Bio 417 Homework 3

(for the week of Feb 25th, due Mar 25th)

1. (Limiting similarity) Consider the Lotka-Volterra competition equation:

$$\frac{dN_1}{dt} = r_1 N_1 \left(\frac{K_1 - N_1 - \alpha_{12} N_2}{K_1} \right) \tag{1}$$

$$\frac{dN_2}{dt} = r_2 N_2 \left(\frac{K_2 - N_2 - \alpha_{21} N_1}{K_2} \right) , \tag{2}$$

where α_{12} is the competitive effect of species 2 on 1, and α_{21} is the competitive effect of 1 on 2.

a) Write down the Jacobian matrix, calculate its trace and determinant, and derive the conditions for the internal equilibrium to be stable.

The Jacobian at the internal equilibrium is:

$$J = \frac{1}{\alpha_{12}\alpha_{21} - 1} \begin{pmatrix} r_1(K_1 - \alpha_{12}K_2)/K_1 & r_1\alpha_{12}(K_1 - \alpha_{12}K_2)/K_1 \\ r_2\alpha_{21}(K_2 - \alpha_{21}K_1)/K_2 & r_2(K_2 - \alpha_{21}K_1)/K_2 \end{pmatrix} \tag{3}$$

Stability requires that the trace is negative and determinant positive:

$$\text{Tr}(J) = \frac{r_1(K_1 - \alpha_{12}K_2) + r_2(K_2 - \alpha_{21}K_1)}{(\alpha_{12}\alpha_{21} - 1)K_1K_2} = -r_1\frac{N_1^*}{K_1} - r_2\frac{N_2^*}{K_2} ,$$

where $N_1^* = \frac{K_1 - \alpha_{12}K_2}{1 - \alpha_{12}\alpha_{21}}$ and $N_2^* = \frac{K_2 - \alpha_{21}K_2}{1 - \alpha_{12}\alpha_{21}}$ are the internal equilibria found by setting equations 1 and 2 equal to zero. Hence, whenever there is a positive internal equilibrium (both N_1^* and N_2^* are positive), the trace of the Jacobian is negative. The determinant is:

$$\det(J) = \frac{r_1r_2(K_1 - \alpha_{12}K_2)(K_2 - \alpha_{21}K_1)}{(1 - \alpha_{12}\alpha_{21})K_1K_2} = r_1r_2\frac{N_1^*}{K_1}\frac{N_2^*}{K_2}(1 - \alpha_{12}\alpha_{21}) ,$$

Hence, the internal equilibrium is only stable if $\alpha_{12}\alpha_{21} < 1$.

b) Suppose that $K_2 > K_1$ and that $\alpha_{12} = \alpha_{21} = \chi(d)$, where $\chi(d)$ is a strictly decreasing function of a variable d, which measures the difference in the ecological requirements of the two species (e.g., d might be the difference between the optimal soil temperature for growth of two plant species). In other words, the competitive effect of the two species on each other is symmetric, and the more different they are, the less competition they experience from each other. What is the condition for the two species to coexist stably (i.e., for the internal equilibrium to be stable)?

For coexistence, we need two things: both N_1^* and N_2^* to be positive, and $1-\chi(d)^2 > 0$, i.e., $\chi(d) < 1$. Since $K_2 > K_1$, we only have to check $N_1^* > 0$:

$$K_1 - \chi(d)K_2 > 0 \, .$$

This means there is the two species have to be differentiated from each other at least to the minimum d value given by the above inequality (since $\chi(d)$ is strictly decreasing). This is the concept of limiting similarity: two very similar species cannot coexist stably with each other.

2. (Community stability) Consider again the symmetric Lotka-Volterra competition as above, with $\alpha_{12} = \alpha_{21} = \alpha$. To make things simpler, assume $r_1 = r_2 = K_1 = K_2 = 1$.

a) Show that in this special case, the condition for stable co-existence is $\alpha < 1$.

Straightforward from question 1a) above

b) Now imagine a third species, that competes with both of the existing species. In particular, assume that the competition coefficient for the third species is $\alpha_{13} = \alpha_{31} = \alpha_{23} = \alpha_{32} = \beta$, and that $K_3 = r_3 = 1$. Write down the new system of equations.

$$\frac{dN_1}{dt} = N_1 \left(1 - N_1 - \alpha N_2 - \beta N_3\right) \tag{4}$$
$$\frac{dN_2}{dt} = N_2 \left(1 - N_2 - \alpha N_1 - \beta N_3\right) \tag{5}$$
$$\frac{dN_3}{dt} = N_3 \left(1 - N_3 - \beta N_1 - \beta N_2\right) \, , \tag{6}$$

c) Now, write the Jacobian for this three-dimensional system, and evaluate the Jacobian at the equilibrium where $N_3 = 0$ but $N_1, N_2 > 0$ (which corresponds to the internal equilibrium for the two-dimensional case in part (a).)

The Jacobian evaluated at the internal equilibrium with N_1 and N_2 positive is:

$$J = \frac{1}{1+\alpha} \begin{pmatrix} -1 & -\alpha & -\beta \\ -\alpha & -1 & -\beta \\ 0 & 0 & \alpha - 2\beta + 1 \end{pmatrix} \tag{7}$$

d) Show that the condition for the third species to invade (increase from zero) is:

$$\frac{1+\alpha}{2} > \beta\,.$$

(Hint, you don't have to [but can] use the Jacobian to answer this question.)

The eigenvalues of the Jacobian matrix above are:

$$-1, \qquad -\frac{1-\alpha}{1+\alpha} \qquad \frac{1+\alpha-2\beta}{1+\alpha}\,,$$

of which the first two are guaranteed to be negative, and the third one is negative only when $1+\alpha < 2\beta$. If the third eigenvalue is positive, that means the equilibrium with species 3 at zero is unstable, so it can invade. In biological terms, the upper limit for the newly invading species' competition coefficient with the existing ones is lower than the first two species' coexistence condition (i.e., 1).

The other way to arrive at this condition is to simply write down the condition for $dN_3/dt > 0$ when $N_3 \approx 0$, which amounts to the term in the parenthesis in the third equation of the system being positive.