EE222 Course Notes

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Disclaimer: These notes reflect EE222 when I took the course (Fall 2021). They may not accurately reflect current course content, so use at your own risk. If you find any typos, errors, etc, please raise an issue on the GitHub repository.

Contents

1 Real Analysis

Definition 1 *The extended real line is the set*

$$\{-\infty\} \cup \mathbb{R} \cup \{\infty\}$$

Definition 2 *The supremum of a set $S \subset \mathbb{R}$ is a value $a \in \mathbb{R}_e$ such that $\forall s \in S,\ s \leq a$ and if $b \in \mathbb{R}_e$ such that $\forall s \in S,\ s \leq b$, then $a \leq b$.*

Supremum is essentially the "least upper bound" in a set. It always exists, and is called $\sup S$. The opposite of supremum is the infinimum.

Definition 3 *The infinimum of a set $S \subset \mathbb{R}$ is a value $a \in \mathbb{R}_e$ such that $\forall s \in S,\ s \geq a$ and if $b \in \mathbb{R}_e$ such that $\forall s \in S,\ s \geq b$, then $a \geq b$.*

The infinimum is the "greatest upper bound". Like the supremum, it always exists, and it is denoted $\inf S$. Supremum and Infinimum can be applied to scalar function $f : S \to \mathbb{R}$ by letting

$$\sup_{x \in S} f(x) = \sup\{f(x) | x \in S\}.$$

1.1 Norms

Definition 4 *Let V be a vector space of $\mathbb{R}$, then $\|\cdot\| : V \to \mathbb{R}$ is a norm if $\forall \boldsymbol{x}, \boldsymbol{y} \in V, \alpha \in \mathbb{R}$,*

$$\|\boldsymbol{x}\| \geq 0 \qquad \boldsymbol{x} = 0 \Leftrightarrow \|\boldsymbol{x}\| = 0 \qquad \|\alpha\boldsymbol{x}\| = |\alpha|\|\boldsymbol{x}\| \qquad \|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$$

Definition 5 *A normed space $(V, \|\cdot\|)$ is a vector space which is equipped with a norm $\|\cdot\| : V \to \mathbb{R}$.*

If we have an operator A which takes as inputs vectors from normed space $(X, \|\cdot\|_X)$ and outputs vectors in normed space $(Y, \|\cdot\|_Y)$, then we can define another norm on the vector space of operators from $X \to Y$.

Definition 6 *Let $A : X \to Y$ be an operator between normed spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$, then the induced norm of A is*

$$\|A\|_i = \sup_{\|\boldsymbol{x} \neq 0\|_X} \frac{\|A\boldsymbol{x}\|_Y}{\|\boldsymbol{x}\|_X}$$

The induced norm can be thought of as the maximum gain of the operator.

Definition 7 *Two norms $\|\cdot\|$ and $|||\cdot|||$ on a vector space V if $\exists k_1, k_2 > 0$ suh that*

$$\forall \boldsymbol{x} \in V,\ k_1\|\boldsymbol{x}\| \leq |||\boldsymbol{x}||| \leq k_2\|\boldsymbol{x}\|$$

If V is a finite dimensional vector space if and only if all norms of V are equivalent.

1.2 Sets

Definition 8 *Let $(V, \|\cdot\|)$ be a normed space, $a \in \mathbb{R}$, $a > 0$, $\boldsymbol{x}_0 \in V$, then the open ball of radius a centered around x_0 is given by*

$$B_a(\boldsymbol{x}_0) = \{\boldsymbol{x} \in V \mid \|\boldsymbol{x} - \boldsymbol{x}_0\| < a\}$$

Definition 9 *A set $S \subset V$ is open if $\forall \boldsymbol{s}_0 \in V,\ \exists \epsilon > 0$ such that $B_\epsilon(\boldsymbol{s}_0) \subset S$.*

Open sets have a boundary which is not included in the set. By convention, we say that the empty set is open.

The opposite of an open set is a closed set.

Definition 10 *A set S is closed if $\sim S$ is open.*

Closed sets have a boundary which is included in the set.

1.3 Convergence

Definition 11 *A sequence of points $\boldsymbol{x}_k$ in normed space $(V, \|\cdot\|)$ converges to a point $\bar{\boldsymbol{x}}$ if*

$$\forall \epsilon > 0,\ \exists N < \infty,\ \textit{ such that } \forall k \geq N, \|\boldsymbol{x}_k - \bar{\boldsymbol{x}}\| < \epsilon$$

Convergence means that we can always find a finite time such that after that time, all points in the sequence stay within a specicied norm ball.

Definition 12 *A sequence $\boldsymbol{x}_k$ is cauchy if*

$$\forall \epsilon > 0,\ \exists N < \infty \textit{ such that } \forall n, m \geq N, \|\boldsymbol{x}_m - \boldsymbol{x}_n\| < \epsilon$$

A Cauchy sequence has a looser type of convergence than a convergent sequence since it only requires all elements to in the sequence to be part of the same norm ball after some time instead of requiring the sequence to get closed and closer to a single point.

Theorem 1 *If $\boldsymbol{x}_n$ is a convergent sequence, then $\boldsymbol{x}_n$ is a also a Cauchy sequence.*

Definition 13 *A normed space $(V, \|\cdot\|)$ is complete if every Cauchy sequence converges to a point in V*

Because a complete space requires that Cauchy sequences converge, all cauchy sequences are convergent in a complete space. Two important complete spaces are

1. Every finite dimensional vector space
2. $(C[a,b], \|\cdot\|_\infty)$, the set of continuously differentiable functions on the closed interval $[a,b]$ equipped with the infinity norm.

A complete normed space is also called a **Banach Space**.

1.4 Contractions

Definition 14 *A point x^* is a fixed point of a function $P : X \to X$ if $P(x^*) = x^*$.*

Definition 15 *A function $P : X \to X$ is a contract if there exists a constant $0 \leq c < 1$ such that*

$$\forall \boldsymbol{x}, \boldsymbol{y} \in X,\ \|P(\boldsymbol{x}) - P(\boldsymbol{y})\| \leq c\|\boldsymbol{x} - \boldsymbol{y}\|$$

Informally, a contraction is a function which makes distances smaller

Suppose we look at a sequence defined by iterates of a function

$$\boldsymbol{x}_{k+1} = P(\boldsymbol{x}_k).$$

where P is a function $P : X \to X$. When does this sequence converge, and to what point will it converge?

Theorem 2 (Contraction Mapping Theorem) *If $P : X \to X$ is a contraction on the Banach space $(X, \|\cdot\|)$, then there is a unique $\boldsymbol{x}^* \in X$ such that $P(\boldsymbol{x}^*) = \boldsymbol{x}^*$ and $\forall x_0 \in X$, the sequence $\boldsymbol{x}_{n+1} = P(\boldsymbol{x}_n)$ is converges to $\boldsymbol{x}^*$.*

The contraction mapping theorem that contractions will have a unique fixed points and that repeatedly applying the contraction will converge to the fixed point.

1.5 Continuity

Definition 16 *A function $h : V \to W$ on normed spaces $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ is continuous at a point $\boldsymbol{x}_0$ if $\forall \epsilon > 0, \exists \delta > 0$ such that*

$$\|\boldsymbol{x} - \boldsymbol{x}\|_V < \delta \implies \|h(\boldsymbol{x}) - h(\boldsymbol{x_0})\|_W < \epsilon$$

Continuity essentially means that given an $\epsilon-$ball in W, we can find a $\delta-$ball in V which is mapped to the ball in W. If a function is continuous at all points $\boldsymbol{x}_0$, then we say the function is continuous.

We can make the definition of continuity more restrictive by restraining the rate of growth of the function.

Definition 17 *A function $h : V \to W$ on normed spaces $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ is Lipschitz continuous at $\boldsymbol{x}_0 \in V$ if $\exists r > 0$ and $L < \infty$ such that*

$$\forall \boldsymbol{x}, \boldsymbol{y} \in B_r(\boldsymbol{x}_0),\ \|h(\boldsymbol{x}) - h(\boldsymbol{y})\|_W \leq L\|\boldsymbol{x} - \boldsymbol{y}\|_V$$

A good interpretation of Lipschitz Continuity is that given two points in a ball around $\boldsymbol{x}_0$, the slope of the line connecting those two points is less than L. It means that the function is growings slower than linear for some region around $\boldsymbol{x}_0$. Lipschitz continuity implies continuity. If a function is lipschitz continuous with respect to one norm, it is also lipschitz continuous with respect to all equivalent norms.

When the function h is a function on $\mathbb{R}^n$ and is also differentiable, then Lipschitz continuity is easy to determine.

Theorem 3 *For a differentiably function $h : \mathbb{R}^n \to \mathbb{R}^n$,*

$$\exists r > 0, L < \infty, \boldsymbol{x}_0 \in \mathbb{R}^n,\ \forall \boldsymbol{x} \in B_r(\boldsymbol{x}), \left|\left|\frac{\partial h}{\partial \boldsymbol{x}}\right|\right|_2 \leq L$$

implies Lipschitz Continuity at $\boldsymbol{x}_0$.

Definition 18 *A function $h : \mathbb{R} \to V$ is piecewise continuous if $\forall k \in \mathbb{Z}$, $h : [-k, k] \to V$ is continuous except at a possibly finite number of points, and at the points of discontinuity t_i, $\lim_{s \to 0^+} h(t_i + s)$ and $\lim_{s \to 0^-}$ exist and are finite.*

2 Solutions to Nonlinear Systems

Consider the nonlinear system

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = f(\boldsymbol{x}, t),\ \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \in \mathbb{R}^n.$$

Definition 19 *A function $\boldsymbol{\Phi}(t)$ is a solution to $\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = f(x, t),\ \boldsymbol{x}(t_0) = \boldsymbol{x}_0$ on the closed interval $[t_0, t]$ if $\boldsymbol{\Phi}(t)$ is defined on the interval $[t_0, t]$, $\frac{\mathrm{d}\boldsymbol{\Phi}}{\mathrm{d}t} = f(\boldsymbol{\Phi}(t), t)$ on the interval $[t_0, t]$, and $\boldsymbol{\Phi}(t_0) = \boldsymbol{x}_0$.*

We say that $\Phi(t)$ is a solution in the sense of Caratheodory if

$$\Phi(t) = \boldsymbol{x}_0 + \int_{t_0}^{t} f(\boldsymbol{\Phi}(\tau), \tau) d\tau.$$

Because the sytem is non-linear, it could potentially have no solution, one solution, or many solutions. These solutions could exist locally, or they could exist for all time. We might also want to know when there is a solution which depends continuously on the initial conditions.

Theorem 4 (Local Existence and Uniqueness) *Given $\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = f(\boldsymbol{x}, t),\ \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \in \mathbb{R}^n$ where f is piecewise continuous in t and $\exists T > t_0$ such that $\forall t \in [t_0, T], f$ is L-Lipschitz Continuous, then $\exists \delta > 0$ such that a solution exists and is unique $\forall t \in [t_0, t_0 + \delta]$.*

Theorem 4 can be proved using the Contraction Mapping Theorem (theorem 2) by finding δ such that the function $P : C_n[t_0, t_0 + \delta] \to C_n[t_0, t_0 + \delta]$ given by

$$P(\boldsymbol{\Phi})(t) = \boldsymbol{x}_0 + \int_{t_0}^{t_0+\delta} f(\boldsymbol{\Phi}(\tau), \tau) d\tau$$

is a contraction under the norm $\|\boldsymbol{\Phi}\|_\infty = \sup_{t_0 \le t \le t_0+\delta} \|\boldsymbol{\Phi}(t)\|$.

Theorem 5 (Global Existence and Uniqueness) *Suppose $f(\boldsymbol{x}, t)$ is piecewise continuous in t and $\forall T \in [t_0, \infty)$, $\exists L_T < \infty$ such that f is L_T Lipshitz continuous for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, then the nonlinear system has exactly one solution on $[t_0, T]$.*

Once we know that solutions to a nonlinear system exist, we can sometimes bound them.

Theorem 6 (Bellman-Gronwall Lemma) *Suppose $\lambda \in \mathbb{R}$ is a constant and $\mu : [a, b] \to \mathbb{R}$ is continuous and non-negative, then for a continuous function $y : [a, b] \to \mathbb{R}$*

$$y(t) \leq \lambda + \int_a^t \mu(\tau)y(\tau)d\tau \implies y(t) \leq \lambda exp\left(\int_a^t \mu(\tau)d\tau\right)$$

Another thing we might want to do is understand how the non-linear system reacts to changes in the initial condition.

Theorem 7 *Suppose the system $\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = f(\boldsymbol{x}, t),\ \boldsymbol{x}(t_0) = \boldsymbol{x}_0$ satisfies the conditions of global uniqueness and existence. Fix $T \in [t_0, \infty]$ and suppose $\boldsymbol{x}(\cdot)$ and $\boldsymbol{z}(\cdot)$ are two solutions satisfying $\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = f(\boldsymbol{x}, t), \boldsymbol{x}(t_0) = \boldsymbol{x}_0$ and $\frac{\mathrm{d}\boldsymbol{z}}{\mathrm{d}t} = f(\boldsymbol{z}(t), t),\ \boldsymbol{z}(t_0) = \boldsymbol{z}_0$, then $\forall \epsilon > 0, \exists \delta > 0$ such that*

$$\|\boldsymbol{x}_0 - \boldsymbol{z}_0\| < \delta \implies \|\boldsymbol{x} - \boldsymbol{z}\|_\infty < \epsilon.$$

Theorem 7 is best understood by defining a function $\Psi : \mathbb{R}^n \to C_n[t_0, t]$ where $\Psi(\boldsymbol{x}_0)(t)$ returns the solution to the system given the initial condition. If the conditions of $Theorem$ 7 are satisfied, then the function Ψ will be continuous.

3 Nonlinear System Dynamics

Consider the non-linear system

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = f(\boldsymbol{x}, t).$$

Definition 20 *The system is autonomous if $f(x, t)$ is not explicitly dependent on time t.*

Definition 21 *A point x_0 is an equilibrium point at time t_0 if*

$$\forall t \geq t_0,\ f(x_0, t) = 0$$

3.1 Planar Dynamical Systems

Planar dynamical systems are those with 2 state variables. Suppose we linearize the system $\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = f(\boldsymbol{x})$ at an equilibrium point.

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = D_f|_{\boldsymbol{x}=\boldsymbol{x_0}}\boldsymbol{x}$$

Depending on the eigenvalues of D_f, the Jacobian, we get several cases for how this linear system behaves. We'll let z_1 and z_2 be the eigenbasis of the *phase space*.

1. The eigenvalues are real, yielding solutions $z_1 = z_1(0)e^{\lambda_1 t}, z_2 = z_2(0)e^{\lambda_2 t}$. If we eliminate the time variable, we can plot the trajectories of the system.

$$\frac{z_1}{z_1(0)} = \left(\frac{z_2}{z_2(0)}\right)^{\frac{\lambda_1}{\lambda_2}}$$

   (a) When $\lambda_1, \lambda_2 < 0$, all trajectories converge to the origin, so we call this a **stable node**.

   (b) When $\lambda_1, \lambda_2 > 0$, all trajectories blow up, so we call this an **unstable node**.

   (c) When $\lambda_1 < 0 < \lambda_2$, the trajectories will converge to the origin along the axis corresponding to λ_1 and diverge along the axis corresponding to λ_2, so we call this a **saddle node**.

2. There is a single repeated eigenvalue with one eigenvector. As before, we can eliminate the time variable and plot the trajectories on the z_1, z_2 axes.

   (a) When $\lambda < 0$, the trajetories will converge to the origin, so we call it an **improper stable node**

   (b) When $\lambda > 0$, the trajetories will diverge from the origin, so we call it an **improper unstable node**

3. When there is a complex pair of eigenvalues, the linear system will have oscillatory behavior. The Real Jordan form of D_f will look like

$$D_f = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}.$$

   The parameter β will determine the direction of the trajectories (clockwise if positive).

   (a) When $\alpha < 0$, the trajectories will spiral towards the origin, so we call it a **stable focus**.

(b) When $\alpha = 0$, the trajectories will remain at a constant radius from the origin, so we call it a **center**.

(c) When $\alpha > 0$, the trajectories will spiral away from the origin, so we call it an **unstable focus**.

It turns out that understanding the linear dynamics at equilibrium points can be helpful in understanding the nonlinear dynamics near equilibrium points.

Theorem 8 (Hartman-Grobman Theorem) *If the linearization of a planar dynamical system* $\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = f(\boldsymbol{x})$ *at an equilibrium point* $\boldsymbol{x_0}$ *has no zero or purely imaginary eigenvalues, then there exists a homeomorphism from a neighborhood* U *of* x_0 *into* $\mathbb{R}^2$ *which takes trajectories of the nonlinear system and maps them onto the linearization where* $h(\boldsymbol{x_0}) = 0$*, and the homeomorphism can be chosen to preserve the parameterization by time.*

Theorem 8 essentially says that the linear dynamics predict the nonlinear dynamics around equilibria, but only for a neighborhood around the equilibrium point. Outisde of this neighborhood, the linearization may be very wrong.

Non-linear systems can also have periodic solutions.

Definition 22 *A closed orbit* γ *is a trajectory of the system such that* $\gamma(0) = \gamma(T)$ *for finite* T*.*

Suppose that we have a simply connected region D (meaning D cannot be contracted to a point) and we want to know if it contains a closed orbit.

Theorem 9 (Bendixon's Theorem) *If* $\mathrm{div}(f)$ *is not identically zero in a sub-region of* D *and does not change sign in* D*, then* D *contains no closed orbits.*

Theorem 9 lets us rule out closed orbits from regions of $\mathbb{R}^2$.

Definition 23 *A region* $M \subset \mathbb{R}^2$ *is positively invariant for a trajectory* $\phi_t(\boldsymbol{x})$ *if* $\forall x \in M, \forall t \geq 0, \phi_t(\boldsymbol{x}) \in M$*.*

A positively invariant set essentially means that once a trajectory enters the set, it cannot leave. That means all of the vector field lines must point inside the set. If we have a positively invariant region, then we can determine whether it contains closed orbits.

Theorem 10 (Poincare-Bendixson Theorem) *If* M *is a compact, positively invariant set for the flow* $\phi_t(\boldsymbol{x})$*, then if* M *contains no equilibrium points, then* M *has a limit cycle.*