0.1 Figuring out the effective competition

From the main document (Eq. 5.37), we have for the effective system:

$$\frac{dS_i}{dt} = \left[\sum_\nu \left(\frac{\sigma_{i\nu}\gamma_{i\nu}l_\nu}{m_\nu + \sum_k \gamma_{k\nu}S_k} - \alpha_{\nu i}\right) - d_i + \sum_{\nu j} \frac{\sigma_{i\nu}\gamma_{i\nu}\alpha_{\nu j}}{m_\nu + \sum_k \gamma_{k\nu}S_k} S_j\right] S_i \tag{1}$$

The goal is to regroup terms such that we may rewrite this equation in the form of:

$$\frac{dS_i}{dt} = \left(\tilde{K}_i - \sum_j C_{ij}S_j + \text{higher order interactions ...}\right) S_i \tag{2}$$

Following standard Lotka-Volterra reasoning (see e.g. [1]), we may interpret C_{ij} as the **effective competition matrix** and define its average ρ_{eff} as the **effective competition**:

$$\rho_{\text{eff}} = \frac{1}{N_S^2}\sum_{ij} C_{ij}. \tag{3}$$

The tricky part is that we have S_k at the denominator of some terms of Eq.(1) and we somehow want them to appear at the numerator, i.e. we want to do a Taylor expansion. We write:

$$m_\nu + \sum_k \gamma_{k\nu}S_k = m_\nu + \sum_k \gamma_{k_\nu} S_k^* - \sum_k \gamma_{k_\nu}\left(S_k^* - S_k\right) \tag{4}$$

We may safely assume that $S_k^* - S_k$ is small compared to the other term, which allows us to Taylor expand:

$$\frac{1}{m_\nu + \sum_k \gamma_{k\nu}S_k} \approx \frac{1}{m_\nu + \gamma_{k\nu}S_k^*}\left[\left(1 + \sum_k \frac{\gamma_{k\nu}S_k^*}{m_\nu + \sum_l \gamma_{l\nu}S_l^*}\right) - \sum_j \frac{\gamma_{j\nu}}{m_\nu + \sum_l \gamma_{l\nu}S_l^*}S_j\right] \tag{5}$$

We would like to get rid of m_ν, since we do not directly control it (m and d are the only parameters that we do not choose, see p.33 of the main document). For this we use Eq.(3.2b) of the main document:

$$m_\nu + \sum_k \gamma_{k\nu}S_k^* = \frac{l_\nu + \sum_k \alpha_{\nu k}S_k^*}{R_\nu^*} = D_\nu \tag{6}$$

This is interesting: D_ν is a strictly positive quantity which appears naturally in the jacobian and we see here that it somehow plays a role in the effective one too! Inserting this in the previous equation yields:

$$\frac{1}{m_\nu + \sum_k \gamma_{k\nu}S_k} \approx \frac{1}{D_\nu}\left[\left(1 + \sum_j \frac{\gamma_{j\nu}S_j^*}{D_\nu}\right) - \sum_j \frac{\gamma_{j\nu}}{D_\nu}S_j\right] = \left(\frac{1}{D_\nu} + \sum_j \frac{\gamma_{j\nu}S_j^*}{D_\nu^2}\right) - \sum_j \frac{\gamma_{j\nu}}{D_\nu^2}S_j \tag{7}$$

This allows us to write Eq.(1) as:

$$\frac{dS_i}{dt} = \left[\tilde{K}_i - \sum_{\nu j} \frac{\sigma_{i\nu}\gamma_{i\nu}}{D_\nu^2}\left(\gamma_{j\nu}l_\nu - \alpha_{\nu j}\left(D_\nu + \sum_k \gamma_{k\nu}S_k^*\right)\right) S_j + \dots\right] S_i \tag{8}$$

Which gives us the following expression for the effective competition matrix:

$$C_{ij} = \sum_{\nu} \frac{\sigma_{i\nu}\gamma_{i\nu}}{D_{\nu}^2} \left(\gamma_{j\nu} l_{\nu} - \alpha_{\nu j} \left(D_{\nu} + \sum_{k} \gamma_{k\nu} S_k^* \right) \right) \tag{9}$$

However we would like an expression which depends only on metaparameters and on the G, A topology matrices, so that we may get an intuition about how this effective competition matrix behaves. The first step is to notice that $D_{\nu} \gg \sum_k \gamma_{k\nu} S_k^*$. Indeed, it is easy to show with basic metaparameters considerations that $m_{\nu} \gg \sum_k \gamma_{k\nu} S_k^*$ which implies $D_{\nu} \gg \sum_k \gamma_{k\nu} S_k^*$ and hence:

$$D_{\nu} + \sum_{k} \gamma_{k\nu} S_k^* \approx D_{\nu} \tag{10}$$

With that approximation, Eq.(9) becomes:

$$C_{ij} \approx \sum_{\nu} \frac{\sigma_{i\nu}\gamma_{i\nu}}{D_{\nu}} \left(\gamma_{j\nu} \frac{l_{\nu}}{D_{\nu}} - \alpha_{\nu j} \right) \tag{11}$$

We then try to make sense of the l_{ν}/D_{ν} term:

$$\frac{l_{\nu}}{D_{\nu}} = \frac{R_{\nu}^*}{1 + \sum_k \frac{\alpha_{\nu k} S_k^*}{l_{\nu}}} \approx \frac{R_{\nu}^*}{1 + \frac{N_S \kappa_A S_0}{l_0}} \tag{12}$$

When building the optimal syntrophy matrices, we take $\gamma_0 = 1$, which means that S_0 **must** be small (remember that feasible systems all respect a relation $S_0 < K\gamma_0^{-1}$), typically we use $S_0 = 0.05$ during the A-optimization process. We also decided to set $l_0 = 1$, $N_S = 25$. An outcome of the optimization is that $\kappa_A \approx \kappa_G$ which is fairly small. These arguments allow to say that $N_S \kappa_A S_0 / l_0$ will be smaller than 1 which means:

$$\frac{l_{\nu}}{D_{\nu}} \approx R_{\nu}^*. \tag{13}$$

Finally, we try to simplify D_{ν}:

$$D_{\nu} = \frac{l_{\nu} + \sum_k \alpha_{\nu k} S_k^*}{R_{\nu}^*} \approx \frac{l_0 + N_S \kappa_A \alpha_0 S_0}{R_0} \approx \frac{l_0}{R_0} \tag{14}$$

Inserting this in Eq.(11) yields:

$$C_{ij} \approx \frac{R_0}{l_0} \sum_{\nu} \frac{\sigma_{i\nu}\gamma_{i\nu} S_i^*}{S_i^*} \left(\gamma_{j\nu} R_{\nu}^* - \alpha_{\nu j} \right) \approx \frac{R_0}{l_0 S_0} \sum_{\nu} \sigma_{i\nu}\gamma_{i\nu} S_i^* \left(\gamma_{j\nu} R_{\nu}^* - \alpha_{\nu j} \right) \tag{15}$$

This expression can be made clearer using the B and Γ matrices:

$$\boxed{C_{ij} = -\frac{R_0}{l_0 S_0} \left(\mathrm{B}\Gamma \right)_{ij}} \tag{16}$$

It is important to remember that this approximation is only true for systems with low S_0 and we should expect a slight departure from it for high S_0 (i.e low γ_0), which can be easily computed. We also assumed κ_A was small so this won't be a good measure for $e.g.$ the fully connected case. However, note that Eq.(9) is always true (as long as the assumptions for the fully effective system are fulfilled) and should be used in these extreme cases.

0.2 Mathematical computations

0.2.1 Simplifying the denominator

We stated earlier that $m_\nu \gg \sum_k \gamma_{k\nu} S_k^*$. This is true under certain assumptions. Using metaparameters considerations we can approximate both of these terms:

$$m_\nu = \frac{l_\nu + \sum_k \alpha_{\nu k} S_k^*}{R_\nu^*} - \sum_k \gamma_{k\nu} S_k^* \approx \frac{l_0 + N_S \kappa_A \alpha_0 S_0}{R_0} - N_S \gamma_0 \kappa_G S_0 \tag{17}$$

And :

$$\sum_k \gamma_{k\nu} S_k^* \approx N_S \gamma_0 \kappa_G S_0 \tag{18}$$

Hence :

$$m_\nu - \sum_k \gamma_{k\nu} S_k^* \approx \frac{l_0 + N_S \kappa_A \alpha_0 S_0}{R_0} \tag{19}$$

Bibliography

[1] Keith W. Dorschner et al. "Lotka-Volterra Competition Revisited: The Importance of Intrinsic Rates of Increase to the Unstable Equilibrium". In: *Oikos* 48.1 (1987). Publisher: [Nordic Society Oikos, Wiley], pp. 55–61. ISSN: 0030-1299. DOI: 10.2307/3565688. URL: https://www.jstor.org/stable/3565688 (visited on 08/04/2021).