Filtering in the Frequency Domain

- As we move away from the origin of transform, the low frequencies correspond to the slowly varying intensity components of the image
- As we move further away from the origin of the transform, the higher frequencies correspond to faster and faster intensity changes in the image. These are the edges of the objects and other components.
- Filtering techniques in the frequency domain are based on modifying the fourier transform to achieve a specific objective, and then computing the inverse DFT to get us back the spatial domain.

$$g(x, y) = Real\{\Im^{-1}[H(u, v)F(u, v)]\}$$

$$g(x, y) = output\ image,\ H(u, v) = filter\ function$$

product of $H,\ F$ is element-wise

The task of specifying $H(u, v)$ is simplified considerably by using functions that are symmetric about their center, which requires that $F(u, v)$ be centered also (hence, why we shift). That is accomplished by multiplying the input image by $(-1)^{x+y}$ prior to computing its transform.

The simplest filter is in which $H(u, v)$ is 0 at the center of the transform, and 1 elsewhere. The dc term is responsible for the average intensity, and the above function will drop that down to 0.

$$g(x, y) = \Im^{-1}\{H(u, v)F(u, v)\}$$

$$F(u, v) = R(u, v) + \iota I(u, v)$$

$$g(x, y) = \Im^{-1}[H(u, v)R(u, v) + \iota H(u, v)I(u, v)]$$

- Filters affect the real and imaginary parts equally.
- These filters are called **zero-phase-shift** filters because they have no effect on phase.

Summary: Steps for filtering in the frequency domain

1. Given an input image $f(x, y)$ of size $M \times N$, obtain the padding parameters P and Q. Typically, $P = 2M$ and $Q = 2N$.
2. Form a padded image, $f_p(x, y)$ of size $P \times Q$ by appending the necessary number of zeroes to $f(x, y)$
3. Multiply $f_p(x, y)$ by $(-1)^{x+y}$ to center its transform.
4. Compute the DFT, $F(u, v)$ of the image from step 3.
5. Generate a real, symmetric filter function, $H(u, v)$, of size $P \times Q$, with center at coordinates $(P/2, Q/2)$, we first pad the spatial filter and then center it, and then compute the DFT of the result to obtain $H(u, v)$.
6. Form the product $G(u, v) = H(u, v)F(u, v)$ using array multiplication.

7. Obtain the processed image $g_p(x, y) = real\{\Im^{-1}[G(u, v)]\}(-1)^{x+y}$
8. Obtain the final processed result, $g(x, y)$, by extracting the $M \times N$ region from the top left quadrant of $g_p(x, y)$.

Correspondence between Filtering in the frequency domains and spatial domains

$H(u) =$ 1D frequency domain gaussian filter

$$H(u) = Ae^{-u^2/2\sigma^2}$$

The corresponding filter in the frequency domain,

$$h(x) = \sqrt{2\pi}\sigma Ae^{-2\pi^2\sigma^2x^2}$$

- Both functions are gaussian and real.
- The functions behave reciprocally. When $H(u)$ have a broad profile (large value of σ), $h(x)$ has a narrow profile. In fact, as $\sigma \to \infty$, $H(u)$ tends toward a constant function and $h(x)$ tends toward an impulse, which implies no filtering in either domain.

Generating $H(u, v)$

- Multiply $h_p(x, y)$ by $(-1)^{x+y}$ to center the frequency domain filter.
- Compute the forward DFT in step 1 to generate $H(u, v)$
- Set the real parts of $H(u, v)$ to 0 to account for parasitic real parts (since h_p is real and odd, H has to be purely imaginary).
- Multiply the result with $(-1)^{u+v}$.

Image smoothing using frequency domain filters: Ideal Lowpass Filters

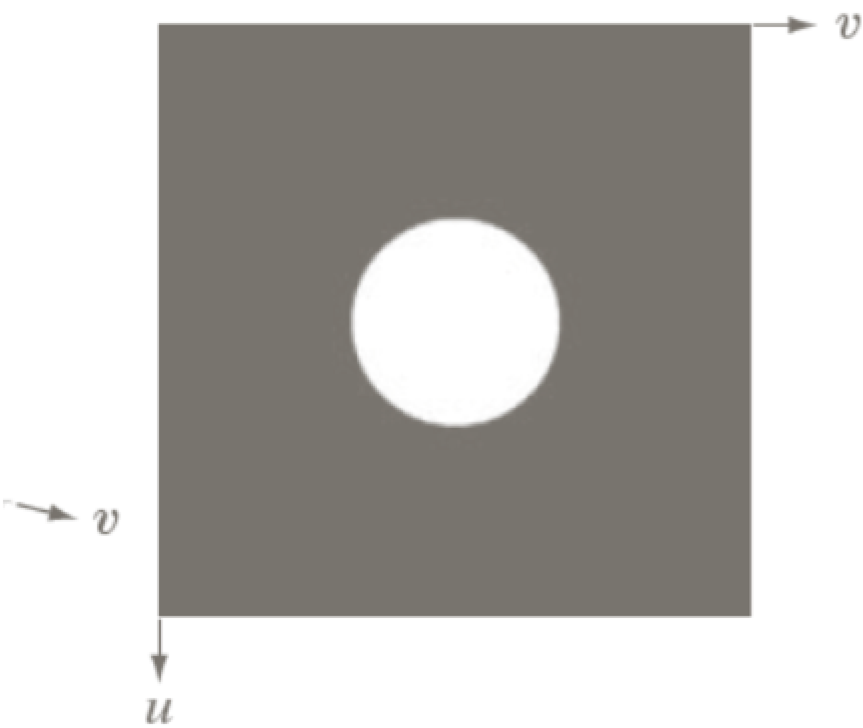
Ideal Lowpass filters (ILPF)

$$H(u, v) = \begin{cases} 1 & D(u, v) \leq D_0 \\ 0 & D(u, v) > D_0 \end{cases}$$

$$D_0 = positive\ constant$$

$D(u, v)$ is distance between a point (u, v) in the frequency domain and the center of the frequency rectangle

$$D(u, v) = \sqrt{(u - P/2)^2 + (v - Q/2)^2}$$

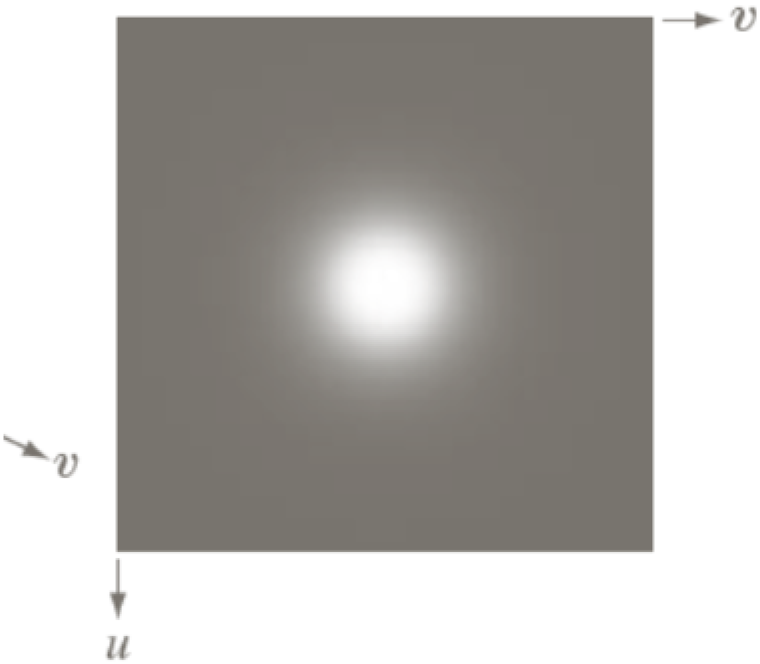


In the image, points closer to the center of the rectangle are passed (intensity = 1) and points farther are given intensity values `0`.

Butterworth Lowpass Filter (BLPF)

$H(u,v) = \frac{1}{1+[D(u,v)/D_0]^{2n}}$

$n = order$



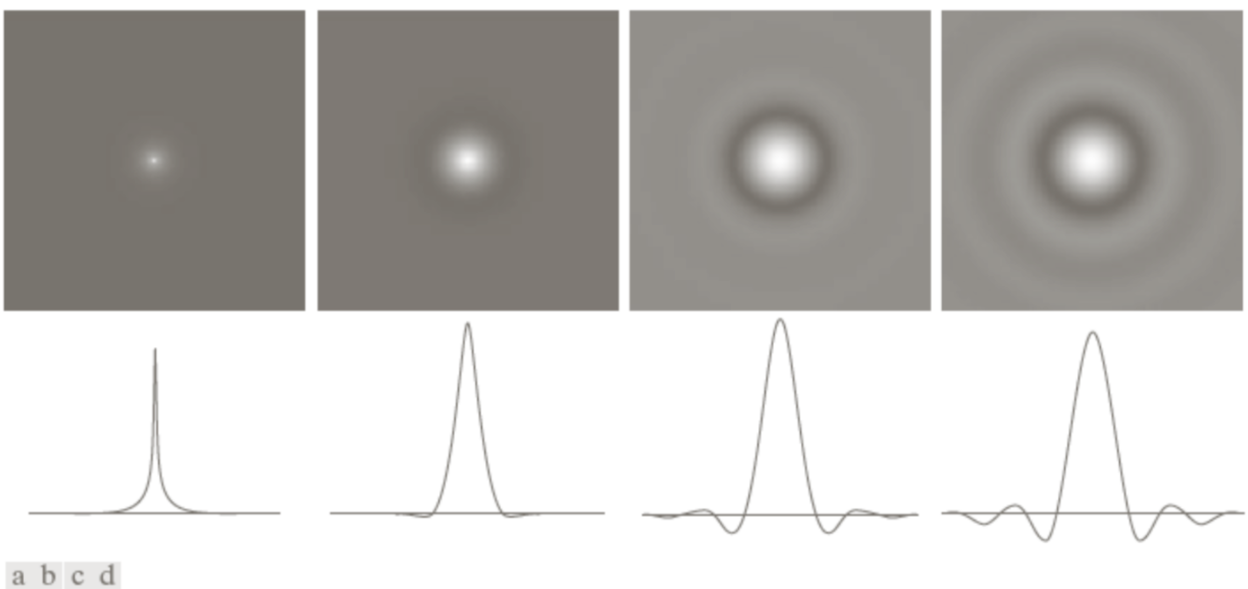


FIGURE 4.46 (a)–(d) Spatial representation of BLPFs of order 1, 2, 5, and 20, and corresponding intensity profiles through the center of the filters (the size in all cases is 1000 × 1000 and the cutoff frequency is 5). Observe how ringing increases as a function of filter order.

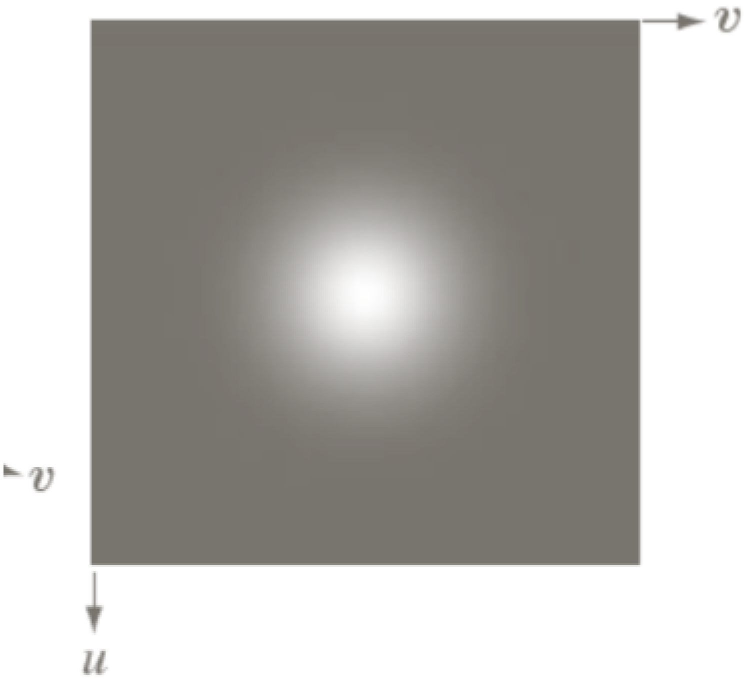
Gaussian Lowpass Filters (GLPF)

$$H(u,v) = e^{-D^2(u,v)/2\sigma^2}$$

By letting $\sigma = D_0$

$$H(u,v) = e^{-D^2(u,v)/2D_0^2}$$

$D^2(u,v) = (u - P/2)^2 + (v - Q/2)^2$



Highpass Filters

Highpass filter from a lowpass filter:

$H_{HP}(u,v) = 1 - H_{LP}(u,v)$

2D ideal highpass filter

$H(u,v) = \begin{cases} 0 & D(u,v) \leq D_0 \\ 1 & D(u,v) > D_0 \end{cases}$

Butterworth Highpass filter (BHPL)

$H(u,v) = \frac{1}{1+[D_0/D(u,v)]^{2n}}$

Gaussian Highpass filter

$H(u,v) = 1 - e^{-D^2(u,v)/2D_0^2}$

Laplacian in the frequency domain

$H(u,v) = -4\pi^2(u^2 + v^2)$

$$H(u,v) = -4\pi^2((u-P/2)^2 + (v-Q/2)^2)$$

$$H(u,v) = -4\pi^2 D^2(u,v)$$

The laplacian image,

$$\nabla^2 f(x,y) = \Im^{-1}\{H(u,v)F(u,v)\}$$

Enchancement is obtained,

$$g(x,y) = f(x,y) + c\nabla^2 f(x,y), \quad c = -1$$

The enchanced image,

$$\begin{aligned} g(x,y) &= \Im^{-1}\{F(u,v) - H(u,v)F(u,v)\} \\ &= \Im^{-1}\{[1 - H(u,v)]F(u,v)\} \\ &= \Im^{-1}\{[1 + 4\pi^2 D^2(u,v)]F(u,v)\} \end{aligned}$$

Unsharp Masking and highboost filtering

$$g_{mask}(x,y) = f(x,y) - f_{LP}(x,y)$$

$$f_{LP}(x,y) = \Im^{-1}[H_{LP}(u,v)F(u,v)]$$

$$g(x,y) = f(x,y) + k \times g_{mask}(x,y)$$

$$g(x,y) = \Im^{-1}\{[1 + k \times [1 - H_{LP}(u,v)]]F(u,v)\}$$

$$g(x,y) = \Im^{-1}\{[1 + k \times H_{HP}(u,v)]F(u,v)\}, \quad H_{HP}(u,v) = 1 - H_{LP}(u,v)$$

$$g(x,y) = \Im^{-1}\{[k_1 + k_2 \times H_{HP}(u,v)]F(u,v)\}$$

where $k_1 \geq 0$ offsets the value transfer function so that the dc term is not zeroed-out, and $k_2 \geq 0$ controls the contribution of high frequencies.

Homomorphic Filtering

$$f(x,y) = i(x,y)r(x,y)$$

$$z(x,y) = \ln f(x,y) = \ln i(x,y) + \ln r(x,y)$$

$$\Im\{z(x,y)\} = \Im\{\ln f(x,y)\} = \Im\{\ln i(x,y)\} + \Im\{\ln r(x,y)\}$$

$$Z(u,v) = F_i(u,v) + F_r(u,v)$$

$$S(u,v) = H(u,v)Z(u,v) = H(u,v)F_i(u,v) + H(u,v)F_r(u,v)$$

$$\begin{aligned} s(x,y) &= \Im^{-1}\{S(u,v)\} \\ &= \Im^{-1}\{H(u,v)F_i(u,v) + H(u,v)F_r(u,v)\} \\ &= \Im^{-1}\{H(u,v)F_i(u,v)\} + \Im^{-1}\{H(u,v)F_r(u,v)\} \\ &= i'(x,y) + r'(x,y) \end{aligned}$$

$$g(x,y) = e^{s(x,y)} = e^{i'(x,y)}e^{r'(x,y)} = i_0(x,y)r_0(x,y)$$

Summary

$$f(x,y) \Rightarrow \ln \Rightarrow \textbf{DFT} \Rightarrow H(u,v) \Rightarrow \textbf{(DFT)}^{-1} \Rightarrow \exp \Rightarrow g(x,y)$$

These characteristics lead to associating the low frequencies of the Fourier transform of the logarithm of an image with illumination and the high frequencies with reflectance.

$$H(u,v) = (\gamma_H - \gamma_L)[1 - e^{-c[D^2(u,v)/D_0^2]}]$$

Selective and non-selective filters

Non-selective filters

Operate over the entire frequency rectangle (in the frequency domain).

Selective filters

Operate over some part , not entire frequency rectangle

- **Bandreject** or **Bandpass**: process specific bands
- **Notch** filters: process small regions of frequency rectangle

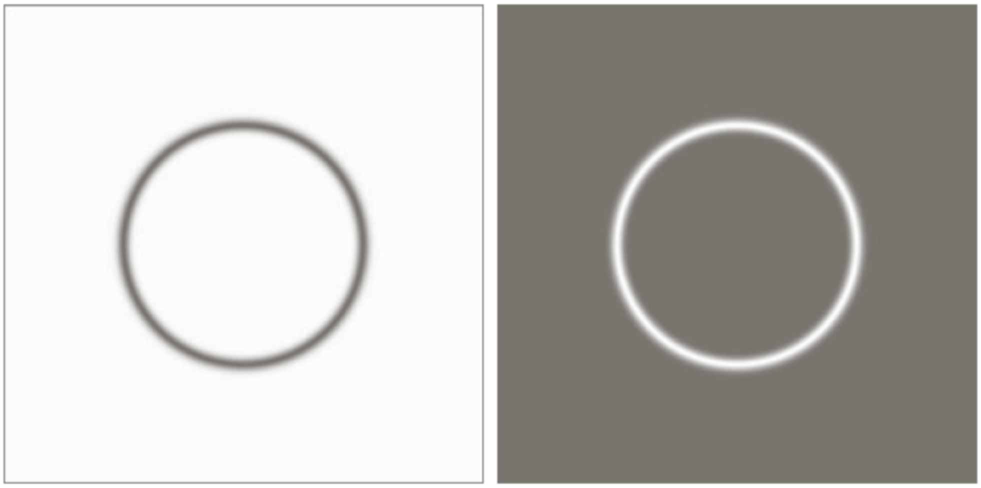
Bandreject filters

Ideal	Butterworth	Gaussian
$H(u,v) = \begin{cases} 0 & \text{if } D_0 - \frac{W}{2} \le D \le D_0 + \frac{W}{2} \\ 1 & \text{otherwise} \end{cases}$	$H(u,v) = \dfrac{1}{1 + \left[\dfrac{DW}{D^2 - D_0^2}\right]^{2n}}$	$H(u,v) = 1 - e^{-\left[\frac{D^2 - D_0^2}{DW}\right]^2}$

$$W = width\ of\ the\ band$$

Bandpass filters

$H_{BP}(u,v) = 1 - H_{BR}(u,v)$



a b

FIGURE 4.63 (a) Bandreject Gaussian filter. (b) Corresponding bandpass filter. The thin black border in (a) was added for clarity; it is not part of the data.

Notch Filters

Zero-phase-shift filtersmust be symmetric about the origin. A notch with center at (u_0, v_0) must have a corresponding notch at location $(-u_0, -v_0)$

Notch reject filters are constructed as products of highpass filters whose centers have been translated to the centers of the notch.

$$H_{NR}(u,v) = \prod_{k=1}^{Q} H_k(u,v) H_{-k}(u,v)$$

where $H_k(u,v)$ and $H_{-k}(u,v)$ are highpass filters whose centers are at (u_k, v_k) and $(-u_k, -v_k)$ respectively.

BUTTERWORTH NOTCH REJECT FILTER

$$H_{NR}(u,v) = \prod_{k=1}^{3} \left[\frac{1}{1+[D_{0k}/D_k(u,v)]^{2n}}\right]\left[\frac{1}{1+[D_{0k}/D_{-k}(u,v)]^{2n}}\right]$$

$$D_k(u,v) = \sqrt{(u - u_k - M/2)^2 + (v - v_k - N/2)^2}$$

$$D_{-k}(u,v) = \sqrt{(u + u_k - M/2)^2 + (v + v_k - N/2)^2}$$