(a) Let X be a bernoulli random variable with $p = 1/2$.

Let Y be a random variable that is -1 with probability $\frac{1}{2}$ and 1 with probability $\frac{1}{2}$.

Let $Z = XY$

X and Y are uncorrelated but dependent, shown by proof below:

PROOF

$$\mathbb{E}[Z] = \mathbb{E}[XY] = \mathbb{E}[X]\,\mathbb{E}[Y].$$

Since $\mathbb{E}[Y] = -1 * \frac{1}{2} + 1 * \frac{1}{2} = \underline{\underline{0}}$.

Then $\mathbb{E}[Z] = 0$

$$\mathbb{E}[X] = p = \frac{1}{2}.$$

$$\begin{aligned}
\mathrm{cov}(Z, X) &= \mathbb{E}\left[\left(Z - \mathbb{E}(Z)\right)\left(X - \mathbb{E}(X)\right)\right] \\
&= \mathbb{E}\left[\left(XY - 0\right)\left(X - \tfrac{1}{2}\right)\right] \\
&= \mathbb{E}\left[X^2Y - \tfrac{1}{2}XY\right] \\
&= \mathbb{E}\left[Y\left(X^2 - \tfrac{1}{2}X\right)\right] \\
&= \mathbb{E}[Y]\,\mathbb{E}\left[X^2 - \tfrac{1}{2}X\right] = \underline{\underline{0}}.
\end{aligned}$$

This shows X and Z are uncorrelated.

For X and Z to be independent, the following must be true:

$$P(Z = z \mid X = x) = P(Z = z)$$

However, this fails if $z = 1$ and $x = 0$.

$P(Z = 1 \mid X = 0) = 0$, while $P(Z = 1) = \frac{1}{4}$.

Thus, X and Z are independent.

(b) Let x and y be 2 continous random variable.

Let $y = x^2$.

To prove : Find a best fit linear regression $y = \beta_1 x + \beta_0$ which has as optimal $\tilde{y} = \beta_1 x + \beta_0$.

Squared loss error $= (y - \tilde{y})^2$

$$= (x^2 - \beta_1 x - \beta_0)^2 .$$

$$\int_{-1}^{1} (x^2 - \beta_1 x - \beta_0)^2 \cdot dx .$$

$$= \int_{-1}^{1} x^4 + \beta_0^2 + \beta_1^2 x^2 - 2x^2\beta_0 - 2x^3\beta_1 + 2\beta_0\beta_1 x \, dx$$

$$= \frac{x^5}{5} - \frac{2x^3\beta_0}{3} - \frac{2x^4\beta_1}{4} + 2\frac{\beta_0\beta_1 x^2}{2} + \beta_0^2 x + \frac{\beta_1^2 x^3}{3}$$

Substituting 1 & -1 :

$$\left(\frac{1}{5} - \frac{2\beta_0}{3} - \frac{2\beta_1}{4} + \frac{2\beta_0\beta_1}{2} + \beta_0^2 + \frac{\beta_1^2}{3}\right) -$$

$$\left(\frac{-1}{5} + \frac{2\beta_0}{3} - \frac{2\beta_1}{4} + 2\frac{\beta_0\beta_1}{2} - \beta_0^2 - \frac{\beta_1^2}{3}\right)$$

Scanned with CamScanner

$$= \left(\frac{1}{5} + \beta_0^2 + \frac{\beta_1^2}{3} - \frac{2\beta_0}{3} - \frac{\cancel{2}\beta_1}{\cancel{4}} + \frac{\cancel{2}\beta_0\beta_1}{\cancel{2}} \right.$$

$$\left. + \frac{1}{5} + \beta_0^2 + \frac{\beta_1^2}{3} - \frac{2\beta_0}{3} + \frac{\cancel{2}\beta_1}{\cancel{4}} - \frac{\cancel{2}\beta_0\beta_1}{\cancel{2}} \right)$$

$$= \frac{2}{5} + 2\beta_0^2 + \frac{2\beta_1^2}{3} - \frac{4\beta_0}{3}$$

Partial derivatives :-

(1) $$\frac{\partial}{\partial \beta_0}\left(\frac{2}{5} + 2\beta_0^2 + \frac{2\beta_1^2}{3} - \frac{4\beta_0}{3} \right) \Rightarrow 0 + 4\beta_0 + 0 - \frac{4}{3} = 0.$$

$$4\beta_0 = \frac{4}{3}$$

$$\beta_0 = \frac{4}{3} \times \frac{1}{4} = \frac{1}{3}.$$

(2) $$\frac{\partial}{\partial \beta_1}\left(\frac{2}{5} + 2\beta_0^2 + \frac{2\beta_1^2}{3} - \frac{4\beta_0}{3} \right) \Rightarrow 0 + 0 + \frac{4\beta_1}{3} - 0 = 0$$

$$\frac{4\beta_1}{3} = 0.$$

$$\beta_1 = 0.$$