Bio 417 Homework 2

for the weeks of Feb 11th and 18th, due Feb 25th

1. (Life history evolution) Suppose you have the a stage structured population, whose dynamics are described by the projection matrix:

$$A = \begin{pmatrix} 0 & 1.5 & 2.5 \\ 0.5 & 0.2 & 0.5 \\ 0 & 0.4 & 0.1 \end{pmatrix}$$

The answers are given in the Jupyter Notebook

a) Calculate the asymptotic growth rate, stable stage distribution, and reproductive values of the classes.

b) Calculate the matrix of sensitivities of the leading eigenvalue to each element.

c) Suppose that individuals in age class 3 can trade-off investment into survival and reproduction, such that the elements a_{13}, a_{23}, and a_{33} tradeoff as follows:

$$a_{23} = 5a_{33}$$
$$a_{13} = 2.55 - a_{33}/2$$

Given your answer in b), do you expect this species evolve to increase or decrease its reproductive output in age-class 3?

2. (Separation of variables) Take equation (17) in the lecture notes for stochastic population dynamics, and show how you can achieve the separation of variables in equation (18). (Hint: consider defining two functions $A(x)$ and $B(x)$ for which $\frac{d\ln(A(x)}{dx}$ and $\frac{d\ln(B(x)}{dx}$ gives the require forms.)

Rewrite (17) as:

$$du = \frac{dz}{(bz-d)(1-z)} \, ,$$

Let's define $A(z) = bz - d$ and $B(z) = 1 - z$, and see if we can separate the fraction into two fractions of the form $K_1/A(z) - K_2/B(z)$, where K_1 and K_2 are constants independent of z or u. If we can, then we are done, since $1/A(z)$ where $A(z)$ is linear in z, is proportional to $d\ln(A(z))/dz$:

$$\frac{K_1}{bz-d} - \frac{K_2}{1-z} = \frac{K_1(1-z) - K_2(bz-d)}{(bz-d)(1-z)} \, .$$

If we set $K_1 = b/(b-d)$ and $K_2 = 1/(b-d)$, the numerator will equal to 1. Hence, we can write

$$du = \frac{dz}{(bz-d)(1-z)} = \frac{1}{b-d}\left[\frac{b}{bz-d} - \frac{1}{1-z}\right] dz \ ,$$

which is equivalent to equation (18).

3. (Sheep on an island). On an island, some settlers have introduced 100 sheep. This breed of sheep has a rate of birth $b = 0.1$ and death $d = 0.05$.

a) Compute the mean and variance of the population size at time t using the PGFs.

Remember that if $F(z)$ is the PGF of the random variable X, then the expectation of X, $\mathrm{E}(X)$ is

$$\mathrm{E}(X) = F'(1) = n_0 e^{(b-d)t} = 100e^{0.05t}$$

Likewise, the variance of X is:

$$\begin{aligned} \mathrm{var}(X) &= F''(1) + F'(1) - F''(1)^2 \\ &= \frac{(b+d)e^{(b-d)t}(e^{(b-d)t}-1)n_0}{b-d} \\ &= 300e^{0.05t}(e^{0.05t}-1) \end{aligned}$$

b) Compute the time you should wait to be 95% sure that the population size has doubled. (Hint: consider each initial sheep as the initiator of a separate birth-death process, and use the central limit theorem.[1])

Let's consider the descendants of each animal separately (we can use the expressions above, setting $n_0 = 1$ instead of 100). Then, the number of descendants of each animal is a random variable with mean $e^{0.05t}$ and variance $3e^{0.05t}(e^{0.05t}-1)$. Since the total population size the sum of 100 of these independent identical RVs, CLT implies that it is distributed as a normal distribution with mean $100e^{0.05t}$ and variance $300e^{0.05t}(e^{0.05t}-1)$. The cumulative density function of the normal distribution (defined as $F(x) = P(X \leq x)$ is:

$$F(x) = \frac{1}{2}(1 + \mathrm{Erf}((\mu - x)/\sqrt{2}\sigma) \ ,$$

[1]The central limit theorem states that sums of n independent, identical random variables with mean μ and variance σ^2 are distribution tends to a normal distribution with mean $n\mu$ and variance $n\sigma^2$.

where μ and σ are the mean and standard deivation (square root of the variance), respectively, and Erf is the error function. We want $F(200)$ to be 0.05 (i.e., only 5% chance that the population size is below 200). Substituting μ, σ, the parameters, and solving for t, we get $t \approx 18.9$.
Alternatively, you can also compute the standard normal deviate $Z = (200 - \mu)/\sigma$ and look up the Z that gives the one-tailed probability 0.05, and solve for t.
To check the value we found is a good approximation, you can run an ensemble of populations using the function "gillespie" in the ipython notebook, and count up how many are below 200.

4. (Quasi-stationary distribution in the discrete time birth-death with extinction) Take the discrete-time birth death process provided in the iPython notebook.

See iPython notebook for the answers.

a) Modify the transition matrix such that the "birth probability at zero" (i.e., the probability a population transitions from 0 to 1) is zero. (Hint: use Kroenecker deltas again; make sure your matrix stays stochastic.) Set d=0.3. Leave the rest of the parameters as before.

b) Calculate the eigenvalues and eigenvectors of this new matrix, and take the eigenvectors corresponding to the largest 2 eigenvalues (Hint: use np.argsort(); look it up at numpy documentation)

c) Set the zeroth element of $\vec{\pi}_2$ (the eigenvector associated with the second largest eigenvalue) equal to zero, and normalize the resulting vector (make it sum to 1). Plot it together with $\vec{\pi}_1$. What do you observe? What is the interpretation of this re-normalized vector?

d) Now modify the discrete time simulation (function discretes) so that the birth probability when pop=0 is zero (again, use Kroenecker delta). Simulate an ensemble of 200 populations for 100 time steps starting with 10 individuals, and plot the histogram of population sizes. Also plot π_1 and the re-normalized vector you constructed in part c). What do you observe?