Lecture 5A: Cointegration — Using OLS

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EESP-FGV

Econometrics 2

Administrative

- Recommended Reading: Hamilton's Chapters 19.1 and 19.2
- Problem Set 4 - Deadline: June 13th at 9:00 am

Outline

Motivation

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Cointegration Models are useful to understand the relationship between many economic variables and test economic theories.

- Marginal Propensity to Consume: Consumption and Income
- Purchasing Power Parity: Price indexes in different countries and exchange rate
- Law of One Price
- Commodities' Spot and Future Prices

Definitions and Examples

Outline

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- a is called a cointegrating vector.
- If $n > 2$, then there may be more than one linearly independent cointegrating vector.

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In this case, we say that there are exactly h cointegrating relations among the elements of $\{Y_t\}$ and that $(a_1, \ldots, a_h)$ form a basis for the space of cointegrating vectors.

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- We normalize the first element in a_1 to be 1.

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Claim: The following system is a cointegrated vector process:

$$Y_{1,t} = \gamma \cdot Y_{2,t} + U_{1,t}$$
$$Y_{2,t} = Y_{2,t-1} + U_{2,t}$$

with $U_{1,t}$ and $U_{2,t}$ uncorrelated white noise processes.

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   $Y_{1,t} = Y_{1,t-1} - U_{1,t-1} + \gamma \cdot U_{2,t} + U_{1,t}$
   Hence, $\{Y_{1,t}\}$ is an $I(1)$ process too.
3. $a' := (1, -\gamma)$ is a cointegrating vector because $Y_{1,t} - \gamma Y_{2,t} = U_{1,t}$ is an $I(0)$ process.

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- there is some long-run equilibrium relation tying the individual components together.
- This long-run equilibrium is represented by the linear combination $a'Y_t$.

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- Over the long-run, consumption tends to be a roughly constant proportion of income.
- The difference between the log of consumption and the log of income appears to be a stationary process.

Empirical Example 2

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- A weaker version of this hypothesis imposes that $Z_t := p_t - s_t - p_t^*$ is stationary.

Empirical Example 3

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Zero Lower Bound (ZLM): Duffy et al. [2022]

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- IS Curve + Phillips Curve + Taylor Rule
- Interest rate has to be positive.
- Unconventional monetary policies when the economy reaches the ZLM
- Non-linear Cointegration between interest rate, unconventional policy and inflation.

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    - Intuitively, the level of $Y_{j,t}$ contains information that is useful for forecasting $Y_{j',t}$ beyond that contained in a finite number of lagged changes in $Y_{j,t}$ alone.

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$$Y_t = \alpha + \Phi_1 \cdot Y_{t-1} + \Phi_2 \cdot Y_{t-2} + \ldots + \Phi_p \cdot Y_{t-p} + \epsilon_t.$$

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$$Y_t = \zeta_1 \cdot \Delta Y_{t-1} + \zeta_2 \cdot \Delta Y_{t-2} + \ldots + \zeta_{p-1} \cdot \Delta Y_{t-p+1} + \alpha + \rho \cdot Y_{t-1} + \epsilon_t,$$

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where

- $\rho := \Phi_1 + \Phi_2 + \ldots + \Phi_p$
- $\zeta_s := -\left[\Phi_{s+1} + \Phi_{s+2} + \ldots + \Phi_p\right]$ for any $s \in \{1, 2, \ldots, p-1\}$ Mathematical Details

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Observation: It looks like an Augmented Dickey-Fuller test for vectors!

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Unproven Lemma (See Hamilton's page 579 for a proof.)

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There exists a $(n \times h)$ matrix B such that $\zeta_0 = -BA'$, where A is a basis for the cointegrating vector space.

We can rewrite our model as

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Define the stationary $(h \times 1)$ vector $Z_t := A'Y_t$. Then, we have that

$$\Delta Y_t = \zeta_1 \cdot \Delta Y_{t-1} + \zeta_2 \cdot \Delta Y_{t-2} + \ldots + \zeta_{p-1} \cdot \Delta Y_{t-p+1} + \alpha - BZ_{t-1} + \epsilon_t.$$

The last equation is known as the vector error-correction representation of the cointegrated system.

Testing the Null Hypothesis of No Cointegration

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In those cases, we should use this information to directly test for cointegration.

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3. Test whether $\{Z_t\}$ is an $I(1)$ process.

Testing for Cointegration when the Cointegrating Vector is Known

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- The purchasing power parity implies that $a = (1, -1, -1)'$.

In those cases, we should use this information to directly test for cointegration.

1. Test whether each of the elements of $\{Y_t\}$ is individually $I(1)$.
2. If the null hypothesis of a unit root in each series is individually not rejected, construct the scalar $Z_t := a' Y_t$.
3. Test whether $\{Z_t\}$ is an $I(1)$ process.
4. If the null hypothesis that $\{Z_t\}$ is $I(1)$ is rejected, then a is a cointegrating vector.

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4. If the null hypothesis that $\{Z_t\}$ is $I(1)$ is rejected, then a is a cointegrating vector.
5. If the null hypothesis that $\{Z_t\}$ is $I(1)$ is not rejected, then a is not a cointegrating vector.

Testing for Cointegration when the Cointegrating Vector is Known

We will implement this test in Question 1 in Problem Set 4.

In this question, we will test the purchasing power parity between Brazil and the United States.

Testing the Null Hypothesis of No Cointegration

What to do when the cointegrating vector is unknown?

Outline

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1. Motivation

2. Definitions and Examples

3. Testing the Null Hypothesis of No Cointegration

3.1 Testing for Cointegration when the Cointegrating Vector is Known

3.2 What to do when the cointegrating vector is unknown?

Estimating the Cointegration Vector by OLS

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This approach will give you consistent estimates.

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In those cases, we can try to estimate the cointegrating vector a by OLS and, then, use this estimate in our cointegration test.

This approach will give you consistent estimates. But we will learn alternative methods to estimate the cointegrating vector a.

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- The OLS approach may be best if your goal is estimating IRFs. You will impose exclusion (or ordering) assumptions anyway.
- The alternative approach may be better if your focus is on cointegration by itself (e.g., testing a theory).

Estimating the Cointegration Vector by OLS

Step-by-step procedure:

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1. Assuming that the cointegrating vector has a nonzero coefficient for the first element of Y_t ($a_1 \neq 0$), normalize $a_1 = 1$ and represent the subsequent entries as the negatives of a set of unknown parameters:

$$(a_1, a_2, \ldots, a_n)' = (1, -\gamma_2, \ldots, -\gamma_n)' .$$

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$$(a_1, a_2, \ldots, a_n)' = (1, -\gamma_2, \ldots, -\gamma_n)' .$$

2. Consistent estimates of $\gamma_2, \ldots, \gamma_n$ are obtained by estimating the following model by OLS:

$$Y_{1,t} = \alpha + \gamma_2 \cdot Y_{2,t} + \ldots + \gamma_n \cdot Y_{n,t} + U_t. \tag{1}$$

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The previous procedure consistently estimate the cointegrating vector whose residuals are uncorrelated with any $I(1)$ linear combination of $Y_{2,t}, \ldots, Y_{n,t}$.

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Our null hypothesis is that there is no cointegrating relation among our variables Y_t.

We will estimate the OLS regression (1) and use its residuals to test for our null hypothesis of no cointegration.

Testing for Cointegration when the Cointegrating Vector is Unknown

Under the null of no cointegration, Equation (1) is a regression of an $I(1)$ variable on a set of $I(1)$ variables for which no coefficients produce an $I(0)$ error term.

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1. We estimate Equation (1) and save the estimated residuals $\hat{U}_t$.
2. We test whether $\hat{U}_t$ is a $I(1)$ process using the Augmented Dickey-Fuller test.

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2. We test whether $\hat{U}_t$ is a $I(1)$ process using the Augmented Dickey-Fuller test.
3. The critical values of this test are different from the standard critical values of the Augmented Dickey-Fuller test.
4. The null of our test is that $\hat{U}_t$ is a $I(1)$ process, i.e., there is no cointegrating relation among our variables.

Testing for Cointegration when the Cointegrating Vector is Unknown

The Phillips-Ouliaris-Hansen test uses the Augumented Dickey-Fuller test with no constant term,

$$\hat{U}_t = \rho \cdot \hat{U}_{t-1} + \sum_{i=2}^{p} \zeta_{i-1} \cdot \Delta \hat{U}_{t-i+1} + V_t,$$

and tests the null hypothesis that

$$H_0 : \rho = 1.$$

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Critical values for this test depend on whether the original stochastic processes $\{Y_t\}$ had a time trend or not.

Phillips-Ouliaris-Hansen Critical Values: No Time Trend

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TABLE B.9
Critical Values for the Phillips Z_t Statistic or the Dickey-Fuller t Statistic When Applied to Residuals from Spurious Cointegrating Regression

Number of right-hand variables in regression, excluding trend or constant $(n-1)$	Sample size (T)	Probability that $(\hat{\rho}-1)/\hat{\sigma}_{\hat{\rho}}$ is less than entry						
		0.010	0.025	0.050	0.075	0.100	0.125	0.150
		Case 2						
1	500	−3.96	−3.64	−3.37	−3.20	−3.07	−2.96	−2.86
2	500	−4.31	−4.02	−3.77	−3.58	−3.45	−3.35	−3.26
3	500	−4.73	−4.37	−4.11	−3.96	−3.83	−3.73	−3.65
4	500	−5.07	−4.71	−4.45	−4.29	−4.16	−4.05	−3.96
5	500	−5.28	−4.98	−4.71	−4.56	−4.43	−4.33	−4.24

Phillips-Ouliaris-Hansen Critical Values: With Time Trend

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		0.010	0.025	0.050	0.075	0.100	0.125	0.150
				Case 3				
1	500	−3.98	−3.68	−3.42	—	−3.13	—	—
2	500	−4.36	−4.07	−3.80	−3.65	−3.52	−3.42	−3.33
3	500	−4.65	−4.39	−4.16	−3.98	−3.84	−3.74	−3.66
4	500	−5.04	−4.77	−4.49	−4.32	−4.20	−4.08	−4.00
5	500	−5.36	−5.02	−4.74	−4.58	−4.46	−4.36	−4.28

Testing for Cointegration when the Cointegrating Vector is Unknown

We implement the Phillips-Ouliaris-Hansen in Question 2 in Problem Set 4.

In this question, we will test whether the following variables cointegrate: Brazilian IPCA, American CPI and the R$/US$ exchange rate.

Phillips-Ouliaris-Hansen: Drawbacks

Phillips-Ouliaris-Hansen: Drawbacks

The Phillips-Ouliaris-Hansen test can give different answers depending on which variable is labeled Y_1.

In the next lecture, we will learn a test for cointegration that is invariant to the ordering of our variables.

This procedure is based on a full-information maximum likelihood test and is known as Johansen's test.

Thank you!

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References

J. A. Duffy, S. Mavroeidis, and S. Wycherley. Cointegration with Occasionally Binding Constraints. Available at https://arxiv.org/abs/2211.09604., Nov. 2022.

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We will show that the former representation is valid for a $VAR(3)$.

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Back