Lecture 1: Stationary ARMA(p,q) Models

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Econometrics 2

Administrative

- Recommended Reading: Hamilton - Chapter 3
- Optional Reading (Forecasting): Hamilton - Chapter 4
- Optional Reading (Estimation and Inference): Hamilton - Chapter 5
- Problem Set 1 - Deadline: May 14 at 9:00 am

Outline

1. Motivation
2. Basic Setup
3. Moving Average Process - MA(q)
4. Autoregressive Process - AR(p)
5. Autoregressive Moving Average Process - ARMA(p,q)
6. Estimation and Inference

Motivation

Income Dynamics through the Life Cycle [Meghir and Pistaferri, 2011]

Income Dynamics through the Life Cycle [Meghir and Pistaferri, 2011]

Goal: Individual income varies through the life cycle

- Experience
- Promotions
- Temporary shocks
- Employment shocks
- Health shocks
- Retirement

Understanding those shocks is policy-relevant.

- Unemployment and Disability insurance
- Conditional Cash Transfers and Social Security

Income Dynamics through the Life Cycle [Meghir and Pistaferri, 2011]

Literature started with fairly simple models: ARMA(1,1) with an unit root

$$y_t = y_{t-1} + \theta \cdot \epsilon_{t-1} + \epsilon_t$$

- y_t denotes annual earnings.
- ϵ_t is an error term.
- y_{t-1} with an implicit coefficient equal to 1 captures permanent shocks.
- $\theta \cdot \epsilon_{t-1}$ with $|\theta| < 1$ captures transitory shocks.
- y_0 is given and captures individual heterogeneity

Forecasting GDP Growth

Forecasting GDP Growth

- ARMA(p,q) are simple forecasting tools.
- They are surprisingly hard to beat.
- At least, they work as a benchmark for more complicated tools.
- Coding example (`code01_gdp-growth.R`): AR(p)

$$y_t = \beta_0 + \beta_1 \cdot y_{t-1} + \ldots + \beta_p \cdot y_{t-p} + \epsilon_t$$

Basic Setup

Outline

Basic Setup

Definition: Stochastic Process

A stochastic process is a collection of random variables that is indexed by some mathematical set.

Let $\{Y_t\}$ be a stochastic process and Y_t be one of its random variables.

Let $\{y_1, y_2, \ldots, y_T\}$ be a sample of size T.

- Just one realization of the underlying stochastic process even if $T = \infty$
- **Loki:** The multiverse is the stochastic process and one universe is one sample.

Basic Setup

Densities:

- $f_{Y_t}(\cdot)$ is the unconditional density of Y_t.
- $f_{Y_{t-j},\dots,Y_t}(\cdot,\dots,\cdot)$ is the joint density of $Y_{t-j},\dots,Y_t$.

Unconditional Mean (Expected Value):

$$\mathbb{E}[Y_t] = \int_{-\infty}^{+\infty} y_t \cdot f_{Y_t}(y_t)\, dy_t =: \mu_t$$

Variance:

$$\gamma_{0,t} := \mathbb{E}\left[(Y_t - \mu_t)^2\right] = \int_{-\infty}^{+\infty} (y_t - \mu_t)^2 \cdot f_{Y_t}(y_t)\, dy_t$$

Basic Setup

Autocovariance:

$$\begin{aligned}\gamma_{j,t} &:= \mathbb{E}\left[(Y_t - \mu_t) \cdot (Y_{t-j} - \mu_{t-j})\right] \\ &= \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} (y_t - \mu_t) \cdot (y_{t-j} - \mu_{t-j}) \cdot f_{Y_{t-j},\ldots,Y_t}(y_{t-j}, \ldots, y_t)\, dy_{t-j} \cdots dy_t\end{aligned}$$

Autocorrelation:

$$\rho_{j,t} := \frac{\gamma_{j,t}}{\gamma_{0,t}}$$

Basic Setup

Definition: Weak Stationarity

The process $\{Y_t\}$ is weakly stationary (a.k.a. covariance-stationary) if

$$\mathbb{E}\left[Y_t\right] = \mu \qquad \text{for all } t$$

$$\mathbb{E}\left[(Y_t - \mu_t) \cdot (Y_{t-j} - \mu_{t-j})\right] = \gamma_j \qquad \text{for all } t \text{ and } j.$$

Definition: Strict Stationarity

The process $\{Y_t\}$ is strictly stationary if, for any values of $j_1, j_2, \ldots, j_n$, the joint distribution of $Y_t, Y_{t+j_1}, \ldots, Y_{t+j_n}$ depends only on the intervals separating the dates $(j_1, j_2, \ldots, j_n)$ and not on the date itself (t).

Basic Setup

Definition: Gaussian Process

A process $\{Y_t\}$ is said to be Gaussian if the joint density $f_{Y_t, Y_{t+j_1}, \ldots, Y_{t+j_n}}$ is Gaussian for any $j_1, j_2, \ldots, j_n$.

- A covariance-stationary Gaussian process is strictly stationary.

Basic Setup

Definition: Mean Ergodicity

A weakly stationary process $\{Y_t\}$ is said to be ergodic for the mean if

$$\operatorname*{plim}_{T \to +\infty} \left(\frac{\sum_{t=1}^{T} Y_t}{T} \right) = \mu.$$

Definition: Second Moment Ergodicity

A weakly stationary process $\{Y_t\}$ is said to be ergodic for second moments if

$$\operatorname*{plim}_{T \to +\infty} \left(\frac{\sum_{t=j+1}^{T} (Y_t - \mu) \cdot (Y_{t-j} - \mu)}{T - j} \right) = \gamma_j.$$

Basic Setup

Definition: White Noise Process

A process $\{\epsilon_t\}$ is white noise if

$$\begin{aligned} \mathbb{E}\left[\epsilon_t\right] &= 0 && \text{for all } t \\ \mathbb{E}\left[\epsilon_t^2\right] &= \sigma^2 && \text{for all } t \\ \mathbb{E}\left[\epsilon_t \cdot \epsilon_{t'}\right] &= 0 && \text{for all } t \neq t' \end{aligned}$$

Basic Setup

Definition: Independent White Noise Process

An independent white noise process is a white noise process that also satisfies:

$$\epsilon_t \perp\!\!\!\perp \epsilon_{t'} \text{ for all } t \neq t'.$$

Definition: Gaussian White Noise Process

A Gaussian white noise process is an independent white noise process that also satisfies:

$$\epsilon_t \sim N\left(0, \sigma^2\right).$$

Moving Average Process - MA(q)

Outline

First-Order Moving Average Process

Definition: First-Order Moving Average Process, MA(1)

A stochastic process $\{Y_t\}$ is a MA(1) process if

$$Y_t = \mu + \epsilon_t + \theta \cdot \epsilon_{t-1},$$

where $\{\epsilon_t\}$ is white noise and μ and θ are constants.

Expectation: $\mathbb{E}\left[Y_t\right] = \mu$

Variance: $\mathbb{E}\left[(Y_t - \mu)^2\right] = \left(1 + \theta^2\right) \cdot \sigma^2$

First autocovariance: $\mathbb{E}\left[(Y_t - \mu) \cdot (Y_{t-1} - \mu)\right] = \theta \cdot \sigma^2$

Higher autocovariances: $\mathbb{E}\left[(Y_t - \mu) \cdot (Y_{t-j} - \mu)\right] = 0$ for $j > 1$.

An MA(1) process is weakly stationary.

qth-Order Moving Average Process

qth-Order Moving Average Process

Definition: qth-Order Moving Average Process, MA(q)

A stochastic process $\{Y_t\}$ is a MA(q) process if

$$Y_t = \mu + \epsilon_t + \theta_1 \cdot \epsilon_{t-1} + \theta_2 \cdot \epsilon_{t-2} + \ldots + \theta_q \cdot \epsilon_{t-q},$$

where $\{\epsilon_t\}$ is white noise and μ, $\theta_1, \ldots, \theta_q$ are constants.

Expectation: $\mathbb{E}[Y_t] = \mu$

Variance: $\mathbb{E}\left[(Y_t - \mu)^2\right] = \left(1 + \theta_1^2 + \theta_2^2 + \ldots + \theta_q^2\right) \cdot \sigma^2$

Autocovariance:

$$\gamma_j = \begin{cases} (\theta_j + \theta_{j+1} \cdot \theta_1 + \theta_{j+2} \cdot \theta_2 + \ldots + \theta_q \cdot \theta_{q-j}) \cdot \sigma^2 & \text{for } j \in \{1, \ldots, q\} \\ 0 & \text{for } j > q \end{cases}$$

An MA(q) process is weakly stationary and ergodic for all moments.

Autoregressive Process - AR(p)

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3. Moving Average Process - MA(q)
4. Autoregressive Process - AR(p)
5. Autoregressive Moving Average Process - ARMA(p,q)
6. Estimation and Inference

First-Order Autoregressive Process

Definition: First-Order Autoregressive Process, AR(1)

A stochastic process $\{Y_t\}$ is an AR(1) process if

$$Y_t = c + \phi \cdot Y_{t-1} + \epsilon_t,$$

where $\{\epsilon_t\}$ is white noise and c and ϕ are constants.

To ensure that our stochastic process is well-behaved (finite first and second moments), we must impose $|\phi| < 1$.

Let's understand why.

First-Order Autoregressive Process

We have that

$$\begin{aligned}
Y_t &= c + \phi \cdot Y_{t-1} + \epsilon_t \\
&= c + \phi \cdot (c + \phi \cdot Y_{t-2} + \epsilon_{t-1}) + \epsilon_t \\
&= c \cdot (1 + \phi) + \phi^2 \cdot Y_{t-2} + \epsilon_t + \phi \cdot \epsilon_{t-1} \\
&= c \cdot (1 + \phi) + \phi^2 \cdot (c + \phi \cdot Y_{t-3} + \epsilon_{t-2}) + \epsilon_t + \phi \cdot \epsilon_{t-1} \\
&= c \cdot \left(1 + \phi + \phi^2\right) + \phi^3 \cdot Y_{t-3} + \epsilon_t + \phi \cdot \epsilon_{t-1} + \phi^2 \epsilon_{t-2} \\
&\vdots \\
&= c \cdot \left(1 + \phi + \phi^2 + \dots\right) + \epsilon_t + \phi \cdot \epsilon_{t-1} + \phi^2 \epsilon_{t-2} + \dots
\end{aligned}$$

If $|\phi| < 1$, then: $Y_t = \dfrac{c}{1 - \phi} + \epsilon_t + \phi \cdot \epsilon_{t-1} + \phi^2 \epsilon_{t-2} + \dots$

First-Order Autoregressive Process

Expectation:

$$\mu = \mathbb{E}\left[Y_t\right] = \mathbb{E}\left[\frac{c}{1-\phi} + \epsilon_t + \phi \cdot \epsilon_{t-1} + \phi^2 \epsilon_{t-2} + \ldots\right] = \frac{c}{1-\phi}$$

Variance:

$$\begin{aligned}
\gamma_0 &= \mathbb{E}\left[(Y_t - \mu)^2\right] \\
&= \mathbb{E}\left[\left(\epsilon_t + \phi \cdot \epsilon_{t-1} + \phi^2 \epsilon_{t-2} + \ldots\right)^2\right] \\
&= \left(1 + \phi^2 + \phi^4 + \ldots\right) \cdot \sigma^2 \\
&= \frac{\sigma^2}{1-\phi^2} \text{ if } |\phi| < 1
\end{aligned}$$

First-Order Autoregressive Process

Autocovariance:

$$
\begin{aligned}
\gamma_j &= \mathbb{E}\left[(Y_t - \mu) \cdot (Y_{t-j} - \mu)\right] \\
&= \mathbb{E}\left[\begin{array}{c} \left(\epsilon_t + \phi \cdot \epsilon_{t-1} + \phi^2 \epsilon_{t-2} + \ldots + \phi^j \cdot \epsilon_{t-j} + \phi^{j+1} \cdot \epsilon_{t-j-1} + \phi^{j+2} \epsilon_{t-j-2} + \ldots\right) \cdot \\ \left(\epsilon_{t-j} + \phi \cdot \epsilon_{t-j-1} + \phi^2 \epsilon_{t-j-2} + \ldots\right) \end{array}\right] \\
&= \left(\phi^j + \phi^{j+2} + \phi^{j+4} + \ldots\right) \cdot \sigma^2 \\
&= \left(1 + \phi^2 + \phi^4 + \ldots\right) \cdot \phi^j \cdot \sigma^2 \\
&= \frac{\phi^j \cdot \sigma^2}{1 - \phi^2} \text{ if } |\phi| < 1
\end{aligned}
$$

Note that an AR(1) process is weakly stationary if $|\phi| < 1$.

Second-Order Autoregressive Process

Second-Order Autoregressive Process

Definition: Second-Order Autoregressive Process, AR(2)

A stochastic process $\{Y_t\}$ is an AR(2) process if

$$Y_t = c + \phi_1 \cdot Y_{t-1} + \phi_2 \cdot Y_{t-2} + \epsilon_t,$$

where $\{\epsilon_t\}$ is white noise and c, ϕ_1 and ϕ_2 are constants.

To ensure that our stochastic process is well-behaved (finite first and second moments), we must impose that the roots of

$$\left(1 - \phi_1 \cdot z - \phi_2 \cdot z^2\right)$$

lie outside the unit circle. Under this condition, an AR(2) process is weakly stationary.

Proof

We will use a Monte Carlo Simulation (`code02_ar2-stationarity.R`) to have an informal understanding of this condition.

We simulate the following processes:

- Weakly stationary: $Y_t = c + 0.25 \cdot Y_{t-1} + 0.25 \cdot Y_{t-2} + \epsilon_t$.
- Non-stationary: $Y_t = c + 1 \cdot Y_{t-1} + 1 \cdot Y_{t-2} + \epsilon_t$.

We look at the mean and at the variance of those processes over time ($T = 100$).

As an optional homework exercise, you can also look at the autocovariances of those processes over time.

To derive the mean, variance and autocovariances of an AR(2) process, we will assume that it is weakly stationary.

Expectation:

$$\begin{aligned}
\mathbb{E}[Y_t] &= c + \phi_1 \cdot \mathbb{E}[Y_{t-1}] + \phi_2 \cdot \mathbb{E}[Y_{t-2}] + \mathbb{E}[\epsilon_t] \\
\mu &= c + \phi_1 \cdot \mu + \phi_2 \cdot \mu \\
\mu &= \frac{c}{1 - \phi_1 - \phi_2}
\end{aligned}$$

To find second moments, note that

$$Y_t = c + \phi_1 \cdot Y_{t-1} + \phi_2 \cdot Y_{t-2} + \epsilon_t$$

$$Y_t = \mu \cdot (1 - \phi_1 - \phi_2) + \phi_1 \cdot Y_{t-1} + \phi_2 \cdot Y_{t-2} + \epsilon_t$$

$$Y_t - \mu = \phi_1 \cdot (Y_{t-1} - \mu) + \phi_2 \cdot (Y_{t-2} - \mu) + \epsilon_t \tag{1}$$

Now, we multiply Equation (1) by $(Y_t - \mu)$ and take expectations:

$$\mathbb{E}\left[(Y_t - \mu)^2\right] = \phi_1 \cdot \mathbb{E}\left[(Y_t - \mu) \cdot (Y_{t-1} - \mu)\right] + \phi_2 \cdot \mathbb{E}\left[(Y_t - \mu) \cdot (Y_{t-2} - \mu)\right]$$

$$+ \mathbb{E}\left[\epsilon_t \cdot (Y_t - \mu)\right]$$

$$\gamma_0 = \phi_1 \cdot \gamma_1 + \phi_2 \cdot \gamma_2 + \mathbb{E}\left[\epsilon_t \cdot (\phi_1 \cdot (Y_{t-1} - \mu) + \phi_2 \cdot (Y_{t-2} - \mu) + \epsilon_t)\right]$$

$$\gamma_0 = \phi_1 \cdot \gamma_1 + \phi_2 \cdot \gamma_2 + \sigma^2 \tag{2}$$

Now, we multiply Equation (1) by $(Y_{t-j} - \mu)$ and take expectations:

$$\begin{aligned} \mathbb{E}\left[(Y_t - \mu) \cdot (Y_{t-j} - \mu)\right] = \phi_1 \cdot \mathbb{E}\left[(Y_{t-j} - \mu) \cdot (Y_{t-1} - \mu)\right] \\ + \phi_2 \cdot \mathbb{E}\left[(Y_{t-j} - \mu) \cdot (Y_{t-2} - \mu)\right] + \mathbb{E}\left[\epsilon_t \cdot (Y_{t-j} - \mu)\right] \\ \gamma_j = \phi_1 \cdot \gamma_{j-1} + \phi_2 \cdot \gamma_{j-2} \end{aligned} \tag{3}$$

From Equation (3), we have that

$$\gamma_1 = \phi_1 \cdot \gamma_0 + \phi_2 \cdot \gamma_1 \tag{4}$$

$$\gamma_2 = \phi_1 \cdot \gamma_1 + \phi_2 \cdot \gamma_0 \tag{5}$$

Combining the last two equations with Equation (2), we have a linear system of three equations and three unknowns.

Solving this system,[1] we find that:

$$\gamma_0 = \frac{(1-\phi_2)\cdot\sigma^2}{(1+\phi_2)\cdot\left[(1-\phi_2)^2-\phi_1^2\right]}$$

$$\gamma_1 = \frac{\phi_1\cdot\sigma^2}{(1+\phi_2)\cdot\left[(1-\phi_2)^2-\phi_1^2\right]}$$

$$\gamma_j = \phi_1\cdot\gamma_{j-1}+\phi_2\cdot\gamma_{j-2} \qquad \text{for all } j>2$$

[1] As an optional homework exercise, work through the algebra here.

pth-Order Autoregressive Process

pth-Order Autoregressive Process

Definition: pth-Order Autoregressive Process, AR(p)

A stochastic process $\{Y_t\}$ is an AR(p) process if

$$Y_t = c + \phi_1 \cdot Y_{t-1} + \phi_2 \cdot Y_{t-2} + \ldots + \phi_p \cdot Y_{t-p} + \epsilon_t,$$

where $\{\epsilon_t\}$ is white noise and c, $\phi_1, \phi_2, \ldots, \phi_p$ are constants.

To ensure that our stochastic process is well-behaved (finite first and second moments), we must impose that the roots of

$$\left(1 - \phi_1 \cdot z - \phi_2 \cdot z^2 - \ldots - \phi_p \cdot z^p\right)$$

lie outside the unit circle. Under this condition, an AR(p) process is weakly stationary.

pth-Order Autoregressive Process

Expectation: $\mu = \dfrac{c}{1 - \phi_1 - \phi_2 - \ldots - \phi_p}$

Second-moments:

$$\gamma_j = \begin{cases} \phi_1 \cdot \gamma_{j-1} + \phi_2 \cdot \gamma_{j-2} + \ldots + \phi_p \cdot \gamma_{j-p} & \text{for } j \in \mathbb{N} \\ \phi_1 \cdot \gamma_1 + \phi_2 \cdot \gamma_2 + \ldots + \phi_p \cdot \gamma_p + \sigma^2 & \text{for } j = 0. \end{cases}$$

As an optional homework exercise, you can solve the system of equations created above by $j \in \{0, 1, \ldots, p\}$ and write $\gamma_0, \gamma_1, \ldots, \gamma_p$ as functions of $\sigma^2, \phi_1, \phi_2, \ldots, \phi_p$. The algebra here will be very annoying.

Autoregressive Moving Average Process - ARMA(p,q)

Outline

Mixed Autoregressive Moving Average Processes

Definition: Mixed Autoregressive Moving Average Processes, ARMA(p, q)

A stochastic process $\{Y_t\}$ is an ARMA(p, q) process if

$$Y_t = c + \phi_1 \cdot Y_{t-1} + \phi_2 \cdot Y_{t-2} + \ldots + \phi_p \cdot Y_{t-p} + \epsilon_t$$
$$+ \theta_1 \cdot \epsilon_{t-1} + \theta_2 \cdot \epsilon_{t-2} + \ldots + \theta_q \cdot \epsilon_{t-q},$$

where $\{\epsilon_t\}$ is white noise and $c, \phi_1, \phi_2, \ldots, \phi_p, \theta_1, \theta_2, \ldots, \theta_q$ are constants.

Mixed Autoregressive Moving Average Processes

To ensure that our stochastic process is well-behaved (finite first and second moments), we must impose that the roots of

$$\left(1 - \phi_1 \cdot z - \phi_2 \cdot z^2 - \ldots - \phi_p \cdot z^p\right)$$

lie outside the unit circle. Under this condition, an ARMA(p,q) process is weakly stationary.

Expectation: $\mu = \dfrac{c}{1 - \phi_1 - \phi_2 - \ldots - \phi_p}$

Redundant Parametrization with ARMA(p,q)

Redundant Parametrization with ARMA(p,q)

Consider the white noise process:

$$
\begin{aligned}
Y_t &= \epsilon_t \\
\alpha \cdot Y_{t-1} &= \alpha \cdot \epsilon_{t-1} \\
Y_t &= \alpha \cdot Y_{t-1} + \epsilon_t - \alpha \cdot \epsilon_{t-1}.
\end{aligned}
$$

Consequently, an white noise process is equivalent to an ARMA(1,1) process with $\phi_1 = -\theta_1$.

We should avoid such an over-parametrization. If you estimate a ARMA(1,1) where ϕ_1 is close to $-\theta_1$, you are better off with a white noise model.

Similar problems show up when we compare ARMA(p,q) processes against ARMA(p - 1, q - 1) processes. Check Hamilton's page 61.

Estimation and Inference

Outline

Estimation and Inference

- To estimate an ARMA(p,q) model, we use an ML estimator based on Gaussian assumptions.
- Its theory (numerical methods and asymptotic inference) is detailed in Hamilton's Chapter 5.
- In practice, we use the function `arima` in R to estimate ARMA(p,q) models.
- We will use a Monte Carlo Simulation (`code03_MC-ma1.R` and Question 2 in Problem Set 1) to have an informal understanding of the asymptotic behavior of this estimator.
    - ML estimation works reasonably well even when the error distribution is non-Normal.
- Hamilton's Chapter 4 discusses tricks to select a forecasting model.
    - Some of those tricks are illustrated in Question 1 in Problem Set 1.

Thank you!

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References

C. Meghir and L. Pistaferri. Chapter 9 - Earnings, Consumption and Life Cycle Choices. In D. Card and O. Ashenfelter, editors, *Handbook of Labor Economics*, volume 4 of *Handbook of Labor Economics*, pages pp. 773–854. Elsevier, 2011. doi: https://doi.org/10.1016/S0169-7218(11)02407-5. URL https://www.sciencedirect.com/science/article/pii/S0169721811024075.

Proof of Stationarity of an AR(2) Process

Let's start by thinking carefully about an AR(1) process:

$$Y_t = c + \phi \cdot Y_{t-1} + \epsilon_t.$$

We have that

$$\begin{aligned} Y_t - \phi \cdot Y_{t-1} &= c + \epsilon_t \\ (1 - \phi \cdot L)\, Y_t &= c + \epsilon_t, \end{aligned}$$

implying that

$$Y_t = \frac{1}{1 - \phi \cdot L} \cdot (c + \epsilon_t)$$

is weakly stationary if $|\phi| < 1$.

Proof of Stationarity of an AR(2) Process

Note that $(1 - \phi \cdot z) = 0$ implies that $z = \frac{1}{\phi}$. So, our AR(1) process is weakly stationary if the root of

$$1 - \phi \cdot z$$

lies outside the unit circle.

Proof of Stationarity of an AR(2) Process

Now let's look at our AR(2) process:

$$Y_t = c + \phi_1 \cdot Y_{t-1} + \phi_2 \cdot Y_{t-2} + \epsilon_t.$$

We have that

$$\begin{aligned} Y_t - \phi_1 \cdot Y_{t-1} - \phi_2 \cdot Y_{t-2} &= c + \epsilon_t \\ \left(1 - \phi_1 \cdot L - \phi_2 \cdot L^2\right) \cdot Y_t &= c + \epsilon_t \\ (1 - \lambda_1 \cdot L) \cdot (1 - \lambda_2 \cdot L) \cdot Y_t &= c + \epsilon_t \\ &\text{where } \lambda_1 + \lambda_2 = \phi_1 \text{ and } \lambda_1 \cdot \lambda_2 = \phi_2 \\ (1 - \lambda_2 \cdot L) \cdot Y_t &= \frac{1}{1 - \lambda_1 \cdot L} \cdot (c + \epsilon_t). \end{aligned}$$

Proof of Stationarity of an AR(2) Process

According to our AR(1) result, we know that

$$W_t := \frac{1}{1 - \lambda_1 \cdot L} \cdot (c + \epsilon_t)$$

is stationary if $|\lambda_1| < 1$. Consequently, W_t is stationary if the root of

$$1 - \lambda_1 \cdot z$$

lies outside the unit circle.

Proof of Stationarity of an AR(2) Process

Now, we have that

$$Y_t = \lambda_2 \cdot Y_{t-1} + W_t$$

is an AR(1) process where W_t is weakly stationary.

According to our AR(1) results, this process is stationary if $|\lambda_2| < 1$. Consequently, this process is weakly stationary if the root of

$$1 - \lambda_2 \cdot z$$

lies outside the unit circle.

Proof of Stationarity of an AR(2) Process

Summing up, we have that

$$Y_t = c + \phi_1 \cdot Y_{t-1} + \phi_2 \cdot Y_{t-2} + \epsilon_t.$$

is stationary if $|\lambda_1| < 1$ and $|\lambda_2| < 1$, implying that Y_t is stationary if the roots of

$$1 - \phi_1 \cdot z - \phi_2 \cdot z^2 = (1 - \lambda_1 \cdot z) \cdot (1 - \lambda_2 \cdot z)$$

lie outside the unit circle.

Back