0.1 Building the model numerically

We want to be able to build feasible models numerically, *i.e.* we would like to generate a set of constant numbers $\{l_\nu, m_\nu, R_\nu^*, S_j^*, \gamma_{j\nu}, \alpha_{\nu j}, \sigma_{j\nu}, \tau_{j\nu}\}$ such that the equilibria equations Eqs.(**??**) are fulfilled.

0.1.1 Algorithmic procedure

We hereby detail the procedure used to numerically build feasible systems. It goes like this:

1. We first draw randomly R_ν^* and S_i^* as a uniform distribution of mean equal to the corresponding metaparameter, *i.e.* :

$$\sum_\nu R_\nu^* = N_R R_0 \text{ and } \sum_i S_i^* = N_S S_0. \tag{1}$$

2. The efficiency matrix $\sigma_{i\nu}$ is then drawn similarly, on a uniform distribution such that σ_0 is the average of the matrix :

$$\sum_{i,\nu} \sigma_{i\nu} = N_S N_R \sigma_0. \tag{2}$$

3. We build gamma using the desired *food matrix* F. F is a binary matrix given by the user (in the `configuration.in` file) and is defined as the adjacency matrix of the consumption network (*i.e.* it tells which species eats which resource). We then build γ with the same network structure as F (*i.e.* both matrices have the same zero elements). The consumption rates are then randomly drawn from a uniform distribution and γ is rescaled such that γ_0 represents the average consumption rate of the system :

$$\sum_{i,\nu} \gamma_{i\nu} = N_S N_R \gamma_0. \tag{3}$$

4. We then need to build $\alpha_{\nu i}$. This is the tricky part of the algorithm because there are constraints on α, for instance energy conservation/dissipation Eq.(5). The general strategy is to assume that the metaparameters are chosen in a way that those constraints will practically always be satisfied (see above). We can then build α from a random uniform distribution such that:

$$\sum_{i,\nu} \alpha_{\nu i} = N_S N_R \alpha_0. \tag{4}$$

   If for some reason the algorithm fails to build a feasible system this way after a given number of attempts, the $\alpha_{\nu i}$ are drawn by the algorithm and the initial α_0 is rescaled accordingly.

5. We build $\tau_{\nu i}$. It usually is equal to $\alpha_{\nu i}$ or 0.

6. With all of these parameters drawn, we can solve Eq.(**??**) for the species death rate d_i (with the caveat that $d_i > 0$, this is one of the constraints on τ and hence α).

7. Finally, we solve Eq.(**??**) for l_ν and m_ν imposing the constraint $l_\nu, m_\nu > 0$. In practice this means one of them is drawn randomly (in the code, l_ν comes from an exponential distribution) with constraints (in the code the minimum value of l_ν) such that both l_ν and m_ν are positive.

0.2 Conditions on the model parameters

Although many studies focus on the study of systems described by random γ, σ or α matrices [**insert ref**], we will focus on systems that respect physical or biological constraints given below. Those systems are called *feasible*.

0.2.1 Energy conservation/dissipation

The first condition we will impose on our systems is that they do not create new matter.

Remember that in our model, the total amount of biomass given to the species i by the resources is $\sum_\nu \gamma_{i\nu} R_\nu S_i$. However species i will not allocate all of this biomass to growth. As seen in Eq.(**??**), only a fraction $\sigma_{i\nu}$ will be used in this purpose. This means species i disposes of $\sum_\nu (1-\sigma_{i\mu})\gamma_{i\mu} R_\mu S_i$ biomass to complete other processes. We know that one of these is producing byproducts (*i.e.* the syntrophic interaction) at a total rate $\sum_\nu \alpha_{\nu i} S_i$. Because this biomass is produced in the cell, it has to come from the biomass the cell disposes of, which naturally leads to the condition[1]:

$$\sum_\nu (1-\sigma_{i\nu})\,\gamma_{i\nu} R_\nu^* \geqslant \sum_\nu \alpha_{\nu i} \tag{5}$$

The above equation will be referred to as the *conservation of biomass constraint*.

The idea is to find metaparameters such that this constraint is automatically satisfied (which eases building the system numerically). This is easily done by finding the minimum of the LHS and maximum of RHS of Eq.(5). Indeed :

$$\sum_\nu (1-\sigma_{i\nu})\,\gamma_{i\nu} R_\nu^* \geqslant (1-\widehat{\sigma})\check{\gamma}\widecheck{R^*}, \tag{6}$$

where $\check{}$ denotes the minimum value of the random variable and $\widehat{}$ its maximum value. On the other hand,

$$\sum_\nu \alpha_{\nu i} \leqslant \widehat{\alpha} N_R. \tag{7}$$

This means that if we take metaparameters such that

$$\widehat{\alpha} N_R < (1-\widehat{\sigma})\check{\gamma}\widecheck{R^*}, \tag{8}$$

then Eq.(5) is automatically followed.

Because of the way we choose our variables we have for every random variable in the problem,

$$\check{X} = (1-\epsilon)\langle X\rangle \text{ and } \widehat{X} = (1+\epsilon)\langle X\rangle \tag{9}$$

[1]Note that the condition is a bit relaxed here. Biomass cannot be created at equilibrium. However we allow some transient regimes where this momentarily can occur, *e.g.* after a big shock inflicted to the system.

where $\langle X \rangle$ denotes the mean of X. This means Eq.(8) is equivalent to, in terms of meta-parameters:

$$\alpha_0 < \frac{(1-\epsilon)^2}{1+\epsilon}\left(1-(1+\epsilon)\,\sigma_0\right)\frac{\gamma_0 R_0}{N_R}. \tag{10}$$

In the $\epsilon \ll 1$ limit, this is equivalent to:

$$\alpha_0 < (1-3\epsilon)\left(1-(1+\epsilon)\sigma_0\right)\frac{\gamma_0 R_0}{N_R}. \tag{11}$$

0.2.2 Positivity of the parameters

Feasability means at least that every physical parameter defined here must be positive. In particular, this implies:

$$d_i > 0 \implies \sum_\mu \sigma_{i\mu}\gamma_{i\mu}R^*_\mu > \sum_\mu \tau_{\mu i} \tag{12}$$

If $\tau_{i\mu} = 0$. this is trivially satisfied because $\sigma_{i\mu}$, $\gamma_{i\mu}$ and R^*_μ have all been drawn positive. However if $\tau_{\mu i} = \alpha_{\mu i}$ this is not always the case and we have to get parameters satisfying :

$$\sum_\mu \sigma_{i\mu}\gamma_{i\mu}R^*_\mu > \sum_\mu \alpha_{\mu i}\ \forall i. \tag{13}$$

We can try to estimate the value of some metaparameters that would satisfy this. We have :

$$\sum_\mu \sigma_{i\mu}\gamma_{i\mu}R^*_\mu \geqslant \breve{\sigma}\breve{\gamma}\breve{R^*}. \tag{14}$$

Using this boundary and Eq.(8), we know that $d_i > 0$ if

$$\widehat{\alpha}N_R \leqslant \breve{\sigma}\breve{\gamma}\breve{R^*}, \tag{15}$$

i.e.

$$\alpha_0 < \frac{(1-\epsilon)^3}{1+\epsilon}\frac{\sigma_0\gamma_0 R_0}{N_R}, \tag{16}$$

or in the $\epsilon \ll 1$ limit :

$$\alpha_0 < (1-4\epsilon)\,\frac{\sigma_0\gamma_0 R_0}{N_R}. \tag{17}$$

Similarly we must have a positive death rate for the resources, *i.e.* :

$$m_\nu = \frac{l_\nu - \sum_j \gamma_{j\nu}R^*_\nu S^*_j + \sum_j \alpha_{\nu j}S^*_j}{R^*_\nu} > 0. \tag{18}$$

This means we have to impose parameters that verify:

$$l_\nu + \sum_j \alpha_{\nu j}S^*_j > \sum_j \gamma_{j\nu}R^*_\nu S^*_j. \tag{19}$$

We can do a reasoning similar to before, *i.e.* find a lower boundary for the LHS and an upper boundary for the RHS. We have

$$l_\nu + \sum_j \alpha_{\nu j}S^*_j \geqslant \breve{l} + \breve{\alpha}\breve{S^*} \tag{20}$$

and

$$\sum_j \gamma_{j\nu} R^*_\nu S^*_j \leqslant N_S \widehat{\gamma} \widehat{R^*} \widehat{S^*}. \tag{21}$$

Hence if we get parameters satisfying

$$\breve{l} + \breve{\alpha} \widetilde{S^*} > N_S \widehat{\gamma} \widehat{R^*} \widehat{S^*}, \tag{22}$$

then Eq.(18) will be immediately satisfied. In terms of metaparameters this is equivalent to:

$$\alpha_0 > \frac{N_S \gamma_0 R_0 S_0 (1+\epsilon)^3 - l_0(1-\epsilon)}{S_0 (1-\epsilon)^2}. \tag{23}$$

In the $\epsilon \ll 1$ limit this is equivalent to:

$$\alpha_0 > (1+5\epsilon)\, N_S \gamma_0 R_0 - (1+\epsilon)\, \frac{l_0}{S_0}. \tag{24}$$

(Interesting, if l_0/S_0 is large enough, *i.e.* "there is enough food for everyone" then this condition is irrelevant).

0.2.3 Combining conditions

If we combine both upperbounds we get a restriction on the metaparameters:

$$\alpha_0 < \min\left(\frac{(1-\epsilon)^2}{1+\epsilon} \left(1 - (1+\epsilon)\,\sigma_0\right) \frac{\gamma_0 R_0}{N_R}, \frac{(1-\epsilon)^3}{1+\epsilon} \frac{\sigma_0 \gamma_0 R_0}{N_R} \right). \tag{25a}$$

We of course also get a restriction on the lowerbound of α_0 through Eq.(23):

$$\alpha_0 > \frac{N_S \gamma_0 R_0 S_0 (1+\epsilon)^3 - l_0(1-\epsilon)}{S_0 (1-\epsilon)^2}. \tag{25b}$$

To get an idea on the order of magnitude of α_0 (which will be our order parameter if $\gamma_0 = 1$), we have for $N_R = 25$, $\sigma_0 = 0.2$, $R_0 = 1$ and $\epsilon = 0.1$:

$$\alpha_0 < 5.3 \times 10^{-3}. \tag{26}$$

So what we see in Eq.(25a) is that α_0 has an upper bound which is dictated either by energy conservation or system feasability. What relations do the metaparameters have to fulfill in these two different regimes?

Suppose that the limiting factor is system feasability. That means:

$$\begin{aligned} &\frac{(1-\epsilon)^3}{1+\epsilon} \frac{\sigma_0 \gamma_0 R_0}{N_R} \leqslant \frac{(1-\epsilon)^2}{1+\epsilon} \left(1 - (1+\epsilon)\,\sigma_0\right) \frac{\gamma_0 R_0}{N_R} \\ &\iff (1-\epsilon)\sigma_0 \leqslant 1 - (1+\epsilon)\sigma_0 \\ &\iff \sigma_0 \leqslant \frac{1}{2} \end{aligned} \tag{27}$$

This means if $\sigma_0 \leqslant \frac{1}{2}$, the limiting factor will be system feasability while if $\sigma_0 \geqslant \frac{1}{2}$, it will be energy conservation.

0.3 Effective system

Models which involve the dynamics of species only are in general better known than consumers-resources models [**insert reference**]. In particular, a huge body of literature exists on the study of Lotka-Volterra systems [**insert reference**]. We may profit from this knowledge by transforming the effect of the resources dynamics into an effective consumers-only system.

This can be done by assuming that the resources reach an equilibrium way faster than the consumers. Mathematically, that is equivalent to

$$\frac{dR_\mu}{dt} \approx 0, \forall \mu. \tag{28}$$

Using Eq.(**??**), we get an explicit value for the resources:

$$R_\mu \approx \frac{l_\mu + \sum_j \alpha_{\mu j} S_j}{m_\mu + \sum_k \gamma_{k\mu} S_k}. \tag{29}$$

This expression can be used in Eq.(**??**) to get an effective system which describes the dynamics of the N_S consumers :

$$\frac{dS_i}{dt} = \left(\sum_\nu \left(\frac{\sigma_{i\nu}\gamma_{i\nu} l_\nu}{m_\nu + \sum_k \gamma_{k\nu} S_k} - \alpha_{\nu i} \right) - d_i + \sum_{\nu j} \frac{\sigma_{i\nu}\gamma_{i\nu}\alpha_{\nu j}}{m_\nu + \sum_k \gamma_{k\nu} S_k} S_j \right) S_i. \tag{30}$$

This can be rewritten in a more compact way:

$$\frac{dS_i}{dt} = p_i(S) S_i + \sum_j M_{ij}(S) S_i S_j \tag{31}$$

with

$$p_i(S) = -\left(d_i + \sum_\nu \alpha_{\nu i} \right) + \sum_\nu \frac{\sigma_{i\nu}\gamma_{i\nu} l_\nu}{m_\nu + \sum_k \gamma_{k\nu} S_k} \text{ and } M_{ij}(S) = \sum_\nu \frac{\sigma_{i\nu}\gamma_{i\nu}\alpha_{\nu j}}{m_\nu + \sum_k \gamma_{k\nu} S_k}. \tag{32}$$

If we assume the species S_k are not too far away from their equilibrium values[2], *i.e.*

$$S_k \approx S_k^* \ \forall k, \tag{33}$$

then using Eq.(18) we can simplify p_i. Indeed,

$$m_\nu + \sum_k \gamma_{k\nu} S_k \approx m_\nu + \sum_k \gamma_{k\nu} S_k^* = \frac{l_\nu + \sum_k \alpha_{\nu k} S_k^*}{R_\nu^*} \tag{34}$$

Hence, the explicit dynamical dependence on S can be removed from p_i and M_{ij}:

$$p_i(S) \approx p_i \equiv -\left(d_i + \sum_\nu \alpha_{\nu i} \right) + \sum_\nu \frac{\sigma_{i\nu}\gamma_{i\nu} l_\nu R_\nu^*}{l_\nu + \sum_k \alpha_{\nu k} S_k^*}, \tag{35}$$

and

$$M_{ij}(S) \approx M_{ij} \equiv \sum_\nu \frac{\sigma_{i\nu}\gamma_{i\nu} R_\nu^* \alpha_{\nu j}}{l_\nu + \sum_k \alpha_{\nu k} S_k^*}. \tag{36}$$

[2]Note that this is very rarely true, especially in the context of the study of structural stability, where entire species sometimes die out.

0.3.1

0.4 Numerical simulations

0.4.1 Testing the algorithm

0.4.2 Time evolution

0.4.3 Allowed parameters : syntrophy range

0.4.4 Studying the impact of the food network structure

0.4.5 Studying the impact of syntrophy

We run a bunch of simulations with the following metaparameters. We made sure that these are compatible with the bounds on α_0 Eqs.(25).

γ_0	σ_0	α_0	R_0	S_0	l_0
1	1	0	300	1	11091
	0.75	0			
		0.5			
	0.5	0			
		0.5			
		1			
	0.25	0			
		0.5			
		1			
		1.5			

Table 1: Metaparameters used for the simulations.

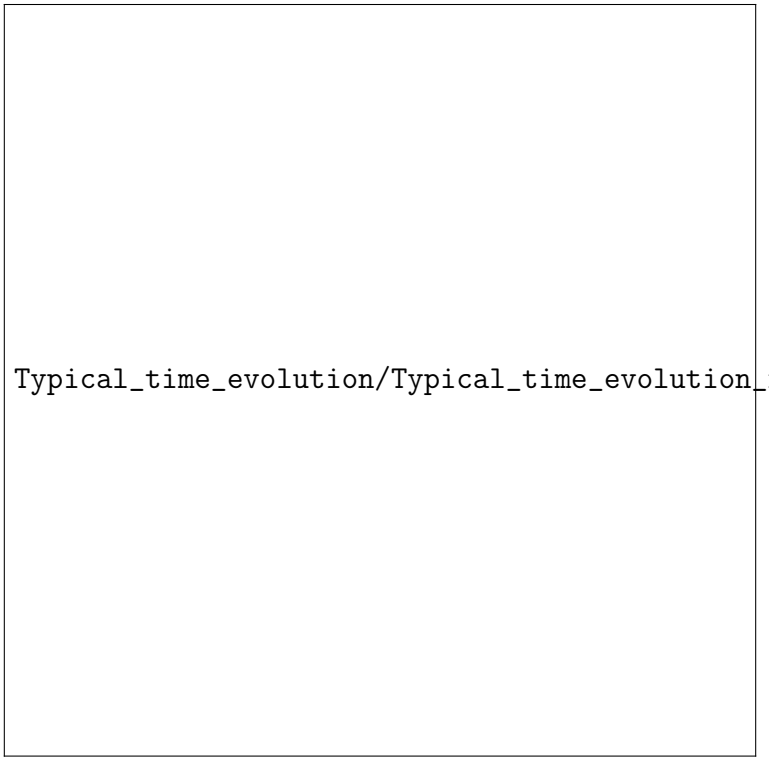

Figure 1: Time evolution for high coefficient threshold ($\epsilon_{\text{conv}} = 10^{-1}$)

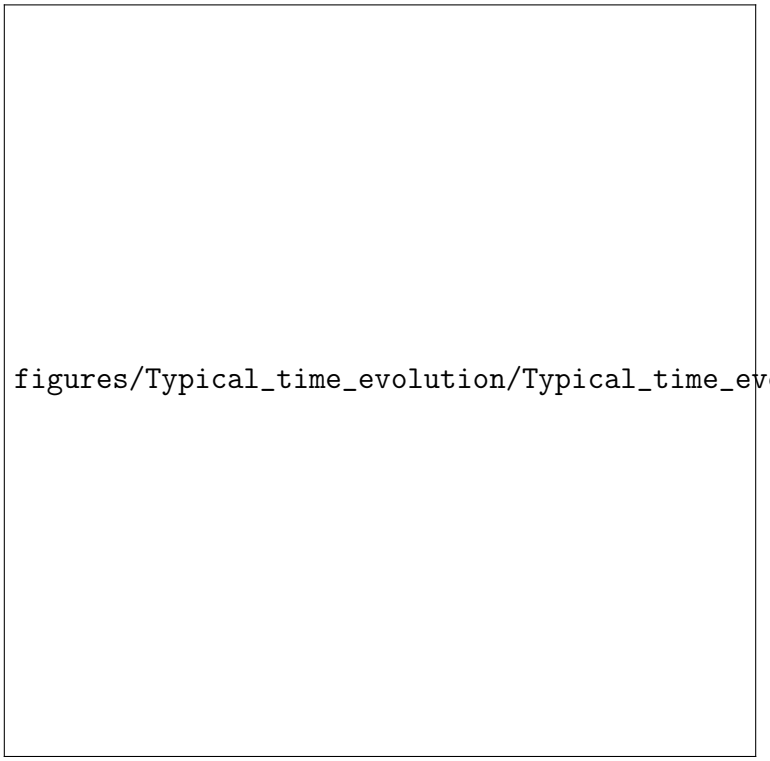

Figure 2: Typical convergence to judge equilibrium, we see the simulation stops at $\epsilon_{\text{conv}} = 10^{-1}$

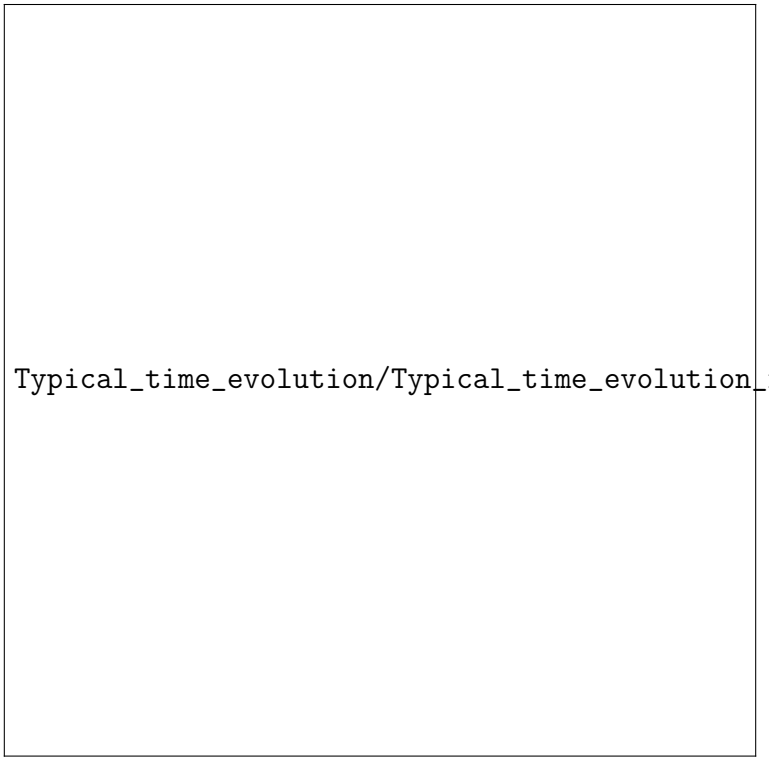

Figure 3: Time evolution for low coefficient threshold (more accuracy) ($\epsilon_{\text{conv}} = 10^{-5}$)

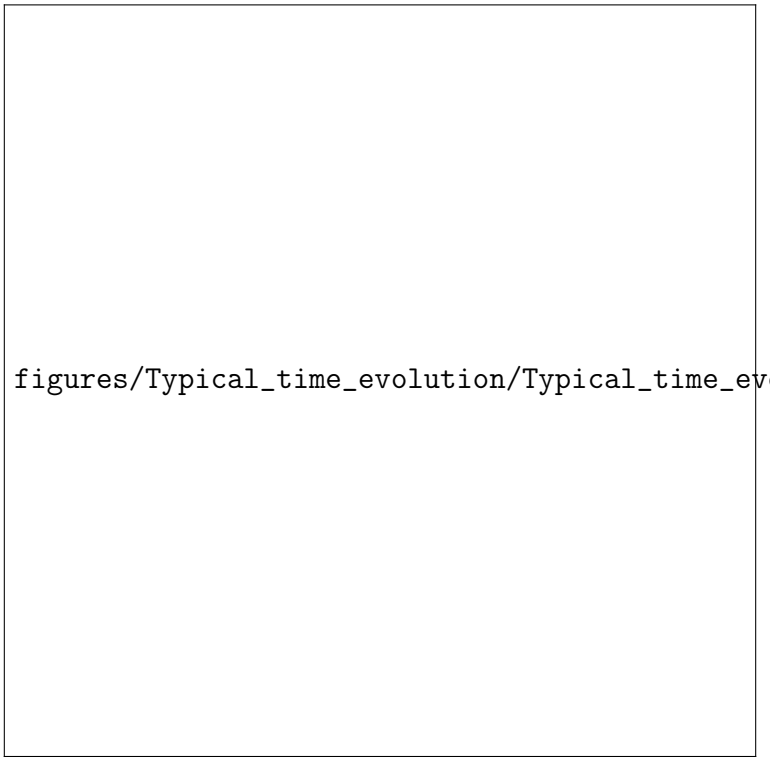

Figure 4: Typical convergence to judge equilibrium, we see the simulation stops at $\epsilon_{\text{conv}} = 10^{-5}$

Figure 5: Typical shape of the probability of feasability for every metaparameter fixed except varying α_0. We see that the probability of drawing a feasible system decreases sharply as α_0 increases. A typical sigmoidal curve (here an erf function) fits the numerical data quite well.