Fast Fourier Transform

$F(\mu, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{-\iota 2\pi(\mu x/M + vy/N)}, \quad \mu = 0,1,2,\ldots,M-1 \; and \; v = 0,1,2,\ldots,N$

Brute force Time $= \mathcal{O}((MN)^2)$

17 trillion calculations for a 2048×2048 image.

Using FFT, Time $= \mathcal{O}(MN \log_2 (MN))$

17 trillon calculations reduces to 92 million calculations for the same 2048×2048 image.

FFT of one variable

$F(u) = \sum_{x=0}^{M-1} f(x)(W_M)^{ux}$

where,

$W_M = e^{-\iota 2\pi/M}$

and M is assumed of the form

$M = 2^p$

which can be expressed as

$M = 2K$

Now, on substituting,

$F(u) = \sum_{x=0}^{2K-1} f(x)(W_{2K})^{ux}$

$F(u) = \sum_{x=0}^{K-1} f(2x)(W_{2K})^{u(2x)} + \sum_{x=0}^{K-1} f(2x+1)(W_{2K})^{u(2x+1)}$

Now, $(W_{2K})^{2ux} = (W_K)^{ux}$

$F(u) = \sum_{x=0}^{K-1} f(2x)(W_K)^{ux} + \sum_{x=0}^{K-1} f(2x+1)(W_K)^{ux}(W_{2K})^u$

Defining

$F_{even}(u) = \sum_{x=0}^{K-1} f(2x)(W_K)^{ux}$

and

$F_{odd}(u) = \sum_{x=0}^{K-1} f(2x+1)(W_K)^{ux}$

Now,

$$F(u) = F_{even}(u) + F_{odd}(u)(W_{2K})^u$$

Also, because $-(W_{2K})^u = (W_{2K})^{u+K}$

$$F(u+K) = F_{even}(u) - F_{odd}(u)(W_{2K})^u$$

Computations

Let $m(p)$ and $a(p)$ be number of complex multiplications and additions, respectively.

then,

$$m(p) = 2m(p-1) + 2^{p-1}$$

and

$$a(p) = 2a(p-1) + 2^p$$

By substitution method, $(a(0) = 0 \; and \; m(0) = 0)$

$$m(p) = \frac{1}{2} M \log_2 M$$

$$a(n) = M \log_2 M$$