Probability

Aparna Sakode

Contents

1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Download the following files and execute the C program.

```
codes/coeffs.h
codes/uni_var_mean.c
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \tag{1.2.1}$$

Solution: The following code plots Fig. 1.2

```
codes/uni_cdf_plot.py
```

1.3 Find a theoretical expression for $F_U(x)$.
Solution: Probability density function is given by:

$$\begin{cases} 1 & \text{for } U \in (0,1) \\ 0 & otherwise \end{cases}$$

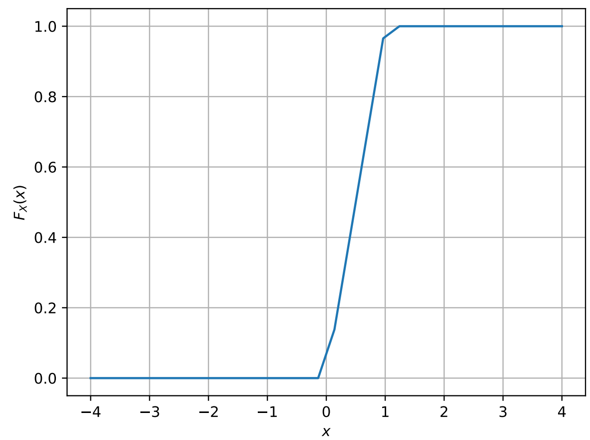

Fig. 1.2. The CDF of U

CDF of random variable U in range $\in [0.1)$

$$F_U(x) = \int_0^U f(x)dx \tag{1.3.1}$$

$$= \int_0^U \frac{1}{1-0}dx \tag{1.3.2}$$

$$= \frac{[U]_0^U}{1-0} \tag{1.3.3}$$

$$= \frac{U}{1} \tag{1.3.4}$$

Cumulative density function:

$$\begin{cases} 0 & \text{for } U < 0 \\ \text{U} & \text{for } U \in [0,1) \\ 1 & \text{for } U \geq b \end{cases}$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.4.1}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{1.4.2}$$

Write a C program to find the mean and variance of U.
Solution:

```
codes/uni_var_mean.c
```

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.5.1}$$

Solution:

$$\begin{aligned} &= E[U - E[U]]^2 \\ &= E[U^2 - 2UE[U] + E[U]^2] \\ &= E[U^2] - E[U]^2 \end{aligned}$$

$$Given, E[U]^k = \int_{-\infty}^{\infty} x^k dF_U(x)$$

For E[U], k=1 and limits are from 0-1.

$$\begin{aligned} E[U] &= \int_0^1 x dF_U(x) \\ &= \int_0^1 x dF_U(x) \end{aligned}$$

$dF_U(x)$ for uniformly distributed function is $\frac{1}{b-a}$ where b=1 and a=0.

$$\begin{aligned} &= \frac{1}{1-0}\int_0^1 x.dx \\ &= \frac{1}{1-0} * [\frac{x^2}{2}]_0^1 \\ &= \frac{1}{2} = 0.5. \end{aligned}$$

$$\begin{aligned} E[U^2] &= \frac{1}{1-0}\int_0^1 x^2 dx \\ &= \frac{1}{1-0} * [\frac{x^3}{3}]_0^1 \\ &= \frac{1}{3}. \end{aligned}$$

Variance will be

$$\begin{aligned} var[U] &= E[U^2] - E[U]^2 \\ &= \frac{1}{3} - [\frac{1}{2}]^2 \\ &= \frac{1}{12} \\ &= 0.0833 \end{aligned}$$

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1.1}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat.
Solution:

```
codes/coeffs.h
codes/gauss_var_mean.c
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: The CDF of X is plotted in Fig. 2.2 using the code below

```
codes/gauss_cdf_plot.py
```

Properties of CDF
1.Every CDF function is right continuous and it is non increasing,where
$lim_{x\to\infty} f(x) = 1, lim_{x\to-\infty} f(x) = 0$
2.If the CDF of a real-valued function is said to be continuous, then 'X' is called a continuous random variable
$F_x(b) - F_x(a) = P(a < X \leq b) = \int_a^b f_x(x)dx$

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx}F_X(x) \tag{2.3.1}$$

What properties does the PDF have?
Solution: The PDF of X is plotted in Fig. 2.3 using the code below

```
codes/gauss_pdf_plot.py
```

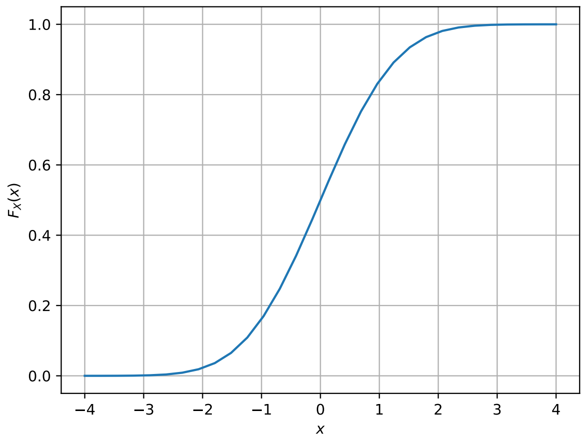

Fig. 2.2. The CDF of X

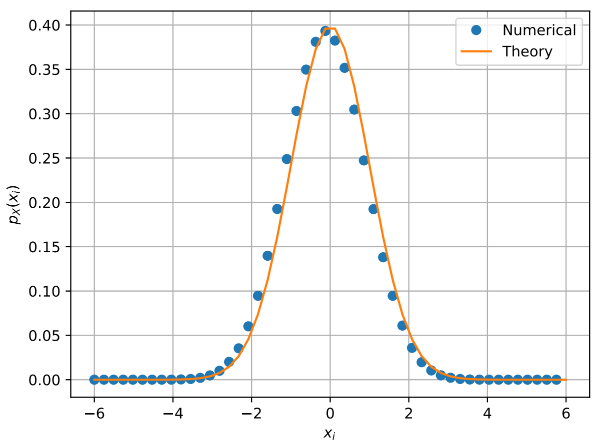

Fig. 2.3. The PDF of X

Properties of PDF is
1. $P(x) = \int_a^b f(x)dx$
2. $\int_{-\infty}^{\infty} f(x)dx = 1$
3. $f(x)$ is non-negative for all possible values.

2.4 Find the mean and variance of X by writing a C program.
Solution: The code give below gives the mean and variance of X

```
codes/gauss_var_mean.c
```

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.5.1}$$

repeat the above exercise theoretically.
Solution: We have

$$\begin{aligned}
E[X] &= \int_{-\infty}^{\infty} xf(x)dx \\
&= \int_{-\infty}^{0} xf(x)dx + \int_{0}^{\infty} xf(x)dx \\
&= 0
\end{aligned}$$

f(x) is an even function, so we can put $-x = x$ in the first half of the integral.
Variance is given by

$$\begin{aligned}
Var(X) &= \int_{-\infty}^{\infty} (x - E[X])^2 f(x)dx \\
&= \int_{-\infty}^{\infty} x^2 f(x)dx \\
&= \int_{-\infty}^{\infty} \frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \\
&= \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) xdx \\
y &= \frac{x^2}{2} \\
\frac{dy}{dx} &= \frac{2x}{2} \\
dy &= xdx \\
x &= \sqrt{2y} \\
E[X^2] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{2y} \exp(-y)dy \\
&= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \sqrt{y} \exp(-y)dy \\
&= \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} \sqrt{y} \exp(-y)dx
\end{aligned}$$

above equation is a Gamma function of form $\Gamma(a) = \int_0^{\infty} y^{a-1} \exp(-y)dy$ for $a > 0$.

$$\begin{aligned}
E[X^2] &= \frac{2}{\sqrt{\pi}} \Gamma(\frac{1}{2} + 1) \\
E[X^2] &= \frac{2}{2\sqrt{\pi}} \Gamma(\frac{1}{2}) \\
&= \frac{2}{2\sqrt{\pi}} * \sqrt{\pi} \\
&= 1 \\
Var[X] &= 1 - 0 \\
Var[X] &= 1
\end{aligned}$$

3 From Uniform to Other

3.1 Generate samples of

$$V = -2\ln(1-U) \tag{3.1.1}$$

and plot its CDF.

3.2 Find a theoretical expression for $F_V(x)$.

4 Triangular Distribution

4.1 Generate

$$T = U_1 + U_2 \tag{4.1.1}$$

4.2 Find the CDF of T.

4.3 Find the PDF of T.

4.4 Find the theoretical expressions for the PDF and CDF of T.

4.5 Verify your results through a plot.

5 Maximum Likelihood

5.1 Generate equiprobable $X \in \{1, -1\}$.

5.2 Generate

$$Y = AX + N, \tag{5.2.1}$$

where $A = 5$ dB, and $N \sim \mathcal{N}(0,1)$.

5.3 Plot Y using a scatter plot.

5.4 Guess how to estimate X from Y.

5.5 Find

$$P_{e|0} = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{5.5.1}$$

and

$$P_{e|1} = \Pr\left(\hat{X} = 1 | X = -1\right) \tag{5.5.2}$$

5.6 Find P_e assuming that X has equiprobable symbols.

5.7 Verify by plotting the theoretical P_e with respect to A from 0 to 10 dB.

5.8 Now, consider a threshold δ while estimating X from Y. Find the value of δ that maximizes the theoretical P_e.

5.9 Repeat the above exercise when

$$p_X(0) = p \tag{5.9.1}$$

5.10 Repeat the above exercise using the MAP criterion.

6 Gaussian to Other

6.1 Let $X_1 \sim \mathcal{N}(0,1)$ and $X_2 \sim \mathcal{N}(0,1)$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \tag{6.1.1}$$

6.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases} \tag{6.2.1}$$

find α.

6.3 Plot the CDF and PDf of

$$A = \sqrt{V} \tag{6.3.1}$$

7 Conditional Probability

7.1 Plot

$$P_e = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{7.1.1}$$

for

$$Y = AX + N, \tag{7.1.2}$$

where A is Raleigh with $E\left[A^2\right] = \gamma, N \sim \mathcal{N}(0,1), X \in (-1,1)$ for $0 \leq \gamma \leq 10$ dB.

7.2 Assuming that N is a constant, find an expression for P_e. Call this $P_e(N)$

7.3 For a function g,

$$E\left[g(X)\right] = \int_{-\infty}^{\infty} g(x) p_X(x)\, dx \tag{7.3.1}$$

Find $P_e = E\left[P_e(N)\right]$.

7.4 Plot P_e in problems 7.1 and 7.3 on the same graph w.r.t γ. Comment.

8 Two Dimensions

Let

$$\mathbf{y} = A\mathbf{x} + \mathbf{n}, \tag{4.1}$$

where

$$x \in (\mathbf{s}_0, \mathbf{s}_1), \mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{4.2}$$

$$\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, n_1, n_2 \sim \mathcal{N}(0,1). \tag{4.3}$$

8.1 Plot

$$\mathbf{y}|\mathbf{s}_0 \text{ and } \mathbf{y}|\mathbf{s}_1 \tag{8.1.1}$$

on the same graph using a scatter plot.

8.2 For the above problem, find a decision rule for detecting the symbols $\mathbf{s}_0$ and $\mathbf{s}_1$.

8.3 Plot

$$P_e = \Pr\left(\hat{\mathbf{x}} = \mathbf{s}_1 | \mathbf{x} = \mathbf{s}_0\right) \tag{8.3.1}$$

with respect to the SNR from 0 to 10 dB.

8.4 Obtain an expression for P_e. Verify this by comparing the theory and simulation plots on the same graph.