FTCS Solver

number density

time step

$$\left(\frac{\partial n}{\partial t}\right)^{K} = \frac{n^{K+1} - n^{K}}{\Delta t} + O(1)$$

Diffusion coefficient

source term

$$\frac{n_{ij}^{K+1} - n_{ij}^{K}}{\Delta t} = D\left[\frac{n_{ij}^{K} - 2n_{ij}^{K} + n_{i+1,j}^{K}}{\Delta^2 x} + \frac{n_{i,j-1}^{K} - 2n_{i,j}^{K} + n_{i,j+1}^{K}}{\Delta^2 y}\right] + R_{i,j}^{K}$$

Forward time: forward Euler approximation for first order approx time

Central Space: central difference stencil (2nd order accurate in space)

Unsteady Diffusion Equation

$$\frac{\partial n}{\partial t} = \underbrace{-\nabla\cdot(n\vec{v})}_{\text{advection term}} + \underbrace{\nabla\cdot(D\nabla n)}_{\text{diffusion term}} + \underbrace{R}_{\text{source term}}$$

advection term: Movement caused by flow field (fluid carrying particles)

diffusion term: Spread due to random molecular motion

source term: local creation or destruction

Steady Diffusion Equation

1. $\vec{v} = 0$ no flow: removes advection term
2. $R = 0$ no sources: removes the source term
3. $\nabla D = 0$ constant diffusion coefficient

$$\nabla\cdot(D\nabla n) = D\nabla^2 n$$

4. $\frac{\partial n}{\partial t} = 0$ Steady State: no time dependence

$$0 = D\nabla^2 n$$

$D \neq 0 \quad \therefore \quad \nabla^2 n = 0$ ← identical to steady heat equation

Connection to FTCS

Unsteady diffusion equation

$$\frac{\partial n}{\partial t} = D\nabla^2 n + R$$

$$\frac{n_{i,j}^{K+1} - n_{i,j}^{K}}{\Delta t} = D \ \ldots$$

Heat-Sink Cooling for Pulsed Rocket Firing

$$\frac{dT}{dt} = \alpha \nabla^2 T$$

insulating far side wall

$$\frac{dT}{dt} = 0 \quad \text{(zero-derivative Neumann BC)}$$

hot side boundary condition $(x=0)$

$$h_g\left[T_{og} - T(x=0)\right] = k\frac{dT}{dx}$$

convective heat transfer coefficient

stagnation temperature of hot combustion gasses

thermal conductivity of the wall

$$\alpha = \frac{k}{\rho c_p}$$

density

specific heat capacity of wall material

When not firing

$$\frac{dT}{dx} = 0$$

When firing

h_g, T_{og} constant $\quad$ k, α constant throughout the wall

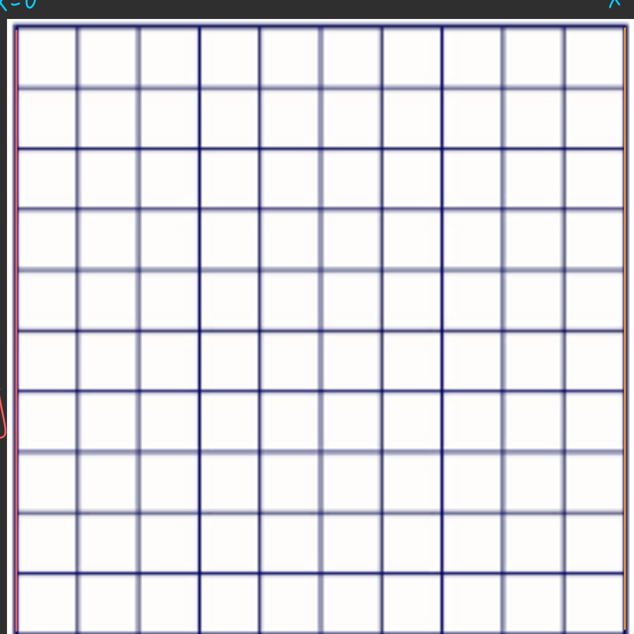
Questions:

- Does this simulation have several frames like a video?
- What dimensions does the 2D plot have, 1cm x 1cm?
- 1D to 2D?

Recipe:

1. Special part is finite difference
2. Time part is forward euler method

$x=0$ $\qquad$ $x = L = 1\,cm = 0.01\,m$



$$\frac{\partial T(t,L)}{\partial x} = 0$$

$y\,(m)$

Thruster on:

$$\frac{\partial T(t,0)}{\partial x} = \frac{h_g}{k}\left[T_{og} - T(t,0)\right]$$

Thruster off:

$$\frac{\partial T(t,0)}{\partial x} = 0$$

$x\,(m)$

Walkthrough

For Laplace label

a

b f d

c

$\Delta x = \Delta y = \Delta h$

- Initial Condition: @ $t=0$ "step 0"
  $T = 300\ K$, thruster on, whole grid same
- Step 1: ($dt = 1e{-}4s$)

Left boundary (if thruster on) | Right boundary

$$T_1 - T_0 = \frac{hg}{K}\left[2500-300\right]$$

$$T_1 = \frac{hg}{K}\left[2200\right] + 300$$

$$T_1 = T_0 = 300$$

Inside (based on node i,j)

$$T_1 = \alpha \cdot dt \cdot \left[\frac{T_{0,b} + T_{0,c} + T_{0,d} + T_{0,a} - 4T_{0,f}}{(\Delta h)^2}\right] + T_0$$

- Step 2: ($dt = 1e{-}4s$)

(i=0) Left boundary (if thruster on)

$$T_2 - T_1 = \frac{hg}{K}\left[2500 - T_{1,x=0}\right]$$

$$T_2 = \frac{hg}{K}\left[2500 - T_{1,x=0}\right] + T_{1,x=0}$$

Right boundary (i=10)

$$T_2 = T_1 = 300$$

Inside (based on node i,j)

$$T_2 = \alpha \cdot dt \cdot \left[\frac{T_{1,b} + T_{1,c} + T_{1,d} + T_{1,a} - 4T_{1,f}}{(\Delta h)^2}\right] + T_1$$

$$\frac{\partial T}{\partial x} = \frac{hg}{K}\left[T_{0g} - T(x=0)\right]$$

$$T_i^{K+1} = T_i^K + \frac{hg}{K} \cdot dx \cdot \left[T_{0g} - \alpha \cdot dt \cdot \left[\frac{-2T_i^K + T_{i+1}^K}{dx^2}\right]\right]$$

$$T_i^{K+1} = T_i^K + \alpha \cdot \Delta t \cdot \left[\frac{T_{i-1}^K - 2T_i^K + T_{i+1}^K}{\Delta^2 x}\right]$$

Pseudo Code Part I Heat Sink Cooling for Pulsed Rocket Firing

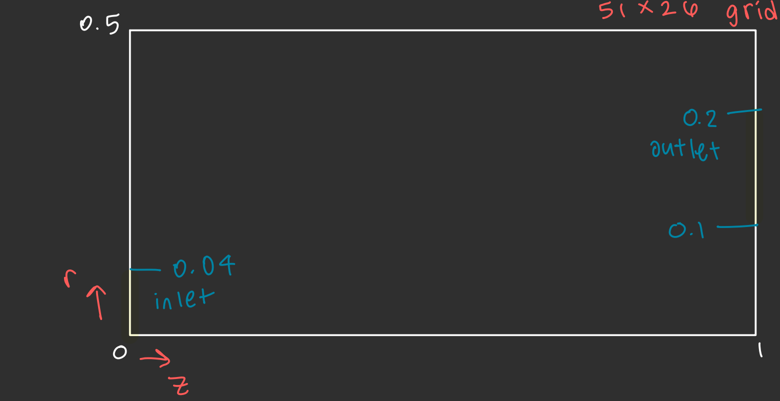
```
// initialization
double  T[k+1][n+1]  ← 2D Matrix initialize w/ 300K
          ↑    ↑
         row  column

for [k=0 to k=15,000], k++ :
    // logic to determine if thruster is on
    ...
    // main loop
    for [i=0 to i=100], i++ :
        // boundary condition handling
        if i==0 :
            if thruster_on:
                dTdx = (hg/K) * (Tog - T[k][i])
                T[k+1][i] = T[k][i] + Δx · dTdx
                continue
            else :
                T[k+1][i] = T[k][i]
                continue

        if i == 100 :
            T[k+1][i] = T[k][i]
            continue
        // interior node handling
        T[k+1][i] = T[k][i] + α · Δt · [ (T[k][i-1] - 2T[k][i] + T[k][i+1]) / Δ²x ]
    end
end
```

Crank-Nicholson Notes

50 x 25 cells
51 x 26 grid points $\Delta z = \Delta r = 0.02$



\# of time steps: 5,000

Δt: 0.01 s

D: 0.1

boundary condition

inlet
$n = 10$

outlet
$n = 0$

Plot @ time steps:

$K = \{100, 400, 1000, 2000, 5000\}$

$$\frac{\partial n}{\partial t} = D \nabla^2 n$$

$$\frac{n_i^{K+1} - n_i^K}{\Delta t} = \frac{1}{2}\left(D \nabla^2 n_i^{K+1} + D \nabla^2 n_i^K\right)$$

$$\Rightarrow n_i^{K+1} = n_i^K + \frac{D\Delta t}{2}\left(\nabla^2 n_i^{K+1} + \nabla^2 n_i^K\right)$$

In Matrix Form

$$\left(I - \frac{D\Delta t}{2} L\right)\vec{n}^{K+1} = \left(I + \frac{D\Delta t}{2} L\right)\vec{n}^K$$

identity matrix

finite differencing matrix

Laplacian finite differencing Matrix L

	a	
b	f	d
	c	

top: if $j \neq nr-1$
$t = j+1$
idx_top = i*nr + t
a = nd[idx_top]
else a = 0

center:
f = nd[idx]

left: if $i \neq 0$
b = nd[idx - 1]
else b = 0

bottom: if $j \neq 0$
$l = j - 1$
idx_bot = i*nr + l
c = nd[idx_bot]
else c = 0

right: if $i \neq nz-1$
d = nd[idx + 1]
else d = 0

$$L[idx] = \frac{1}{r\_val}\left(\frac{a-c}{2dr}\right) + \left[\frac{a-2f+c}{dr^2}\right] + \left[\frac{b-2f+d}{dz^2}\right]$$

logic @ inlet

- (0, 0.04) i = 100
- (0, 0.02) i = 50
- (0, 0) i = 0

all set to 100

Each Interior Node:

$$\frac{\Psi(x-\Delta x) - 2\Psi(x) + \Psi(x+\Delta x)}{(\Delta x)^2} + \frac{\Psi(y-\Delta y) - 2\Psi(y) + \Psi(y+\Delta y)}{(\Delta y)^2} = 0$$