Filtering in the Frequency Domain

Fourier Series and Fourier Transform

Fourier Series

Any **periodic function** can be expressed as the sum of sines and/or cosines of different frequencies, each multiplied by different coefficients.

A function $f(t)$ of a continuous variable t that is periodic with period, T, can be expressed as the sum of sines and cosines muliplied by appropriate coefficients

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{\iota \frac{2\pi n}{T} t}$$

where,

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-\iota \frac{2\pi n}{T} dt} \quad for\ n = 0, \pm 1, \pm 2, \ldots$$

Fourier Transform

Any **function that is not periodic** can be expressed as the integral of sines and/or cosines multiplied by a weighing function.

Preliminary Concepts

$$iota = \iota = \sqrt{-1}$$

$$C = R + \iota I,\ conjugate = C^* = R - \iota I$$

$$\mid C \mid = \sqrt{R^2 + I^2},\ \theta = \arctan(I/R)$$

Also,

$$C = \mid C \mid (\cos\theta + \iota \sin\theta)$$

Using Euler's formula,

$$C = \mid C \mid e^{\iota\theta} \Rightarrow e^{\iota\theta} = \cos\theta + \iota \sin\theta$$

Impulses and Sifting Property

A *unit* impulse of continuous variable t located at $t = 0$, denoted by $\delta(t)$, defines as

$$\delta(t) = \begin{cases} \infty & if \;\; t = 0 \\ 0 & if \;\; t \neq 0 \end{cases}$$

and is constrained to also satisfy:

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

The **sifting property**,

$$\int_{-\infty}^{\infty} f(t)\delta(t - t_0)dt = f(t_0)$$

$$for \; t_0 = 0, \; \int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0)$$

Physically, **if we interpret t as time, an impulse may be viewed as a spike of ∞ amplitude and 0 duration, having unit area.**

Discrete Form

A unit impulse of a discrete variable x located at $x = 0$, denoted by $\delta(x)$, defined as:

$$\delta(x) = \begin{cases} 1 & if \;\; x = 0 \\ 0 & if \;\; x \neq 0 \end{cases}$$

and is also constrained to satisfy:

$$\sum_{x=-\infty}^{\infty} \delta(x) = 1$$

The **Sifting property**,

$$\sum_{x=-\infty}^{\infty} f(x)\delta(x - x_0) = f(x_0)$$

$$for \; x_0 = 0, \; \sum_{x=-\infty}^{\infty} f(x)\delta(x) = f(0)$$

Impulse train $s_{\triangle T}(t)$,

$$s_{\triangle T}(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\triangle T)$$

$$for \; \delta(t - t_0) = 1, \; when \; t - t_0 = 0 \Rightarrow t = t_0$$

Fourier Transform: One continuous variable

The **fourier transform** of continous variable $f(t)$

$$F(\mu) = \Im\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{-\iota 2\pi\mu t}dt$$

Inverse Fourier Tranform is of the form,

$f(t) = \Im^{-1}\{F(\mu)\} = \int_{-\infty}^{\infty} F(\mu)e^{\iota 2\pi\mu t}d\mu$

The Fourier transform can be computed as,

$F(\mu) = \int_{-\infty}^{\infty} f(t)e^{-\iota 2\pi\mu t}dt = \int_{-W/2}^{W/2} Ae^{-\iota 2\pi\mu t}dt$

$= \frac{-A}{\iota 2\pi\mu}[e^{-\iota 2\pi\mu t}]_{-W/2}^{W/2} = \frac{-A}{\iota 2\pi\mu}[e^{-\iota\pi\mu W} - e^{\iota\pi\mu W}]$

$= \frac{A}{\iota 2\pi\mu}[e^{\iota\pi\mu W} - e^{-\iota\pi\mu W}]$

$= \frac{A}{\iota 2\pi\mu}[\cos(\pi\mu W) + \iota\sin(\pi\mu W) - \cos(\pi\mu W) + \iota\sin(\pi\mu W)]$

$= \frac{A}{\iota 2\pi\mu}[2\iota\sin(\pi\mu W)]$

$= \frac{A\sin(\pi\mu W)}{\pi\mu} = AW\frac{\sin(\pi\mu W)}{(\pi\mu W)}$

This is called the $sinc$ function,

$sinc(m) = \frac{\sin(\pi m)}{(\pi m)}$

where $sinc(0) = 1,\ sinc(m) = 0\ \ (\lim_{x\to 0}\frac{\sin x}{x} = 1)$

Customary for display purposes to work with magintude of the transform (since the transform is complex), which is called the **Fourier spectrum** or the **frequency spectrum**.

$\mid F(\mu) \mid = AW \left| \frac{\sin(\pi\mu W)}{(\pi\mu W)} \right|$

Key Properties:

- Locations of zeroes of both $F(\mu)$ and $\mid F(\mu) \mid$ are inversely proportional to the width, W, of the "box" function
- The function extends to ∞ for both positive and negative values of μ.

The fourier transform of a unit impulse located at the origin,

$F(\mu) = \int_{-\infty}^{\infty} \delta(t)e^{-\iota 2\pi\mu t}dt$

$= e^{-\iota 2\pi\mu 0} = 1$

Similarly, for a unit impulse located at $t = t_0$:

$F(\mu) = e^{-\iota 2\pi\mu t_0} = \cos(2\pi\mu t_0) - \iota\sin(2\pi\mu t_0)$

Impulse Train as a Fourier Series

$s_{\triangle T}(t) = \sum_{n=-\infty}^{\infty} c_n e^{\iota \frac{2\pi n}{\triangle T} t}$

where,

$c_n = \frac{1}{\triangle T}$

so,

$s_{\triangle T}(t) = \frac{1}{\triangle T} \sum_{n=-\infty}^{\infty} e^{\iota \frac{2\pi n}{\triangle T} t}$

$\Im\{e^{\iota \frac{2\pi n}{\triangle T} t}\} = \int_{-\infty}^{\infty} e^{\iota \frac{2\pi n}{\triangle T} t} e^{-\iota 2\pi\mu t} dt$

$= \int_{-\infty}^{\infty} e^{-\iota 2\pi(\mu - \frac{n}{\triangle T})t} dt = \delta(\mu - \frac{n}{\triangle T})$

Let $S(\mu)$ denote the Fourier transform of the periodic impulse train $S_{\triangle T}(t)$,

$S(\mu) = \frac{1}{\triangle T} \sum_{n=-\infty}^{\infty} \delta(\mu - \frac{n}{\triangle T})$

Fourier transform and Convolution

$f(t) * h(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau$

$\Im\{(t) * h(t)\} = \int_{-\infty}^{\infty} [\int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau] e^{-\iota 2\pi\mu t} dt$

pair dt with $h(t-\tau)$ and $d\tau$ with $f(\tau)$

$\Im\{(t) * h(t)\} = H(\mu)F(\mu)$

where $H(\mu)$ is the Fourier transform of $h(t)$,

$\Im\{h(t-\tau)\} = H(\mu)^{-\iota 2\pi\mu\tau}$

Fourier Transform Pairs

$f(t) * h(t) \iff H(\mu)F(\mu)$

$f(t)h(t) \iff H(\mu) * F(\mu)$

Fourier Transform of sampled functions

$\tilde{F}(\mu) = F(\mu) * S(\mu) = \int_{-\infty}^{\infty} F(\tau)S(t-\tau)d\tau$

$= \frac{1}{\triangle T} \sum_{n=-\infty}^{\infty} F(\mu - \frac{n}{\triangle T}),\ since\ S(\mu) = \frac{1}{\triangle T} \sum_{n=-\infty}^{\infty} \delta(\mu - \frac{n}{\triangle T})$

Bandlimited Signal

A signal whose Fourier transform is zero above a certain finite frequency. In other words, if the Fourier transform has finite support then the signal is said to bandlimited.

Example:

$$x(t) = \sin(2\pi ft + \theta)$$

Over-sampling and under-sampling

$$\tilde{F}(\mu) = \frac{1}{\triangle T} \sum_{n=-\infty}^{\infty} F(\mu - \frac{n}{\triangle T})$$

Over-sampling

$$\frac{1}{\triangle T} > 2\mu_{max}$$

Critically sampling

$$\frac{1}{\triangle T} = 2\mu_{max}$$

Under-sampling

$$\frac{1}{\triangle T} < 2\mu_{max}$$

Nyquist-Shannon Sampling theorem

A continuous, band-limited function can be **recovered completely** from a set of its samples if the samples are **acquired at a rate exceeding twice the highest frequency** content of the function.

Sufficient Seperation is guaranteed if

$$\frac{1}{\triangle T} > 2\mu_{max}$$

Aliasing

If the band-limited function is sampled at a **rate less than twice its highest frequency**, then the inverse transform will yield a **corrupted function**. This effect is known as **frequency aliasing** or simply as **aliasing**.

The under-sampled function looks like the sine wave having a frequency much lower than the frequency of the continuous signal.

Function reconstruction from sampled data

$$F(\mu) = H(\mu)\tilde{F}(\mu)$$

$$f(t) = \Im^{-1}\{F(\mu)\}$$

$$= \Im^{-1}\{H(\mu)\tilde{F}(\mu)\}$$

$$= h(t) * \tilde{f}(t)$$

Discrete Fourier Transform (DFT) of one variable

$$F(\mu) = \sum_{x=0}^{M-1} f(x) e^{-\iota 2\pi\mu x/M}, \quad \mu = 0, 1, \ldots, M-1$$

$$f(x) = \frac{1}{M} \sum_{x=0}^{M-1} F(\mu) e^{\iota 2\pi\mu x/M}, \quad x = 0, 1, \ldots, M-1$$

Relationship between Sampling and frequency intervals

If $f(x)$ consists of M samples of a function $f(t)$ taken $\triangle T$ units apart, the length of the record comprising the set $\{f(x)\},\ x = 0, 1, 2, \ldots, M-1$, is

$$T = M\triangle T$$

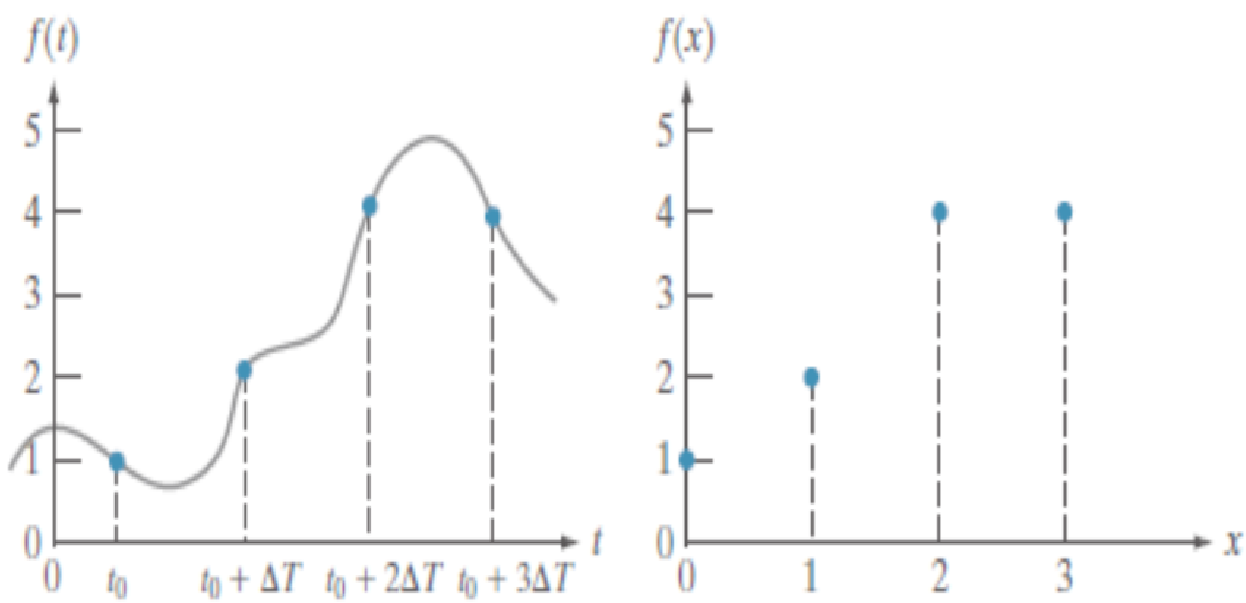
The corresponding spacing, $\triangle u$,

$$\triangle u = \frac{1}{T} = \frac{1}{M\triangle T}$$

The entire frequency range spanned by the M components of DFT is

$$R = M\triangle u = \frac{1}{\triangle T}$$

Example



$$F(\mu) = \sum_{x=0}^{M-1} f(x) e^{-\iota 2\pi\mu x/M}, \quad \mu = 0, 1, \ldots, M-1$$

$$F(0) = \sum_{x=0}^{3} f(x) e^{0} = [f(0) + f(1) + f(2) + f(3)] = 1 + 2 + 4 + 4 = 11$$

$F(1) = \sum_{x=0}^{3} f(x)e^{-\iota 2\pi(1)x/4} = 1e^0 + 2e^{-\iota 2\pi/4} + 4e^{-\iota 4\pi/4} + 4e^{-\iota 6\pi/4} = -3 + 2\iota$

2D Impulse and Sifting Property: Continuous

Impulse is denoted by $\delta(t, z)$

$$\delta(t, z) = \begin{cases} \infty & t = z = 0 \\ 0 & otherwise \end{cases}$$

and

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(t, z) dtdz = 1$$

The **sifting property**,

$$\sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} f(x, y)\delta(x, y) = f(0, 0)$$

and

$$\sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} f(x, y)\delta(x - x_0, y - y_0) = f(x_0, y_0)$$

2D Fourier Transform

$$F(\mu, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t, z)e^{-\iota 2\pi(\mu t + vz)} dtdz$$

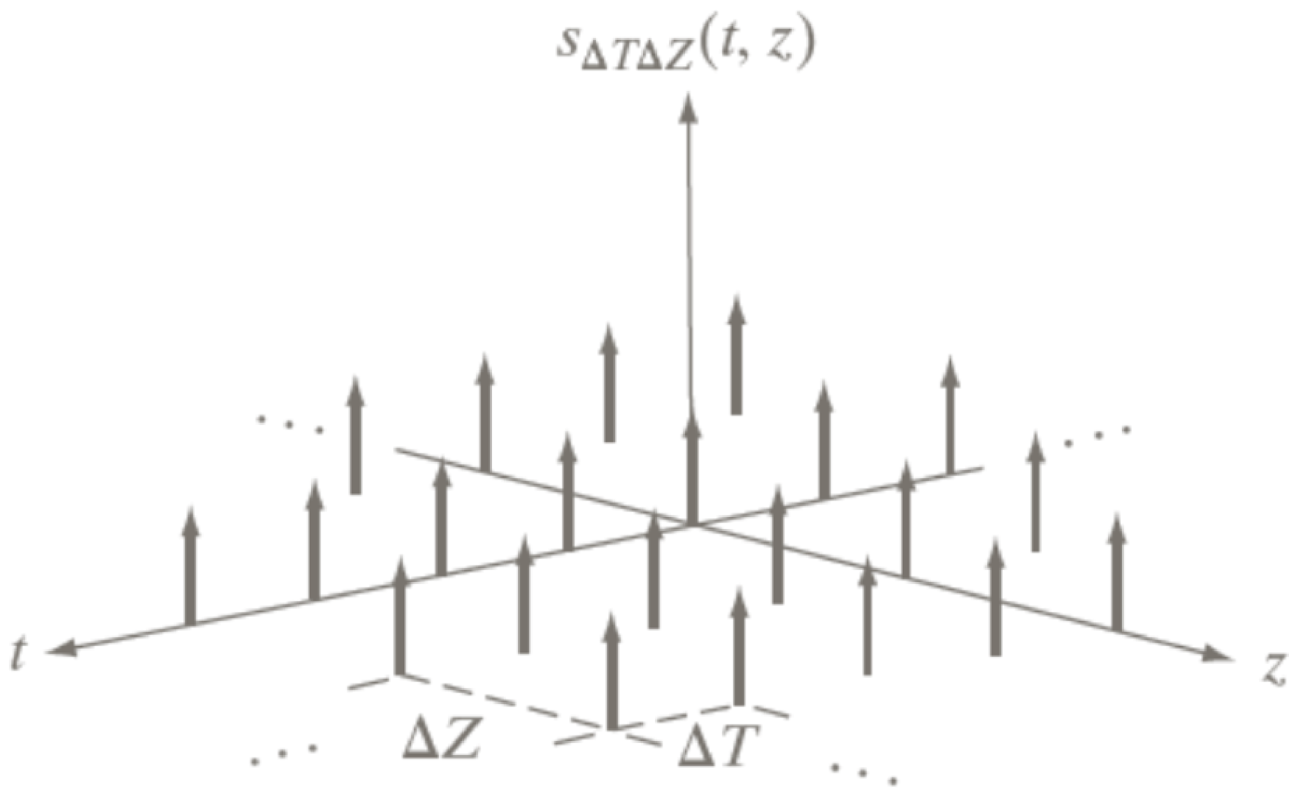
$$f(t, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(\mu, v)e^{\iota 2\pi(\mu t + vz)} d\mu dv$$

$$F(\mu, v) = \int_{-T/2}^{T/2} \int_{-Z/2}^{Z/2} Ae^{-\iota 2\pi(\mu t + vz)} dtdz$$

$$= ATZ[\frac{\sin(\pi\mu T)}{(\pi\mu T)}][\frac{\sin(\pi v T)}{(\pi v T)}]$$

2D Impulse Train

$$s_{\triangle T \triangle Z}(t, z) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(t - m\triangle T, z - n\triangle Z)$$



2D Band-limited function

A function $f(t,z)$ is said to be band-limited if its Fourier Transform is 0 outside a rectangle established by the intervals $[-\mu_{max},\ \mu_{max}]$ and $[-v_{max}, v_{max}]$

2D Sampling Theorem

A continuous, band-limited function $f(t,z)$ can be recovered with no error from a set of its samples if the sampling intervals are

$$\triangle T < \frac{1}{2\mu_{max}}\ and\ \triangle Z < \frac{1}{2v_{max}}$$

Moire Patterns

Moire Patterns are often an undesired artifact of images produced by various digital imaging and computer graphics techniques.

Example: When scanning a halftone picture, or ray tracing a checkered plane

It is a special case of aliasing, formed due to under-sampling a fine regular pattern.

2D DFT and its inverse

$$F(\mu, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{-\iota 2\pi(\mu x/M + vy/N)},\ \ \mu = 0,1,2,\ldots,M-1\ and\ v = 0,1,2,\ldots,N$$

Inverse:

$f(x,y) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} F(\mu,v)e^{\iota 2\pi(\mu x/M + vy/N)}$

Let $\triangle T\ and\ \triangle Z$ denote the seperations between the samples, then the seperations between corresponding discrete, frequency domian variables is given by

$\triangle\mu = \frac{1}{M\triangle T}$

$\triangle v = \frac{1}{N\triangle Z}$

$f(x,y)e^{\iota 2\pi(\mu_0 x/M + v_0 y/N)} \iff F(\mu-\mu_0, v-v_0)$

and

$f(x-x_0, y-y_0) \iff F(\mu,v)e^{-\iota 2\pi(\mu_0 x/M + v_0 y/N)}$

Multiplying $f(x,y)$ by $e^{\iota\cdots}$ shown shifts the origin of DFT to (μ_0, v_0), and, conversely, multiplying $F(\mu,v)$ by the negative of that exponent shifts the origin of $f(x,y)$ to (x_0, y_0) . This translation has no effect on the spectrum (magnitude) of the $F(\mu,v)$

Properties of 2D DFT: Rotation

Rotating $f(x,y)$ by an angle θ_0 rotates $F(\mu,v)$ by the same angle. Converse is also true.

Properties of 2D DFT: Periodicity

2D Fourier transform and its inverse are infinitely periodic.

$f(x)e^{\iota 2\pi(\mu_0 x/M)} \iff F(\mu-\mu_0)$

Shift origin to μ_0. If we let $\mu_0 = M/2$, the exponential term becomes $e^{\iota\pi x}$, which is equal to $(-1)^x$, if we use the sine and cosine form. In this case,

$f(x)(-1)^x \iff F(\mu - M/2)$

Now $F(\mu)$ is centered on the interval $[0, M-1]$.

In 2D,

$f(x,y)(-1)^{x+y} \iff F(\mu - M/2, v - N/2)$

Now it is centered on the rectangle defined by intervals $[0, M-1]$ and $[0, N-1]$.

Fourier Spectrum and Phase angle (2D)

2D DFT in Polar Form

$F(u,v) = \mid F(u,v) \mid e^{\iota\phi(u,v)}$

Fourier Spectrum

$$| F(u,v) |= \sqrt{R^2(u,v) + I^2(u,v)}$$

Power Spectrum

$$P(u,v) =| F(u,v) |^2= R^2(u,v) + I^2(u,v)$$

Phase Angle

$$\phi(u,v) = \arctan[\frac{I(u,v)}{R(u,v)}]$$

$$F(0,0) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)e^0 = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)$$

$$F(0,0) = MN \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)$$

$$= MN\bar{f}$$

where $\bar{f}$ (a scalar) denotes the average value of $f(x,y)$, then

$$| F(0,0) |= MN \mid \bar{f} \mid$$

$F(0,0)$ is called the **dc component** of the transform.

Convolution

1D convolution

$$f(x) * h(x) = \sum_{m=0}^{M-1} f(m)h(x-m)$$

2D convolution

$$f(x,y) * h(x,y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m,n)h(x-m,y-n)$$

Important properties

$$f(x,y) * h(x,y) \iff F(u,v)H(u,v)$$

$$f(x,y)h(x,y) \iff F(u,v) * H(u,v)$$

Zero Padding

Consider 2 functions $f(x)$ and $h(x)$ composed of A and B respectively.

Append zeroes to both functions such that they have the same length, denoted by P, then wrap around is avoided by choosing

$$P \geq A + B - 1$$

Let $f(x, y)$ and $h(x, y)$ be two image arrays of sizes $A \times B$ and $C \times D$ pixels, respectively. Wraparound error in their convolution can be avoided by padding these functions with $0s$.

$$f_p(x, y) = \begin{cases} f(x, y) & 0 \leq x \leq A - 1 \; and \; 0 \leq y \leq B - 1 \\ 0 & A \leq x \leq P \; or \; B \leq y \leq Q \end{cases}$$

$$h_p(x, y) = \begin{cases} h(x, y) & 0 \leq x \leq C - 1 \; and \; 0 \leq y \leq D - 1 \\ 0 & C \leq x \leq P \; or \; D \leq y \leq Q \end{cases}$$

here, $P \geq A + C - 1 \; and \; Q \geq B + D - 1$

NOTE: When we perform convolution and choose $(0, 0)$ as the center pixel, we run into trouble because there are no pixels above that row, so what we can do to avoid that is take intensity of those pixels to be 0 by padding the images with $0s$.