Lecture 2B: Deterministic Time Trends (Estimation and Inference)

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EESP-FGV

Econometrics 2

Administrative

- Recommended Reading: Hamilton - Chapter 16
- Problem Set 2 - Deadline: May 23rd at 9:00 am

Outline

1. Nonstationary models: General Aspects
2. Simple Time Trend Model: OLS Estimator's Asymptotic Distribution
3. Simple Time Trend Model: Hypothesis Testing
4. AR(p) around a Deterministic Time Trend

Nonstationary models: General Aspects

Nonstationary models: General Aspects

- Nonstationary models (unit root or deterministic time trend): estimated by OLS
  - When estimating ARMA(p,q) models, we used MLE due to the MA(q) component.
  - We now focus on AR(p) models with a nonstationary component $\Rightarrow$ OLS is enough.
- OLS estimator's asymptotic distribution: Nonstandard
  - Estimators of different coefficients may have different asymptotic rates of convergence
  - Asymptotic Distribution may be nonnormal
- Lecture 2B: Deterministic Time Trends
- Lecture 2C: Unit Root Models

Simple Time Trend Model: OLS Estimator's Asymptotic Distribution

Outline

Model and Estimator

Consider the model

$$Y_t = \alpha + \delta \cdot t + \epsilon_t$$

where $\{\epsilon_t\}$ is a white noise process.

OLS Estimator:

$$b_T := \begin{bmatrix} \hat{\alpha}_T \\ \hat{\delta}_T \end{bmatrix} = \left[\sum_{t=1}^{T} X_t X_t'\right]^{-1} \left[\sum_{t=1}^{T} X_t Y_t\right],$$

where $X_t := \begin{bmatrix} 1 \\ t \end{bmatrix}$ and $\beta := \begin{bmatrix} \alpha \\ \delta \end{bmatrix}$.

Problems with the usual approach

Problems with the usual approach

Usual approach:

$$\sqrt{T}\,(b_T - \beta) = \left[\frac{\sum_{t=1}^{T} X_t X_t'}{T}\right]^{-1} \left[\frac{\sum_{t=1}^{T} X_t \epsilon_t}{\sqrt{T}}\right]$$

It requires, among other things, that $\frac{\sum_{t=1}^{T} X_t X_t'}{T}$ converges in probability to a nonsingular matrix Q.

But $\frac{\sum_{t=1}^{T} X_t X_t'}{T}$ diverges!

Problems with the usual approach

$$
\begin{aligned}
\frac{\sum_{t=1}^{T} X_t X_t'}{T} &= \frac{1}{T} \cdot \sum_{t=1}^{T} \left(\begin{bmatrix} 1 \\ t \end{bmatrix} \begin{bmatrix} 1 & t \end{bmatrix} \right) = \frac{1}{T} \cdot \sum_{t=1}^{T} \begin{bmatrix} 1 & t \\ t & t^2 \end{bmatrix} \\
&= \frac{1}{T} \cdot \begin{bmatrix} \sum_{t=1}^{T} 1 & \sum_{t=1}^{T} t \\ \sum_{t=1}^{T} t & \sum_{t=1}^{T} t^2 \end{bmatrix} = \frac{1}{T} \cdot \begin{bmatrix} T & \frac{T \cdot (T+1)}{2} \\ \frac{T \cdot (T+1)}{2} & \frac{T \cdot (T+1) \cdot (2 \cdot T + 1)}{6} \end{bmatrix}
\end{aligned}
$$

proof by induction (Hamilton's page 456)

$$
= \begin{bmatrix} 1 & \frac{(T+1)}{2} \\ \frac{(T+1)}{2} & \frac{(T+1) \cdot (2 \cdot T + 1)}{6} \end{bmatrix} \underset{T \to +\infty}{\longrightarrow} \begin{bmatrix} 1 & +\infty \\ +\infty & +\infty \end{bmatrix}
$$

Problems with the usual approach

$\hat{\alpha}_T$ and $\hat{\delta}_T$ have different rates of convergence!

Solution

Solution

We need to multiply $\hat{\alpha}_T$ by $\sqrt{T}$ and $\hat{\delta}_T$ by $T^{3/2}$.

Define $\Upsilon_T := \begin{bmatrix} \sqrt{T} & 0 \\ 0 & T^{3/2} \end{bmatrix}$.

We have that

$$\begin{bmatrix} \sqrt{T} \cdot (\hat{\alpha}_T - \alpha) \\ T^{3/2} \cdot \left(\hat{\delta}_T - \delta\right) \end{bmatrix} = \Upsilon_T \left[\sum_{t=1}^{T} X_t X_t'\right]^{-1} \left[\sum_{t=1}^{T} X_t \epsilon_t\right]$$

$$= \left\{\Upsilon_T^{-1} \left[\sum_{t=1}^{T} X_t X_t'\right] \Upsilon_T^{-1}\right\}^{-1} \left\{\Upsilon_T^{-1} \left[\sum_{t=1}^{T} X_t \epsilon_t\right]\right\}$$

Solution

First term converges to a nonsingular matrix Q.

$$
\left\{ \Upsilon_T^{-1} \left[\sum_{t=1}^{T} X_t X_t' \right] \Upsilon_T^{-1} \right\}
$$

$$
= \left\{ \begin{bmatrix} T^{-1/2} & 0 \\ 0 & T^{-3/2} \end{bmatrix} \begin{bmatrix} \sum_{t=1}^{T} 1 & \sum_{t=1}^{T} t \\ \sum_{t=1}^{T} t & \sum_{t=1}^{T} t^2 \end{bmatrix} \begin{bmatrix} T^{-1/2} & 0 \\ 0 & T^{-3/2} \end{bmatrix} \right\}
$$

$$
= \left\{ \begin{bmatrix} T^{-1/2} \cdot \sum_{t=1}^{T} 1 & T^{-1/2} \cdot \sum_{t=1}^{T} t \\ T^{-3/2} \cdot \sum_{t=1}^{T} t & T^{-3/2} \cdot \sum_{t=1}^{T} t^2 \end{bmatrix} \begin{bmatrix} T^{-1/2} & 0 \\ 0 & T^{-3/2} \end{bmatrix} \right\}
$$

$$
= \left\{ \begin{bmatrix} T^{-1} \cdot \sum_{t=1}^{T} 1 & T^{-2} \cdot \sum_{t=1}^{T} t \\ T^{-2} \cdot \sum_{t=1}^{T} t & T^{-3} \cdot \sum_{t=1}^{T} t^2 \end{bmatrix} \right\}
$$

Solution

$$\left\{\Upsilon_T^{-1}\left[\sum_{t=1}^{T} X_t X_t'\right]\Upsilon_T^{-1}\right\}$$

$$= \left\{\begin{bmatrix} 1 & \frac{1}{2}\cdot\left(1+\frac{1}{T}\right) \\ \frac{1}{2}\cdot\left(1+\frac{1}{T}\right) & \frac{1}{6}\cdot\left(2+\frac{3}{T}+\frac{1}{T^2}\right) \end{bmatrix}\right\}$$

$$\underset{T\to+\infty}{\longrightarrow} \left\{\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix}\right\} =: Q$$

This is our LLN-like result.

Solution

Now, we need a CLT-like result! That's our second term.

$$\left\{ \Upsilon_T^{-1} \left[\sum_{t=1}^{T} X_t \epsilon_t \right] \right\} \xrightarrow{d} N\left(0, \sigma^2 \cdot Q\right)$$

if ϵ_t is i.i.d. with mean zero, variance σ^2 and finite fourth moment.

Solution

We can combine the last two result in a theorem about asymptotic convergence.

Theorem 1 (Simple Time Trend Model: Asymptotic Convergence)

Let $\{Y_t\}$ be a stochastic process satisfying $Y_T = \alpha + \delta \cdot t + \epsilon_t$ where ϵ_t is i.i.d. with $\mathbb{E}[\epsilon_t] = 0$, $\mathbb{E}\left[\epsilon_t^2\right] = \sigma^2$ and $\mathbb{E}\left[\epsilon_t^4\right] < +\infty$. Then,

$$\begin{bmatrix} \sqrt{T} \cdot (\hat{\alpha}_T - \alpha) \\ T^{3/2} \cdot \left(\hat{\delta}_T - \delta\right) \end{bmatrix} \xrightarrow{d} N\left(0, \sigma^2 \cdot \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{bmatrix}^{-1}\right).$$

Notational Remark

Notational Remark

Definition: Rate of Convergence

A sequence of random variables $\{X_T\}$ is said to converge at rate T^k or to be $O_P\left(T^{-k}\right)$ if, for every $\epsilon > 0$, there exists an $M > 0$ such that

$$\mathbb{P}\left[|X_T| > \frac{M}{T^k}\right] < \epsilon.$$

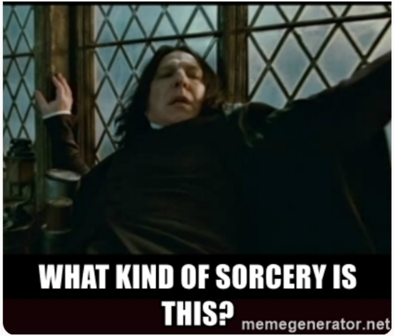
- Estimator $\hat{\alpha}_T$ is $O_P\left(T^{-1/2}\right)$.
- Estimator $\hat{\delta}_T$ is $O_P\left(T^{-3/2}\right)$.

Simple Time Trend Model: Hypothesis Testing

Outline

Simple Time Trend Model: Hypothesis Testing

Although OLS estimators $\hat{\alpha}_T$ and $\hat{\delta}_T$ have different rates of convergence, **the usual** t **and** F **tests are asymptotically valid**.



Simple Time Trend Model: Hypothesis Testing

Intuition: Although OLS estimators $\hat{\alpha}_T$ and $\hat{\delta}_T$ have different rates of convergence, the corresponding standard errors also incorporate different orders of T.

Estimator of σ^2:

$$s_T^2 := \frac{\sum_{t=1}^{T}\left(Y_t - \hat{\alpha}_T - \hat{\delta}_T \cdot t\right)^2}{T-2}$$

Simple Time Trend Model: Hypothesis Testing

Intercept: $H_0 : \alpha = \alpha_0$

$$\frac{\hat{\alpha}_T - \alpha_0}{\left\{ s_T^2 \cdot \left[\begin{array}{cc} 1 & 0 \end{array} \right] \left(\sum_{t=1}^{T} X_t X_t' \right)^{-1} \left[\begin{array}{c} 1 \\ 0 \end{array} \right] \right\}^{1/2}} \xrightarrow{d} N(0,1)$$

Simple Time Trend Model: Hypothesis Testing

Trend coefficient: $H_0 : \delta = \delta_0$

$$\frac{\hat{\delta}_T - \delta_0}{\left\{ s_T^2 \cdot \begin{bmatrix} 0 & 1 \end{bmatrix} \left(\sum_{t=1}^{T} X_t X_t' \right)^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}^{1/2}} \xrightarrow{d} N(0,1)$$

Let's understand why this works!

Simple Time Trend Model: Hypothesis Testing

$$
\begin{aligned}
t_{\delta_T} &:= \frac{\hat{\delta}_T - \delta_0}{\left\{ s_T^2 \cdot \begin{bmatrix} 0 & 1 \end{bmatrix} \left(\sum_{t=1}^T X_t X_t' \right)^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}^{1/2}} \\
&= \frac{T^{3/2} \cdot \left(\hat{\delta}_T - \delta_0 \right)}{\left\{ s_T^2 \cdot \begin{bmatrix} 0 & T^{3/2} \end{bmatrix} \left(\sum_{t=1}^T X_t X_t' \right)^{-1} \begin{bmatrix} 0 \\ T^{3/2} \end{bmatrix} \right\}^{1/2}} \\
&= \frac{T^{3/2} \cdot \left(\hat{\delta}_T - \delta_0 \right)}{\left\{ s_T^2 \cdot \begin{bmatrix} 0 & 1 \end{bmatrix} \Upsilon_T \left(\sum_{t=1}^T X_t X_t' \right)^{-1} \Upsilon_T \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}^{1/2}}
\end{aligned}
$$

Simple Time Trend Model: Hypothesis Testing

$$t_{\delta_T} = \frac{T^{3/2} \cdot \left(\hat{\delta}_T - \delta_0\right)}{\left\{ s_T^2 \cdot \left[\begin{array}{cc} 0 & 1 \end{array} \right] \Upsilon_T \left(\sum_{t=1}^{T} X_t X_t' \right)^{-1} \Upsilon_T \left[\begin{array}{c} 0 \\ 1 \end{array} \right] \right\}^{1/2}} \xrightarrow{p} \frac{T^{3/2} \cdot \left(\hat{\delta}_T - \delta_0\right)}{\sigma \cdot \sqrt{q_{22}}},$$

where q_{22} is the (2,2)-element of Q^{-1}.

Since $T^{3/2} \cdot \left(\hat{\delta}_T - \delta_0\right) \xrightarrow{d} N\left(0, \sigma^2 \cdot q_{22}\right)$ according to Theorem 1, we have that $t_{\delta_T} \xrightarrow{d} N(0,1)$.

Simple Time Trend Model: Hypothesis Testing

Joint Hypothesis Testing: $H_0 : \begin{bmatrix} \alpha \\ \delta \end{bmatrix} = \begin{bmatrix} \alpha_0 \\ \delta_0 \end{bmatrix}$ or, in vector form, $H_0 : \beta = \beta_0$

$$(b_T - \beta_0)' \left[s_T^2 \left(\sum_{t=1}^{T} X_t X_t' \right)^{-1} \right]^{-1} (b_T - \beta_0) \xrightarrow{d} \chi^2 (2)$$

AR(p) around a Deterministic Time Trend

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AR(p) around a Deterministic Time Trend

Let $\{Y_t\}$ be a stochastic process satisfying

$$Y_t = \alpha + \delta \cdot t + \phi_1 \cdot Y_{t-1} + \phi_2 \cdot Y_{t-2} + \ldots + \phi_p \cdot Y_{t-p} + \epsilon_t$$

where ϵ_t is i.i.d with mean zero, variance σ^2 and finite fourth moment, and the roots of

$$1 - \phi_1 \cdot z - \ldots - \phi_p \cdot z^p = 0$$

lie outside the unit circle.

AR(p) around a Deterministic Time Trend

Matrix notation:

$$Y_t = X_t'\beta + \epsilon_t$$

,

where $X_t = \begin{bmatrix} Y_{t-1} & Y_{t-2} & \cdots & Y_{t-p} & 1 & t \end{bmatrix}'$ and $\beta = \begin{bmatrix} \phi_1 & \phi_2 & \cdots & \phi_p & \alpha & \delta \end{bmatrix}'$

OLS estimators:

$$b_T := \begin{bmatrix} \hat{\phi}_{1,T} & \hat{\phi}_{2,T} & \cdots & \hat{\phi}_{p,T} & \hat{\alpha}_T & \hat{\delta}_T \end{bmatrix}' = \left[\sum_{t=1}^{T} X_t X_t'\right]^{-1} \left[\sum_{t=1}^{T} X_t Y_t\right]$$

Useful Transformation

Useful Transformation

- We want to transform our model so that it includes only a constant term, a time trend, and zero-mean weakly stationary random variables.

- It provides a general technique for finding the asymptotic distribution of of regressions involving nonstationary variables.

- General result: if such a transformed equation were estimated by OLS, the coefficient on zero-mean weakly stationary random variables would converge at rate $\sqrt{T}$ to a Gaussian distribution.

Useful Transformation

Define:

$$G' := \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ -\alpha + \delta & -\alpha + 2 \cdot \delta & \cdots & -\alpha + p \cdot \delta & 1 & 0 \\ -\delta & -\delta & \cdots & -\delta & 0 & 1 \end{bmatrix}$$

Useful Transformation

$$X_t^* := GX_t = \begin{bmatrix} Y_{t-1}^* \\ Y_{t-2}^* \\ \vdots \\ Y_{t-p}^* \\ 1 \\ t \end{bmatrix} = \begin{bmatrix} Y_{t-1} - \alpha - \delta \cdot (t-1) \\ Y_{t-2} - \alpha - \delta \cdot (t-2) \\ \vdots \\ Y_{t-p} - \alpha - \delta \cdot (t-p) \\ 1 \\ t \end{bmatrix}$$

Useful Transformation

$$\beta^* := \left(G'\right)^{-1} \beta = \begin{bmatrix} \phi_1^* \\ \phi_2^* \\ \vdots \\ \phi_p^* \\ \alpha^* \\ \delta^* \end{bmatrix} = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_p \\ \alpha \cdot \left(1 + \sum_{k=1}^{p} \phi_k\right) - \delta \cdot \left(\sum_{k=1}^{p} k \cdot \phi_k\right) \\ \delta \cdot \left(1 + \sum_{k=1}^{p} \phi_k\right) \end{bmatrix}$$

$$b_T^* = \left[\sum_{t=1}^{T} X_t^* \left(X_t^*\right)'\right]^{-1} \left[\sum_{t=1}^{T} X_t^* Y_t\right]$$

Useful Transformation

We have that:

$$Y_t = (X_t^*)' \beta^* + \epsilon_t$$

and

$$b_T = G' b_T^*.$$

- G is unknown because it depends on the true values of the parameters α and δ.
- But analyzing the transformed model is much easier.
- If we can find the asymptotic properties of b_T^*, then we can find the asymptotic properties of b_T.

Asymptotic Properties of the Transformed Model

Asymptotic Properties of the Transformed Model

Lemma 2 (Asymptotic Distribution of b_T^*)

$$\Upsilon_T \left(b_T^* - \beta^* \right) \xrightarrow{d} N\left(0, \sigma^2 \left(Q^*\right)^{-1}\right)$$

where

$$\Upsilon_T := \begin{bmatrix} \sqrt{T} & 0 & \cdots & 0 & 0 & 0 \\ 0 & \sqrt{T} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \sqrt{T} & 0 & 0 \\ 0 & 0 & \cdots & 0 & \sqrt{T} & 0 \\ 0 & 0 & \cdots & 0 & 0 & T^{3/2} \end{bmatrix}$$

Asymptotic Properties of the Transformed Model

$$Q^* := \begin{bmatrix} \gamma_0^* & \gamma_1^* & \gamma_2^* & \cdots & \gamma_{p-1}^* & 0 & 0 \\ \gamma_1^* & \gamma_0^* & \gamma_1^* & \cdots & \gamma_{p-2}^* & 0 & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \vdots \\ \gamma_{p-1}^* & \gamma_{p-2}^* & \gamma_{p-3}^* & \cdots & \gamma_0^* & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1/2 \\ 0 & 0 & 0 & \cdots & 0 & 1/2 & 1/3 \end{bmatrix}$$

$$\gamma_j^* := \mathbb{E}\left[Y_t^* \cdot Y_{t-j}^*\right]$$

Asymptotic Properties of the Original Model

Asymptotic Properties of the Original Model

- Since OLS estimators $\hat{\phi}_j$ are identical to $\hat{\phi}_j^*$, their asymptotic distribution is immediately given by Lemma 2.

- Estimator $\hat{\alpha}_T$ is a linear combination of variables that converge to a Gaussian distribution at rate $\sqrt{T}$, implying that

$$\sqrt{T} \cdot (\hat{\alpha}_T - \alpha) \xrightarrow{d} N\left(0, \sigma^2 g_\alpha' \left(Q^*\right)^{-1} g_\alpha\right),$$

where $g_\alpha' := \begin{bmatrix} -\alpha + \delta & -\alpha + 2 \cdot \delta & \cdots & -\alpha + p \cdot \delta & 1 & 0 \end{bmatrix}$.

Asymptotic Properties of the Original Model

- Estimator $\hat{\delta}_T = g'_\delta b^*_T + \hat{\delta}^*_T$ is a linear combination of variables converging at different rates and its asymptotic distribution is governed by the variables with the slowest rate of convergence:

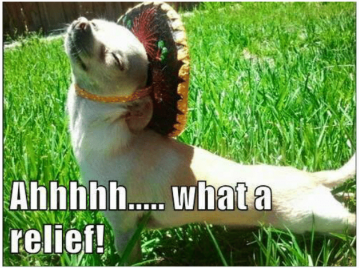
$$\sqrt{T} \cdot \left(\hat{\delta}_T - \delta\right) \xrightarrow{d} N\left(0, \sigma^2 g'_\delta \left(Q^*\right)^{-1} g_\delta\right),$$

where $g'_\delta := \begin{bmatrix} -\delta & -\delta & \cdots & -\delta & 0 & 0 \end{bmatrix}$.

Hypothesis Testing

Hypothesis Testing

- Although we relied on the transformed model to derive asymptotic distributions, there is no need to know G to conduct hypothesis testing.
- The usual t and F tests about β calculated in the usual way on the original model are asymptotically valid.



Hypothesis Testing

When testing

$$H_0 : R\beta = r$$

with m restrictions, we have that

$$(Rb_T - r)' \left(s_T^2 \cdot R \left(\sum_{t=1}^{T} X_t X_t' \right)^{-1} R' \right)^{-1} (Rb_T - r) \xrightarrow{d} \chi^2 (m) ,$$

where $s_T^2 := \dfrac{\sum_{t=1}^{T} (Y_t - X_t' b_T)^2}{T - p - 2}$.

Thank you!

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References