Lecture 2A: Introducing Nonstationarity

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Econometrics 2

Administrative

- Recommended Reading: Hamilton - Chapter 15
- Optional Reading (Structural Breaks): *Introduction to Econometrics with R - Section 14.8*
- Problem Set 2 - Deadline: May 23rd at 9:00 am

Outline

Motivation

Motivation

Last lecture: Stationary ARMA(p,q)

$$Y_t = c + \phi_1 \cdot Y_{t-1} + \phi_2 \cdot Y_{t-2} + \ldots + \phi_p \cdot Y_{t-p} + \epsilon_t$$
$$+ \theta_1 \cdot \epsilon_{t-1} + \theta_2 \cdot \epsilon_{t-2} + \ldots + \theta_q \cdot \epsilon_{t-q},$$

where $\{\epsilon_t\}$ is white noise and c, $\phi_1, \phi_2, \ldots, \phi_p, \theta_1, \theta_2, \ldots, \theta_q$ are constants.

We impose that the roots of $\left(1 - \phi_1 \cdot z - \phi_2 \cdot z^2 - \ldots - \phi_p \cdot z^p\right)$ lie outside the unit circle.

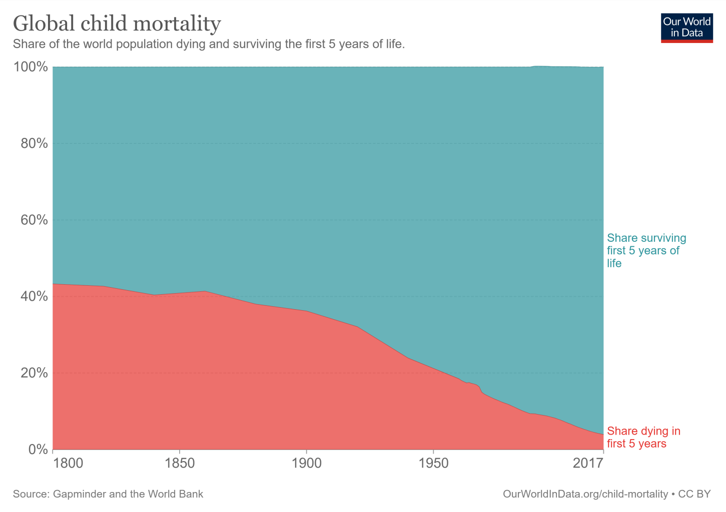
- Expectation: $\mu = \dfrac{c}{1 - \phi_1 - \phi_2 - \ldots - \phi_p}$

Motivation

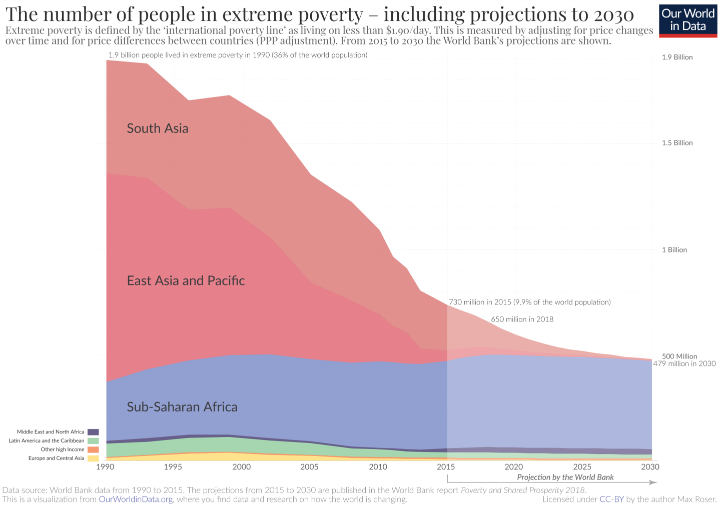
Expected value does not depend on time.

- Is this a positive feature of our model?
- Is it too restrictive?
- Is this model enough for most of the relevant time series?
- What other features would we like to capture?

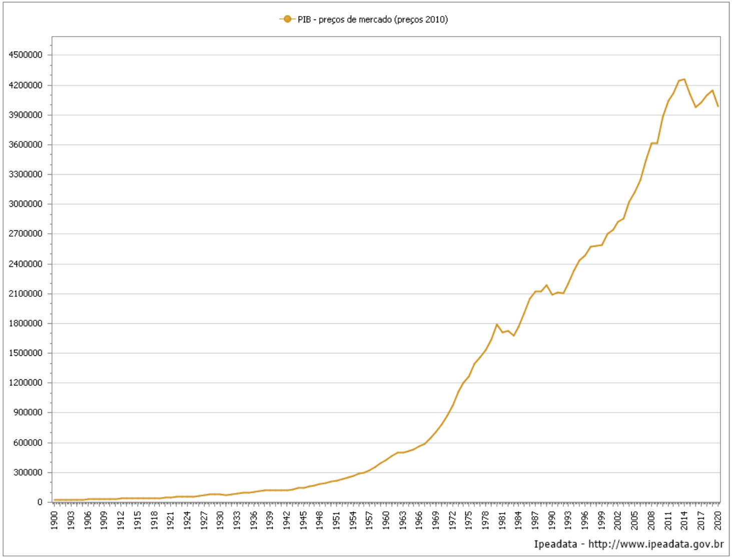
Child Mortality - Source: *Our World in Data*



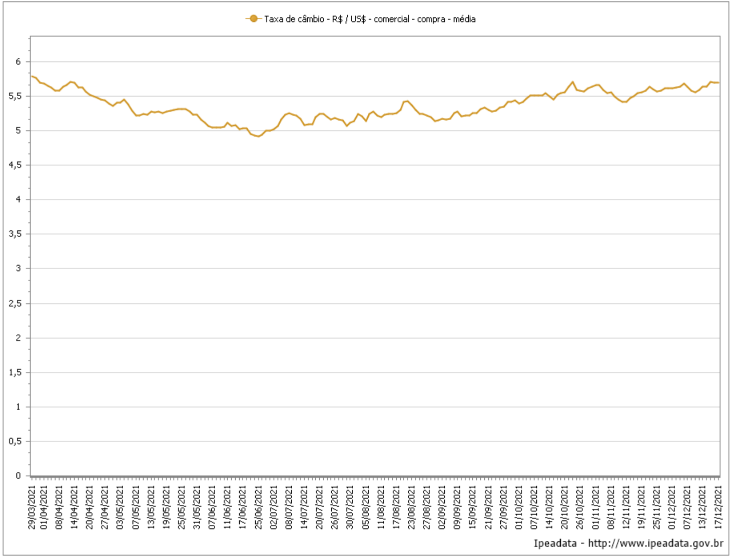
Extreme Poverty - Source: *Our World in Data*



Brazilian GDP - Market Prices (Reais of 2010 [Millions]) - Source: Ipeadata



Exchange Rate: Real v. Dollar - Source: Ipeadata



Solutions: Deterministic Time Trends and Unit Root Processes

Outline

Solutions: Deterministic Time Trends and Unit Root Processes

Solution 1: Deterministic Time Trend Model

A stochastic process $\{Y_t\}$ is a deterministic time trend model if

$$Y_t = \alpha + \delta \cdot t + \epsilon_t + \theta_1 \cdot \epsilon_{t-1} + \theta_2 \cdot \epsilon_{t-2} + \ldots + \theta_q \cdot \epsilon_{t-q},$$

where $\{\epsilon_t\}$ is white noise and $\alpha, \delta, \theta_1, \theta_2, \ldots, \theta_q$ are constants.

This type of process is also known as **trend-stationary**.

Solutions: Deterministic Time Trends and Unit Root Processes

Solution 2: Unit Root Process

A stochastic process $\{Y_t\}$ is a unit root process if

$$\Delta Y_t := Y_t - Y_{t-1} = \delta + \epsilon_t + \theta_1 \cdot \epsilon_{t-1} + \theta_2 \cdot \epsilon_{t-2} + \ldots + \theta_q \cdot \epsilon_{t-q},$$

where $\{\epsilon_t\}$ is white noise and $\delta, \theta_1, \theta_2, \ldots, \theta_q$ are constants such that $1 + \theta_1 + \theta_2 + \ldots + \theta_q \neq 0$.

The last restriction rules out that the original time series is stationary.

This type of process is also known as **difference-stationary**. Moreover, it is known as **integrated of order 1** and denoted by I(1).

Solutions: Deterministic Time Trends and Unit Root Processes

Example of a I(1) process: Random Walk with Drift

$$Y_t = Y_{t-1} + \delta + \epsilon_t$$

Example of a more complicated I(1) process: ARIMA(1,1,1)

$$Y_t = (1 + \phi) \cdot Y_{t-1} - \phi \cdot Y_{t-2} + \epsilon_t + \theta \cdot \epsilon_{t-1},$$

where $|\phi| < 1$.

Definition: Autoregressive Integrated Moving Average Process - ARIMA(p,d,q)

A stochastic process $\{Y_t\}$ is an ARIMA(p,d,q) process if taking d-th differences produces a stationary ARMA(p,q).

Example of a more complicated I(1) process: ARIMA(1,1,1)

$$Y_t = (1+\phi) \cdot Y_{t-1} - \phi \cdot Y_{t-2} + \epsilon_t + \theta \cdot \epsilon_{t-1}$$

with $|\phi| < 1$ implies that

$$Y_t - Y_{t-1} = \phi \cdot (Y_{t-1} - Y_{t-2}) + \epsilon_t + \theta \cdot \epsilon_{t-1}$$

is an ARMA(1,1).

Solutions: Deterministic Time Trends and Unit Root Processes

Some models have a higher order of integration. The second most common one is a I(2) process (integrated of order 2):

$$Y_t = 2 \cdot Y_{t-1} - Y_{t-2} + \delta + \epsilon_t,$$

implying that

$$\Delta Y_t = \Delta Y_{t-1} + \delta + \epsilon_t$$

and

$$\Delta^2 Y_t := \Delta \left(\Delta Y_t \right) = \delta + \epsilon_t$$

This process is also denoted by ARIMA(0,2,0).

Comparing Solutions

Outline

Comparing Solutions

Comparing Forecasts

Comparing Forecasts: Deterministic Trend Model

Let $\{Y_t\}$ be a deterministic time trend process:

$$Y_t = \alpha + \delta \cdot t + \epsilon_t,$$

where $\{\epsilon_t\}$ is white noise and α, δ are constants.

Our forecast s periods ahead is given by

$$\mathbb{E}\left[\, Y_{t+s} \middle| \, Y_t \right] = \alpha + \delta \cdot (t + s)$$

Comparing Forecasts: Unit Root Process

Comparing Forecasts: Unit Root Process

Let $\{Y_t\}$ be a Random Walk with Drift:

$$Y_t = Y_{t-1} + \delta + \epsilon_t,$$

where $\{\epsilon_t\}$ is white noise and δ is a constant.

Our forecast s periods ahead is given by

$$\begin{aligned}
\mathbb{E}\left[\, Y_{t+s} \middle|\, Y_t = y_t\right] &= \mathbb{E}\left[\, Y_{t+s-1} + \delta + \epsilon_{t+s} \middle|\, Y_t = y_t\right] \\
&= \delta + \mathbb{E}\left[\, Y_{t+s-1} \middle|\, Y_t = y_t\right] \\
&\vdots \\
&= \delta \cdot s + y_t
\end{aligned}$$

Comparing Forecasts

Comparing Forecasts

Deterministic time trend model: Forecast's intercept does not depend on y_t

Random Walk with Drift: Forecast's intercept depends on y_t

- Unit root processes have infinite memory. One shock in t stays forever through a change in the intercept.
- For more details on this issue, check Hamilton's discussion about dynamic multipliers (page 442).

Comparing Forecasts

Why should I care about this discussion about intercepts?

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- If GDP follows a deterministic time trend process, then recessions represent temporary downturns with the lost output eventually made up during recovery.

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- If GDP follows a deterministic time trend process, then recessions represent temporary downturns with the lost output eventually made up during recovery.
- If GDP follows a Random Walk with Drift, then recessions have a permanent impact on the level of future GDP. Society will be poorer forever.

Comparing Solutions

Comparing Forecast Errors

Comparing Forecast Errors: Deterministic Trend Model

Let $\{Y_t\}$ be a deterministic time trend process: $Y_t = \alpha + \delta \cdot t + \epsilon_t$, where $\{\epsilon_t\}$ is white noise and α, δ are constants.

Forecast s periods ahead: $\mathbb{E}\left[Y_{t+s} \mid Y_t\right] = \alpha + \delta \cdot (t + s)$

Mean Squared Error:

$$\begin{aligned} MSE &= \mathbb{E}\left[\left(Y_{t+s} - \mathbb{E}\left[Y_{t+s} \mid Y_t\right]\right)^2\right] \\ &= \mathbb{E}\left[\left(\alpha + \delta \cdot (t+s) + \epsilon_{t+s} - \alpha - \delta \cdot (t+s)\right)^2\right] \\ &= \sigma^2 \end{aligned}$$

Comparing Forecast Errors: Unit Root Process

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Let $\{Y_t\}$ be a Random Walk with Drift: $Y_t = Y_{t-1} + \delta + \epsilon_t$, where $\{\epsilon_t\}$ is white noise and δ is a constant.

Forecast s periods ahead: $\mathbb{E}[Y_{t+s} | Y_t] = \delta \cdot s + Y_t$

Mean Squared Error:

$$
\begin{aligned}
MSE &= \mathbb{E}\left[(Y_{t+s} - \mathbb{E}[Y_{t+s} | Y_t])^2\right] \\
&= \mathbb{E}\left[(Y_{t+s-1} + \delta + \epsilon_{t+s} - \delta \cdot s - Y_t)^2\right] \\
&\vdots \\
&= \mathbb{E}\left[(\epsilon_{t+s} + \epsilon_{t+s-1} + \ldots + \epsilon_{t+1})^2\right] \\
&= s \cdot \sigma^2
\end{aligned}
$$

Comparing Forecast Errors

Comparing Forecast Errors

Deterministic time trend model: Forecast's MSE does <u>not</u> depend on s.

$\Rightarrow$ MSE reaches a finite bound as the forecast horizon becomes larger.

Random Walk with Drift: Forecast's MSE depends on s.

$\Rightarrow$ MSE grows to infinity when the forecast horizon becomes larger.

- (Assume $\delta > 0$) Since the standard deviation of the forecast error grows at rate $\sqrt{s}$ while the forecast itself grows at rate s, the data from a Random Walk with Drift are certain to exhibit an upward trend if observed for a sufficiently long period.
  - Important for testing (Lecture 2C): The trend introduced by the drift asymptotically dominates the increasing variability introduced by the unit root component.

Alternative Approaches

Outline

Fractional Integration

Fractionally Integrated Processes parsimoniously capture long-run multipliers that decay very slowly.

- These models may work better than large-order ARMA models to explain long-memory (but not infinite memory) series.
- See Hamilton's Section 15.5.

Structural Breaks

Structural Breaks

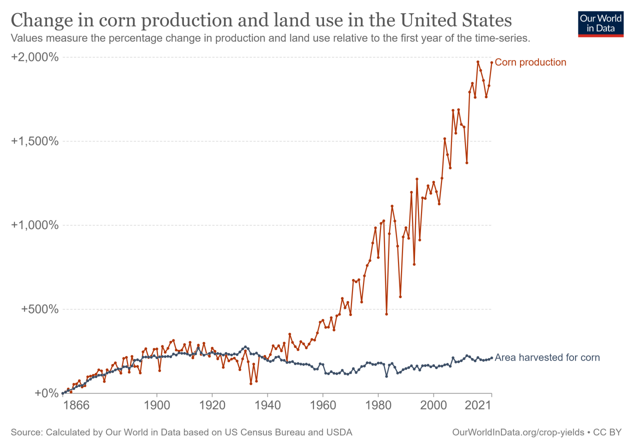
A trend-stationary process $\{Y_t\}$ presents a structural break if its intercept changes at some point in time:

$$Y_t = \begin{cases} \alpha_1 + \delta \cdot t + \epsilon_t & \text{for } t < T_0 \\ \alpha_2 + \delta \cdot t + \epsilon_t & \text{for } t \geq T_0 \end{cases}$$

Important for testing (Lecture 2C): This series would appear to exhibit a unit root based on the usual tests.

Structural Breaks

Structural breaks can happen in any coefficient in your model. For example,



Source: Our World in Data

Structural Breaks

For more information, see Section 14.8 of *Introduction to Econometrics with R*.

Structural Breaks are also useful with cross-sectional data:

- Song [2021] estimates zoning restrictions based on a structural break detection algorithm.
- Wage determination processes for part-time and full-time workers: *Blogpost*

Thank you!

Contact Information:

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References

J. Song. The Effects of Residential Zoning in U.S. Housing Markets. Available at `https://jaeheesong.com/research`., Nov. 2021.