Lecture 4A: Vector Autoregression (VAR) — Theory

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EESP-FGV

Econometrics 2

Administrative

- Recommended Reading: Hamilton's Chapters 10.1, 10.2, 10.5, 11.1, 11.4-11.6
- Problem Set 3 - Deadline: June 4th at 9:00 am

Outline

Motivation

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Perennial Questions in Macroeconomics:

Does money cause output? Does printing money only increase prices?

We want to understand the impact of monetary shocks on unemployment and inflation.

- Central Bank's Response Function: Taylor rule.

Christiano et al. [1999] summarizes the evidence about this question.

- Detailed discussion about the identification assumptions behind a VAR.
- If you are into monetary policy, this chapter is a mandatory reading. It has over 4,000 citations.

Definitions and Useful Results

Outline

Definition: $VAR\,(p)$

A p-th order vector autoregression process ($VAR\,(p)$) is given by:

$$Y_t = c + \Phi_1 Y_{t-1} + \Phi_2 Y_{t-2} + \ldots + \Phi_p Y_{t-p} + \epsilon_t,$$

where

- Y_t: $(n \times 1)$-vector of variables — $Y_{1,t} =$ inflation, $Y_{2,t} =$ unemployment etc.
- c: $(n \times 1)$-vector of constants
- Φ_j: $(n \times n)$-matrix of coefficients for $j \in \{1, 2, \ldots, p\}$
- ϵ_t: $(n \times 1)$-vector generalization of white noise
  - $\mathbb{E}\left[\epsilon_t\right] = 0$
  - $\underbrace{\mathbb{E}\left[\epsilon_t \epsilon_\tau'\right]}_{n \times n} = \begin{cases} \Omega & \text{for } t = \tau \\ 0 & \text{otherwise} \end{cases}$ with Ω an $(n \times n)$ symmetric positive definite matrix.

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Plain English: A vector autoregression is a system in which each variable is regressed on a constant and p of its own lags as well as on p lags of each of the other variables in the VAR.

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Formally, we impose that ϵ_t is uncorrelated with $Y_{t-p-1}, Y_{t-p-2}, \ldots$.

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A vector process $\{Y_t\}$ is said to be covariance-stationary if its first and second moments $\left(\mathbb{E}\left[Y_t\right] \text{ and } \mathbb{E}\left[Y_t Y_{t-j}'\right]\right)$ are independent of date t.

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Condition for Stationarity

A $VAR\,(p)$ is covariance-stationary if all values of z satisfying

$$\left|I_n - \Phi_1 z - \Phi_2 z^2 - \ldots - \Phi_p z^p\right| = 0$$

lie outside the unit circle.

Useful Result: A Type of LLN

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Proposition 1

Let $\{Y_t\}$ be a covariance-stationary process with moments given by

$$\mathbb{E}[Y_t] = \mu,$$
$$\mathbb{E}\left[(Y_t - \mu)(Y_{t-j} - \mu)'\right] = \Gamma_j$$

and with absolutely summable autocovariances (i.e., $\left(\sum_{v=-\infty}^{+\infty} \Gamma_v\right) \in \mathbb{R}^{n\times n}$). Assume we have a sample of size T drawn from $\{Y_t\}$. Then, the sample mean $\bar{Y}_T := \frac{\sum_{t=1}^{T} Y_t}{T}$ satisfies

1. $\bar{Y}_T \xrightarrow{p} \mu$
2. $\lim_{T\to+\infty} \left\{T \cdot \mathbb{E}\left[\left(\bar{Y}_T - \mu\right)\left(\bar{Y}_T - \mu\right)'\right]\right\} = \sum_{v=-\infty}^{+\infty} \Gamma_v.$

Estimating the Variance of the Sample Mean

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Let S represent the Variance of the Sample Mean:

$$S := \lim_{T \to +\infty} \left\{ T \cdot \mathbb{E}\left[\left(\bar{Y}_T - \mu \right) \left(\bar{Y}_T - \mu \right)' \right] \right\}.$$

Estimating S is a necessary step for hypothesis testing.

Our goal is to estimate S consistently.

Estimating the Variance of the Sample Mean

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$$\hat{S} := \hat{\Gamma}_0 + \sum_{v=1}^{q} \left(\hat{\Gamma}_v + \hat{\Gamma}'_v \right),$$

where $\hat{\Gamma}_v = \dfrac{\sum_{t=v+1}^{T} \left(Y_t - \bar{Y}_T\right)\left(Y_{t-v} - \bar{Y}_T\right)'}{T}$.

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Intuition: If $\{Y_t\}$ is a vector $MA(q)$ process, then $S = \sum_{v=-q}^{q} \Gamma_v$. Moreover, from PSet 3, $\Gamma_v' = \Gamma_{-v}$.

Estimator 1:

$$\hat{S}_T^1 := \hat{\Gamma}_0 + \sum_{v=1}^{q} \left(\hat{\Gamma}_v + \hat{\Gamma}_v' \right),$$

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- It consistently estimates S in the presence of heteroskedasticity and autocorrelation up through order q.

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Properties:

- It consistently estimates S in the presence of heteroskedasticity and autocorrelation up through order q.
- If $\Gamma_j \underset{|j| \to +\infty}{\longrightarrow} 0$ sufficiently quickly, $\hat{S}_T^1$ still consistently estimates S if q grows with T. Specifically, if $q \to +\infty$, $T \to +\infty$ and $\dfrac{q}{T^{1/4}} \to 0$, then $\hat{S}_T^1 \xrightarrow{p} S$.

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- $\hat{S}_T^1$ is not necessarily positive semidefinite in small samples.
- We can estimate that a variance is negative!

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We need a better estimator.

Estimating the Variance of the Sample Mean

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$$\hat{S}_T^2 := \hat{\Gamma}_0 + \sum_{v=1}^{q} \left[\left(1 - \frac{v}{q+1}\right) \cdot \left(\hat{\Gamma}_v + \hat{\Gamma}_v'\right) \right].$$

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Properties:

- $\hat{S}_T^2$ is positive semidefinite by construction.
- $\hat{S}_T^2$ has the same consistency properties that were noted for $\hat{S}_T^1$.

Estimation and Inference

Outline

1. Motivation

2. Definitions and Useful Results

3. Estimation and Inference

4. Impulse Response Functions

5. Connecting VARs to Structural Models

Estimation and Inference

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Let $\{Y_t\}$ be a $VAR(p)$ process such that:

$$Y_t = c + \Phi_1 Y_{t-1} + \Phi_2 Y_{t-2} + \ldots + \Phi_p Y_{t-p} + \epsilon_t,$$

where $\epsilon_t \sim$ i.i.d. $N(0, \Omega)$.

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Define $\Pi' := (c, \Phi_1, \Phi_2, \ldots, \Phi_p)$.

We want to estimate this model by MLE.

- MLE has nice efficiency properties.

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- MLE estimator of the coefficients of the j-th equation are found by an OLS regression of $Y_{j,t}$ on a constant term and p lags of all the of the variables in the system.

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- MLE estimator of the variance Ω is connected to OLS too. Given that $\hat{\epsilon}_{j,t}$ is the OLS estimated residual of the j-th equation, we have that:
  - $\hat{\sigma}_i^2 = \frac{\sum_{t=1}^{T} \hat{\epsilon}_{i,t}^2}{T}$

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  - $\hat{\sigma}_{i,j}^2 = \frac{\sum_{t=1}^{T} (\hat{\epsilon}_{i,t} \cdot \hat{\epsilon}_{j,t})}{T}$

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Take-home Lesson: Standard OLS t and F statistics applied to the coefficients of any single equation in the *VAR* are asymptotically valid and can be evaluated in the usual way.

OBS: Cross-equations restrictions can be tested with Wald tests too thanks to the following result.

Estimation and Inference: Formal Result

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lie outside the unit circle. Let $k := n \cdot p + 1$ and let X_t' be the $(1 \times k)$-vector

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where $\hat{\epsilon}_t = \left(\hat{\epsilon}_{1,t}, \hat{\epsilon}_{2,t}, \ldots, \hat{\epsilon}_{n,t}\right)$ and $\hat{\epsilon}_{i,t} = Y_{i,t} - X_t'\hat{\pi}_{i,T}$.

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where ϵ_t is independent and identically distributed with mean 0, variance Ω. Let $k := n \cdot p + 1$ and let X_t' be the $(1 \times k)$-vector $X_t' := \left(1, Y_{t-1}', Y_{t-2}', \ldots, Y_{t-p}'\right)$. Let $\hat{\pi}_T := vec\left(\hat{\Pi}_T\right)$ denote the $(k \times 1)$-vector of estimated coefficients and π denote the $(k \times 1)$-vector of corresponding population coefficients. Let $\hat{\Omega}_T = \frac{\sum_{t=1}^{T}\left(\hat{\epsilon}_t \hat{\epsilon}_t'\right)}{T}$. Then

2. $\hat{\pi}_T \xrightarrow{p} \pi$.

Estimation and Inference: Formal Result

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4. $\sqrt{T} \cdot (\hat{\pi}_T - \pi) \xrightarrow{d} N\left(0, \Omega \otimes \{\mathbb{E}\left[X_t X_t'\right]\}^{-1}\right)$.

Impulse Response Functions

Outline

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A Impulse Response Function (IRF) is a plot of

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In other words, it describes the response of $Y_{i,t+s}$ to an one-time impulse in $Y_{j,t}$ with all other variables dated t or earlier held constant.

Impulse Response Functions

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5. Repeat this procedure for every $j \in \{1, 2, \ldots, n\}$ to calculate all columns of Ψ_s.

So, if we can estimate Π, we can estimate Ψ_s.

Interpreting Impulse Response Functions

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If Ω is a diagonal matrix, the increase in $\epsilon_{j,t}$ can be interpreted as an independent shock in equation j with no current effect in the other equations.

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In other words, when we increase $\epsilon_{j,t}$, we must change $\epsilon_{i,t}$ too. But derivatives keep everything else constant!

It is impossible in this framework to interpret what the impulse response in a simple VAR means. In other words, the IRF is a response to something that does not exist because, in the real world, these shocks do not move on their own.

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This transformation creates what is known as a structural VAR or an orthogonilized VAR.

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We can see A as a function of Ω. (So, if we can estimate Ω, we can estimate A.)

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$$\begin{aligned} Y_t &= c + \Phi_1 Y_{t-1} + \Phi_2 Y_{t-2} + \ldots + \Phi_p Y_{t-p} + \epsilon_t \\ &= c + \Phi_1 Y_{t-1} + \Phi_2 Y_{t-2} + \ldots + \Phi_p Y_{t-p} + AA^{-1}\epsilon_t \\ &= c + \Phi_1 Y_{t-1} + \Phi_2 Y_{t-2} + \ldots + \Phi_p Y_{t-p} + A\nu_t \\ &\quad \text{where } \nu_t := A^{-1}\epsilon_t \end{aligned}$$

Structural Impulse Response Functions

Note that

$$
\begin{aligned}
\mathbb{E}\left[\nu_t \nu_t'\right] &= \mathbb{E}\left[A^{-1}\epsilon_t\left(A^{-1}\epsilon_t\right)'\right] \\
&= \mathbb{E}\left[A^{-1}\epsilon_t \epsilon_t'\left(A^{-1}\right)'\right] \\
&= A^{-1}\mathbb{E}\left[\epsilon_t \epsilon_t'\right]\left(A'\right)^{-1} \\
&= A^{-1}\Omega\left(A'\right)^{-1} \\
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Then, a shock to $v_{j,t}$ can be interpreted as a shock to equation j independently from the other equations.

Structural Impulse Response Functions

The Structural Impulse Response Function of $Y_{i,t}$ to a shock in $\nu_{j,t}$ plots

$$\frac{\partial Y_{i,t+s}}{\partial \nu_{j,t}}$$

as a function of s.

Structural Impulse Response Functions

We can compute $\dfrac{\partial Y_{i,t+s}}{\partial \nu_{j,t}}$ by connecting it to the reduced-form impulse response function Ψ_s and matrix A.

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Computing the Structural Impulse Response Function

We have that

$$\frac{\partial Y_{t+s}}{\partial \nu_{j,t}} = \Psi_s \begin{bmatrix} a_{1,j} \\ a_{2,j} \\ \vdots \\ a_{n,j} \end{bmatrix},$$

where $a_{i,j}$ is the cell in row i and column j of matrix A and $a_{i,j} = 0$ if $j > i$.

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We can estimate the Structural Impulse Response Function because we can estimate A and Ψ_s by estimating Π and Ω.

Computing the Structural Impulse Response Function: Proof

Note that

$$\begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \\ \vdots \\ \epsilon_{n,t} \end{bmatrix} = \epsilon_t = A\nu_t = \begin{bmatrix} a_{1,1} & 0 & \cdots & 0 \\ a_{2,1} & a_{2,2} & \cdots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix} \begin{bmatrix} \nu_{1,t} \\ \nu_{2,t} \\ \vdots \\ \nu_{n,t} \end{bmatrix}$$

implies that

$$\frac{\partial \epsilon_t}{\partial \nu_{j,t}} = \begin{bmatrix} a_{1,j} \\ a_{2,j} \\ \vdots \\ a_{n,j} \end{bmatrix}.$$

Computing the Structural Impulse Response Function: Proof

Note also that

$$\frac{\partial Y_{i,t+s}}{\partial \nu_{j,t}} = \frac{\partial Y_{i,t+s}\left(\epsilon_t\left(\nu_{1,t},\ldots,\nu_{j,t},\ldots,\nu_{n,t}\right)\right)}{\partial \nu_{j,t}} = \frac{\partial Y_{i,t+s}}{\partial \epsilon_t'}\frac{\partial \epsilon_t}{\partial \nu_{j,t}}$$

$$= \begin{bmatrix} \frac{\partial Y_{i,t+s}}{\partial \epsilon_{1,t}} & \frac{\partial Y_{i,t+s}}{\partial \epsilon_{2,t}} & \cdots & \frac{\partial Y_{i,t+s}}{\partial \epsilon_{n,t}} \end{bmatrix} \begin{bmatrix} a_{1,j} \\ a_{2,j} \\ \vdots \\ a_{n,j} \end{bmatrix}$$

$$= \frac{\partial Y_{i,t+s}}{\partial \epsilon_{1,t}} \cdot a_{1,j} + \frac{\partial Y_{i,t+s}}{\partial \epsilon_{2,t}} \cdot a_{2,j} + \cdots + \frac{\partial Y_{i,t+s}}{\partial \epsilon_{n,t}} \cdot a_{n,j}$$

Computing the Structural Impulse Response Function: Proof

Consequently, we have that

$$\frac{\partial Y_{t+s}}{\partial \nu_{j,t}} = \begin{bmatrix} \dfrac{\partial Y_{1,t+s}}{\partial \nu_{j,t}} \\ \dfrac{\partial Y_{2,t+s}}{\partial \nu_{j,t}} \\ \vdots \\ \dfrac{\partial Y_{n,t+s}}{\partial \nu_{j,t}} \end{bmatrix} = \Psi_s \begin{bmatrix} a_{1,j} \\ a_{2,j} \\ \vdots \\ a_{n,j} \end{bmatrix}.$$

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- The n-th variable reacts to current shocks to any variable. $(\epsilon_{n,t} = a_{n,1} \cdot \nu_{1,t} + a_{n,2} \cdot \nu_{2,t} + \ldots + a_{n,n} \cdot \nu_{n,t})$

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I will illustrate how to "defend" those exclusion restrictions with a concrete example.

Connecting VARs to Structural Models

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- P_t is the log of the aggregate price level,
- Y_t is the log of real GNP,
- I_t is a nominal interest rate,
- V_t^D represents factors other than income and interest rates that influence money demand.

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- β_1 and β_2 represent the effect of income and interest rates on desired cash holdings.
- Part of the adjustment in money balances to a change in income is thought to take effect immediately, with further adjustments coming in subsequent periods.
- β_3 characterizes this partial adjustment.

Connecting VARs to Structural Models

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Connecting VARs to Structural Models

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$$M_t - P_t = \beta_0 + \beta_1 \cdot Y_t + \beta_2 \cdot I_t + \beta_3 \cdot (M_{t-1} - P_{t-1}) + V_t^D$$

We assume that $V_t^D = \rho \cdot V_{t-1}^D + \eta_t^D$, where η_t^D is white noise.

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$$\begin{aligned} M_t - P_t = (1-\rho) \cdot \beta_0 + \beta_1 \cdot Y_t - \beta_1 \cdot \rho \cdot Y_{t-1} + \beta_2 \cdot I_t - \beta_2 \cdot \rho \cdot I_{t-1} \\ + (\beta_3 + \rho) \cdot (M_{t-1} - P_{t-1}) - \beta_3 \cdot \rho \cdot (M_{t-2} - P_{t-2}) + \eta_t^D \end{aligned}$$

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Our money demand function fits, now, a restricted *ADL* model.

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- $\beta_{32}^{(0)}$ and $\beta_{34}^{(0)}$ are the effects of current income and interest rate on desired money holdings.

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- $\beta_{32}^{(0)}$ and $\beta_{34}^{(0)}$ are the effects of current income and interest rate on desired money holdings.
- η_t^D represents factors influencing money demand other than inflation, income and interest rates.
- This *ADL* model generalizes the dynamic behavior for the error term V_t^D, the partial adjustment process and the influence of the price level on desired money holdings.

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We cannot estimate this model by *OLS* due to simultaneous equation bias.

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For example, each period, the Central Bank may be adjusting I_t to a level consistent with its policy objectives, which may depend on current and lagged values of income, interest rates, price levels and money supply:

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$$
\begin{aligned}
I_t = k_4 &+ \beta_{41}^{(0)} \cdot P_t + \beta_{42}^{(0)} \cdot Y_t + \beta_{43}^{(0)} \cdot M_t \\
&+ \beta_{41}^{(1)} \cdot P_{t-1} + \beta_{42}^{(1)} \cdot Y_{t-1} + \beta_{43}^{(1)} \cdot M_{t-1} + \beta_{44}^{(1)} \cdot I_{t-1} + \dots \\
&+ \beta_{41}^{(p)} \cdot P_{t-p} + \beta_{42}^{(p)} \cdot Y_{t-p} + \beta_{43}^{(p)} \cdot M_{t-p} + \beta_{44}^{(p)} \cdot I_{t-p} + \eta_t^C .
\end{aligned}
$$

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Taylor Rule:

$$\begin{aligned} I_t = k_4 &+ \beta_{41}^{(0)} \cdot P_t + \beta_{42}^{(0)} \cdot Y_t + \beta_{43}^{(0)} \cdot M_t \\ &+ \beta_{41}^{(1)} \cdot P_{t-1} + \beta_{42}^{(1)} \cdot Y_{t-1} + \beta_{43}^{(1)} \cdot M_{t-1} + \beta_{44}^{(1)} \cdot I_{t-1} + \dots \\ &+ \beta_{41}^{(p)} \cdot P_{t-p} + \beta_{42}^{(p)} \cdot Y_{t-p} + \beta_{43}^{(p)} \cdot M_{t-p} + \beta_{44}^{(p)} \cdot I_{t-p} + \eta_t^C. \end{aligned}$$

Connecting VARs to Structural Models

Moreover, money demand disturbances and changes in Central Bank policy also have effects on aggregate output and the price level.

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Aggregate demand equation:

$$\begin{aligned} Y_t = k_2 &+ \beta_{21}^{(0)} \cdot P_t + \beta_{23}^{(0)} \cdot M_t + \beta_{24}^{(0)} \cdot I_t \\ &+ \beta_{21}^{(1)} \cdot P_{t-1} + \beta_{22}^{(1)} \cdot Y_{t-1} + \beta_{23}^{(1)} \cdot M_{t-1} + \beta_{24}^{(1)} \cdot I_{t-1} + \dots \\ &+ \beta_{21}^{(p)} \cdot P_{t-p} + \beta_{22}^{(p)} \cdot Y_{t-p} + \beta_{23}^{(p)} \cdot M_{t-p} + \beta_{24}^{(p)} \cdot I_{t-p} + \eta_t^A. \end{aligned}$$

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\end{aligned}
$$

Aggregate supply curve:

$$
\begin{aligned}
P_t = k_1 &+ \beta_{12}^{(0)} \cdot Y_t + \beta_{13}^{(0)} \cdot M_t + \beta_{14}^{(0)} \cdot I_t \\
&+ \beta_{11}^{(1)} \cdot P_{t-1} + \beta_{12}^{(1)} \cdot Y_{t-1} + \beta_{13}^{(1)} \cdot M_{t-1} + \beta_{14}^{(1)} \cdot I_{t-1} + \dots \\
&+ \beta_{11}^{(p)} \cdot P_{t-p} + \beta_{12}^{(p)} \cdot Y_{t-p} + \beta_{13}^{(p)} \cdot M_{t-p} + \beta_{14}^{(p)} \cdot I_{t-p} + \eta_t^S.
\end{aligned}
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Writing our model in vector form, we have that:

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$$B_0 X_t = k + B_1 X_{t-1} + \ldots + B_p X_{t-p} + \eta_t,$$

where $X_t = (P_t, Y_t, M_t, I_t)'$, $\eta_t = \left(\eta_t^S, \eta_t^A, \eta_t^D, \eta_t^C\right)'$, $(k_1, k_2, k_3, k_4)'$,

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$$B_0 = \begin{bmatrix} 1 & -\beta_{12}^{(0)} & -\beta_{13}^{(0)} & -\beta_{14}^{(0)} \\ -\beta_{21}^{(0)} & 1 & -\beta_{23}^{(0)} & -\beta_{24}^{(0)} \\ -\beta_{31}^{(0)} & -\beta_{32}^{(0)} & 1 & -\beta_{34}^{(0)} \\ -\beta_{41}^{(0)} & -\beta_{42}^{(0)} & -\beta_{43}^{(0)} & 1 \end{bmatrix},$$

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for any $s \in \{1, 2, \ldots, p\}$

Connecting VARs to Structural Models

Making exclusion restrictions about B_0:

$$B_0 = \begin{bmatrix} 1 & -\beta_{12}^{(0)} & -\beta_{13}^{(0)} & -\beta_{14}^{(0)} \\ -\beta_{21}^{(0)} & 1 & -\beta_{23}^{(0)} & -\beta_{24}^{(0)} \\ -\beta_{31}^{(0)} & -\beta_{32}^{(0)} & 1 & -\beta_{34}^{(0)} \\ -\beta_{41}^{(0)} & -\beta_{42}^{(0)} & -\beta_{43}^{(0)} & 1 \end{bmatrix}$$

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Prices respond to other economic variables only with a lag because they are sticky.

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Money and interest rates influence aggregate demand only with a lag due to adjustment costs.

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Since most Central Banks monitor current economic conditions quite carefully, all the current vales should be included in the equation for I_t.

Connecting VARs to Structural Models

Our model

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Connecting VARs to Structural Models

Moreover, assume that the disturbances η_t are serially uncorrelated and uncorrelated with each other:

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$$\mathbb{E}\left[\eta_t \eta_\tau'\right] = \begin{cases} D^{1/2} D^{1/2} & \text{for } t = \tau \\ 0 & \text{otherwise,} \end{cases}$$

where D is a diagonal matrix and the diagonal elements of $D^{1/2}$ are the standard deviations of each disturbance.

Connecting VARs to Structural Models

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can be rewritten as

$$\begin{aligned}
X_t &= (B_0)^{-1} k + (B_0)^{-1} B_1 X_{t-1} + \ldots + (B_0)^{-1} B_p X_{t-p} + (B_0)^{-1} D^{1/2} D^{-1/2} \eta_t, \\
&= (B_0)^{-1} k + (B_0)^{-1} B_1 X_{t-1} + \ldots + (B_0)^{-1} B_p X_{t-p} + (B_0)^{-1} D^{1/2} \nu_t, \\
&\qquad \text{where } \nu_t := D^{-1/2} \eta_t \\
&= (B_0)^{-1} k + (B_0)^{-1} B_1 X_{t-1} + \ldots + (B_0)^{-1} B_p X_{t-p} + A \nu_t, \\
&\qquad \text{where } A := (B_0)^{-1} D^{1/2} \text{ is a lower triangular matrix.}
\end{aligned}$$

Connecting VARs to Structural Models

Our model is now a Structural VAR:

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since

$$\begin{aligned}
\mathbb{E}\left[\nu_t \nu_t'\right] &= \mathbb{E}\left[D^{-1/2}\eta_t \left(D^{-1/2}\eta_t\right)'\right] \\
&= D^{-1/2}\mathbb{E}\left[\eta_t \eta_t'\right] D^{-1/2} \\
&= D^{-1/2} D^{1/2} D^{1/2} D^{-1/2} \\
&= I_n
\end{aligned}$$

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- There is an unique decomposition $A = PQ$ where P is a lower triangular matrix and Q is a diagonal matrix.
- We have that $B_0 = P^{-1}$ and $D^{1/2} = Q$.
- We also have that $B_s = P^{-1}\Phi_s$ for any $s \in \{1, 2, \ldots, p\}$.

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Interpreting the Structural Impulse Response Functions

Connecting VARs to Structural Models

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Interpreting the Structural Impulse Response Functions

- Provided that our exclusion restrictions are valid, the structural IRFs would give the dynamic consequences of the structural events represented by ν_t.

Thank you!

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