We want to be able to build feasible models numerically, *i.e.* we would like to generate a set of constant numbers $\{l_\nu, m_\nu, R^*_\nu, S^*_j, \gamma_{j\nu}, \alpha_{\nu j}, \sigma_{j\nu}, \tau_{j\nu}\}$ such that the equilibria equations Eqs.(??) are fulfilled.

0.0.1 Algorithmic procedure

We hereby detail the procedure used to numerically build feasible systems. It goes like this:

1. We first draw randomly each R^*_ν and S^*_i from a uniform distribution of mean equal to the corresponding metaparameter, *i.e.* :

$$R^*_\nu = \mathcal{R} \ \forall \nu = 1, \ldots, N_R \text{ and } S^*_i = \mathcal{S} \ \forall i = 1, \ldots, N_S, \qquad (1)$$

where $\mathcal{R}$ and $\mathcal{S}$ are random variables coming from a distribution of mean equal to the corresponding metaparameter and relative standard deviation[1] ϵ. In our simulations, we chose uniform distributions :

$$\mathcal{R} \sim \text{Unif}(R_0, \epsilon) \text{ and } \mathcal{S} \sim \text{Unif}(S_0, \epsilon). \qquad (2)$$

2. The efficiency matrix $\sigma_{i\nu}$ is then drawn similarly, from a distribution with average σ_0. In order to simplify the problem[2], we will take a zero-variance à la Butler and O'Dwyer in [1], *i.e.* all species consume resources at the same global efficiency :

$$\sigma_{i\nu} = \sigma_0. \qquad (3)$$

3. We build the consumption matrix $\gamma_{i\nu}$. Its adjacency matrix G is loaded through a user-provided file. While G gives the structure of γ, *i.e.* if a given $\gamma_{i\nu}$ is zero or not, the actual values of $\gamma_{i\nu}$ need then to be determined. They are drawn from a uniform distribution of mean γ_0 and relative standard deviation ϵ:

$$\gamma_{i\nu} = \text{Unif}(\gamma_0, \epsilon)G_{i\nu}. \qquad (4)$$

4. We draw the resources external feeding rates, similarly to the other parameters :

$$l_\mu = \text{Unif}(l_0, \epsilon) \ \forall \mu = 1, \ldots, N_R. \qquad (5)$$

5. The last free parameter is the syntrophy matrix $\alpha_{\nu i}$, the d_i and l_μ are determined through the equations of evolution at equilibrium. This is the tricky part of the algorithm because α has to follow three constraints, namely energy conservation/dissipation Eq.(??) and positiveness of d_i and l_μ [**insert reference to equation**]. The general strategy is to choose the metaparameters in a way that these constraints should *almost always* be satisfied, *i.e.* we pick metaparameters that follow the feasibility constraint Eq.(??). The adjacency matrix A of α needs then to be specified. At the moment, it can be chosen in three different ways : fully connected, or in a way that no resource eaten

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[1]By relative standard deviation, we mean the standard deviation measured in units of the average value.
[2]Indeed, a non uniform σ_0 introduces instability in the system

by a given species can be released by that same species (*i.e.* $G_{i\mu} > 0 \iff A_{\mu i} = 0$) or by a user provided matrix. After the adjacency matrix is loaded, we can build α from a uniform distribution of mean α_0 and relative standard deviation ϵ:

$$\alpha_{\nu i} = \text{Unif}(\alpha_0, \epsilon) A_{\nu i}. \tag{6}$$

6. With all of these parameters drawn, we can solve Eq.(**??**) for the species death rate d_i.

7. We solve Eq.(**??**) for m_ν. All the parameters of the model are now fully determined.

8. We check if the constraints Eq.(**insert reference**) on the parameters are fulfilled. If they are not, we go back to step 1. Otherwise, we can exist the algorithm, a feasible system has been built.

0.0.2 The feasibility volume

The algorithmic procedure above explains how feasible systems can be built. However, it implies that we first found a combination of metaparameters that will most of the time lead to the realisation of feasible systems when they are taken as an input of the algorithm.

Overall we have six metaparameters that we can play with : γ_0, α_0, l_0, σ_0, S_0 and R_0. However, following the analysis of [2], we notice that our system (**??**) is arbitrary on some level. Indeed we have a "scale freedom", that means we decide in which set of units we work. There are two physical quantities at stake here : biomass and time, and we may choose, however we want it, a specific set of units describing both of them.

We will measure biomass in units of the average resource abundance at equilibrium[3], that means :

$$\langle R_\mu \rangle = R_0 = 1. \tag{7}$$

Similarly, we will measure time such that the average external resource uptake rate is one, that is :

$$\langle l_\mu \rangle = l_0 = 1. \tag{8}$$

After this manipulation, our number of metaparameters is reduced from six to four : only γ_0, S_0, α_0 and σ_0 remain.

For the sake of simplicity, we will keep the same σ_0 throughout our whole study. We take a value close to the efficiency of real microbial systems [**insert ref**], that is $\sigma_0 = 0.3$.

Overall, we need to choose the last three remaining metaparameters: α_0, γ_0 and S_0. What Eq.(**??**) tells us is that as soon if we choose γ_0 and S_0, we will get a feasibility range for α_0. We will then choose γ_0 and S_0 such that they lead to feasible systems for every consumption matrix considered here **when there is no syntrophy**, *i.e.* $\alpha_0 = 0$. We will then study the impact of varying α_0 at those values of γ_0 and S_0.

Formally, we can define for a consumption adjacency matrix G the volume $\mathcal{V}_x^G$ of the metaparameters space that will lead to at least a ratio x of feasible systems *i.e.* :

$$\mathcal{V}_x^G \equiv \{P \in \mathcal{M} : \text{feasibility}(P, G) \geqslant x\}. \tag{9}$$

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[3]Note that this is not a completely innocent choice. Indeed we will see later that the matrix $\alpha_{\nu i} - \gamma_{i\nu} R_\nu^*$ is a crucial quantity here. Setting $\langle R^* \rangle = 1$ allows us to simply study the impact of γ against α instead of the more complicated γR^* versus α.

It is clear that $\mathcal{V}_0^G = \mathcal{M}$ $\forall G$ and $\mathcal{V}_x^G \leqslant \mathcal{V}_y^G$ $\forall x > y, G$. We can similarly define for a set $S = \{G_1, G_2, \ldots, G_N\}$ of N matrices their *common feasibility* volume $\mathcal{V}_x^S$, which is the region of the metaparameters space where feasibility is at least x for every matrix in the set:

$$\mathcal{V}_x^S = \bigcap_{G \in S} \mathcal{V}_x^G. \tag{10}$$

We also define for a matrix set S, its critical feasibility $x^*(S)$, which is the largest feasibility we can get while still having a non-zero common volume :

$$x^*(S) \equiv \max_{x \in [0,1]} \left\{ x : \mathcal{V}_x^S > 0 \right\}. \tag{11}$$

For actual computations, we will choose a matrix set S_M, stick to it during the whole thesis, and work in its critical feasibility volume $\mathcal{V}^*$, defined as :

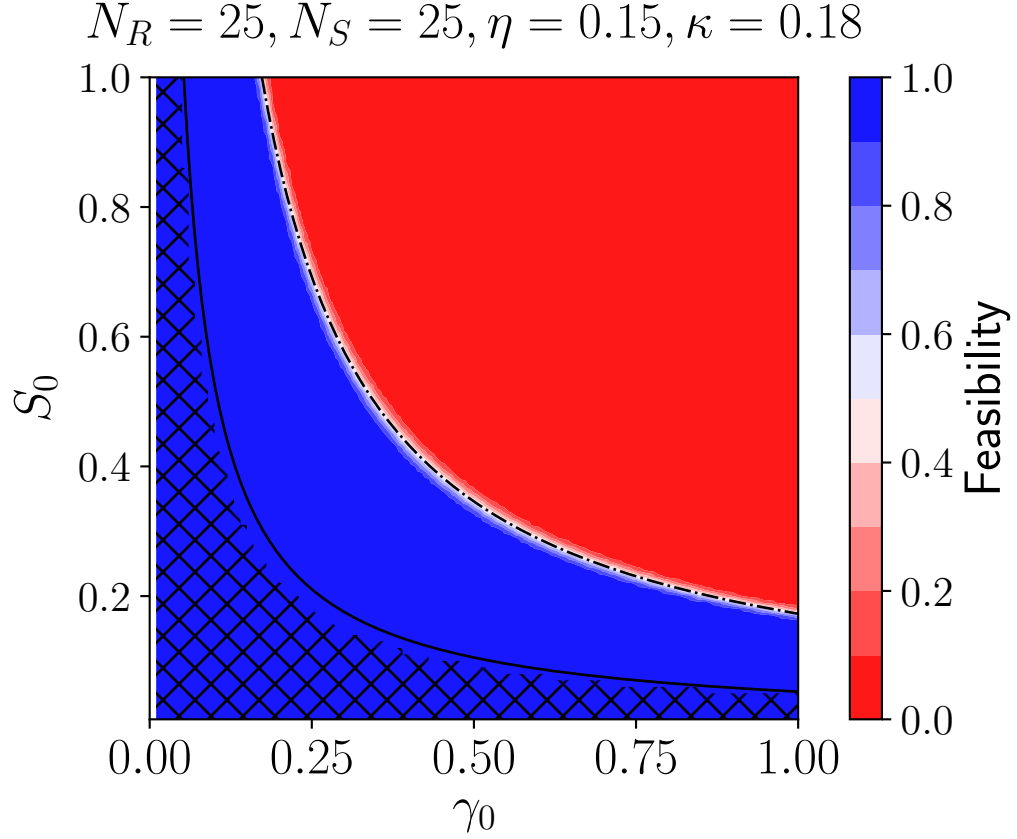$$\boxed{\mathcal{V}^* \equiv \mathcal{V}_{x^*(S_M)}^{S_M}.} \tag{12}$$

Figure 1: Plot of the feasability area. The color curve indicicates the feasibility function $\mathcal{F}$ for the matrix of our set with connectance 0.18 and nestedness 0.15 . We observe a steep descent from a totally feasible to totally unfeasible regime. A fit of the points $\mathcal{F}(\gamma_0, S_0) \approx 0.5$ yields $S_0 = 0.17310809\gamma_0 - 0.0029606$ (with a relative error of 10^{-6}). The theoretical prediction is $S_0 = 0.2\gamma_0$. The full line indicates the theoretical critical feasibility volume, and the dashed area the measured one. Although not perfect it matches quite well. A fit yields $S_0 = 0.04261718\gamma_0 - 0.00456834$ with a relative error of the order of 10^{-6}. Theoretical predicts $S_0 = 0.0526316\gamma_0$