Escola de Economia de São Paulo - Fundação Getulio Vargas

STUDENT NAME:

Course: Econometria 2

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Final Exam 2022 - Tuesday, 07/12, 9:00 am - 12:50 pm - Total = 360 points

Instructions:

This exam lasts 3 hours and 50 minutes. No extra time will be provided and you must hand me your exam before 12:50 pm. I will display a clock using the projector and this clock will be our official time.

While taking the exam, you can eat your own food, drink your own water or any other type of liquid, and go to the restroom. However, you cannot leave the floor to buy food or drinks.

You cannot check any notes, books or electronic devices. You cannot talk to your colleagues.

The exam is worth 360 points distributed in the following way:

- Question 1 - Taylor Rule Estimation by OLS: 220 points
  - Item a: 70 points
  - Item b: 20 points
  - Item c: 130 points
- Question 2 - The Rise of the Taylor Principle: 100 points
  - Item a: 10 points
  - Item b: 30 points
  - Item c: 40 points
  - Item d: 20 points
- Question 3 - Estimating an ARMA(1,2) using GMM: 40 points
  - Item a: 10 points
  - Item b: 10 points

- Item c: 20 points

Each pair question-item has an assigned space for its answer. This assigned space starts immediately below the pair question-item and ends at the bottom of the page before the page with the next pair question-item. I will not grade anything that is not properly located in its correct place.

You can write your answers using any writing material, including quills, pieces of charcoal or a lipstick. I recommend using a pencil, a mechanic pencil or a standard pen. Have in mind that I can only grade what I am actually able to read and understand.

You can write answers in any language that I can fluently understand. Since I am dumb, you can only write in Portuguese or English. If you write in Spanish, I can try to understand, but I cannot promise you anything.

At the end of the exam, there are 10 blank pages. You can use them as you please. You can even write a poem or draw a cartoon. I recommend using them to sketch your own work. Importantly, I will not read nor grade anything in those sketching pages.

At the top of the first page, you must write your FULL NAME in the assigned space. You must write your FULL NAME even if you are Pedro I.

You cannot detach any sheet of paper from this exam. All sheets must be kept attached as they were handed to you.

Question 1 (Taylor Rule Estimation by OLS - 220 points)

In this question, we will understand the results derived by Carvalho et al. (2021). They discuss one of the most important topics in applied macroeconometrics: Taylor Rule Estimation.

Ordinary Least Squares (OLS) estimation of monetary policy rules produces potentially inconsistent estimates of policy parameters. This problem arises because Central Banks react to variables (inflation and output gap) that are impacted by monetary shocks. This phenomenon generates an endogeneity problem known as simultaneity and implies that our OLS estimator is asymptotically biased.

A common way to solve this problem is through the use of instrumental variables (IV). However, IV estimation is challenging too because its validity depends on unobservable characteristics of the economy.

Since both methods have caveats, understanding their biases is relevant. In this question, we will deepen our understanding of the OLS asymptotic bias when our goal is estimating a Taylor Rule.

To do so, we will analyze a simple three-equation New Keynesian model that allows us to derive the asymptotic bias of the OLS estimator when we try to estimate a Taylor Rule using only one equation.

In our model, equilibrium inflation, output and the policy interest rate evolve as functions of technology and monetary shocks. Our models consists of:

1. *a Phillips curve (Equation (1)) connecting inflation,* π_t*, to the current output gap,* $\tilde{y}_t$*, and to the expected inflation,* $\mathbb{E}_t\left[\pi_{t+1}\right]$*.*

2. *a dynamic IS curve (Equation (2)) connecting the output gap to the expected output gap,* $\mathbb{E}_t\left[\tilde{y}_{t+1}\right]$*, and to the gap between the ex-ante real interest rate,* $i_t - \mathbb{E}_t\left[\pi_{t+1}\right]$*, and the natural rate of interest,* r_t^n*.*

3. *a simplified policy rule (Equation (3)) connecting the nominal interest rate,* i_t*, to inflation and a monetary shock,* v_t*.*

The natural interest rate is determined by the dynamics of output in the model's flexible-price equilibrium, which is a function of the technology shock, a_t*. Technology and monetary shocks follow autoregressive processes.*

Formally, we have that

$$\pi_t = \beta \mathbb{E}_t\left[\pi_{t+1}\right] + \kappa \tilde{y}_t \tag{1}$$

$$\tilde{y}_t = \mathbb{E}_t\left[\tilde{y}_{t+1}\right] - \frac{1}{\sigma}\left(i_t - \mathbb{E}_t\left[\pi_{t+1}\right] - r_t^n\right) \tag{2}$$

$$i_t = \phi_\pi \pi_t + v_t, \tag{3}$$

where $a_t = \rho_a a_{t-1} + \epsilon_t^a$*,* $v_t = \rho_v v_{t-1} + \epsilon_t^v$*,* $\epsilon_t^a \perp\!\!\!\perp \epsilon_s^v$ *for any* $(t,s) \in \mathbb{N}^2$*,* $\rho_a \in (0,1)$*,* $\rho_v \in (0,1)$*,* $\sigma > 0$*,* $\kappa > 0$ *and* $\beta \in (0,1)$*.*

The policy parameter of interest is ϕ_π*. Assume that the Taylor Principle holds, i.e.,* $\phi_\pi > 1$*.*

In equilibrium, inflation is given by

$$\pi_t = -\kappa \Lambda_v v_t - \sigma \psi_{ya}^n \left(1-\rho_a\right) \kappa \Lambda_a a_t, \tag{4}$$

where $\Lambda_j = \dfrac{1}{\left(1-\beta\rho_j\right)\sigma\left(1-\rho_j\right) + \kappa\left(\phi_\pi - \rho_j\right)}$ *for* $j \in \{v, a\}$ *and* $\psi_{ya}^n = \dfrac{1-\beta\rho_a}{\left(1-\beta\rho_a\right)\sigma\left(1-\rho_a\right) + \kappa\left(\phi_\pi - \rho_a\right)}$*.*

We denote the OLS estimator of Equation (3) by $\hat{\phi}_\pi$*.*

Question 1.a (70 points): *Assume that the observable variables in Equation (3) are jointly stationary and weakly dependent. Moreover, assume that* $\text{Var}(\pi_t) > 0$. *Let* plim *denote the probability limit of a sequence of random variables. Derive the OLS asymptotic bias,*

$$Bias \coloneqq \text{plim}\left(\hat{\phi}_\pi\right) - \phi_\pi,$$

as a function of the structural parameters in our New Keynesian Model.

Hint: *Carvalho et al. (2021) derive an expression that depends on the fraction of the variance of inflation that is due to monetary policy shocks.*

Question 1.b (20 points): *Is the OLS asymptotic bias positive, zero or negative? Justify your answer mathematically and provide the economic intuition for this result.*

Question 1.c (130 points): *Assume that the monetary shocks are less persistent than technology shocks, i.e.,* $\rho_v < \rho_a$. *What happens to the absolute value of the OLS asymptotic bias when the policy response to inflation* (ϕ_π) *becomes stronger? Does it increase or decrease?*

Answer 1.a:

We have that

$$\begin{aligned}
\text{plim}\,\hat{\phi}_\pi &= \frac{\text{Cov}(i_t,\pi_t)}{\text{Var}(\pi_t)} \\
&\quad \text{because the observable variables are jointly stationary and weakly dependent,} \\
&\quad \text{and } \text{Var}(\pi_t) > 0 \\
&= \frac{\text{Cov}(\phi_\pi \pi_t + v_t, \pi_t)}{\text{Var}(\pi_t)} \\
&\quad \text{because of Equation (3)} \\
&= \phi_\pi + \frac{\text{Cov}(v_t,\pi_t)}{\text{Var}(\pi_t)} \\
&= \phi_\pi + \frac{\text{Cov}\left(v_t, \left\{-\kappa\Lambda_v v_t - \sigma\psi_{ya}^n(1-\rho_a)\kappa\Lambda_a a_t\right\}\right)}{\text{Var}\left(\left\{-\kappa\Lambda_v v_t - \sigma\psi_{ya}^n(1-\rho_a)\kappa\Lambda_a a_t\right\}\right)} \\
&\quad \text{because of Equation (4)} \\
&= \phi_\pi - \frac{\kappa\Lambda_v \text{Var}(v_t)}{(\kappa\Lambda_v)^2\text{Var}(v_t) + \left(\sigma\psi_{ya}^n(1-\rho_a)\kappa\Lambda_a\right)^2\text{Var}(a_t)},
\end{aligned}$$

implying that

$$Bias = \operatorname{plim} \hat{\phi}_\pi - \phi_\pi = -\frac{1}{\kappa \Lambda_v} \gamma_v,$$

where

$$\gamma_v := \frac{(\kappa \Lambda_v)^2 \operatorname{Var}(v_t)}{(\kappa \Lambda_v)^2 \operatorname{Var}(v_t) + \left(\sigma \psi_{ya}^n (1 - \rho_a) \kappa \Lambda_a\right)^2 \operatorname{Var}(a_t).}$$

Note that γ_v is the fraction of the variance of inflation that is due to monetary policy shocks.

Answer 1.b:

Since $\rho_a \in (0,1)$, $\rho_v \in (0,1)$, $\sigma > 0$, $\kappa > 0$ and $\beta \in (0,1)$ and $\phi_\pi > 1$, we know that $\Lambda_a > 0$, $\Lambda_v > 0$ and $\psi_{ya}^n > 0$. Consequently, $Bias = \operatorname{plim} \hat{\phi}_\pi - \phi_\pi = -\frac{1}{\kappa \Lambda_v} \gamma_v < 0$, i.e., the OLS asymptotic bias is negative.

To understand the economic intuition behind this we will analyze a expansionary monetary shock, i.e., a negative innovation to v_t. This shock increases inflation (Equation (4)), leading to an endogenous increase in the interest rate (Equation (3)). Since the policy shock and the endogenous policy response have different signs, the interest rate seems to respond less intensely to movements in π_t, creating a negative asymptotic bias in $\hat{\phi}_\pi$.

Answer 1.c:

Based on our answer to Question 1.a, we know that

$$\begin{aligned} Bias &= -\frac{\kappa \Lambda_v \operatorname{Var}(v_t)}{(\kappa \Lambda_v)^2 \operatorname{Var}(v_t) + \left(\sigma \psi_{ya}^n (1 - \rho_a) \kappa \Lambda_a\right)^2 \operatorname{Var}(a_t)} \\ &= -\frac{f(\phi_\pi)}{g(\phi_\pi)}, \end{aligned}$$

where we define $f\colon (1, +\infty) \to \mathbb{R}$ and $g\colon (1, +\infty) \to \mathbb{R}$ such that, for any $\phi \in (1, +\infty)$,

$$f(\phi) = \kappa \Lambda_v \operatorname{Var}(v_t)$$

and

$$\begin{aligned} g(\phi) &= (\kappa \Lambda_v)^2 \operatorname{Var}(v_t) + \left(\sigma \psi_{ya}^n (1 - \rho_a) \kappa \Lambda_a\right)^2 \operatorname{Var}(a_t) \\ &= (\kappa \Lambda_v)^2 \operatorname{Var}(v_t) + \left((1 - \beta \rho_a) \sigma (1 - \rho_a) \kappa \Lambda_a^2\right)^2 \operatorname{Var}(a_t) \end{aligned}$$

Moreover, note that

$$
\begin{aligned}
\frac{\partial \Lambda_v}{\partial \phi}(\phi_\pi) &= -\frac{\kappa}{\left[\sigma(1-\rho_v)(1-\beta\rho_v)+\kappa(\phi_\pi-\rho_v)\right]^2} = -\kappa\Lambda_v^2 \\
\frac{\partial \Lambda_a}{\partial \phi}(\phi_\pi) &= -\frac{\kappa}{\left[\sigma(1-\rho_a)(1-\beta\rho_a)+\kappa(\phi_\pi-\rho_a)\right]^2} = -\kappa\Lambda_a^2 \\
\frac{\partial f}{\partial \phi}(\phi_\pi) &= \kappa\left(\frac{\partial \Lambda_v}{\partial \phi_\pi}\right)\operatorname{Var}(v_t) = -(\kappa\Lambda_v)^2\operatorname{Var}(v_t) \\
\frac{\partial g}{\partial \phi}(\phi_\pi) &= 2\kappa^2\Lambda_v\operatorname{Var}(v_t)\left(\frac{\partial \Lambda_v}{\partial \phi_\pi}\right) + 4\left((1-\beta\rho_a)\sigma(1-\rho_a)\kappa\right)^2\Lambda_a^3\operatorname{Var}(a_t)\left(\frac{\partial \Lambda_a}{\partial \phi_\pi}\right) \\
&= -2(\kappa\Lambda_v)^3\operatorname{Var}(v_t) - 4\left((1-\beta\rho_a)\sigma(1-\rho_a)\right)^2(\kappa\Lambda_a)^3\operatorname{Var}(a_t)\Lambda_a^2
\end{aligned}
$$

according to the standard rules of Calculus.

Consequently, we have that

$$
\frac{\partial\ \text{Bias}}{\partial \phi}(\phi_\pi) = -\frac{\overbrace{-(\kappa\Lambda_v)^2\operatorname{Var}(v_t)\,g(\phi_\pi) - f(\phi_\pi)\left[-2(\kappa\Lambda_v)^3\operatorname{Var}(v_t) - 4\left((1-\beta\rho_a)\sigma(1-\rho_a)\right)^2(\kappa\Lambda_a)^3\operatorname{Var}(a_t)\Lambda_a^2\right]}^{\mathcal{N}}}{g(\phi_\pi)^2}
$$

according to the standard rules of Calculus.

Since the denominator of the last expression is always positive, the sign of $\frac{\partial\ \text{Bias}}{\partial \phi}(\phi_\pi)$ depends only on the sign of the numerator, $\mathcal{N}$. Observe that

$$
\mathcal{N} = \overbrace{(f(\phi_\pi)(\kappa\Lambda_v))}^{>0}\left[2(\kappa\Lambda_v)^2\operatorname{Var}(v_t) + 4\left((1-\beta\rho_a)\sigma(1-\rho_a)\kappa\Lambda_a^2\right)^2\operatorname{Var}(a_t)\frac{\Lambda_a}{\Lambda_v} - g(\phi_\pi)\right]
$$

$$
\begin{aligned}
&= (f(\phi_\pi)(\kappa\Lambda_v))\begin{bmatrix} 2(\kappa\Lambda_v)^2\operatorname{Var}(v_t) + 2\left((1-\beta\rho_a)\sigma(1-\rho_a)\kappa\Lambda_a^2\right)^2\operatorname{Var}(a_t) \\ -2\left((1-\beta\rho_a)\sigma(1-\rho_a)\kappa\Lambda_a^2\right)^2\operatorname{Var}(a_t) \\ +4\left((1-\beta\rho_a)\sigma(1-\rho_a)\kappa\Lambda_a^2\right)^2\operatorname{Var}(a_t)\frac{\Lambda_a}{\Lambda_v} - g(\phi_\pi) \end{bmatrix} \\
&= (f(\phi_\pi)(\kappa\Lambda_v))\begin{bmatrix} g(\phi_\pi) - 2\left((1-\beta\rho_a)\sigma(1-\rho_a)\kappa\Lambda_a^2\right)^2\operatorname{Var}(a_t) \\ +4\left((1-\beta\rho_a)\sigma(1-\rho_a)\kappa\Lambda_a^2\right)^2\operatorname{Var}(a_t)\frac{\Lambda_a}{\Lambda_v} \end{bmatrix} \\
&= (f(\phi_\pi)(\kappa\Lambda_v))\left[\underbrace{g(\phi_\pi)}_{>0} + \underbrace{2\left((1-\beta\rho_a)\sigma(1-\rho_a)\kappa\Lambda_a^2\right)^2\operatorname{Var}(a_t)}_{>0}\left\{2\frac{\Lambda_a}{\Lambda_v} - 1\right\}\right]
\end{aligned}
$$

Since $\rho_v < \rho_a$, we know that $\frac{\Lambda_a}{\Lambda_v} > 1$, implying that $\frac{\partial \text{ Bias}}{\partial \phi}(\phi_\pi) < 0$ according to the last expression.

We can, then, conclude that the absolute value of the OLS asymptotic bias increases when the policy response to inflation becomes stronger.

To answer the next question, you need to know the following statement derived by Carvalho et al. (2021) based on the results you derived in the last question.

> The key insight produced by this illustrative model is that the OLS bias depends on the fraction of the variance of inflation that is due to monetary shocks. If these shocks explain only a small fraction of variation of inflation and other macroeconomic variables to which central banks respond, then the OLS bias may be small.

Question 2 (The Rise of the Taylor Principle - 100 points)

The Taylor Principle states that an increase of 1 p.p. point in inflation must be matched by the Central Bank with a larger increase in nominal the interest rate. This principle is a generally accepted policy prescription because many economic models imply that this rule generates low and stable inflation.

Goncalves and Guimaraes (2022) uses annual data on interest rates since 1915 for 18 countries to estimate the interest-rate response to inflation and the output gap. The observed countries are Australia, Belgium, Canada, Denmark, Finland, France, Germany, Ireland, Italy, Japan, Netherlands, Norway, Portugal, Spain, Sweden, Switzerland, United Kingdom and United States. For each country, they create 73 30-year moving windows and run one OLS regression for each one of those 30-year moving windows. By doing so, they obtain a historical perspective on the Taylor Principle in developed countries.

Formally, for each country and each 30-year moving window, Goncalves and Guimaraes (2022) posit that nominal interest rates i_t at time t are determined by

$$i_t = a + \beta \pi_t + \gamma y_t + \rho i_{t-1} + \eta_t \tag{5}$$

where π_t is the inflation rate at time period t and y_t is the output gap. The long-run interest-rate response to inflation is then given by

$$\phi := \frac{\beta}{1-\rho}.$$

The Taylor Principle states that $\phi > 1$.

Goncalves and Guimaraes (2022) estimate Equation (5) using OLS.

Question 2.a (10 points): *Knowing that only 5% of the variance of inflation is due to monetary shocks in a typical quantitative macroeconomic model, do you believe that the OLS asymptotic bias associated with β in Equation (5) is large or small? Justify your answer.*

Question 2.b (30 points): *Figure 1 shows the median, the maximum and the minimum long-term response of interest-rate to inflation across countries for each 30-year moving window.*[1] *The x-axis shows the last year of each 30-year window. To calculate those moments, we discard data points with standard errors larger than 2 or estimated ρ larger than 0.99 (those have little effect on the median but impact the maximum and minimum estimates).*

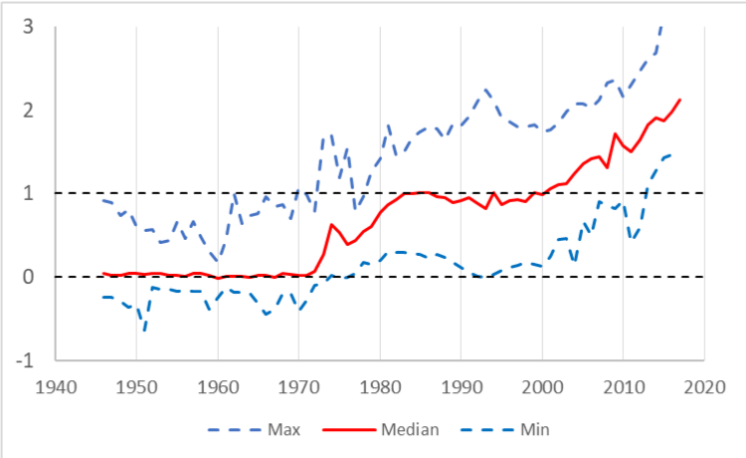

Figure 1: Long-term response of interest rates to inflation in 30-year windows ending at the year in the x-axis. In order to calculate the median, minimum and maximum interest-rate response for every time window, we discard data points with standard errors larger than 2 or estimates of ρ larger than 0.99.

Figure 1: Figure 1 by Goncalves and Guimaraes (2022)

Why do the authors discard data points whose estimated ρ is larger than 0.99? Instead of using the 0.99 threshold, is there an alternative way to choose which data points to include?

Question 2.c (40 points): *Interpret the results in Figure 1.*

Question 2.d (20 points): *Several economic models predict a negative relation between the long-run interest-rate response to inflation (ϕ) and inflation volatility $\left(\sqrt{\text{Var}(\pi_t)}\right)$. Goncalves and Guimaraes (2022) also investigate the relationship between these two variables.*

They calculate the standard deviation of inflation in 30-year moving windows, from 1915-1945 to 1987-

[1]Formally, Figure 1 reports the median, maximum and the minimum of $\hat{\phi}_{i,\tau} = \dfrac{\hat{\beta}_{i,\tau}}{1-\hat{\rho}_{i,\tau}}$ where i indexes the country and τ indexes the 30-year moving windows.

2017, for the 18 countries in their sample. They then run the following regression:

$$V_{i,\tau} = \alpha + \delta\hat{\phi}_{i,\tau} + \mu_i + \mu_\tau + \epsilon_{i,\tau}, \tag{6}$$

where i denotes countries and τ denotes 30-year moving windows, $V_{i,\tau}$ denotes inflation volatility for country i in moving window τ, $\hat{\phi}_{i,\tau}$ is the estimated long-run interest-rate response to inflation based on the OLS estimation of the coefficients in Equation (5), μ_i are country fixed effects and μ_τ are moving-windows fixed effects.

Goncalves and Guimaraes (2022) estimate Equation (6) using the whole sample and only the post-war period. Figure 2 report their post-war results.

	(4)
	post-1945
ϕ	-0.36**
	(0.142)
	0.01
Constant	2.27***
	(0.359)
	0.00
Fixed-Effects	Y
Time-Effects	Y
Observations	691
R-squared	0.692
# countries	18

Table 1: Dependent variable: standard deviation of inflation in each 30-year window; ϕ: estimated response of nominal interest rates to inflation in each 30-year window;
Estimates of ϕ with standard errors larger than 2 or ρ larger than 0.999 are excluded from the sample.
Bootstrapped standard errors in parentheses, p-values below.
*** p<0.01, ** p<0.05, * p<0.1.

Figure 2: Table 1 by Goncalves and Guimaraes (2022)

Interpret the results in Figure 2.

Answer 2.a:

Carvalho et al. (2021) argue that the OLS asymptotic bias associated with β in Equation (5) is proportional to the fraction of the variance of inflation that is due to monetary shocks. Since only 5% of the variance of inflation is due to monetary shocks in a typical quantitative macroeconomic model, the OLS asymptotic bias associated with β in Equation (5) is likely to be small.

Answer 2.b:

Equation (5) is an augmented distributed lag model. To consistently estimate this model's coefficients using OLS, the random variables in the model must be stationary, requiring that $|\rho| < 1$. To avoid using non-stationary random variables, the authors impose the threshold that $\hat{\rho} < 0.99$.

An alternative way to avoid the use of non-stationary random variables is to test for unit roots in each one of the model's variables. To do so, we run an Augmented Dickey-Fuller test for each one of the model's variable and test the null hypothesis that there is a unit root. To choose the type of our Augmented Dickey-Fuller test (i.e., whether to include drift and trend terms), we need to plot our time series.

Answer 2.c:

Goncalves and Guimaraes (2022) interpret the results in the following way.

> The median response of interest rates to inflation is remarkably close to zero for every window between 1915-1945 and 1942-1972. Moreover, all well-estimated coefficients are below 1 in that period. This is consistent with a system with fixed exchange rates where countries have little say on monetary policy.
>
> Between 1943-1973 and 1951-1981, the median long-run response to inflation quickly rises. Then, from 1952-1982 to 1971-2001, the median oscillates around 1, corresponding to real interest rates insensitive to inflation. However, there is a great deal of heterogeneity in this period even though no well-estimated negative coefficient is found. From then on, the median coefficient is clearly above one, and since 1983-2013, all countries in the sample seem to abide by the Taylor Principle.

Answer 2.d:

Goncalves and Guimaraes (2022) interpret the results in the following way.

> An increase in the interest-rate response to inflation from 0 to 1 reduces the standard deviation of inflation by around 0.36 percentage points.
>
> [...]
>
> Overall, the results show that a stronger monetary policy response to inflation is associated with more stable inflation rates, even when controlling for country fixed effects and time effects. [...] Following the Taylor Principle has led to lower inflation volatility.

Question 3 (Estimating an ARMA(1,2) using GMM - 40 points)

ARMA(p,q) models are usually estimated by Maximum Likelihood. However, we can also estimate them by GMM.

Let $\{Y_t\}$ *be a stationary stochastic process such that*

$$Y_t = \phi_1 \cdot Y_{t-1} + U_t \tag{7}$$

and

$$U_t = \epsilon_t + \theta_1 \cdot \epsilon_{t-1} + \theta_2 \cdot \epsilon_{t-2}, \tag{8}$$

where $\phi_1, \theta_1, \theta_2 > 0$ *and* $\epsilon_t \sim i.i.d. N(0,1)$.

Question 3.a (10 points): *Show that you cannot consistently estimate Equation (7) using OLS.*

Question 3.b (10 points): *Show that* Y_{t-2} *is not a valid instrument for* Y_{t-1} *in Equation (7).*

Question 3.c (20 points): *Show that* Y_{t-3} *is a valid instrument for* Y_{t-1} *in Equation (7).*

Answer 3.a:

We must show that the assumption behind OLS estimation do not hold. In particular, we will show that $\mathbb{E}[Y_{t-1} \cdot U_t] \neq 0$.

$$\begin{aligned}
\mathbb{E}[Y_{t-1} \cdot U_t] &= \mathbb{E}[(\phi_1 \cdot Y_{t-2} + U_{t-1}) \cdot U_t] \\
&\quad \text{according to Equation (7)} \\
&= \mathbb{E}\left[\left(\phi_1^2 \cdot Y_{t-3} + \phi_1 \cdot U_{t-2} + U_{t-1}\right) \cdot U_t\right] \\
&\quad \text{according to Equation (7)} \\
&= \mathbb{E}\left[\begin{array}{c} \left(\phi_1^2 \cdot Y_{t-3} + \phi_1 \cdot \{\epsilon_{t-2} + \theta_1 \cdot \epsilon_{t-3} + \theta_2 \cdot \epsilon_{t-4}\} + \{\epsilon_{t-1} + \theta_1 \cdot \epsilon_{t-2} + \theta_2 \cdot \epsilon_{t-3}\}\right) \\ \cdot (\epsilon_t + \theta_1 \cdot \epsilon_{t-1} + \theta_2 \cdot \epsilon_{t-2}) \end{array}\right] \\
&\quad \text{according to Equation (8)} \\
&= \theta_2 \cdot (\phi_1 + \theta_1) \cdot \mathbb{E}\left[\epsilon_{t-2}^2\right] + \theta_1 \cdot \mathbb{E}\left[\epsilon_{t-1}^2\right] \\
&\quad \text{because } \epsilon_t \text{ is independently distributed} \\
&= \theta_2 \cdot (\phi_1 + \theta_1) + \theta_1 \\
&\quad \text{because } \epsilon_t \text{ is identically distributed distributed with variance 1}
\end{aligned}$$

$\neq 0$

because $\phi_1, \theta_1, \theta_2 > 0$.

Answer 3.b:

We must show that the assumption behind IV estimation do not hold when we use Y_{t-1} to instrument for Y_{t-1} in Equation 7. In particular, we will show that $\mathbb{E}\left[Y_{t-2} \cdot U_t\right] \neq 0$.

$$\begin{aligned}\mathbb{E}\left[Y_{t-2} \cdot U_t\right] &= \mathbb{E}\left[(\phi_1 \cdot Y_{t-3} + \epsilon_{t-2} + \theta_1 \cdot \epsilon_{t-3} + \theta_2 \cdot \epsilon_{t-4}) \cdot (\epsilon_t + \theta_1 \cdot \epsilon_{t-1} + \theta_2 \cdot \epsilon_{t-2})\right] \\ &= \theta_2 \neq 0.\end{aligned}$$

Answer 3.c:

We must show that $\mathbb{E}\left[Y_{t-1} \cdot Y_{t-3}\right] \neq 0$ and $\mathbb{E}\left[Y_{t-3} \cdot U_t\right] = 0$.

Note that $\mathbb{E}\left[Y_{t-3} \cdot U_t\right] = 0$ because Y_{t-3} depends only on terms $\{\epsilon_{t-j}\}_{j=3}^{+\infty}$ while U_t depends only on terms ϵ_t, ϵ_{t-1} and ϵ_{t-2}.

Now, we will show that $\mathbb{E}\left[Y_{t-1} \cdot Y_{t-3}\right] \neq 0$.

$$\begin{aligned}\mathbb{E}\left[Y_{t-1} \cdot Y_{t-3}\right] &= \mathbb{E}\left[(\phi_1 \cdot Y_{t-2} + U_{t-1}) \cdot Y_{t-3}\right] \\ &= \mathbb{E}\left[\left(\phi_1^2 \cdot Y_{t-3} + \phi_1 \cdot U_{t-2} + U_{t-1}\right) \cdot Y_{t-3}\right] \\ &= \phi_1^2 \cdot \mathbb{E}\left[Y_{t-3}^2\right] + \mathbb{E}\left[(\phi_1 \cdot U_{t-2} + U_{t-1}) \cdot Y_{t-3}\right] \\ &= \phi_1^2 \cdot \sigma_Y^2 + \mathbb{E}\left[(\phi_1 \cdot U_{t-2} + U_{t-1}) \cdot Y_{t-3}\right] \\ &\qquad \text{because } \{Y_t\} \text{ is a stationary process} \\ &= \phi_1^2 \cdot \sigma_Y^2 + \mathbb{E}\left[(\phi_1 \cdot U_{t-2} + U_{t-1}) \cdot (\phi_1 \cdot Y_{t-4} + U_{t-3})\right] \\ &= \phi_1^2 \cdot \sigma_Y^2 + \mathbb{E}\left[(\phi_1 \cdot U_{t-2} + U_{t-1}) \cdot \left(\phi_1^2 \cdot Y_{t-5} + \phi_1 \cdot U_{t-4} + U_{t-3}\right)\right] \\ &= \phi_1^2 \cdot \sigma_Y^2 + \mathbb{E}\left[(\phi_1 \cdot U_{t-2} + U_{t-1}) \cdot (\phi_1 \cdot U_{t-4} + U_{t-3})\right] \\ &= \phi_1^2 \cdot \sigma_Y^2 + \mathbb{E}\left[\begin{array}{c}(\phi_1 \cdot \{\epsilon_{t-2} + \theta_1 \cdot \epsilon_{t-3} + \theta_2 \cdot \epsilon_{t-4}\} + \{\epsilon_{t-1} + \theta_1 \cdot \epsilon_{t-2} + \theta_2 \cdot \epsilon_{t-3}\}) \\ \cdot (\phi_1 \cdot \{\epsilon_{t-4} + \theta_1 \cdot \epsilon_{t-5} + \theta_2 \cdot \epsilon_{t-6}\} + \{\epsilon_{t-3} + \theta_1 \cdot \epsilon_{t-4} + \theta_2 \cdot \epsilon_{t-5}\})\end{array}\right] \\ &= \phi_1^2 \cdot \sigma_Y^2 + \phi_1 \cdot \theta_1 + \theta_2 + \phi_1 \cdot \theta_2 \cdot (\phi_1 + \theta_1) \\ &> 0.\end{aligned}$$

References

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