-k]h[k]| = \sum_k |x[n-k]||h[k]| \leq B_x \sum_k |h[k]$$

This means as long as $\sum_k |h[k]$ converges, $y[n]$ will be bounded.

2. *Assume $\sum_n |h[n]|$ does not converge. Show that the system is unstable. Choose $x[n] = sgn\{h[-n]\}$*

$$y[n] = \sum_k x[n-k]h[k]$$

so

$$y[0] = \sum_k x[-k]h[k] = \sum_k |h[k]|$$

And this is unbounded, so $y[n]$ is unbounded.

5.3 Frequency Response and Transfer Functions

Definition 33 *The frequency response of a system is the output when passed a purely oscillatory signal*

If we pass a complex exponential into an LTI system, the output signal is the same signal but scaled. In otherwise, it is an eigenfunction of LTI systems.

$$y(t) = \int_{-\infty}^{\infty} e^{s(t-\tau)} h(\tau) d\tau = e^{st} \int_{-\infty}^{\infty} e^{-s\tau} h(\tau)$$

The integral is a constant, and the original function is unchanged. The same analysis can be done in the discrete case.

$$y[n] = \sum_{k=-\infty}^{\infty} z^{n-k} h[k] = z^n \sum_{k=-\infty}^{\infty} z^{-k} h[k]$$

We give these constant terms a special name called the transfer function.

Definition 34 *The frequency response of an LTI system $H(j\omega)$ is how the system scales a pure tone of frequency ω*

$$H(\omega) := \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau, H(\omega) := \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k}$$

Notice: The frequency response is the fourier transform of the impulse response! This means the Fourier Transform takes us from the impulse response of the system to the frequency response. There is no reason to limit ourselves to the Fourier Domain though

Definition 35 *The transfer function of an LTI system $H(s)$ is how the system responds to complex exponentials.*

The transfer function is merely the Laplace Transform of the impulse response. In many ways, this can be more useful than the frequency response.

5.3.1 Stability of $H(s)$

Recall that an LTI system is stable if the impulse response is absolutely integrable. We can determine this from the transfer function.

Theorem 11 *A causal LTI system is stable iff all poles of $H(s)$ have negative real parts.*

The proof of this theorem stems from some facts about the Laplace Transform. If the system is causual, then the ROC is the half place demarcated by the right most pole. When this ROC includes the imaginary axis, the Fourier Transform is well defined, and this only happens when $h(t)$ is absolutely integrable.

5.3.2 Bode Plots

Because transfer functions, and hence the frequency response, can be quite complex, we need a easy way to visualize how a system responds to different frequencies.

Definition 36 *A Bode Plot is a straight-line approximation plot of $|H(j\omega)|$ and $\angle H(j\omega)$ on a log-log scale*

The log-log scale not only allows us to determine the behavior of Bode plots over a large range of frequencies, but they also let us easily figure out what the plot looks like because it converts the frequency response into piecewise linear components.

To see why, lets write our transfer function in polar form.

$$H(j\omega) = K\frac{(j\omega)^{N_{z0}}}{(j\omega)^{N_{p0}}}\frac{\prod_{i=0}^{n}(1+\frac{j\omega}{\omega_{zi}})}{\prod_{k=0}^{m}(1+\frac{j\omega}{\omega_{pk}})} = Ke^{j\frac{\pi}{2}(N_{z0}-N_{p0})}\frac{\prod_{i=0}^{n} r_{zi}}{\prod_{k=0}^{m} r_{pk}}e^{j(\sum_{i=0}^{n} z_i - \sum_{k=0}^{m} p_k)}$$

Each r is the magnitude of a factor $1+\frac{j\omega}{\omega_n}$ where ω_n is either a root or a pole, and the z_i, p_k are the phases of each factor. By writing $H(j\omega)$ this way, it is clear that

$$|H(\omega)| = K\frac{\prod_{i=0}^{n} r_{zi}}{\prod_{k=0}^{m} r_{pk}}$$

If we take the log of this, we get

$$log(|H(\omega)|) = log(K) + \sum_{i=0}^{n} log(r_{zi}) - \sum_{k=0}^{m} log(r_{pk})$$

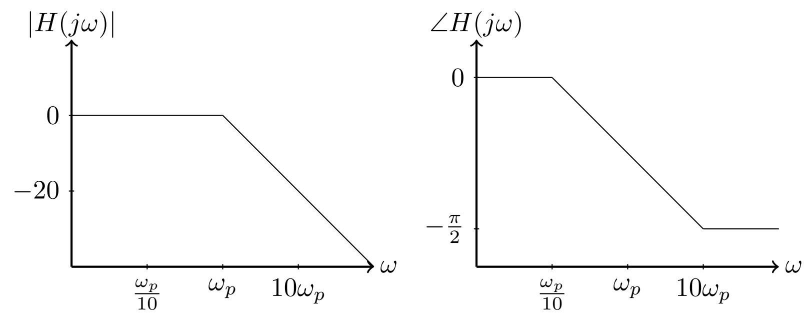
For Bode plots, we use the decibel scale, meaning we will multiply this value by 20 when constructing our plot. The exponential form of $H(j\omega)$ tells us that

$$\angle H(j\omega) = \frac{\pi}{2}(N_{z0} - N_{p0}) + \left(\sum_{i=0}^{n} z_i - \sum_{k=0}^{m} p_k\right)$$

Next, we should verify if we can approximate these equations as linear on a log-log scale. Take the example transfer function $H(j\omega) = \frac{1}{1+\frac{j\omega}{\omega_p}} = \frac{1}{r_p}e^{-j\theta_p}$.

$$\begin{array}{llll} \text{if } \omega = \omega_p & H(j\omega) = \frac{1}{1+j} & r_p = \sqrt{2} & \theta_p = \frac{\pi}{4} \\ \text{if } \omega = 10\omega_p & H(j\omega) = \frac{1}{1+10j} & r_p \approx 10 & \theta_p \approx \frac{\pi}{2} \\ \text{if } \omega = 0.1\omega_p & H(j\omega) = \frac{1}{1+0.1j} & r_p \approx 1 & \theta_p \approx 0 \end{array}$$

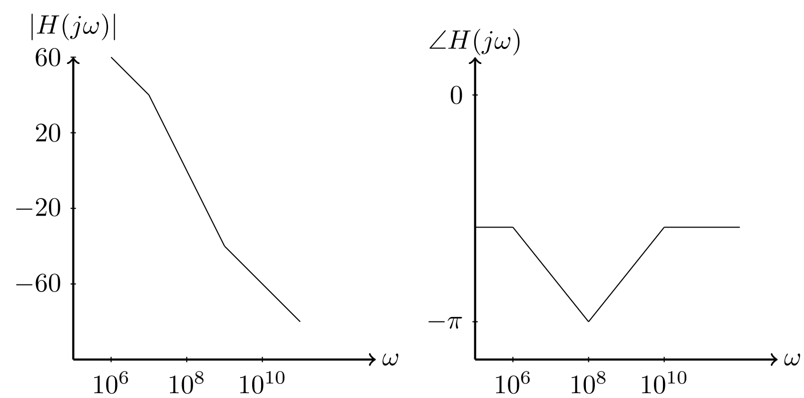
Thus we can see at decades away from the poles and zeros, the magnitudes and the phases will have less of an effect. Let's try constructing the Bode Plot for this transfer function.



For the magnitude plot, since there are no poles or zeros at $\omega = 0$, we draw a straight line until the pole kicks in at $\omega = \omega_p$ at which point the slope of the line will be -1. For the phase plot, we apply the same logic, except the pole kicks in at $\frac{\omega_p}{10}$ (to see why, look above to see how at $\omega = \omega_p$, the phase is $-\frac{\pi}{4}$). We can apply this same logic for more complicated transfer functions too. Lets take

$$H(j\omega) = 10^9 \frac{(1 + \frac{j\omega}{10^9})}{(j\omega)(1 + \frac{j\omega}{10^7})}$$

Notice we have a zero at 10^9, poles at $1, 10^7$, and 9 zeros at $\omega = 0$. With this information, we can see the plots will look like this:



The pole at 0 kicks in immediately, causing the decreasing magnitude and starting the phase at $\frac{-\pi}{2}$. The second pole at 10^7 will kick in next, followed by the zero at 10^9.

5.4 Special LTI Systems

5.4.1 Linear Constant Coefficient Difference/Differential Equations

Definition 37 *A linear constant coefficient difference equation is a system of one of the following forms*

$$\textit{Discrete: } \sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$\textit{Continuous: } \sum_{k=0}^{N} a_k \frac{d^k y}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x}{dt^k}$$

Theorem 12 *Systems described by a linear constant coefficient difference equation are causal LTI iff $a_0 \neq 0$ and the system is initially at rest ($y[n] = 0$ for $n < n_0$ where n_0 is the first instant $x[n] \neq 0$)*

Notice that if $a_1..a_n = 0$, then the system will have a finite impulse response because eventually the signal will die out. It turns out that all causal FIR systems can be written as a linear constant coefficient difference equation.

Theorem 13 *Systems of the form*

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

are causal, FIR LTI systems and their impulse response is

$$h[n] = \sum_{k=0}^{M} b_k \delta[n-k]$$

Theorem 14 *Given a constant coefficient difference/differential equation, the transfer function $H(\omega)$ is*

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k} \text{ [Continuous Case]}$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{\sum_{k=0}^{N} a_k e^{-j\omega k}} \text{ [Discrete Case]}$$

Proof 3
The Continuous Case

$$\sum_{k=0}^{N} a_k \frac{d^k y}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x}{dt^k}$$

Taking the Fourier Transform,

$$\sum_{k=0}^{N} a_k (j\omega)^k Y(\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(\omega)$$

$$\frac{Y(\omega)}{X(\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$$

$$y(t) = (h * x)(t) \leftrightarrow H(\omega)X(\omega)$$

$$\therefore H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$$

The Discrete Case

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

Remember the frequecy response is the impulse response, so let $x[n] = \delta[n]$

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k \delta[n-k]$$

Take the DTFT

$$\sum_{k=0}^{N} a_k e^{-j\omega k} H(\omega) = \sum_{k=0}^{M} b_k e^{-j\omega k}$$

$$H(\omega) = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{\sum_{k=0}^{N} a_k e^{-j\omega k}}$$

5.4.2 State Space Equations

When we have a LCCDE of the form

$$\sum_{i=0}^{N} a_i \frac{d^i y}{dt^i} = b_0 x(t)$$

we can represent the system in state space form where we keep track of a state vector $\vec{z}(t) \in \mathbb{R}^N$.

$$\frac{d}{dt}\vec{z}(t) = A\vec{z}(t) + Bx(t)$$
$$y(t) = C\vec{z}(t) + Dx(t)$$

The matrices A, B, C, D describe the dynamics of the system. If we want to find the transfer function of the system, we can use the Laplace transform.

$$s\vec{Z}(s) = A\vec{Z}(s) + BX(s) \implies \vec{Z}(s) = (sI - A)^{-1}BX(s)$$
$$\therefore Y(s) = C(sI - A)^{-1}BX(s) + DX(s)$$
$$\therefore H(s) = C(sI - A)^{-1}B + D$$

Notice that the poles of the transfer function are simply the eigenvalues of A. This is because if s is an eigenvalue of A, $(sI - A)^{-1}$ is not invertible so the transfer function is undefined just like it is at the poles.

6 Sampling

6.1 Continuous Time

Sampling a continuous-time signal means representing it as a sequence of points measured at regular intervals T. Notice that if we were to take a signal $x(t)$ and multiply it by an impulse train, then we would get a series of impulses equal to $x(t)$ at the sampling points and 0 everywhere else. We can call this signal $x_p(t)$.

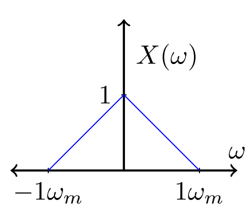
$$p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

$$x_p(t) = x(t)p(t) = \sum_{k=-\infty}^{\infty} x(t)\delta(t - kT)$$

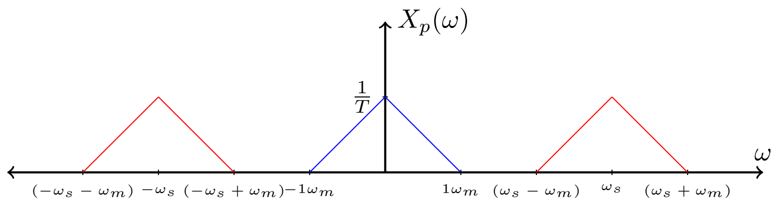
In the Fourier Domain,

$$X_p(\omega) = \frac{1}{2\pi}X(\omega) * P(\omega)$$
$$P(\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0)$$
$$\therefore X_p(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\theta)P(\omega - \theta)d\theta = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(\omega - k\omega_0)$$

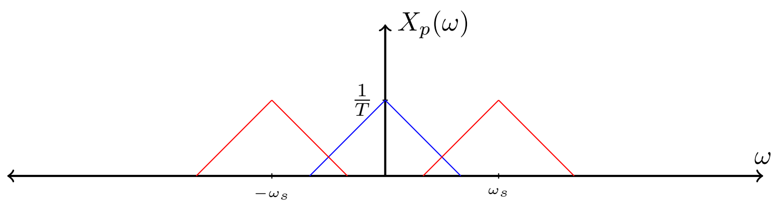
What this tells us is that the Fourier Transform of our sampled signal is a series of copies of $X(\omega)$, each centered at $k\omega_0$ where $\omega_0 = \frac{2\pi}{T}$ For example, lets say that our original signal has the following Fourier Transform. Notice the signal is band-limited by ω_M.



There are two major cases: if $\omega_0 > 2\omega_m$ and $\omega_0 < 2\omega_M$.
Case One: $\omega_s > 2\omega_m$



When $\omega_s > 2\omega_M$, the shifted copies of the original $X(\omega)$ (shown in blue) do not overlap with each other or which the original copy. If we wanted to recover the original signal, we could simply apply a low pass filter to isolate the unshifted copy of $X(\omega)$ and then take the inverse Fourier Transform.
Case Two: $\omega_s < 2\omega_m$



Notice how in this case, the shifted copies overlap with the original $X(\omega)$. This means in our sampled signal, the higher frequency information is bleeding in with the lower frequency information. This phenomenon is known as aliasing. When aliasing occurs, we cannot simply apply a low pass filter to isolate the unshifted copy of $X(\omega)$.

When $\omega_0 = 2\omega_M$, then our ability to reconstruct the original signal depends on the shape of its Fourier Transform. As long as $X_p(k\omega_m)$ are equal to $X(\omega_m)$ and $X(-\omega_m)$, then we can apply an LPF because we can isolate the original $X(\omega)$ and take its inverse Fourier Transform.

Remember that an ideal low pass filter is a square wave in the frequency domain and a sinc in the time domain. Thus if we allow

$$X_r(\omega) = X_p(\omega) \cdot \left\{ \begin{array}{cc} T & |\omega| < \frac{\omega_s}{2} \\ 0 & \text{else} \end{array} \right\}$$

then our reconstructed signal will be

$$x_r(t) = x_p(t) * sinc\left(\frac{t}{T}\right) = \sum_{n=-\infty}^{\infty} X(nT) sinc\left(\frac{t-nT}{T}\right)$$

This is why we call reconstructing a signal from its samples "sinc interpolation." This leads us to formulate the Nyquist Theorem.

Theorem 15 (CT Nyquist Theorem) *Suppose a continuous signal x is bandlimited and we sample it at a rate of $\omega_s > 2\omega_M$, then the signal $x_r(t)$ reconstructed by sinc interpolation is exactly $x(t)$*

6.2 Discrete Time

Sampling in discrete time is very much the same as sampling in continuous time. Using a sampling period of N we construct a new signal by taking an impulse train and multiplying elementwise with the original signal.

$$p[n] = \sum_{n=-\infty}^{\infty} \delta[n - kN]$$

$$x_p[n] = x[n]p[n] = \sum_{n=-\infty}^{\infty} x[kN]\delta[n - kN]$$

$$X_p(\omega) = \frac{1}{N} \sum_{k=0}^{N-1} X(\omega - k\omega_s)$$

Our indices only go from k to $N-1$ in the Fourier Domain because we can only shift a particular number of times before we start to get repeated copies. This is the impulse train sampled signal. It has 0's at the unsampled locations. If we want to, we could simply remove those zeros and get a downsampled signal

$$x_p[n] = x[Nn]$$

Like in continuous time, the reconstructed signal is recovered via sinc interpolation.

$$x_r[n] = \sum_{k=-\infty}^{\infty} x_p[n] sinc\left(\frac{n - kN}{N}\right)$$

The Nyquist Theorem in DT will tell us when this works.

Theorem 16 (DT Nyquist Theorem) *Suppose a discrete signal x is bandlimited by $\frac{\pi}{N}$ and we sample it at a rate of $\omega_s > 2\omega_M$, then the signal $x_r[n]$ reconstructed by sinc interpolation is exactly $x[n]$*

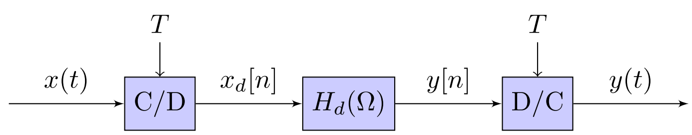
Thus as long as the Nyquist Theorem holds, we can take a downsampled signal and upsample it (i.e reconstruct the missing pieces) by expanding y by a factor of N and putting $0's$ for padding, and then applying sinc-interpolation to it.

6.3 Sampling as a System

Notice that we have two ways of representing our sample signal. We can either write it as a discrete time signal $x_d[n] = x(nT)$ or we can write it as an impulse train $x_p(t) = \sum_{-\infty}^{\infty} x(nT)\delta(t - nT)$. Based on their Fourier Transforms,

$$X_d(\Omega) = \sum_{n=-\infty}^{\infty} x(nT)e^{-j\Omega n}$$
$$X_p(\omega) = \sum_{n=-\infty}^{\infty} x(nT)e^{-j\omega nT}$$

Thus if we let $\Omega = \omega T$, then we see that these two representations of a signal have the same Fourier Transforms and thus contain the same information. This means that for some continuous signals, convert them to discrete time via sampling, use a computer to apply an LTI system, and convert the result back to a CT output.



We must be careful though because as long as the DT system we apply is LTI, the overall CT system will be linear too, but it will not necessarily be time invariant because sampling inherently depends on the signal's timing.

$$Y_d(\Omega) = H_d(\Omega)X_d(\Omega) = H_d(\Omega)X_p\left(\frac{\Omega}{T}\right)$$
$$Y_p(\omega) = Y_d(\omega T) = H_d(\omega T)X_p(\omega)$$
$$Y(\omega) = \left\{ \begin{array}{cc} T & |\omega| < \frac{\omega_s}{2} \\ 0 & |\omega| \geq \frac{\omega_s}{2} \end{array} \right\} \cdot Y_p(\omega) = \left\{ \begin{array}{cc} TH_d(\omega T)X_p(\omega) & |\omega| < \frac{\omega_s}{2} \\ 0 & |\omega| \geq \frac{\omega_s}{2} \end{array} \right\}$$

Assuming that the Nyquist theorem holds,

$$X_p(\omega) = \frac{1}{T}X(\omega)$$
$$\therefore Y(\omega) = \left\{ \begin{array}{cc} H_d(\omega T)X(\omega) & |\omega| < \frac{\omega_s}{2} \\ 0 & |\omega| \geq \frac{\omega_s}{2} \end{array} \right\}$$
$$\therefore H_{system} = \left\{ \begin{array}{cc} H_d(\omega T) & |\omega| < \frac{\omega_s}{2} \\ 0 & |\omega| \geq \frac{\omega_s}{2} \end{array} \right\}$$

This shows us that as long as the Nyquist theorem holds, we can process continuous signals with a disrete time LTI system and still have the result be LTI.

7 Appendix

7.1 Theory of Distributions

The Theory of Distributions is the mathematical framework which underlies the generalized Fourier Transforms.

Definition 38 *Given a test function x, a distribution T operates on x to produce a number $< T, x >$.*

Definition 39 *The distrbution induced by a function g is defined as*

$$< T_g, x >= \int_{-\infty}^{\infty} g(t)^* x(t) dt$$

Notice two things:

- $< T_g, x >$ is linear
- $< \alpha T_g, x >= \alpha^* < T_g, x >$

With these definitions, we can now define the Dirac delta in terms of distrubions. Let g be any function such that

$$\int_{\infty}^{\infty} g(t) dt = 1$$

Define g_ϵ to be

$$g_\epsilon = \frac{1}{\epsilon} g(\frac{t}{\epsilon})$$

Now we can defined $\delta(t) = \lim_{\epsilon \to 0} T_{g_\epsilon}$

$$< \delta, x >= \int_{-\infty}^{\infty} \delta(t) x(t) dt = x(0)$$

which is the property of the Dirac Delta we want. Now we can define the generalized Fourier Transform in terms of distributions.

Definition 40 *The generalized Continuous Time Fourier Transform of a distrubtion T is*

$$< FT, X >= 2\pi < T, x >$$

for test function x whose Fourier Transform is X