Determinate Games and the Gale-Stewart Theorem

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Table des matières

1 Introduction

Game theory is a rapidly growing field of mathematics and computer science that can be defined as the study of the strategies of agents defending their own interests in various games. We will focus in particular on two-player games with perfect information. We will first define the various mathematical objects that we will use, then we will conduct a study of finite games, and finally we will focus on the Gale-Stewart theorem, which makes a very interesting connection between game theory and topology.

2 Notations and definitions

2.1 Notations

Let X be any set. For $x \in X^{\mathbb{N}}$ and $m \in \mathbb{N}$, we denote by $a|m$ the restriction $(x_0, ..., x_{m-1})$ to the first m terms of the sequence.
We denote by $< x_0, x_1, ..., x_n >$ the set of sequences $x \in X^{\mathbb{N}}$ such that $(x_0, ..., x_n) = x|n + 1$, that is to say which begin with $(x_0, x_1, ..., x_n)$

2.2 Definitions

In this TIPE we are interested in infinite two-player games, I to II, known as Gale-Stewart games, that is to say they verify the following axioms :
• Each player has at all times the moves played by the opponent and himself.
• Each player takes turns, without missing a turn.
• The game does not contain any form of randomness (dice rolling, etc.).

Remark 1. *Given the multitude of formalizations proposed in different works and journals, we have chosen a formalization a little different from that proposed in the first paper by Gale and Stewart in 1953.*

Definition 2.1. *Let X be a non-empty set and $A \subseteq X^{\mathbb{N}}$.*
*A **game** is a sequence $(x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$ where $\forall n \in \mathbb{N}$, x_{2n} corresponds to the $n+1$- th move of player I and x_{2n+1} to the n- th move of player II.*
A game (x_n) is won by player I if and only if $(x_n) \in A$.
We denote $G(X, A)$ such a game.

Remark 2. *It is important to note that in this definition there cannot be a tie at the end of the game.*

We can represent the parts of a two-player game using the following diagram :

$$\begin{tikzcd}[column sep=small]
\text{I :} & x_0 \arrow[dr] & & x_2 \arrow[dr] & & \cdots & x_{2n} \arrow[dr] & & x_{2n+2} & \cdots \\
\text{II :} & & x_1 \arrow[ur] & & x_3 & & & x_{2n+1} \arrow[ur] & &
\end{tikzcd}$$

FIGURE 1 – Part $(x_0, x_1, ...)$

The key point of our study is to know whether or not one of the two players has a strategy that will allow him to ensure victory regardless of the moves his opponent plays.

Definition 2.2. *A **strategy** for player I is an application $\sigma : (X^2)^{<\mathbb{N}} \to X$ which to any finite sequence of even length $(x_0, ..., x_{2n-1})$ (which corresponds to a certain instant of the game) associates the move to play $\sigma((x_0, ..., x_{2n-1}))$.*
We similarly define a strategy for player II : $\sigma : (X^2)^{<\mathbb{N}} \times X \to X$.

If σ is a strategy for player I and τ is a strategy for player II, we denote by $\sigma * \tau$ the following game where each player follows his strategy :

$$\begin{tikzcd}[column sep=small]
\text{I :} & \sigma(\emptyset) \arrow[dr] & & \sigma(\tau(\sigma(\emptyset))) \arrow[dr] & & \cdots \\
\text{II :} & & \tau(\sigma(\emptyset)) \arrow[ur] & & \tau(\sigma(\tau(\sigma(\emptyset)))) &
\end{tikzcd}$$

FIGURE 2 – Game where I and II play the strategies σ and τ respectively

Definition 2.3. *Let $G(X, A)$ be a Gale-Stewart game. A strategy σ for player I is said to be **winning** if for every strategy τ, $\sigma * \tau \in A$. We define the notion of a winning strategy for player II in the same way. If one of the two players has a winning strategy, the game is said to be **determined**.*

Corollary 2.3.1. *Let $G(X, A)$ be a Gale-Stewart game. Then I and II cannot both have a winning strategy.*

Démonstration. If by absurdity they have respectively σ and τ winning, then $\sigma * \tau \in A$ and $\sigma * \tau \notin A$. Absurd. □

2.3 Product topology

Since our goal is not to give a presentation on product topology, we will try to present it as succinctly as possible.

Definition 2.4. *Let X be any set. We say that $(X, \mathcal{O})$ where $\mathcal{O} \subset \mathcal{P}(X)$ is a **topological space** when :*
(i) : $X \in \mathcal{O}$ and $\varnothing \in \mathcal{O}$
(ii) : $\mathcal{O}$ is stable by any union
(iii) : $\mathcal{O}$ is stable by finite intersection
*The elements of $\mathcal{O}$ are then called the **open** of X.*

Proposition 2.5. *Let X be a topological space. The space $X^{\mathbb{N}}$ equipped with the topology $\mathcal{O}_\omega$ such that :*

$$O \in \mathcal{O}_\omega \text{ iff } O = \Pi_{j\in J} O_j \times \Pi_{j\notin J} X \text{ with } J \text{ finite}$$

where $O_j \in \mathcal{O}$, provides $X^{\mathbb{N}}$ with a topology that we call product topology.

Subsequently, X will be equipped with the discrete topology, so the generated product topology will be the set of open sets $O_{(a_0,a_1,...,a_n)}$ such that $a \in O_{(a_0,a_1,...,a_n)}$ iff $a|(n+1) = (a_0, a_1, ..., a_n)$, that is, $O_{(a_0,a_1,...,a_n)} =< a_0, a_1, ..., a_n >$

3 Finite Games

3.1 Presentation

Much of game theory is based on the study of finite games ; we will see that Gale-Stewart games can model a large part of them, including chess and Nim 's game .

Definition 3.1. *A Gale - Stewart game $G(A$*

Thus, the first n moves determine the outcome of the game

Example 1. *With these definitions, we can see chess as a finite Gale-Stewart game. Indeed, with the "50-move" rule, a chess game can only have at most about 7000 moves. By setting X the set of possible board positions (which can be code with integers), and with A the set of games consisting only of legal moves until I checkmates and then of any elements (a draw is a defeat for I), we have that after 7000 moves played, we already know whether $(x_n) \in A$ or not.*

3.2 Determination of finite games

Theorem 3.2. *Let $G(X, A)$ be a finite Gale-Stewart game. Then $G(X, A)$ is determined.*

Démonstration. By recurrence on n the order of the game.
• Initialization : For the trivial game that ends at the initial position, either all games are winning for I, in which case all strategies are winning for I, or all games are winning for II.
• Inheritance : Assume the property is true for all finite games of order $n-1$. Let $G(X, A)$ be of order n. Then $\forall a \in X$, the game defined by $G_a(X, A_a)$ where $(u_n) \in A_a \iff (a, u_0, u_1, ...) \in A$ is of order $n-1$:

(Intuitively, it's the same game except that move a has already been played, so there is an inversion between I and II with respect to $G(X, A)$).
It is therefore determined by the recurrence hypothesis.
• If there exists a such that II has a winning strategy for $G_a(X, A_a)$, then the strategy of $G(X, A)$ for I which consists of playing a and then following the above-mentioned strategy (because the roles are reversed) is winning, so the game is determined.
• Otherwise, this means that $\forall a \in X$, $G_a(X, A_a)$ is winning for I, so $G(X, A)$ is winning for II : it suffices,

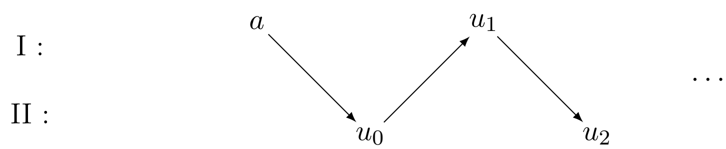

FIGURE 3 – Game u_n where move a was played beforehand

regardless of the move a played by I, to follow I's winning strategy in $G_a(X, A_a)$. G(X,A) is then well determined.

□

This theorem implies that any game satisfying the hypotheses is "solvable", in the sense that it is possible with a computer to know from any position whether it is possible or not to win. It is important to note, however, that the naive algorithm for traversing the game states is generally exponentially complex, which makes it irrelevant in most cases.

3.3 Determination of countable sets

We are interested in another particular case of determination where this time A is countable (and not the number of moves played to finish the game as before), which we will be able to demonstrate using a diagonal argument.

Theorem 3.3. *Let $G(X, A)$ be a Gale-Stewart game where A is countable. Then $G(X, A)$ is determined.*

Démonstration. The idea is similar to Cantor's diagonal argument. We denote A = $a^{(k)}, k \in \mathbb{N}$ and we construct τ a winning strategy for II that will not depend on the moves that I plays. To do this, we set $\forall(x_0, ..., x_{2n-1}) \in X^{<\text{inf}}, \tau((x_0, ..., x_{2n-1})) = a_{2n}^{(n)} + 1$.
We are then sure that the part (x_n) cannot belong in A, so it is indeed a winning strategy for II □

co-countable case A is also determined by reversing roles. Thus, an indeterminate game must not only be far enough from games in real life (we cannot know who wins after a finite number of moves), but the set A must also be sufficiently "complex".

4 Gale-Stewart Theorem

4.1 Existence of indeterminate games

It turns out that for an infinite game, it is possible that it is not determined, so that it is impossible to predict a winner in advance. This is the fundamental observation giving importance to the following part.

Theorem 4.1. *undetermined game $G(X, A)$.*

Démonstration. Let S_I be the set of strategies for I. We admit that we can index S_I by well-ordered J. We define in the same way S_{II} which we admit can also be indexed by J (intuitively, there are about as many strategies for I as for II).
Thus, we have $S_I = \ \sigma_\alpha, \alpha \in J$ and $S_{II} = \ \tau_\alpha, \alpha \in J$
Let $(\sigma, \tau) \in S_I \times S_{II}$. We note : $P_I(\sigma) = \{\sigma * y, y \in X^{\mathbb{N}}\}$ et $P_{II}(\tau) = \{y * \tau, y \in X^{\mathbb{N}}\}$

We construct by induction an indeterminable set A.
Let 0 be the smallest element of J. We choose a_0 any element of $P_{II}(\tau_0)$ and $b_0 \neq a_0$ an element of $P_I(\sigma_0)$
.
Let $\beta \in J$. Suppose that $\forall \alpha < \beta$, a_α and b_α have been chosen. Let us choose a_β and b_β.
We have a bijection (coming from the indexing) between $b_\alpha, \alpha < \beta$ and $\alpha \in J, \alpha < \beta$.

Thus, by definition of J, $| b_\alpha, \alpha < \beta | < |J| = |S_I|$. The set $P_{II}(\tau_\beta) \setminus b_\alpha, \alpha < \beta$ is non-empty. We choose a_β from this set.
Similarly, $P_I(\sigma_\beta) \setminus a_\alpha, \alpha < \beta$ is non-empty. We choose b_β from this set.
We note $A = a_\alpha, \alpha \in J$ and $B = b_\alpha, \alpha \in J$.
We can start by noticing that $A \cap B = \varnothing$ by definition of sets.
Let us show by contradiction that $G(X, A)$ is not determined.
Suppose σ is a winning strategy for I, let $P_I(\sigma) \subset A$. So there exists $\alpha \in J$ such that $\sigma = \sigma_\alpha$. By definition of b_α, $b_\alpha \in P_I(\sigma_\alpha)$, so $b_\alpha \in A$, absurd. Similarly, there cannot exist a winning strategy for II.
We have therefore proven that A is not determined.

□

4.2 Gale-Stewart

We now equip X with the discrete topology and $X^{\mathbb{N}}$ with the product topology

Theorem 4.2 (Gale-Stewart). *Suppose that A is an open set. Then $G(X, a)$ is determined*

Démonstration. Assume that I has no winning strategy.
In this case, it turns out that no matter what position $(x_0, x_1, ..., x_{2n})$ it is II's turn to play, II can "stay out of danger" by staying in a position where I cannot win. Let σ be a strategy of player I. We show by induction on n that there exists τ a strategy for II that I does not have a winning strategy after his n -th move :
-Initialization : After I has played the move (x_0), the fact that I does not have a winning strategy implies that there exists a move for II such that there does not exist a winning strategy for I after the move. We set $\tau(x_0)$ such a move
-Heredity of the same
We now show that τ is a winning strategy.
We set $A =< a_0, ..., a_n >$ by definition of the product topology on X If $\sigma * \tau \in A$, then by definition of the open sets of the product topology, we have a rank n from which $< x_0, ..., x_n > \subset A$, which means that II has already lost at the moment when the nth move has been played, which is absurd. Thus$\sigma * \tau \notin A$

□

We proceed in the same way in the case where A is closed by reversing the roles of I and II.

5 Bibliography

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