3.2 Independence

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1 3.2 Independence

We would like to measure the strength of the relationship between two variables. *Independence* is a way to quantify the intuitive notion that two variables are *unrelated*. Once we have defined independence, we can quantify the relationship between two variables by calculating how far they are from independence.

Formally, two variables X and Y are **independent** if the conditional distributions of Y given X (or vice versa) are all *identical*. In other words, the value of X does not affect the distribution of Y.

```
In [1]: %matplotlib inline
        import numpy as np
        import pandas as pd

        titanic_df = pd.read_csv("https://raw.githubusercontent.com/dlsun/data-science-book/mas
        titanic_df["adult"] = (titanic_df["age"] >= 18)
```

For example, consider the relationship between sex and age group (adult or not). First, let's calculate the contingency table:

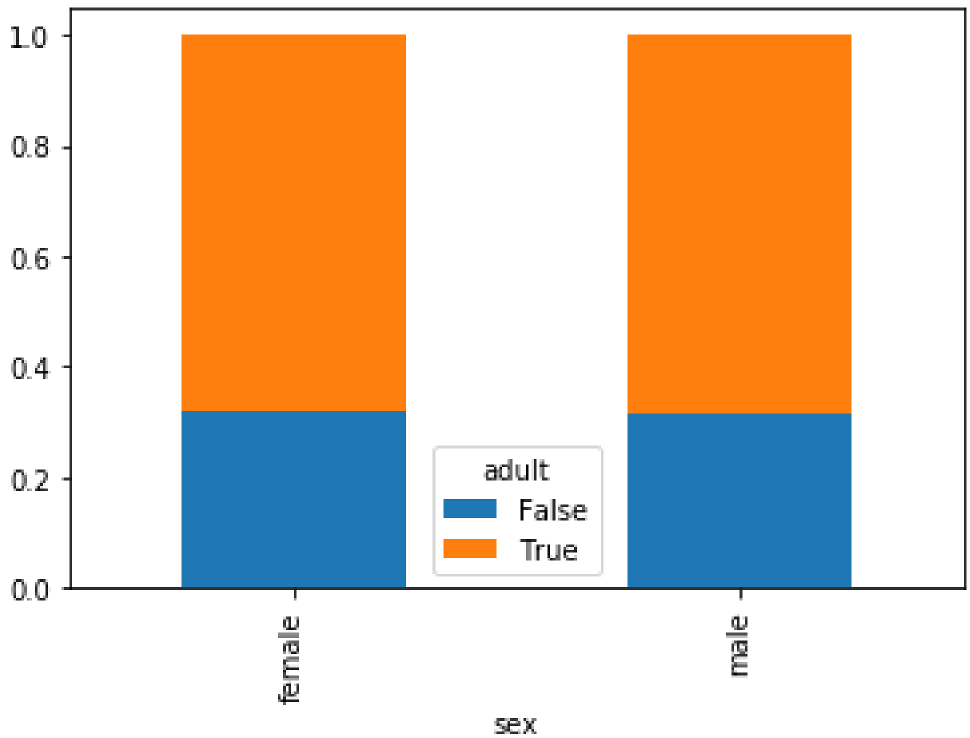
```
In [2]: counts = pd.crosstab(titanic_df.sex, titanic_df.adult)
        counts
```

```
Out[2]: adult   False  True
        sex
        female    150   316
        male      267   576
```

Although there are more male adults (576) than female adults (316), the *conditional proportion* of adults, given sex, are actually very close (about 0.68).

```
In [3]: adult_given_sex = counts.divide(counts.sum(axis=1), axis=0)
        adult_given_sex.plot.bar(stacked=True)
        adult_given_sex
```

```
Out[3]: adult      False     True
        sex
        female  0.321888  0.678112
        male    0.316726  0.683274
```

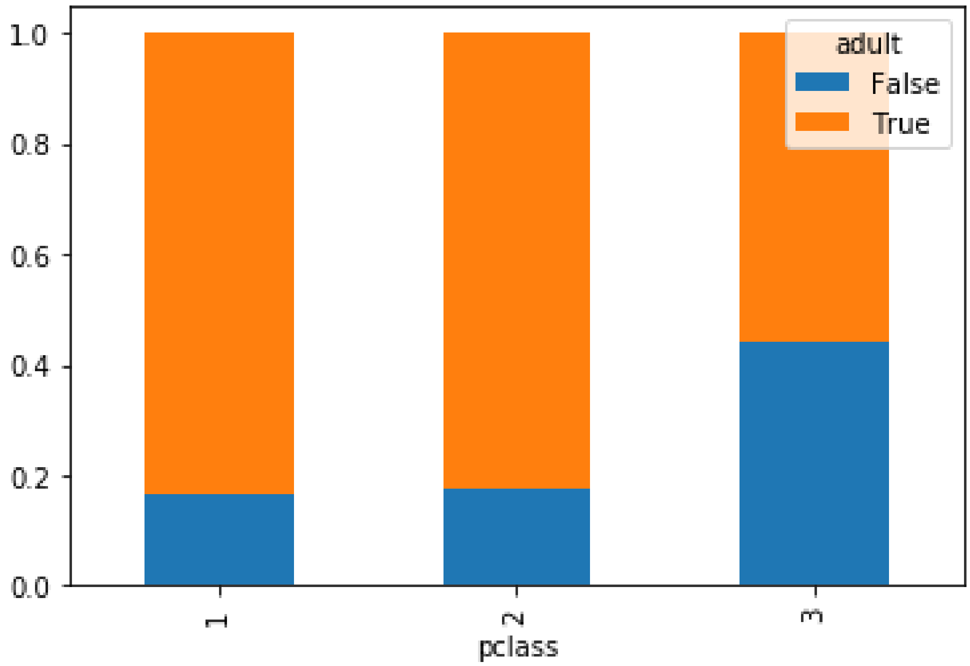


Because the conditional distribution of adult is (approximately) the same, regardless of whether we are conditioning on sex = male or sex = female, we say that the two variables are (approximately) independent.

For an example of two non-independent variables, consider passenger class and age group. If we look at the conditional distributions of adult given pclass, they are not all the same:

```
In [33]: adult_pclass_counts = pd.crosstab(titanic_df.pclass, titanic_df.adult)
         (adult_pclass_counts.divide(
             adult_pclass_counts.sum(axis=1), axis=0)
         ).plot.bar(stacked=True)
         adult_pclass_counts
```

```
Out[33]: adult   False  True
         pclass
         1          54   269
         2          49   228
         3         314   395
```



The conditional distribution of `adult` given `pclass` = 3 is quite different from the other two conditional distributions. Because the conditional distributions are not all equal, the two variables are *not* independent. Note that it only takes *one* conditional distribution to be off to render two variables *not* independent.

1.1 The Joint Distribution Assuming Independence

What would the joint distribution of passenger class (`pclass`) and age group (`adult`) be, if the two variables were independent? If two variables are independent, then their joint distribution is the product of the marginal distributions. That is,

- $P(\text{1st class and adult}) = P(\text{1st class}) \cdot P(\text{adult})$
- $P(\text{2nd class and adult}) = P(\text{2nd class}) \cdot P(\text{adult})$
- $P(\text{3rd class and adult}) = P(\text{3rd class}) \cdot P(\text{adult})$
- $P(\text{1st class and not adult}) = P(\text{1st class}) \cdot P(\text{not adult})$
- $P(\text{2nd class and not adult}) = P(\text{2nd class}) \cdot P(\text{not adult})$
- $P(\text{3rd class and not adult}) = P(\text{3rd class}) \cdot P(\text{not adult})$

We can calculate the marginal distributions:

In [22]:
```
# Calculate the total number of passengers.
N = adult_pclass_counts.sum().sum()

# Calculate the marginal distribution of adult by summing over pclass.
adult = adult_pclass_counts.sum(axis=0) / N
adult
```

```
Out[22]: adult
         False    0.318564
         True     0.681436
         dtype: float64
```

```
In [6]: # Calculate the marginal distributio nof pclass by summing over adult.
        pclass = adult_pclass_counts.sum(axis=1) / N
        pclass
```

```
Out[6]: pclass
        1    0.246753
        2    0.211612
        3    0.541635
        dtype: float64
```

How do we multiply these two distributions to get a 3×2 table of the joint distribution, assuming independence? We can use matrix multiplication. We can think of one `Series` as a matrix with 1 column and the other as a matrix with 1 row. Multiplying the two matrices using the usual definition of matrix multiplication gives the desired joint proportions.

$$\mathbf{u}\mathbf{v}^T = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \begin{pmatrix} v_1 & v_2 \end{pmatrix} = \begin{pmatrix} u_1v_1 & u_1v_2 \\ u_2v_1 & u_2v_2 \\ u_3v_1 & u_3v_2 \end{pmatrix}$$

This is an operation in linear algebra known as an **outer product**. To calculate the outer product of two `numpy` arrays, we can use the function `np.outer`:

```
In [7]: np.outer(pclass, adult)
```

```
Out[7]: array([[ 0.07860665,  0.1681466 ],
               [ 0.06741189,  0.14420002],
               [ 0.17254525,  0.36908959]])
```

Note that this returns a plain `numpy` `array` instead of a `pandas` `DataFrame`. It turns out that this will be good enough for our purposes.

1.2 Measuring Distance from Independence

We now have, for every combination of our two variables, two proportions:

- the proportion that was actually observed, $P(A \text{ and } B)$
- the proportion that we would expect assuming independence, $P(A)P(B)$

To measure the relationship between two variables, we calculate how far the observed proportions are from what we would expect if the variables were independent. It turns out that there are several ways to calculate the "distance" between two distributions.

Total Variation Distance

Total variation distance is probably the first distance metric that comes to mind. We calculate the difference and take absolute values before summing so that negative errors don't cancel out positive ones (the motivation for taking absolute values is the same as in MAD, which we learned in Chapter 1):

$$TV = \sum_{\text{A, B}} \left| P(A \text{ and } B) - P(A)P(B) \right|.$$

```
In [24]: joint = adult_pclass_counts / N
         expected = np.outer(pclass, adult)

         # Total Variation Distance
         (joint - expected).abs().sum().sum()
```

```
Out[24]: 0.26933009470195468
```

Unfortunately, differences turn out to be a bad way to measure distances between proportions. For example, most people would agree that the difference between 0.42 and 0.41 is insignificant, but the difference between 0.01 and 0.00 is vast. But total variation distance treats both differences the same.

Chi-Square Distance

Chi-square distance solves the problem of total variation distance by dividing by the difference by expected proportion, effectively calculating the *relative* difference between the two proportions:

$$\chi^2 = \sum_{\text{A, B}} \frac{(P(A \text{ and } B) - P(A)P(B))^2}{P(A)P(B)}.$$

```
In [9]: (((joint - expected) ** 2) / expected).sum().sum()
```

```
Out[9]: 0.084171579418509584
```

You might be familiar with the chi-square test from a previous statistics class. The chi-square distance is essentially the same as the chi-square test statistic, except for a normalizing constant.

Mutual Information

Another popular distance metric is **mutual information**. Whereas chi-square distance tends to be more popular among statisticians, mututal information tends to be more popular among engineers. (It arises from a field called *information theory*.)

$$I = \sum_{\text{A, B}} P(A \text{ and } B) \log \left(\frac{P(A \text{ and } B)}{P(A)P(B)} \right)$$

```
In [10]: (joint * np.log(joint / expected)).sum().sum()
```

```
Out[10]: 0.043760484527146329
```

There is no best distance metric for measuring departures from independence. All three distance metrics above are used in practice. The distances themselves can also be difficult to interpret. But the distance metric can give a rough sense of how closely two variables are related.

2 Exercises

The following exercise deals with the Tips data set (`https://raw.githubusercontent.com/dlsun/data-science`

Exercise 1. Report a measure of the strength of the relationship between the size of the party and the day of the week.

```
In [11]: tips = pd.read_csv("https://raw.githubusercontent.com/dlsun/data-science-book/master/c
```

```
In [20]: tips_counts = pd.crosstab(tips["size"], tips["day"])
         tips_counts
```

```
Out[20]: day   Fri  Sat  Sun  Thur
         size
         1       1    2    0     1
         2      16   53   39    48
         3       1   18   15     4
         4       1   13   18     5
         5       0    1    3     1
         6       0    0    1     3
```

```
In [29]: tips_size = tips_counts.sum(axis=1) / len(tips)
         tips_size
```

```
Out[29]: size
         1    0.016393
         2    0.639344
         3    0.155738
         4    0.151639
         5    0.020492
         6    0.016393
         dtype: float64
```

```
In [30]: tips_day = tips_counts.sum(axis=0) / len(tips)
         tips_day
```

```
Out[30]: day
         Fri     0.077869
         Sat     0.356557
         Sun     0.311475
         Thur    0.254098
         dtype: float64
```

```
In [38]: tips_expected = np.outer(tips_size, tips_day)
         tips_joint = tips_counts / len(tips)

         (tips_joint - tips_expected).abs().sum().sum()
```

```
Out[38]: 0.25403117441547973
```

```
In [39]: (((tips_joint - tips_expected) ** 2) / tips_expected).sum().sum()
```

Out[39]: 0.12144610629885125

```
In [40]: (tips_joint * np.log(tips_joint / tips_expected)).sum().sum()
```

```
/opt/conda/lib/python3.6/site-packages/ipykernel_launcher.py:1: RuntimeWarning: divide by zero
  """Entry point for launching an IPython kernel.
```

Out[40]: 0.066137592765100339